Highlights from attending the Simons Workshop on Transformers as a Computational Model

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"Be guided by beauty. I really mean that. Pretty much everything I've done has had an aesthetic component, at least to me. Now you might think 'well, building a company that's trading bonds, what's so aesthetic about that?' But, what's aesthetic about it is doing it right. Getting the right kind of people, and approaching the problem, and doing it right ... it's a beautiful thing to do something right."

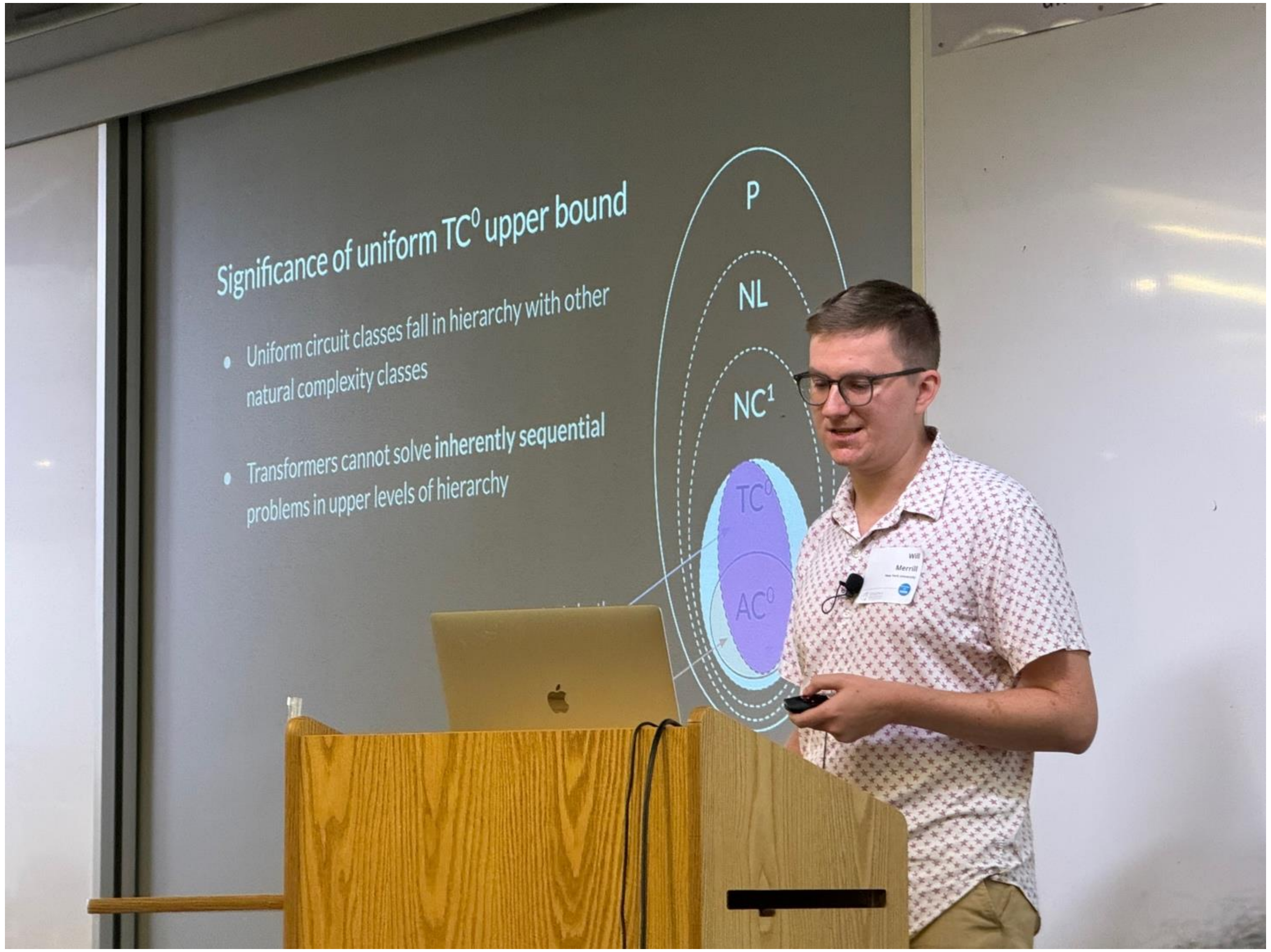
—James Harris Simons (1938–2024)

Three overall themes

- Formal language theory
  - Logical expressivity
  - Circuit complexity
  - Algebraic decomposition
- Mechanistic interpretability
  - RASP programs
  - Training dynamics
- Scaling up inference time compute
  - Recurrence and SSMs
  - Search and planning
  - Generator verifier gap

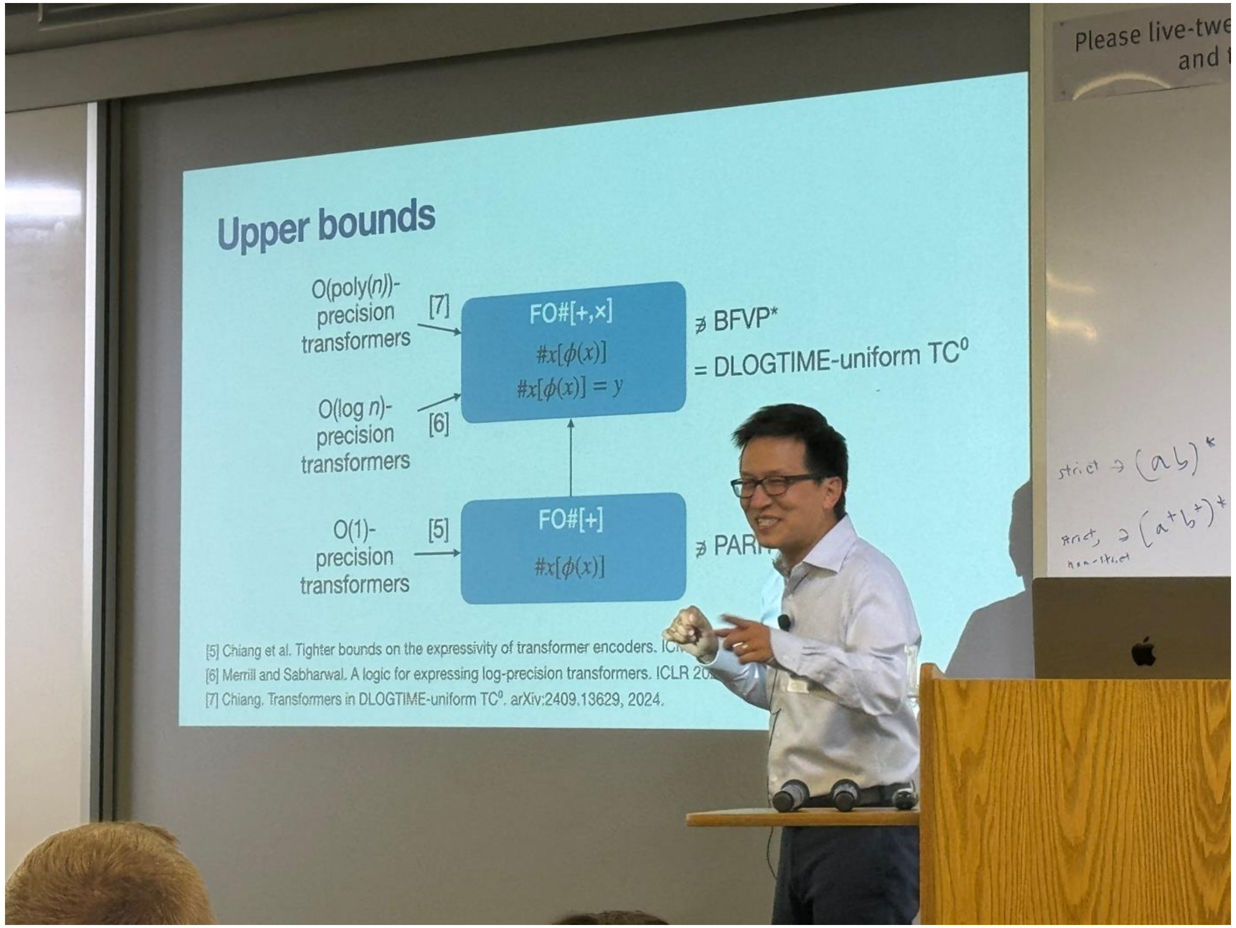
Circuit complexity (Merrill)

- Fixed precision circuits
- Can be studied as logical gates
- Transformer layers = circuit depth
- Computational parallelism vs. sequentiality



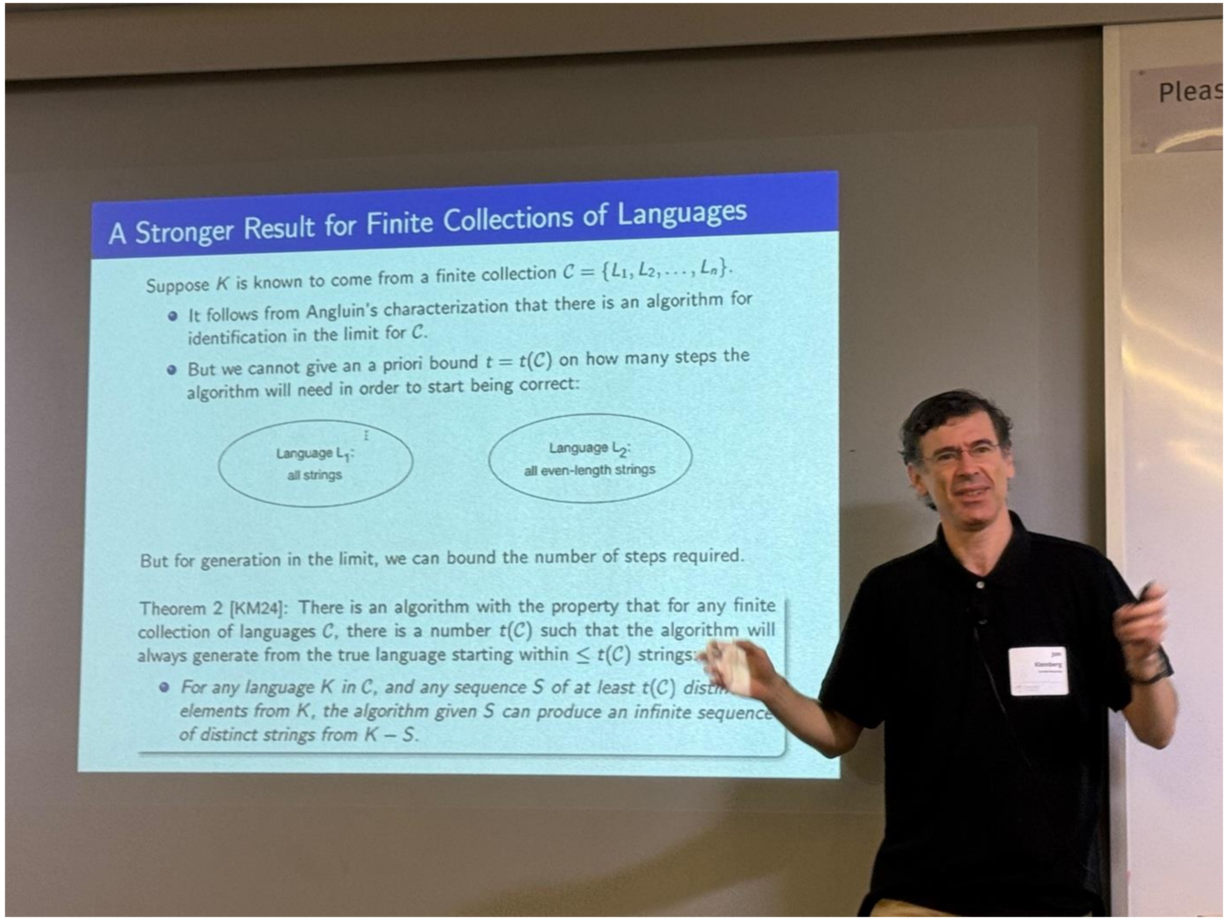
Logical expressivity (Chiang)

- Starts with linear logic
- Permits fine-grained control over resources
- Can be used to study transformer model expressivity
- Many fruitful connections with PL theory



Language generation in the limit (Kleinberg)

- Identifying the grammar for a language is not possible (Gold)
- But generating strings from the language is possible!
- Very elegant proof technique



Learning to reason with LLMs (Brown)

- Starts with his work designing poker bots
- (Re?)discovered the importance of planning
- Generator-verifier gap
- Cue O1, OpenAI's newest LLM with SoTA reasoning capabilities

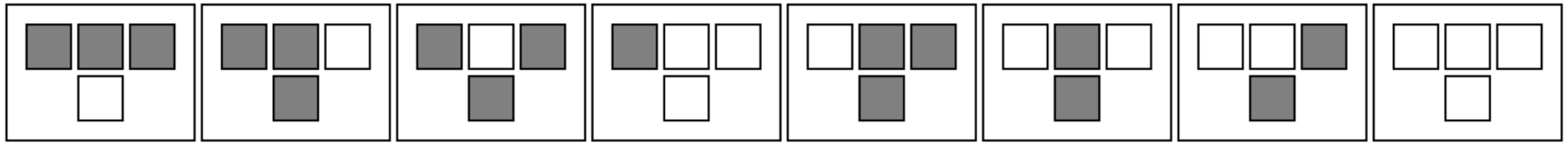
State space models (Gu)

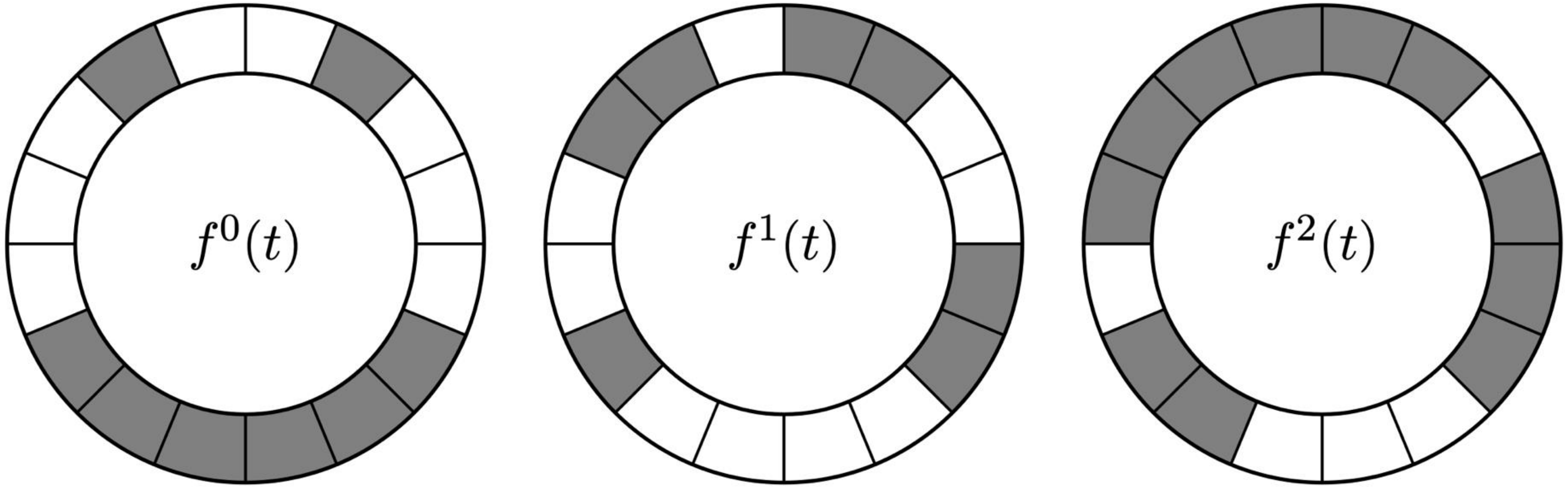
- Combines attention with SSM layers
- S4, Mamba, H3 models
- Gives model variable time to "think"

$$x'(t) = \boldsymbol{A}x(t) + \boldsymbol{B}u(t)$$
$$y(t) = \boldsymbol{C}x(t) + \boldsymbol{D}u(t)$$

State space models (Breandan's primer)

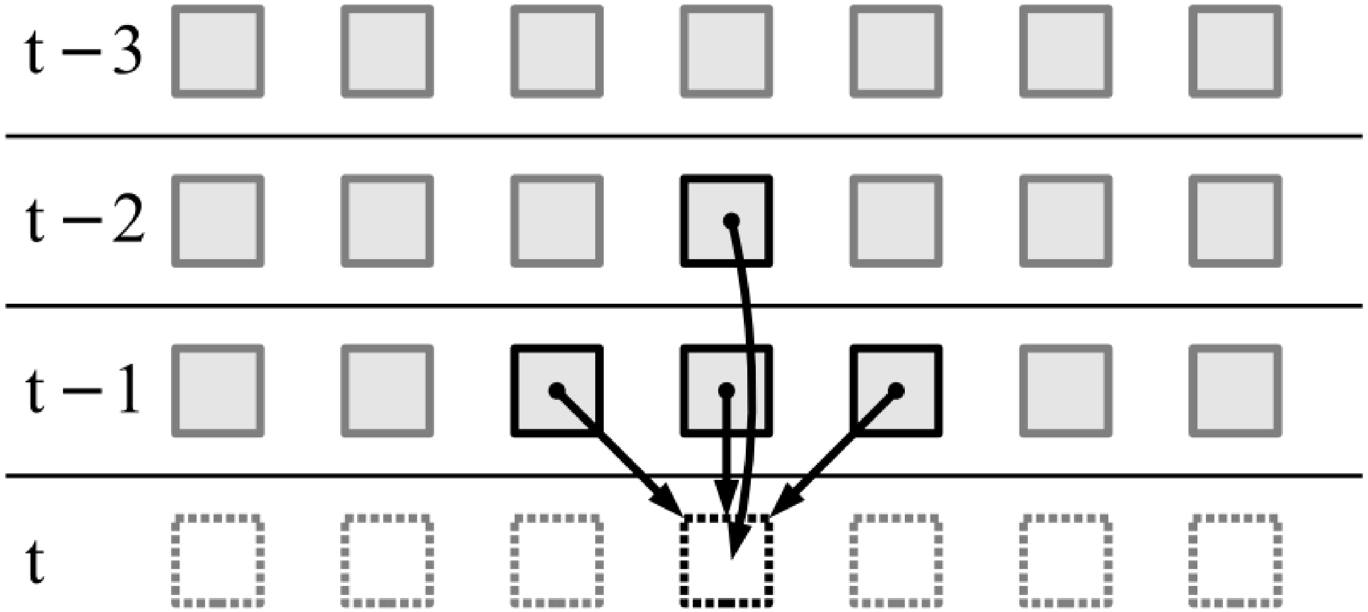
- Consider the first-order discrete dynamical system





State space models (Breandan's primer)

- Now consider a second-order dynamical system



State space models (Breandan’s primer)

- A very natural question to ask is the following:

  How do we represent a non-Markovian
  dynamical system in a Markovian way?

State space models (Breandan's primer)

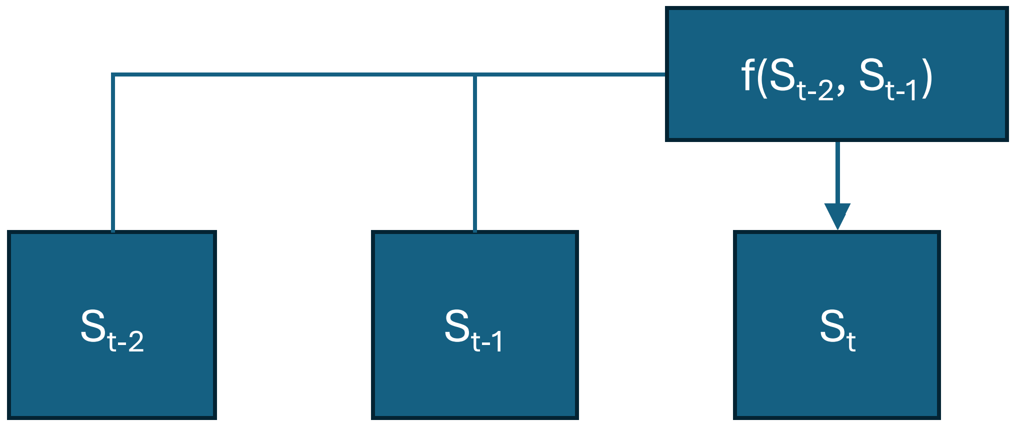
- A very natural question to ask is the following:

  How do we represent a non-Markovian dynamical system in a Markovian way?

- Using the state-space representation!
- Transforms higher-order dependence into a set of 1^{st}-order (i.e., Markovian) differential equations

State space models (Breandan's primer)

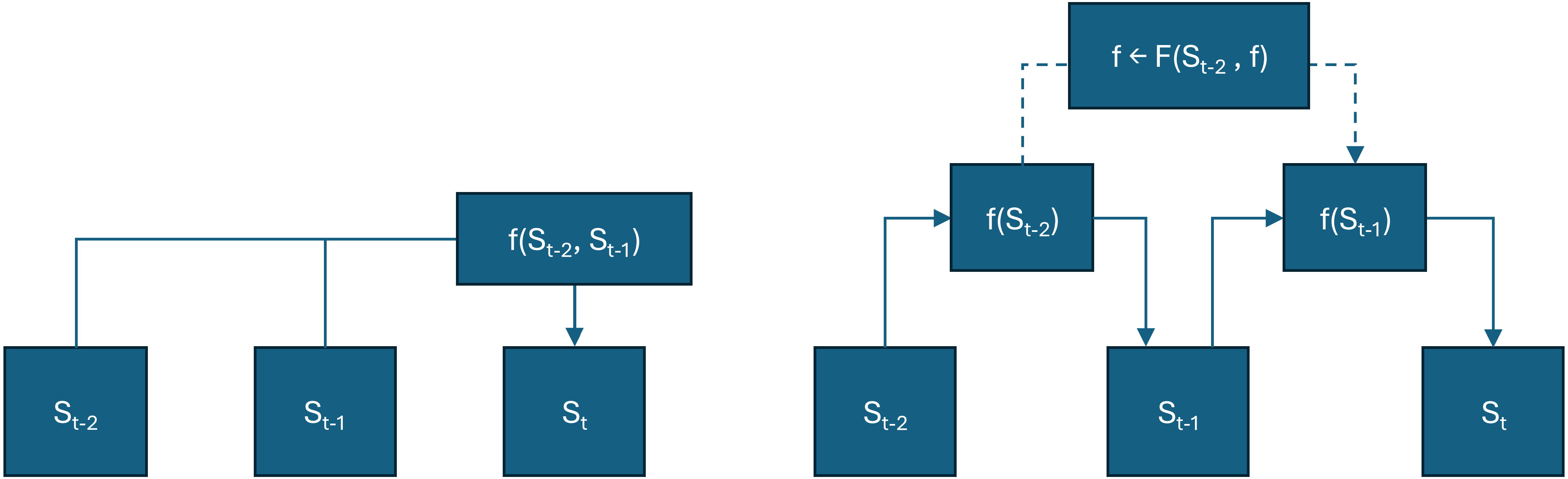
We can take a second-order differential equation...



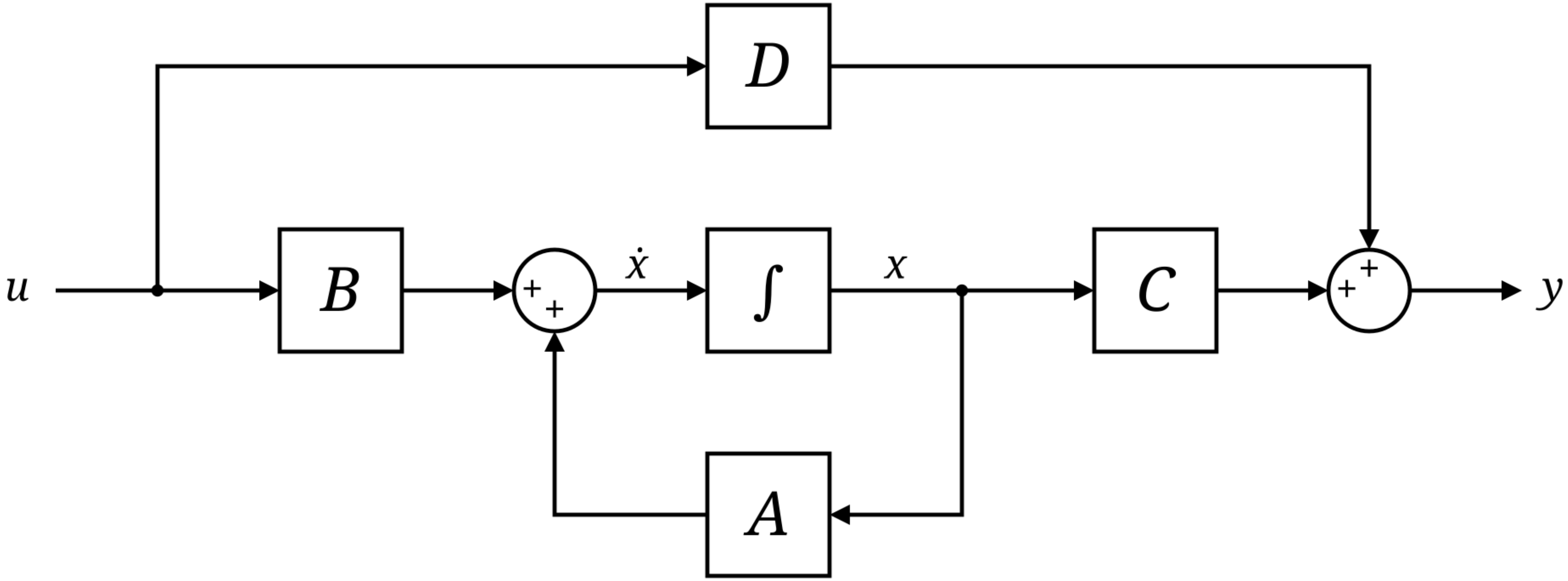
State space models (Breandan's primer)

We can take a second-order differential equation…

…and make it stateful with an evolving transition function.



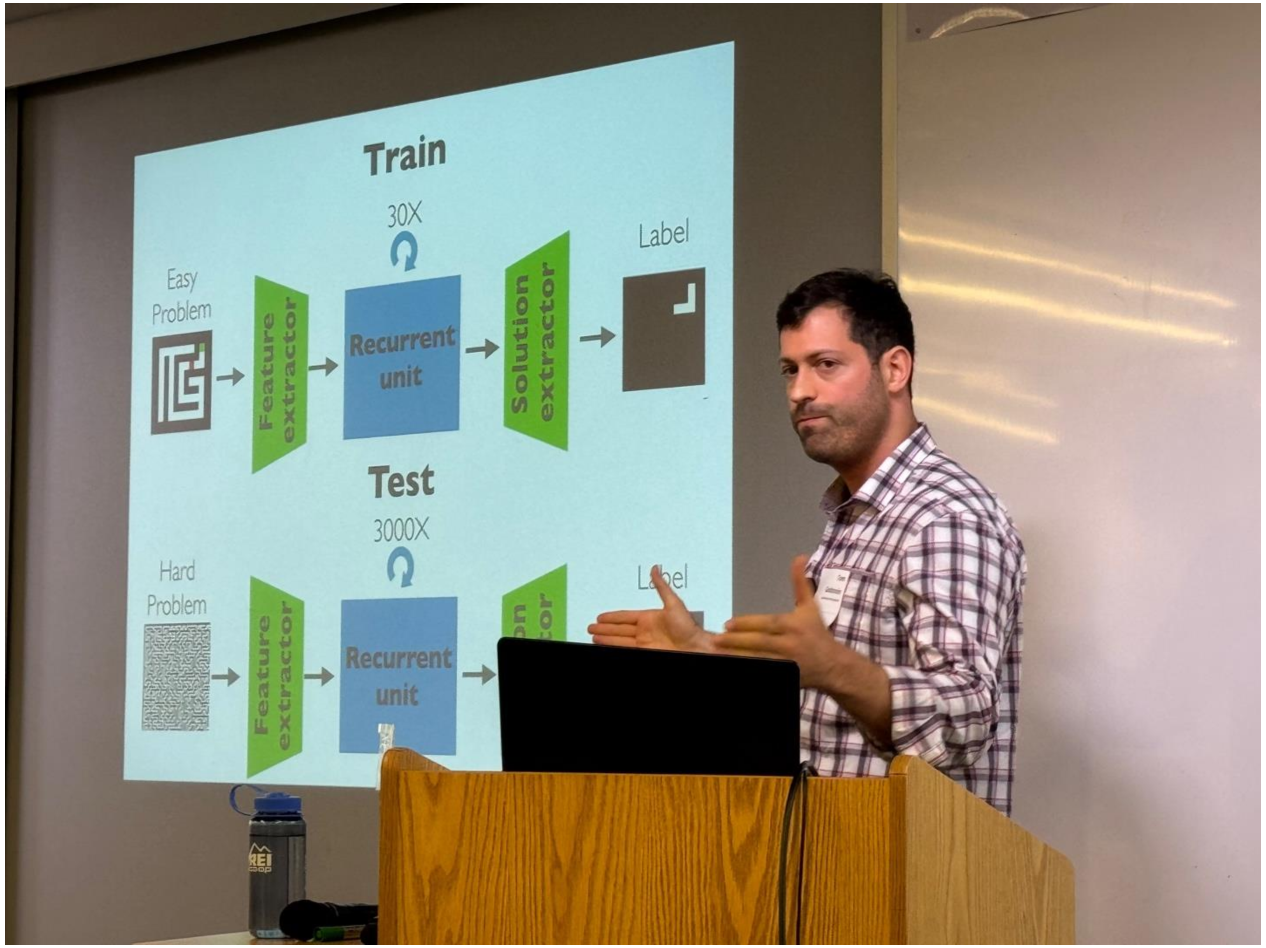
State space models (Breandan's primer)



$$x'(t) = \boldsymbol{A}x(t) + \boldsymbol{B}u(t)$$
$$y(t) = \boldsymbol{C}x(t) + \boldsymbol{D}u(t)$$

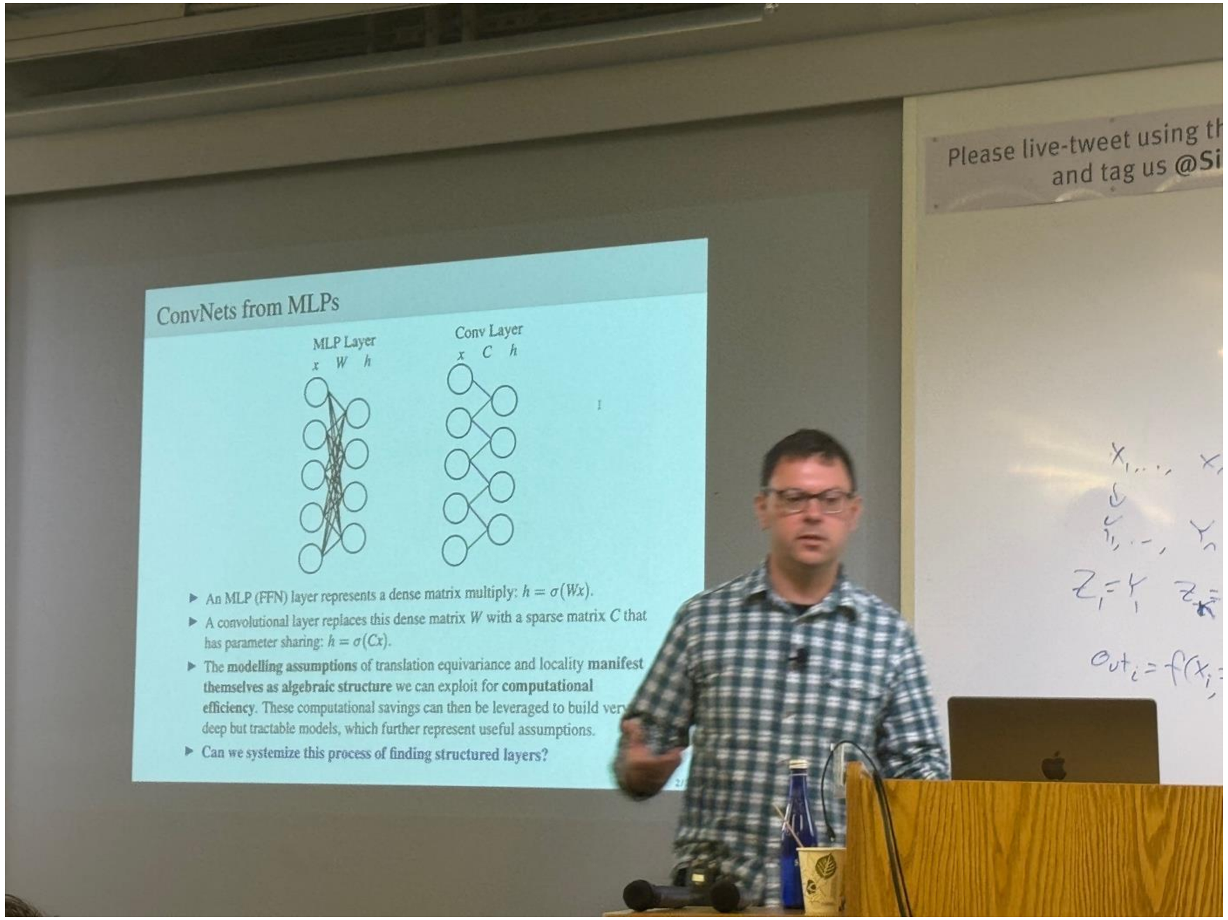
Weak to strong generalization (Goldstein)

- Train on easy reasoning tasks, then evaluate on harder instances of the same task
- Uses recurrent units
- Shows plenty of convincing results on mazes and sudoku

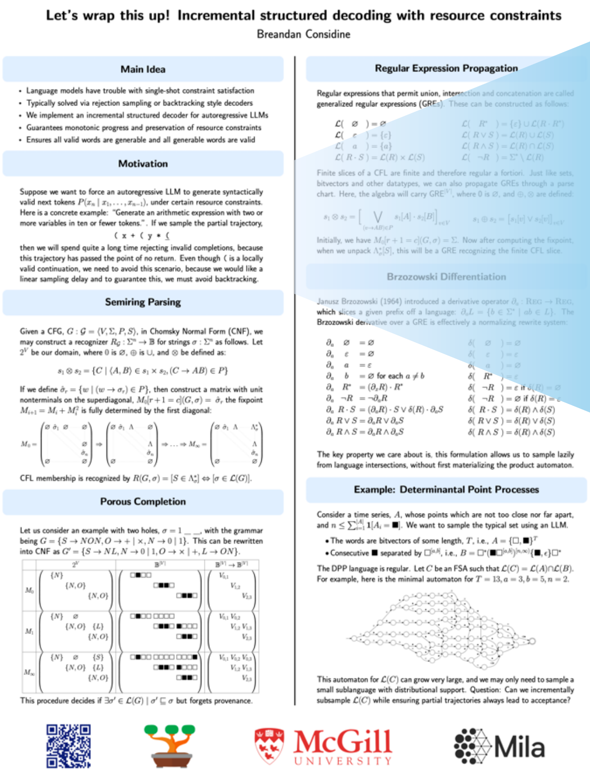


Machine learning is linear algebra (Wilson)

- Inductive biases for pretraining transformers
- Can leverage sparse structure from domain
- Continuous design space of structured matrices and tensors



My poster: Incremental structured decoding



Main Idea

- Language models have trouble with single-shot constraint satisfaction
- Typically solved via rejection sampling or backtracking style decoders
- We implement an incremental structured decoder for autoregressive LLMs
- Guarantees monotonic progress and preservation of resource constraints
- Ensures all valid words are generable and all generable words are valid

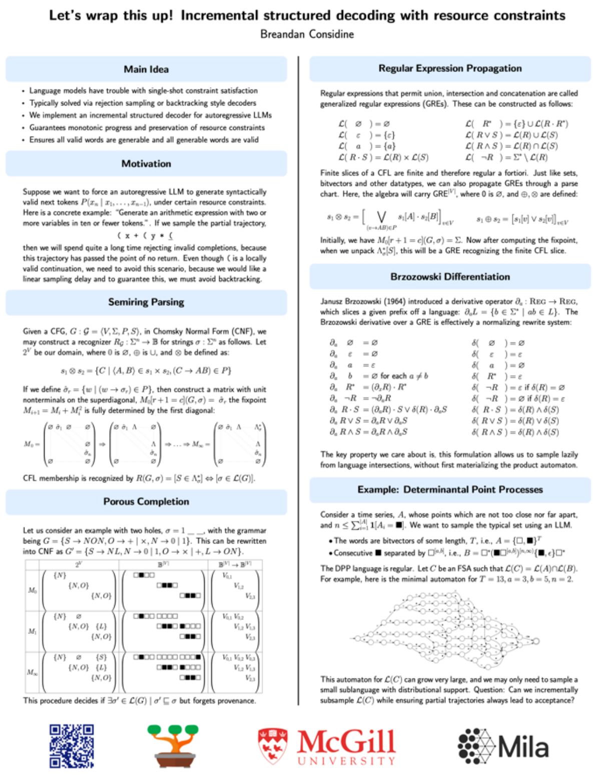
Motivation

Suppose we want to force an autoregressive LLM to generate syntactically valid next tokens $P(x_n \mid x_1, \ldots, x_{n-1})$, under certain resource constraints. Here is a concrete example: "Generate an arithmetic expression with two or more variables in ten or fewer tokens.". If we sample the partial trajectory,

(x + (y * (

then we will spend quite a long time rejecting invalid completions, because this trajectory has passed the point of no return. Even though (is a locally valid continuation, we need to avoid this scenario, because we would like a linear sampling delay and to guarantee this, we must avoid backtracking.

My poster: Incremental structured decoding



Regular Expression Propagation

Regular expressions that permit union, intersection and concatenation are called generalized regular expressions (GREs). These can be constructed as follows:

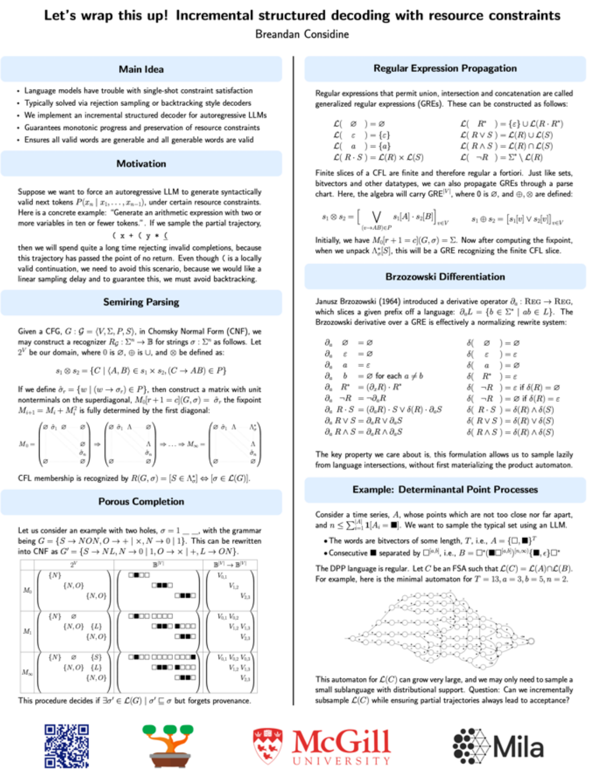
$$
\begin{aligned}
\mathcal{L}(\ \varnothing\ ) &= \varnothing & \mathcal{L}(\ R^*\ ) &= \{\varepsilon\} \cup \mathcal{L}(R \cdot R^*) \\
\mathcal{L}(\ \varepsilon\ ) &= \{\varepsilon\} & \mathcal{L}(\ R \vee S\ ) &= \mathcal{L}(R) \cup \mathcal{L}(S) \\
\mathcal{L}(\ a\ ) &= \{a\} & \mathcal{L}(\ R \wedge S\ ) &= \mathcal{L}(R) \cap \mathcal{L}(S) \\
\mathcal{L}(\ R \cdot S\ ) &= \mathcal{L}(R) \times \mathcal{L}(S) & \mathcal{L}(\ \neg R\ ) &= \Sigma^* \setminus \mathcal{L}(R)
\end{aligned}
$$

Finite slices of a CFL are finite and therefore regular a fortiori. Just like sets, bitvectors and other datatypes, we can also propagate GREs through a parse chart. Here, the algebra will carry $\text{GRE}^{|V|}$, where 0 is $\varnothing$, and $\oplus, \otimes$ are defined:

$$
s_1 \otimes s_2 = \Big[\bigvee_{(v \to AB) \in P} s_1[A] \cdot s_2[B]\Big]_{v \in V} \qquad s_1 \oplus s_2 = \big[s_1[v] \vee s_2[v]\big]_{v \in V}
$$

Initially, we have $M_0[r+1=c](G,\sigma) = \Sigma$. Now after computing the fixpoint, when we unpack $\Lambda_\sigma^*[S]$, this will be a GRE recognizing the finite CFL slice.

My poster: Incremental structured decoding



Brzozowski Differentiation

Janusz Brzozowski (1964) introduced a derivative operator $\partial_a : \textsc{Reg} \rightarrow \textsc{Reg}$, which slices a given prefix off a language: $\partial_a L = \{b \in \Sigma^* \mid ab \in L\}$. The Brzozowski derivative over a GRE is effectively a normalizing rewrite system:

$$
\begin{aligned}
\partial_a \quad \varnothing \quad &= \varnothing \\
\partial_a \quad \varepsilon \quad &= \varnothing \\
\partial_a \quad a \quad &= \varepsilon \\
\partial_a \quad b \quad &= \varnothing \text{ for each } a \neq b \\
\partial_a \quad R^* \quad &= (\partial_x R) \cdot R^* \\
\partial_a \quad \neg R \quad &= \neg \partial_a R \\
\partial_a \quad R \cdot S \quad &= (\partial_a R) \cdot S \vee \delta(R) \cdot \partial_a S \\
\partial_a \quad R \vee S \quad &= \partial_a R \vee \partial_a S \\
\partial_a \quad R \wedge S \quad &= \partial_a R \wedge \partial_a S
\end{aligned}
$$

$$
\begin{aligned}
\delta(\quad \varnothing \quad) &= \varnothing \\
\delta(\quad \varepsilon \quad) &= \varepsilon \\
\delta(\quad a \quad) &= \varnothing \\
\delta(\quad R^* \quad) &= \varepsilon \\
\delta(\quad \neg R \quad) &= \varepsilon \text{ if } \delta(R) = \varnothing \\
\delta(\quad \neg R \quad) &= \varnothing \text{ if } \delta(R) = \varepsilon \\
\delta(\quad R \cdot S \quad) &= \delta(R) \wedge \delta(S) \\
\delta(\quad R \vee S \quad) &= \delta(R) \vee \delta(S) \\
\delta(\quad R \wedge S \quad) &= \delta(R) \wedge \delta(S)
\end{aligned}
$$

The key property we care about is, this formulation allows us to sample lazily from language intersections, without first materializing the product automaton.

Visit to UC Berkeley