Probabilistic Reasoning, from Graphs to Circuits

Breandan Considine

Abstract

- Classical statistical modeling and inference requires pen-and-paper derivation
- Computers are steadily improving at logical reasoning and symbolic processing
- Can we employ computer-aided reasoning to assist with probabilistic modeling?
- Can we design an inference system to derive algorithms from first principles?

Denotational Semantics

The grammar of probabilistic modeling can be described approximately as follows:

$$\begin{array}{llll}
\mathcal{P} \to \text{Gaussian} & \mathcal{V} \to \mathcal{V}, \mathcal{V} & \mathcal{S} \to \mathcal{V} \sim \mathcal{P} & \mathcal{P} \to \prod_{i=1}^{n} \mathcal{P} \\
\mathcal{P} \to \text{Bernoulli} & \mathcal{P} \to \mathcal{P}(\mathcal{V}) & \mathcal{E} \to \mathcal{V} \pm \mathcal{V} & \mathcal{P} \to \sum_{i=1}^{n} \mathcal{P} \\
\mathcal{P} \to \text{Dirichlet} & \mathcal{P} \to \mathcal{P}(\mathcal{V} \mid \mathcal{V}) & \mathcal{E} \to \mathcal{V} \times \mathcal{V} & \mathcal{P} \to \int \mathcal{P} d\mathcal{V} \\
\mathcal{V} \to A \mid \ldots \mid Z & \mathcal{P} \to \mathcal{P}(\mathcal{E}) & \mathcal{E} \to \mathbb{E}[\mathcal{E}] & \mathcal{P} \to \mathcal{P} \div \mathcal{P}
\end{array}$$

Given a distribution over a set X, we can *sample* it to produce a *random variable*:

$$\frac{\Gamma \vdash P(X) : X \times \Sigma \to \mathbb{R}^{+} \quad x \sim P(X)}{\Gamma \vdash x : (X \times \Sigma \to \mathbb{R}^{+}) \rightsquigarrow X}\text{Sample}$$

The joint distribution $P(X, Y)$ is a distribution over the product space $X \times Y$:

$$\frac{\Gamma \vdash P(X) : X \times \Sigma \to \mathbb{R}^{+} \quad \Gamma \vdash P(Y) : Y \times \Sigma \to \mathbb{R}^{+}}{\Gamma \vdash P(X, Y) : X \times Y \times \Sigma \to \mathbb{R}^{+}}\text{Joint}$$

If we have a joint distribution $P(X, Y)$ and see $Y = y$, this observation is called *conditioning* and the resulting distribution over X is called a *conditional distribution*:

$$\frac{\Gamma \vdash P(X, Y) : X \times Y \times \Sigma \to \mathbb{R}^{+} \quad \Gamma \vdash y : Y}{\Gamma \vdash P(X \mid Y = y) : X \times \Sigma \to \mathbb{R}^{+}}\text{Cond}$$

We can use Bayes' rule to exchange the order of a conditional distribution as follows:

$$\frac{\overbrace{P(X \mid Y)}^{\text{Likelihood}} \quad \overbrace{P(Y)}^{\text{Prior}}}{\underbrace{P(Y \mid X)}_{\text{Normalize}} \propto \underbrace{P(X \mid Y)}_{\text{Observe}} \underbrace{P(Y)}_{\text{Sample}}}\text{Bayes}$$

When a conditional distribution $P(X \mid Y)$ does not depend on its prior Y, or may be factorized into $P(X)P(Y)$, we conclude that X and Y are independent RVs:

$$\frac{P(X \mid Y) = P(X)}{X \perp Y}\text{Indep} \qquad \frac{P(X, Y) = P(X)P(Y)}{X \perp Y}\text{Fact}$$

If a joint distribution $P(X, Y \mid Z)$ can be factored as the product of conditionals $P(X \mid Z)P(Y \mid Z)$, X and Y are said to be *conditionally independent given* Z:

$$\frac{P(X, Y \mid Z) = P(X \mid Z)P(Y \mid Z)}{X \perp Y \mid Z}\text{CondIndep}$$

Operational Semantics

To sample, we can use the Kolmogorov-Smirnov transform on a uniform PRNG:

$$\frac{x \sim P(X) \quad \texttt{CDF} : x \mapsto \int P(X = x)dx}{x = \texttt{INV(CDF(PRNG()))}}\text{Draw}$$

There are various ways of combining two probability distributions. For independent RVs, we could combine them using the product or convolution distribution:

$$\frac{P(X) \quad P(Y)}{P(X, Y) = P(X)P(Y)}\text{Join} \qquad \frac{P(X) \quad P(Y)}{P(X + Y) = P(X) * P(Y)}\text{Conv}$$

In general, to combine two arbitrary RVs, we need to know their dependence relation. If two variables are related by a dyadic function $f(x, y)$, we can use Fubini-Tonelli:

$$\frac{\begin{array}{c} x \sim P(X) \quad y \sim P(Y) \quad f : X \times Y \to Z \\ \hline P(Z \mid X = x, Y = y) = \int_{X \times Y} f(x, y) d(x \times y) \end{array}}{\int_{Y} \int_{X} f(x, y) dx dy = \int_{X} \int_{Y} f(x, y) dy dx}\text{FuTon}$$

To remove an RV from a joint distribution $P(X, Y)$ we can *marginalize*, or integrate over the conditional. The resulting distribution is called a *marginal distribution*:

$$\frac{\Gamma \vdash P(X, Y) : X \times Y \times \Sigma \to \mathbb{R}^{+}}{\Gamma \vdash P(X) : X \times \Sigma \to \mathbb{R}^{+} \propto \int_{Y} P(X \mid Y = y) dy}\text{Margin}$$

Probabilistic Circuits

A semiring algebra has two operators, $\oplus$ and $\otimes$ which are closed under distributivity:

$$\frac{X \otimes (Y \oplus Z)}{(X \otimes Y) \oplus (X \otimes Z)}\text{LDist} \qquad \frac{(Y \oplus Z) \otimes X}{(X \otimes Y) \oplus (X \otimes Z)}\text{RDist}$$

The sum-product network (SPN) is a commutative semiring on simple distributions:

$$PC \to v \sim \mathcal{D} \qquad PC \to PC \oplus PC \qquad PC \to PC \otimes PC$$

A Bayesian belief network or Bayes network (BN) is an acyclic DGM of the form:

$$P(x_1, \ldots, x_D) = \prod_{i=1}^{D} P(x_i \mid \texttt{parents}(x_i))$$

Given a BN, we can compile it to an SPN using the procedure from Butz (2019):

procedure BN2SPN(b: BN): SPN
  c ← VariableEliminate(b)
  s ← RedistributeParams(c)
  s ← CompileMarginalized(s)
  return Reduce(s)
end procedure

procedure Reduce(s_0: SPN): SPN
  s_1 ← AddTerminals(s_0)
  s_1 ← MergeProducts(s_1)
  if $s_0 = s_1$ **then return** s_1
  else return Reduce(s_1)
end procedure

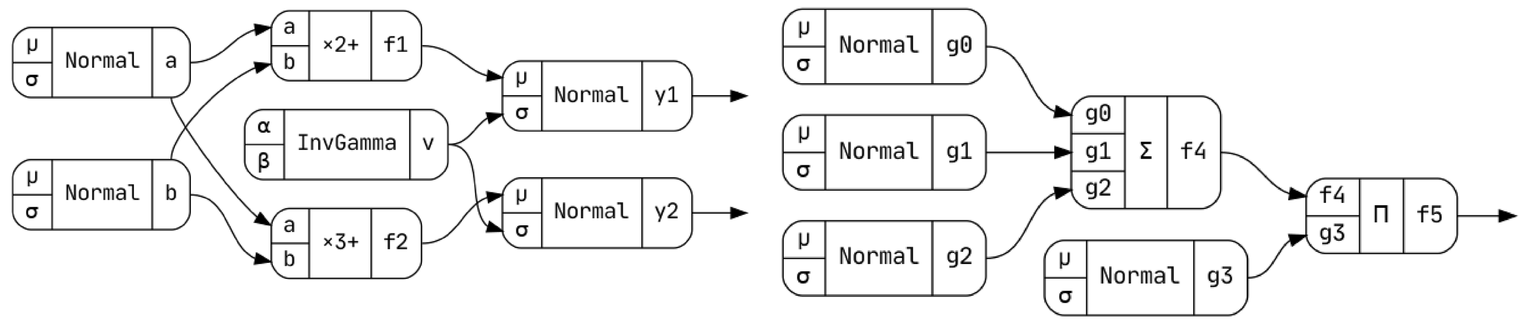
Generative Modeling

Our DSL, Markovian, can construct models in a probabilistic context free grammar. We validate it by implementing some generative models for Bayesian regression:

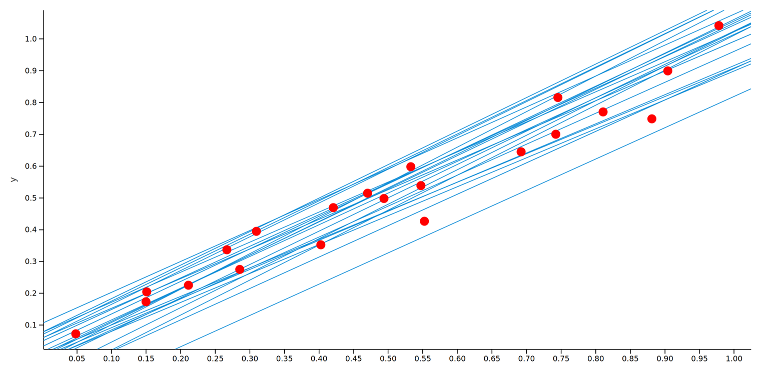
```
val a  by Gaussian(0, 9.0)
val b  by Gaussian(0, 9.0)
val v  by InvGamma(.5, .5)
val f1 by a * 2 + b
val f2 by a * 3 + b
val y1 by Gaussian(f1, v)
val y2 by Gaussian(f2, v)
sample(f1, f2, y3, y4).show()
```

```
val g0 by Gaussian(0.1,  1.0)
val g1 by Gaussian(5.0,  1.0)
val g2 by Gaussian(10.0, 1.0)
val g3 by Gaussian(5.0,  2.0)
val f4 by g0 + g1 + g2
val f5 by g3 * f4
compare(g0, g1, g2, g3
        g4, g5).show()
```

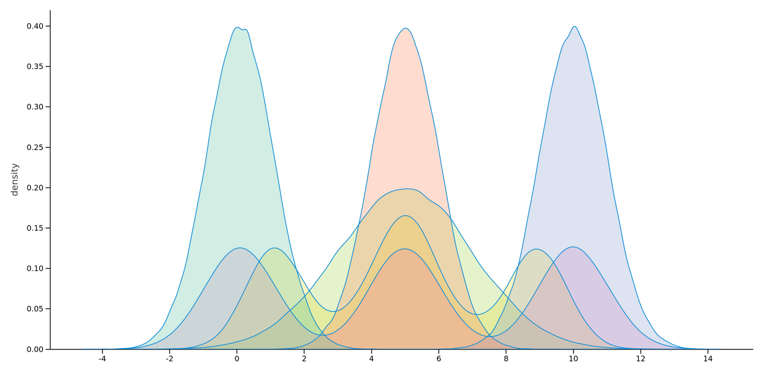
We can depict the sequential programs written above as computational graphs. The topology of this data structure is permutation invariant to instruction reordering.



We can also represent these functions by plotting their probability distributions:



Bayesian linear regression



Sum-product network

We plan extend this DSL to support discriminative models and statistical inference.

Contributions

- Lift numerical computation graph into the domain of probability kernels
- Implements general-purpose combinators for various probability distributions
- Implements domain-specific estimators for the algebra of random variables
- Allows flexible algebraic rewriting to optimize for e.g. latency or numerical stability
- Lowering to numerical values when necessary to perform e.g. sampling/inference

Code available at: https://github.com/breandan/markovian

Paper available at: https://brea.ndan.co/public/probcirc.pdf