Calculus

0 Precalculus

Algebra

Fundamental Theorem of Algebra - An nth degree polynomial has n (not necessary distinct) zeroes. Although all zeroes may be complex, a real polynomial of odd degree must have at least one real zero.

Rational Zero Theorem - If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every rational root of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Binomial Theorem - $(x+y)^n = \Sigma_{i=0}^{n} \binom{n}{i} x^{n-i} y^i = \Sigma_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$

Geometry

Heron's Formula - $A_\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$

Law of sines - $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2A_\Delta}{abc}$

Law of cosines - $c^2 = a^2 + b^2 - 2ab\cos C$

Sector of circle - $A = \frac{1}{2}\theta r^2$ and arc length $s = r\theta$

Ellipse - $A = \pi ab$ and circumference $C \approx 2\pi\sqrt{\frac{a^2+b^2}{2}}$

Right circular cone - $V_{cone} = \frac{1}{3}A_{base}h$ and lateral surface area $S = \pi r\sqrt{r^2+h^2}$

Frustum of right circular cone - $V = \frac{1}{3}\pi(R^2 + Rr + r^2)h$ and lateral surface area $S = \pi s(R+r)$

Trigonometry

Trigonometric identities

Definitions	$\sin\theta = \frac{opp}{hyp} = \frac{y}{r}$	$\cos\theta = \frac{adj}{hyp} = \frac{x}{r}$	$\tan\theta = \frac{opp}{adj} = \frac{y}{x}$
Reciprocal	$\sin x \csc x = 1$	$\cos x \sec x = 1$	$\tan x \cot x = 1$
Quotient	$\tan x = \frac{\sin x}{\cos x}$	$\cot x = \frac{\cos x}{\sin x}$	
Pythagorean	$\sin^2 x + \cos^2 x = 1$	$\tan^2 x + 1 = \sec^2 x$	$1 + \cot^2 x = \csc^2 x$
Cofunction	$\sin(x - \frac{\pi}{2}) = \cos x$	$\cos(x - \frac{\pi}{2}) = \sin x$	$\tan(x - \frac{\pi}{2}) = \cot x$
	$\csc(x - \frac{\pi}{2}) = \sec x$	$\sec(x - \frac{\pi}{2}) = \csc x$	$\cot(x - \frac{\pi}{2}) = \tan x$
Even / odd	$\sin(-x) = -\sin x$	$\cos(-x) = \cos x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$	$\sec(-x) = \sec x$	$\cot(-x) = -\cot x$
Sum / difference	$\sin(x \pm y) =$	$\cos(x \pm y) =$	$\tan(x \pm y) =$
	$\sin x \cos y \pm \cos x \sin y$	$\cos x \cos y \mp \sin x \sin y$	$\frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
Double-angle	$\sin 2x = 2\sin x \cos x$	$\cos 2x = \cos^2 x - \sin^2 x$	$\tan x = \frac{2\tan x}{1-\tan^2 x}$
Power-reducing	$\sin^2 x = \frac{1-\cos 2x}{2}$	$\cos^2 x = \frac{1+\cos 2x}{2}$	$\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$
Sum-to-product	$\sin x \pm \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$ or $2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$		
	$\cos x \pm \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$ or $-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$		
Product-to-sum	$\sin x \sin y$ or $\cos x \cos y = \frac{1}{2}[\cos(x-y) \mp \cos(x+y)]$		
	$\sin x \cos y$ or $\cos x \sin y = \frac{1}{2}[\sin(x-y) \pm \sin(x+y)]$		
Limits	$\lim_{x\to 0}\frac{\sin x}{x} = 1$	$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$	$\lim_{x\to 0}\frac{\tan x}{x} = 1$

1 Limits and continuity

Limit - $[\lim_{x\to c} f(x)]$ The value that $f(x)$ approaches as x approaches some number c from both sides

Epsilon-delta definition	$\lim_{x\to c} f(x) = L$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < \lvert x-c\rvert < \delta$, then $\lvert f(x)-L\rvert < \varepsilon$
Sum / difference property	$\lim_{x\to c}[f(x) \pm g(x)] = [\lim_{x\to c} f(x)] \pm [\lim_{x\to c} g(x)]$
Product property	$\lim_{x\to c}[f(x) \cdot g(x)] = [\lim_{x\to c} f(x)] \cdot [\lim_{x\to c} g(x)]$
Quotient property	$\lim_{x\to c}[f(x) \,/\, g(x)] = [\lim_{x\to c} f(x)] \,/\, [\lim_{x\to c} g(x)]$
Exponent property	$\lim_{x\to c}[f(x)^n] = [\lim_{x\to c} f(x)]^n$
Constant multiple property	$\lim_{x\to c}[cf(x)] = c\,[\lim_{x\to c} f(x)]$
Composition property	$\lim_{x\to c}[f(g(x))] = f(\lim_{x\to c} g(x))$ iff $f(x)$ is continuous at $x = c$
One-sided limit	$\lim_{x\to c^\pm} f(x)$ (The value of $f(x)$ approached from one side only)
Unbounded limits	$\lim_{x\to c} f(x) = \pm\infty$, represented by a vertical asymptote at $x = c$
Limits at infinity	$\lim_{x\to\pm\infty} f(x) = c$, represented by a horizontal asymptote at $y = c$
Rational limits at infinity	Let $f(x) = \frac{ax^m+cx^n+\ldots}{bx^n+\ldots}$, where a and b are leading coefficients, then
	$\lim_{x\to\pm\infty} f(x) = \pm\infty$ if $m > n$
	$\lim_{x\to\pm\infty} f(x) = ax + c$ if $m = n+1$
	$\lim_{x\to\pm\infty} f(x) = \frac{a}{b}$ if $m = n$
	$\lim_{x\to\pm\infty} f(x) = 0$ if $m < n$
Squeeze theorem (sandwich)	If $f(x) \le g(x) \le h(x)$ over an interval, and at some point c in the interval $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$

>	Direct substitution $\lim_{x\to c} f(x) = f(c)$	>	Limit found (likely) Indeterminate form Vertical asymptote (likely)	>	Factorization Rationalization Trigonometric Identities L'Hopital's Rule $[\lim\limits_{x\to c} \frac{f(x)}{g(x)} = \lim\limits_{x\to c} \frac{f'(x)}{g'(x)}]$ Graphical / Numerical Approximations	>

Continuity and discontinuities

Continuity at a point c	iff $\lim_{x\to c} f(x) = f(c)$
Continuity over an open interval (a, b)	iff f is continuous on every point in (a, b)
Continuity over a closed interval $[a, b]$	iff f is continuous over $(a, b), \lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$
Point discontinuity (removable)	$\lim_{x\to c} f(x) \,\exists$ but $\lim_{x\to c} f(x) \neq f(c)$
Jump discontinuity	$\lim_{x\to c} f(x) \nexists$ because $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$
Asymptotic discontinuity (infinite)	$\lim_{x\to c} f(x) \nexists$ because it is unbounded

2 Differential Calculus

Derivative - The instantaneous rate of change of a function (the slope of the tangent line) at a point

Leibniz's notation	$\frac{dy}{dx}, \frac{d^n y}{dx^n}, \frac{df}{dx}, \frac{d}{dx}[f(x)]$ (differential notation)
Lagrange's notation	$y', y'', f'(x), f''(x), f^{(n)}(x)$
Euler's notation	$D_x y, D_x f(x), D_x^n y$
Newton's notation	$\dot{y}, \ddot{y}, \dot{f}, \overset{(n)}{\dot{f}}$ (most commonly used in physics)
Definition	$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$
Differentiability	The derivative $\nexists$ at discontinuities, cusps, or vertical tangent lines Differentiability implies continuity, but not the converse
Constant rule	$\frac{d}{dx}[c] = 0$
Constant multiple rule	$\frac{d}{dx}[cf] = cf'$
Power rule	$\frac{d}{dx}[x^n] = nx^{n-1}$
Sum / difference rule	$\frac{d}{dx}[f \pm g] = f' \pm g'$
Product rule	$\frac{d}{dx}[fg] = f'g + fg'$
Quotient rule	$\frac{d}{dx}[\frac{f}{g}] = \frac{gf'-fg'}{g^2}$
Chain rule	$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = \frac{df}{dg}\frac{dg}{dx}$
Implicit differentiation	$\frac{d}{dx}[f(y)] = f'(y)\frac{dy}{dx}$ (an application of the chain rule)
Related rates	$\frac{d}{dt}[f(x)] = \frac{df}{dx}\frac{dx}{dt}$ (an application of the chain rule)
Trig functions	$\frac{d}{dx}[\sin x] = \cos x$ $\frac{d}{dx}[\cos x] = -\sin x$ $\frac{d}{dx}[\tan x] = \sec^2 x$ $\frac{d}{dx}[\cot x] = -\csc^2 x$ $\frac{d}{dx}[\sec x] = \sec x \tan x$ $\frac{d}{dx}[\csc x] = -\csc x \cot x$ $\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$ $\frac{d}{dx}[\text{arccot}\, x] = -\frac{1}{1+x^2}$ $\frac{d}{dx}[\text{arcsec}\, x] = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ $\frac{d}{dx}[\text{arccsc}\, x] = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
Exponential functions	$\frac{d}{dx}[e^x] = e^x$ $\frac{d}{dx}[a^x] = \ln a \cdot a^x$
Logarithmic functions	$\frac{d}{dx}[\ln x] = \frac{1}{x}$ $\frac{d}{dx}[\log_a x] = \frac{1}{\ln a \cdot x},\ \{a \mid a > 0 \text{ and } a \neq 1\}$
Inverse functions	$f'(x)f^{-1\prime}(y) = 1$ $f(x) : x \mapsto y$ and $f^{-1}(x) : y \mapsto x$
Average rate of change	$\frac{\Delta y}{\Delta x} = \frac{f(b)-f(a)}{b-a}$ (slope of the secant line; approximation of the derivative)
Local linearization	$f(x) \approx f(c) + f'(c)(x-c)$
Local linearity	Whether a tangent line can closely approximate the function over a sufficiently small interval around a point (a function is differentiable at $x = c$ iff it is locally linear at $x = c$)
Newton's Method	Approximating a real zero through repeated iterations of $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ (Converges if $\left\lvert \frac{f(x)f''(x)}{f'(x)^2} \right\rvert < 1$ on an open interval containing the zero)
Optimization	The process of finding quantities that maximize or minimize a function
Motion	Finding position, velocity, and acceleration as $x(t),\ v(t) = \frac{dx}{dt}$, and $a(t) = \frac{d^2x}{dt^2}$

Calculus-based justification - Analysis of properties of a function using its derivative

Interval of increase	$f'(x) > 0$
Interval of decrease	$f'(x) < 0$
Critical points	Defined points in the domain of $f(x)$ where $f'(x) = 0$ or undefined (includes all global extrema, local extrema, and other points)
Absolute extrema	The maximum and minimum values of a function (may also occur at the endpoints of a closed interval) $f(c) \geq f(x)$ or $f(c) \leq f(x)$ $\forall x$
Relative extrema	The maximum and minimum values of an interval (includes all absolute extrema and endpoints of intervals) $f(c) \geq f(x)$ or $f(c) \leq f(x)$ $\forall x$ on an open interval containing c
First derivative test	Finding relative extrema and absolute extrema at critical points using $f'(x)$ - $f'(x)$ changes from positive to negative at a relative maximum - $f'(x)$ changes from negative to positive at a relative minimum - $f'(x)$ has no sign changes at a non-extremum critical number - Absolute extrema are found by comparing all relative extrema and end-points in the interval
Concave upwards	Intervals where $f'(x)$ is increasing and $f''(x) > 0$
Concave downwards	Intervals where $f'(x)$ is decreasing and $f''(x) < 0$
Inflection points	Defined points in the domain of $f(x)$ where $f(x)$ changes concavity ($f''(x) = 0$ or undefined and $f''(x)$ changes signs)
Second derivative test	Finding relative extrema and absolute extrema at critical points using $f''(x)$ (Given that $f'(c) = 0$ and exists in neighborhood around $x = c$) - $f(x)$ has a relative maximum at $x = c$ if $f''(c) < 0$ - $f(x)$ has a relative minimum at $x = c$ if $f''(c) > 0$ - $f(x)$ is inconclusive (potential inflection point) if $f''(c) = 0$ or undefined

Existence theorems - Theorems that guarantee a certain type of point must exist under certain conditions

Intermediate value theorem	Suppose f is continuous on $[a, b]$ For any $\{L \mid f(a) \leq L \leq f(b)\}$, $\exists\, c \in [a, b]$ for which $f(c) = L$
Extreme value theorem	Suppose f is continuous on $[a, b]$ $\exists\, c, d \in [a, b]$ where $f(c) \leq f(x) \leq f(d)$ $\forall x \in [a, b]$
Mean value theorem	Suppose f is continuous on $[a, b]$ and differentiable on (a, b) $\exists\, c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$
Rolle's theorem	Suppose f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$ $\exists\, c \in (a, b)$ such that $f'(c) = 0$

3 Integral Calculus

Riemann sum - The approximation of the area under a curve as the sum of many rectangles

Width $\Delta x = \frac{b-a}{n}$, height $f(x_i)$, endpoint $x_i = a + i\Delta x$

Left Riemann sum	$\sum_{i=1}^{n} f(x_{i-1})\Delta x$	(top-left corners)
Right Riemann sum	$\sum_{i=1}^{n} f(x_i)\Delta x$	(top-right corners)
Midpoint sum	$\sum_{i=1}^{n} f(\frac{x_{i-1}+x_i}{2})\Delta x$	(midpoints)
Trapezoidal sum	$\sum_{i=1}^{n} \frac{f(x_{i-1})+f(x_i)}{2}\Delta x$	(both endpoints)

Definite integrals - [$\int_a^b f(x)\ dx$] The signed area bounded by the curve of a function $f(x)$ and the x-axis, from lower bound $x = a$ to upper bound $x = b$

Definition	$\int_a^b f(x)\ dx = F(b) - F(a)$
Riemann sum	$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\ \Delta x = \lim_{\lVert\Delta\rVert\to 0} \sum_{i=1}^{n} f(c_i)\ \Delta x$
Interval addition property	$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$
Zero-length interval property	$\int_a^a f(x)\ dx = 0$
Reverse interval property	$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
Sum / difference property	$\int_a^b f(x) \pm g(x)\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$
Constant multiple property	$\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$
Improper integrals	Definite integrals over an unbounded area (convergent / divergent) $\int_a^\infty f(x)\ dx = \lim_{n\to\infty}[F(x)]_a^n$ $\int_{-\infty}^\infty f(x)\ dx = \lim_{n\to\infty} F(n) - \lim_{m\to-\infty} F(m)$ $\int_a^b f(x)\ dx = \lim_{n\to c^-}[F(x)]_a^n + \lim_{m\to c^+}[F(x)]_m^b$ (Vertical asymptote in the interval of integration)
Accumulation	$\int_a^b f'(x)\ dx = f(b) - f(a)$ Integration is the accumulation of net change in the described quantity
Cross-sectional volume	$V := \begin{cases} \int_a^b\ A(x)\ dx, & A(x) \perp \text{(x-axis)} \\ \int_a^b\ A(y)\ dy, & A(y) \perp \text{(y-axis)} \end{cases}$
Solid of revolution	Solid constructed by revolving a function around a line Axis of revolution $y = c$ | Axis of revolution $x = c$
Disk method	$\int_a^b\ \pi[f(x) - c]^2\ dx$ | $\int_a^b\ \pi[f(y) - c]^2\ dy$
Washer method	Generalization of the disk method for two functions $\int_a^b\ \pi\lvert(f(x) - c)^2 - (g(x) - c)^2\rvert\ dx$ | $\int_a^b\ \pi\lvert(f(y) - c)^2 - (g(y) - c)^2\rvert\ dy$
Shell method	Volumes of implicit functions as hollow cylinders with surface area $\int_a^b\ 2\pi\,\lvert y - c\rvert \cdot f(y)\ dy$ | $\int_a^b\ 2\pi\,\lvert x - c\rvert \cdot f(x)\ dx$ $\int_a^b\ 2\pi\,\lvert y - c\rvert \cdot \lvert f(y) - g(y)\rvert\ dy$ | $\int_a^b\ 2\pi\,\lvert x - c\rvert \cdot \lvert f(x) - g(x)\rvert\ dx$
Arc length	$s = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\ dx$ for a smooth curve on the interval $[a, b]$

Area between two curves	$A = \int_{x_1}^{x_2} f(x) - g(x)\ dx$ for continuous f and g, $f(x) \geq g(x)\ \forall x \in [x_1, x_2]$ $A = \int_{y_1}^{y_2} f(y) - g(y)\ dy$ for continuous f and g, $f(y) \geq g(y)\ \forall y \in [y_1, y_2]$ (x_1, y_1) and (x_2, y_2) are either adjacent points of intersection or points on the specified boundary lines

Indefinite integral - $[\int f(x)\ dx = F(x) + C]$ The family of antiderivatives for $f(x)$, where $F'(x) = f(x)$

Antiderivative	$[F(x) = \int f(x)\ dx]$ Differentiable function $F(x)$ where $F'(x) = f(x)$	
Integrability	Continuity of $f(x)$ implies integrability of $f(x)$ $(\exists\ F(x))$	
Sum / difference property	$\int f(x)g(x)\, dx = \int f(x)\, dx \pm \int g(x)\, dx$	
Constant multiple property	$\int cf(x)\, dx = c\int f(x)\, dx$	
Reverse power rule	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$	
Trig functions	$\sin \mapsto -\cos$	$\cos \mapsto \sin$
	$\tan \mapsto \ln\lvert\sec\rvert,\ -\ln\lvert\cos\rvert$	$\cot \mapsto \ln\lvert\sin\rvert$
	$\sec \mapsto \ln\lvert\sec + \tan\rvert$	$\csc \mapsto -\ln\lvert\csc + \cot\rvert,\ \ln\lvert\csc - \cot\rvert$
	$\sec^2 \mapsto \tan$	$\csc^2 \mapsto -\cot$
	$\sec \cdot \tan \mapsto \sec$	$\csc \cdot \cot \mapsto -\csc$
	$\frac{dx}{\sqrt{a^2-x^2}} \mapsto \arcsin\frac{x}{a}$	or $-\arccos\frac{x}{a}$
	$\frac{dx}{a^2+x^2} \mapsto \frac{1}{a}\arctan\frac{x}{a}$	or $-\frac{1}{a}\operatorname{arccot}\frac{x}{a}$
	$\frac{dx}{x\sqrt{x^2-a^2}} \mapsto \frac{1}{a}\operatorname{arcsec}\frac{\lvert x\rvert}{a}$	or $-\frac{1}{a}\operatorname{arccsc}\frac{\lvert x\rvert}{a}$
Exponential functions	$\int e^x\, dx = e^x + C$	$\int a^x\, dx = \frac{1}{\ln a}a^x + C$
Logarithmic functions	$\int \ln x\, dx = x\ln x - x + C$	$\int \frac{1}{x}\, dx = \ln\lvert x\rvert + C$
u-substitution	(Reverse chain rule) Integration of a composite function $\int f(x)\, dx = \int w(u(x)) \cdot u'(x)\, dx = \int w(u)\, du = W(u(x)) + C$	
Integration by parts	(Reverse product rule) Integration of a product of functions $\int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx$ or $\int u\, dv = uv - \int v\, du$ $f(x)$ is typically logarithmic, polynomial, exponential, or trigonometric	
Partial fraction expansion	(Rational integrals) Long division and partial fraction decomposition	
Average value	For f continuous on $[a, b]$, $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\ dx = \frac{F(b)-F(a)}{b-a}$	
Mean value theorem	For f continuous on $[a, b]$, $\exists\ c \in [a, b]$ where $f(c) = \frac{1}{b-a}\int_a^b f(x)\ dx$	
Chain rule	$\frac{d}{dx}[F(g(x))] = \frac{d}{dx}\int_a^{g(x)} f(t)\ dt = f(g(x))g'(x)$	

Fundamental theorem of calculus - Relation between differentiation and integration (antidifferentiation)

Let f be a function continuous on $[a, b]$

If $F(x) = \int_a^x f(t)\ dt, x \in [a, b]$, then $F(x)$ is an antiderivative of $f(x)$ and $F'(x) = \frac{d}{dx}[\int_a^x f(t)\ dt] = f(x)$

If $F(x)$ is an antiderivative of $f(x)$, then $\int_a^b f(x)\ dx = F(b) - F(a)$

4 Differential equations

Differential equations - Equations that relate a function with one or more of its derivatives

Solution	A function, set of functions, or class of functions
Separable differential equations	$[\frac{dy}{dx} = f(x)g(y)$ or $\frac{f(x)}{g(y)}]$ Class of differential equations
Separation of variables	$\int f(x)\,dx = \int g(y)\,dy$ Method of solving differential equations for its general solution
Specific solutions	Initial conditions applied to general solutions
Slope fields	Graphical visualization of the set of general solutions
Euler's method	$f(x+\Delta x) \approx f(x) + f'(x,y)\Delta x$ Numerical approximations for particular solutions
Exponential models	$y = Ce^{kx}$ is the general solution to $\frac{dy}{dx} = ky$
Newton's Law of Cooling	$\frac{dT}{dt} = -k(T - T_{ambient})$
Logistic models	$y = \frac{L}{1+(\frac{L}{y_0}-1)e^{-kt}}$ is the general solution to $\frac{dy}{dt} = ky(1 - \frac{L}{y})$ with population y, growth rate k, carrying capacity L $\frac{dy}{dt}$ is largest when $y(t) = \frac{L}{2}$ (inflection point); $\lim_{t\to\infty} \frac{dy}{dt} = 0; \lim_{t\to\infty} y(t) = L$ Annual growth rate $A = (\frac{y_1}{y_0})^{1/(t_1-t_0)} - 1$ Instantaneous growth rate $r = \ln(A+1)$

5 Advanced functions

Inverse function - $[f^{(-1)\prime}(x)]$ A function that is strictly monotonic, one-to-one, and a reflection of the original function across the line $y = x$

Parametric function - $[f(t) = (x, y) = (x(t), y(t))]$ A function that defines a group of quantities as functions of one or more independent variables called parameters

Implicitization - Converting a parametric function into a single implicit equation by eliminating t

Derivative	$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
Second derivative	$\frac{d^2y}{dx^2} = \frac{d}{dt}[\frac{dy}{dx}] \cdot \frac{dt}{dx} = \frac{d}{dt}[\frac{dy}{dx}]/\frac{dx}{dt}$
Arc length	$s = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$

Inverse trig functions - The inverse of the six regular trig functions, on a restricted domain

$y = \arcsin x$ iff $x = \sin y$	$x \in [-1, 1]$	$y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$	
$y = \arccos x$ iff $x = \cos y$	$x \in [-1, 1]$	$y \in [0, \pi]$	
$y = \arctan x$ iff $x = \tan y$	$x \in (-\infty, \infty)$	$y \in (-\frac{\pi}{2}, \frac{\pi}{2})$	
$y = \operatorname{arccot} x$ iff $x = \cot y$	$x \in (-\infty, \infty)$	$y \in (0, \pi)$	$\cot^{-1} x = \tan^{-1}(1/x)$
$y = \operatorname{arcsec} x$ iff $x = \sec y$	$x \in (-\infty, -1] \cup [1, \infty)$	$y \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$	$\sec^{-1} x = \cos^{-1}(1/x)$
$y = \operatorname{arccsc} x$ iff $x = \csc y$	$x \in (-\infty, -1] \cup [1, \infty)$	$y \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$	$\csc^{-1} x = \sin^{-1}(1/x)$

Hyperbolic functions - Analogs of the trigonometric (circular) functions, based upon the unit hyperbola $x^2 - y^2 = 1$ formed by $(\cosh x, \sinh x)$

$\sinh x = \frac{e^x - e^{-x}}{2}$	$x \in \mathbb{R}$	$y \in \mathbb{R}$	$\frac{d}{dx} \mapsto \cosh x$	$\int \mapsto \cosh x$
$\cosh x = \frac{e^x + e^{-x}}{2}$	$x \in \mathbb{R}$	$y \in \mathbb{R}$ and $y \geq 1$	$\frac{d}{dx} \mapsto \sinh x$	$\int \mapsto \sinh x$
$\tanh x = \frac{\sinh x}{\cosh x}$	$x \in \mathbb{R}$	$y \in (-1, 1)$	$\frac{d}{dx} \mapsto \operatorname{sech}^2 x$	$\int \mapsto \ln\lvert\cosh x\rvert$
$\coth x = \frac{1}{\tanh x}$	$x \in \mathbb{R}$ and $x \neq 0$	$y \in \mathbb{R}$ and $y \neq 0$	$\frac{d}{dx} \mapsto -\operatorname{csch}^2 x$	$\int \mapsto \ln\lvert\tanh \frac{x}{2}\rvert$
$\operatorname{sech} x = \frac{1}{\cosh x}$	$x \in \mathbb{R}$	$y \in [0, 1]$ and $y \neq 0$	$\frac{d}{dx} \mapsto -\operatorname{sech} x \tanh x$	$\int \mapsto \arctan(\sinh x)$
$\operatorname{csch} x = \frac{1}{\sinh x}$	$x \in \mathbb{R}$ and $x \neq 0$	$y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $y \neq 0$	$\frac{d}{dx} \mapsto -\operatorname{csch} x \coth x$	$\int \mapsto \ln\lvert\sinh x\rvert$

Vector-valued function - A function whose range is a set of vectors $[\mathbf{r} = \langle x(t), y(t)\rangle = x(t)\,\hat{\mathbf{\imath}} + y(t)\,\hat{\mathbf{\jmath}}]$

Position vector - A standard form vector (directed from the origin to $\langle x(t), y(t)\rangle$), representing a position — $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $a \leq t \leq b$

Derivatives — $\mathbf{r}^{(n)}(t) = \langle x^{(n)}(t)\,, y^{(n)}(t)\,\rangle$

Planar motion - The motion of a particle in the xy-plane at time t

Position vector	$\mathbf{r}(t) = \langle x(t), y(t)\rangle$	Given a rate vector $\mathbf{v}(t) = \langle \mathbf{v_x}(t),\ \mathbf{v_y}(t)\rangle$,
Velocity vector	$\mathbf{v}(t) = \langle x'(t), y'(t)\rangle = \mathbf{r}'(t)$	Displacement $\Delta x = \int_a^b \mathbf{v_x}(t)\ dt$
Acceleration vector	$\mathbf{a}(t) = \langle x''(t), y''(t)\rangle = \mathbf{r}''(t)$	Displacement $\Delta y = \int_a^b \mathbf{v_y}(t)\ dt$

Polar function - $[r = f(\theta),$ where $r = \sqrt{x^2 + y^2},\ \theta = \arctan \frac{y}{x},\ x = r\cos\theta, y = r\sin\theta]$

Derivative	$\frac{dy}{dx} = \frac{dy}{d\theta}/\frac{dx}{d\theta}$
Area bound by a polar curve	$A = \int_\alpha^\beta \frac{1}{2}r^2\, d\theta = \int_\alpha^\beta \frac{d\theta}{2\pi} \cdot \pi r^2 = \int_\alpha^\beta \int_0^r r\, dr\, d\theta$
Area bounded by two polar curves	$A = \int_\alpha^\gamma \frac{1}{2}r_1^2\ d\theta + \int_\gamma^\beta \frac{1}{2}r_2^2\ d\theta$ or $\int_\alpha^\beta \frac{1}{2}[r_2^2 - r_1^2]\ d\theta$

6 Series Calculus

Infinite series - $[S = \Sigma_{n=k}^{\infty} a_n = \lim_{n\to\infty} S_n]$ The sum of the terms of an infinite sequence

Limit of a sequence	If for any $\varepsilon > 0$, there exists a M such that if $n > M$, then $\lvert a_n - L\rvert < \varepsilon$, then $\lim_{n\to\infty} a_n = L$ ($\{a_n\}$ converges to L)
Convergence	$\lim_{n\to\infty} S_n$ approaches a specific value L
Divergence	$\lim_{n\to\infty} S_n$ does not approach a specific value
Absolute convergence	$\Sigma \lvert a_n\rvert$ converges, thus $\Sigma\, a_n$ converges
Conditional convergence	$\Sigma \lvert a_n\rvert$ diverges, but $\Sigma\, a_n$ converges
Geometric series	$S = \Sigma_{i=1}^{\infty} ar^i = \lim_{n\to\infty} a\frac{1-r^n}{1-r}$ If $\lvert r\rvert < 1$, S converges to $\frac{a}{1-r}$ If $\lvert r\rvert \geq 1$, S diverges
Telescoping series	$S = \Sigma_{n=1}^{\infty} a_n - a_{n+1} = a_1 - \lim_{n\to\infty} a_n$ S and $\lim_{n\to\infty} a_n$ either both converge or both diverge
Harmonic series	The divergent series $S = \Sigma_{n=1}^{\infty} \frac{1}{n}$
General harmonic series	The divergent series $S = \Sigma_{n=1}^{\infty} \frac{1}{an+b},\ a > 0,\ b \geq 0$
n^{th} term test	If $\Sigma_{n=k}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$ If $\lim_{n\to\infty} a_n \neq 0$, then $\Sigma_{n=k}^{\infty} a_n$ diverges
Integral test	Given $f(x)$ positive, continuous, and decreasing on $[k, \infty)$, $\int_k^{\infty} f(x)\, dx$ and $\Sigma_{n=k}^{\infty} f(n)$ either both converge or both diverge
p-series	Let $S = \Sigma_{n=1}^{\infty} \frac{1}{n^p}$ If $p > 1$, S converges If $p \leq 1$, S diverges
Direct comparison test	Given $S_a = \Sigma_{n=k}^{\infty} a_n$ and $S_b = \Sigma_{n=k}^{\infty} b_n$ where $b_n \geq a_n \geq 0\ \forall n \geq k$, If S_b converges, S_a must also converge If S_a diverges, S_b must also diverge
Limit comparison test	Given $S_a = \Sigma_{n=k}^{\infty} a_n$ and $S_b = \Sigma_{n=k}^{\infty} b_n$ where $a_n \geq 0,\ b_n > 0\ \forall n \geq k$ If $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, S_a and S_b either both converge or both diverge
Alternating series test	Given $S = \Sigma_{n=k}^{\infty} (-1)^n a_n$ or $(-1)^{n+1} a_n$ where $a_n \geq 0$ If $\lim_{n\to\infty} a_n = 0$ and $a_n > a_{n+1}$ ($\{a_n\}$ is decreasing), S converges Else, the alternating series test is inconclusive
Alternating series remainder	Given convergent alternating series $S = \Sigma_{n=k}^{\infty} (-1)^n a_n$, n^{th} partial sum S_n, and remainder (error bound) R_n, $\lvert S - S_n\rvert = \lvert R_n\rvert \leq a_{n+1}$
Ratio test	Given $S = \Sigma_{n=k}^{\infty} a_n$ and $L = \lim_{n\to\infty} \lvert\frac{a_{n+1}}{a_n}\rvert$, If $L < 1$, S converges absolutely If $L > 1$, S diverges If $L = 1$, the ratio test is inconclusive
Root test	Given $S = \Sigma_{n=k}^{\infty} a_n$ and $L = \lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert}$ If $L < 1$, S converges absolutely If $L > 1$, S diverges If $L = 1$, the root test is inconclusive

Riemann arrangement theorem	For a conditionally convergent series S, the sum of the positive terms is a divergent series and the sum of the negative terms is a divergent series For a conditionally convergent series S, given any real number L, there exists a permutation of S such that S converges to L; there also exist permutations such that S diverges to $\pm\infty$ or fails to approach any limit
Power series	$[f(x) = \Sigma_{n=0}^{\infty} a_n(x-c)^n]$ An infinite series of polynomials $\int \Sigma f(x)\,dx = \Sigma \int f(x)\,dx$ within the open interval of convergence $\frac{d}{dx}\Sigma f(x) = \Sigma \frac{d}{dx} f(x)$ within the open interval of convergence
Interval of convergence	The interval I (set of all x) for which the power series $f(x)$ converges
Radius of convergence	The radius R for the interval of convergence $I = \{x \mid \lvert x-c\rvert \leq R\}$
Geometric series	$[f(x) = \Sigma_{n=0}^{\infty} a(x-c)^n]$ Special case of power series which converges to $\frac{a}{1-(x-c)}$ on the interval of convergence $\lvert x-c\rvert < 1$
Taylor polynomial	$[P_n(x) = \Sigma_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(x-c)^i]$ The approximation of $f(x)$ at c, n times differentiable at $x = c$, by a n^{th} order Taylor polynomial
Maclaurin polynomial	$[P_n(x) = \Sigma_{i=0}^{n} \frac{f^{(i)}(0)}{i!} x^i]$ The n^{th} order Taylor polynomial, centered at $x = 0$
Taylor series	$[\lim_{n\to\infty} P_n(x) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n]$ The approximation of a function $f(x)$ infinitely-differentiable at $x = c$ as an infinite series of polynomials, which converges to $f(x)$ on the interval of convergence containing $x = c$
Maclaurin series	$[\lim_{n\to\infty} P_n(x) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n]$ The Taylor series centered at $x = 0$ $e^x = \Sigma_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + ...$ $R = \infty$ $\sin x = \Sigma_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$ $R = \infty$ $\cos x = \Sigma_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$ $R = \infty$ $\arctan x = \Sigma_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$ $R = 1$ $\frac{1}{1-x} = \Sigma_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$ $R = 1$ $\ln(1+x) = \Sigma_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$ $R = 1$ $(1+x)^k = \Sigma_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + ...$ $R = 1$
Euler's Formula	$e^{ix} = \cos x + i \sin x$
Euler's Identity	$e^{\pi i} + 1 = 0$
Taylor's Theorem	Let f be a function $n+1$ times differentiable on $[a, b]$ containing c. Let $P_n(x)$ be the n^{th} order Taylor polynomial of f at c, $\Sigma_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(x-c)^i$ Let $\lvert R_n(x)\rvert$ be the approximation error (remainder), $f(x) = P_n(x) + R_n(x)$
Peano form	There exists an $h_n(x)$ such that $R_n(x) = h_n(x)(x-c)^n = o(\lvert x-c\rvert^n),\ x \to c$ and $\lim_{x\to c} h_n(x) = 0$
Lagrange form	For each $x \in [a, b]$, $\exists\, z$ on the closed interval between x and c such that $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$
Cauchy form	For each $x \in [a, b]$, $\exists\, z$ on the closed interval between x and c such that $R_n(x) = \frac{f^{(n+1)}(z)}{n!}(x-z)^n(x-c)$
Mean Value Theorem	Taylor's theorem is a generalization of the Mean Value Theorem when $n = 0$ For a differentiable function f, $\exists\, z$ such that $f(x) - f(c) = f'(z)(x-c)$
Lagrange error bound	$\lvert R_n(x)\rvert \leq \lvert \frac{M}{(n+1)!}(x-c)^{n+1}\rvert$, where $M = \max \lvert f^{(n+1)}(z)\rvert$ on $[a, b]$

7 Multivariable Calculus