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汤家凤考研数学基础课程

第一章 极限与连续

Part 1 极限

一、defs

1、极限 　　　　　　　　　　　　($\underline{N(\varepsilon)>0}$)

Case 1. $(\varepsilon-N)$ — 若 $\forall \varepsilon>0$，$\exists N>0$，当 $n>N$ 时，

$$|a_n - A| < \varepsilon$$

$$\lim_{n\to\infty} a_n = A.$$

如：$\lim\limits_{n\to\infty} \frac{n-1}{n+1} = 1$. 但 $\frac{n-1}{n+1} \neq 1$

Q: "若 $\forall \varepsilon>0$，$\exists N>0$，当 $n>N$ 时，$|a_n-A| \leq 3\varepsilon$"？

☆ Case 2. $(\varepsilon-\delta)$

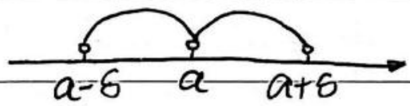
Notes: ① $x\to a$ $\begin{cases} x\neq a \\ x\to a^-、x\to a^+ \end{cases}$

如：$\lim\limits_{x\to0} \frac{0}{x} = 0$

② $\lim\limits_{x\to a} f(x)$ 与 $f(a)$ 无关

如：$\lim\limits_{x\to1} \frac{x^2-1}{x-1} = \lim\limits_{x\to1}(x+1) = 2$

③ $0<|x-a|<\delta$



$(\varepsilon-\delta)$: — 若 $\forall \varepsilon>0$，$\exists \delta>0$，当 $0<|x-a|<\delta$ 时，

$$|f(x)-A| < \varepsilon.$$

$$\lim_{x\to a} f(x) = A.$$

④ $\lim\limits_{x\to a^-} f(x) \triangleq f(a-0)$ — 左极限

$\lim\limits_{x\to a^+} f(x) \triangleq f(a+0)$ — 右极限

☆ $\lim\limits_{x\to a} f(x)$ 存在 $\Longleftrightarrow$ $f(a-0)$、$f(a+0)$ 存在且等

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☆⑤ $f(x)$ 含 $\begin{cases} a^{\frac{?}{x-b}} \\ a^{\frac{?}{b-x}} \end{cases}$ $(x\to b)$ 一定分左右

例1. $f(x)=\left(1-2^{\frac{1}{x-1}}\right)/\left(1+2^{\frac{1}{x-1}}\right)$，$\lim\limits_{x\to 1} f(x)$?

解：$x\to 1^- \Rightarrow x-1\to 0^- \Rightarrow \frac{1}{x-1}\to -\infty$

$\Rightarrow 2^{\frac{1}{x-1}}\to 0 \Rightarrow f(1-0)=1$；

$x\to 1^+ \Rightarrow x-1\to 0^+ \Rightarrow \frac{1}{x-1}\to +\infty \Rightarrow 2^{\frac{1}{x-1}}\to +\infty \Rightarrow f(1+0)=-1$

$\because f(1-0)\neq f(1+0)$ $\therefore \lim\limits_{x\to 1} f(x)$ 不存在

Case 3. $(\varepsilon - X)$ $\begin{cases} x\to+\infty \checkmark \\ x\to-\infty \\ x\to\infty \end{cases}$ — 若 $\forall \varepsilon>0$，$\exists X>0$，

当 $x>X$ 时，$|f(x)-A|<\varepsilon$. $\lim\limits_{x\to+\infty} f(x)=A$.

2、无穷小

① $\lim\limits_{x\to a}\alpha(x)=0$. 称 $\alpha(x)$ 当 $x\to a$ 为无穷小.

Q1. $3(x-1)^2$?

$\lim\limits_{x\to 1} 3(x-1)^2=0$. $3(x-1)^2$ 当 $x\to 1$ 时为无穷小.

Q2. 0? ✓

② 层次：$\alpha\to 0$，$\beta\to 0$

Case 1. $\lim \frac{\beta}{\alpha}=0$. $\beta=o(\alpha)$；

如 $\alpha=3x$，$\beta=x^4$，$(x\to 0)$

Case 2. $\lim\frac{\beta}{\alpha}=k\ (\neq 0, \neq\infty)$

如 $\alpha=x^2$，$\beta=3x^2+x^4$ $(x\to 0)$

$\lim\limits_{x\to 0}\frac{\beta}{\alpha}=\lim\limits_{x\to 0}(3+x^2)=3$

特例：$\lim\frac{\beta}{\alpha}=1$，$\alpha\sim\beta$

二、性质：

(一) 一般性质：

1、(唯一性) 极限存在必唯一.

☆ 2、(保号性) $\lim\limits_{x\to a} f(x)=A\begin{cases}>0\\<0\end{cases}$，则 $\exists\,\delta>0$，当 $0<|x-a|<\delta$ 时. $f(x)\begin{cases}>0\\<0\end{cases}$.

"极限正则去心邻域正，
极限负则去心邻域负."

证明：设 $\lim\limits_{x\to a} f(x)=A>0$. 取 $\varepsilon=\frac{A}{2}>0$，$\exists\,\delta>0$，当 $0<|x-a|<\delta$ 时，$|f(x)-A|<\frac{A}{2}$

$$-\frac{A}{2}<f(x)-A<\frac{A}{2}\ ,\quad f(x)>\frac{A}{2}>0$$

例1、$f'(1)=0$. $\lim\limits_{x\to 1}\frac{f'(x)}{(x-1)^3}=-2<0$. $x=1$?

解：$\exists\,\delta>0$，当 $0<|x-1|<\delta$ 时，

$$\frac{f'(x)}{(x-1)^3}<0$$

$$\begin{cases}f'(x)>0\ ,\ x\in(1-\delta,1)\\ f'(x)<0\ ,\ x\in(1,1+\delta)\end{cases}\quad \therefore\ x=1 \text{为极大点}$$

3、(有界性) $\lim\limits_{n\to\infty} a_n=A \underset{\not\Leftarrow}{\Rightarrow} \exists\, M>0$，使 $|a_n|\le M$

☆ (二) 存在性

准则 I 夹逼定理

$$\begin{cases}a_n\le b_n\le c_n\\ \lim\limits_{n\to\infty} a_n=\lim\limits_{n\to\infty} c_n=A\end{cases}\Rightarrow \lim\limits_{n\to\infty} b_n=A$$

型一 n项和、积极限

① 先和、积后极限

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例1. $\lim\limits_{n\to\infty}\left[\frac{1}{1\times2}+\frac{1}{2\times3}+\cdots+\frac{1}{n(n+1)}\right]$

解: $\frac{1}{1\times2}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$

原式 $=\lim\limits_{n\to\infty}(1-\frac{1}{n+1})=1.$

② 夹逼定理 (不齐)

例2. $\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)$

解: $\frac{n}{\sqrt{n^2+n}}\le b_n\le\frac{n}{\sqrt{n^2+1}}$

$\lim\limits_{n\to\infty}$ 左 $=\lim\limits_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}}=1$, $\lim\limits_{n\to\infty}$ 右 $=\lim\limits_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}}=1$

∴原式 $=1$

例3. $\lim\limits_{n\to\infty}\left(\frac{1}{n^2+1}+\frac{2}{n^2+2}+\cdots+\frac{n}{n^2+n}\right)$

$\frac{1+2+\cdots+n}{n^2+n}\le b_n\le\frac{1+2+\cdots+n}{n^2+1}$

③ 定积分的定义. (齐)

$\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}f(\frac{i}{n})=\int_0^1 f(x)dx$

例4. $\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1^2}}+\cdots+\frac{1}{\sqrt{n^2+n^2}}\right)$

解: 原式 $=\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\frac{1}{\sqrt{1+(\frac{i}{n})^2}}=\int_0^1\frac{1}{\sqrt{1+x^2}}dx=?$

例5. $\lim\limits_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)$

解: 原式 $=\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\frac{1}{1+\frac{i}{n}}=\int_0^1\frac{1}{1+x}dx=\ln 2$

准则Ⅱ 单调有界的数列必有极限

Notes:

① 有界 — $\{a_n\}$, 若 $\exists M>0$, 使 $|a_n|\le M$, 有界

若 $\exists M_1$, 使 $a_n\ge M_1$, 有下界

若 $\exists M_2$, 使 $a_n\le M_2$, 有上界

有界 $\Longleftrightarrow$ 有上、下界

② Case 1. $\{a_n\}\uparrow$

$\begin{cases} \text{无上界} \Rightarrow \lim\limits_{n\to\infty} a_n = +\infty \\ \exists M, \text{使 } a_n \le M \Rightarrow \lim\limits_{n\to\infty} a_n \text{存在} \end{cases}$，令 $\lim\limits_{n\to\infty} a_n = A$，则 $a_n \le A$.

如：$a_n = \frac{n}{n+1} = 1 - \frac{1}{n+1}$

$\{a_n\}\uparrow$ 且 $a_n \le 1 \Rightarrow \lim\limits_{n\to\infty} \frac{n}{n+1} = 1$

Case 2. $\{a_n\}\downarrow$

$\begin{cases} \text{无下界} \Rightarrow \lim\limits_{n\to\infty} a_n = -\infty \\ \exists m, a_n \ge m \Rightarrow \lim\limits_{n\to\infty} a_n \text{存在} \end{cases}$，令 $\lim\limits_{n\to\infty} a_n = A$，则 $a_n \ge A$.

型二 极限存在证明

例 1. $a_1 = \sqrt{2}$，$a_2 = \sqrt{2+\sqrt{2}}$，$a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$，…

证：$\lim\limits_{n\to\infty} a_n$ 存在，求之.

证：$a_{n+1} = \sqrt{2+a_n}$

① $a_1 = \sqrt{2} < a_2 = \sqrt{2+\sqrt{2}}$

设 $a_k < a_{k+1}$，则 $\sqrt{2+a_k} < \sqrt{2+a_{k+1}}$，即 $a_{k+1} < a_{k+2}$

$\therefore \forall n$，有 $a_n \le a_{n+1} \Rightarrow \{a_n\}\uparrow$

② $a_1 = \sqrt{2} \le 2$

设 $a_k \le 2$，则 $a_{k+1} = \sqrt{2+a_k} \le \sqrt{2+2} = 2$

$\therefore \forall n$，$a_n \le 2$　　$\therefore \lim\limits_{n\to\infty} a_n$ 存在

③ 令 $\lim\limits_{n\to\infty} a_n = A$，由 $a_{n+1} = \sqrt{2+a_n} \Rightarrow A = \sqrt{2+A}$

$\Rightarrow A = -1$（舍），$A = 2$

例 2. $a_1 = 2$，$a_{n+1} = \frac{1}{2}(a_n + \frac{1}{a_n})$

证：$\lim\limits_{n\to\infty} a_n$ 存在.

证：① $a_{n+1} \geqslant 1$

② $a_{n+1}-a_n=\frac{1}{2}(a_n+\frac{1}{a_n})-a_n=\frac{1-a_n^2}{2a_n}\leqslant 0$

$\Rightarrow a_{n+1}\leqslant a_n \Rightarrow \{a_n\}\downarrow$ ，∴ $\lim\limits_{n\to\infty} a_n$ 存在

☆（三）无穷小性质：

1、一般性质

① $\alpha\to 0,\ \beta\to 0 \Rightarrow \alpha\pm\beta\to 0,\ \alpha\beta\to 0,\ k\alpha\to 0$

② $|\alpha|\leqslant M,\ \beta\to 0 \Rightarrow \alpha\beta\to 0$

如：$\lim\limits_{x\to 0} x\sin\frac{1}{x}=0$

③ $\lim f(x)=A \Leftrightarrow f(x)=A+\alpha,\ \alpha\to 0$.

2、等价无穷小

① $\begin{cases} \alpha\sim\alpha \\ \alpha\sim\beta\Rightarrow\beta\sim\alpha \\ \alpha\sim\beta,\ \beta\sim\gamma\Rightarrow\alpha\sim\gamma \end{cases}$

② $\alpha\sim\alpha_1,\ \beta\sim\beta_1$，且 $\lim\frac{\beta_1}{\alpha_1}=A$，则 $\lim\frac{\beta}{\alpha}=A$

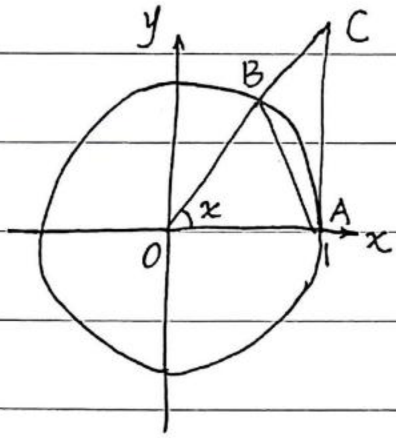
3、$x\to 0$：

① $x\sim\sin x\sim\tan x\sim\arcsin x\sim\arctan x\sim e^x-1\sim\ln(1+x)$；

② $1-\cos x\sim\frac{1}{2}x^2$

③ $(1+x)^a-1\sim ax$

三、两个重要极限



$0<x<\frac{\pi}{2}$

$S_{\triangle OAB}=\frac{1}{2}\sin x$

$S_{扇AOB}=\frac{1}{2}x$

$S_{Rt\triangle AOC}=\frac{1}{2}\tan x$

① $0<x<\frac{\pi}{2}$，则 $\sin x<x<\tan x$

② $x>0$ 时，$\ln(1+x)<x$

(Ⅰ) $\lim\limits_{\Delta\to 0}\frac{\sin\Delta}{\Delta}=1$

(Ⅱ) $\lim\limits_{\Delta\to 0}(1+\Delta)^{\frac{1}{\Delta}}=e$ (1^{∞})

☆型三 不定型极限

$\begin{cases}\frac{0}{0}、1^{\infty}\\ \frac{\infty}{\infty}、0\times\infty、\infty-\infty、\infty^{0}、0^{0}\end{cases}$

$\frac{0}{0}$ 型.

Note:

① 习惯
$$\begin{cases} u(x)^{v(x)} \Rightarrow e^{v(x)\ln u(x)} \\ *-1 \Rightarrow \begin{cases} e^{\Delta}-1\sim\Delta \\ (1+\Delta)^{a}-1\sim a\Delta \end{cases} (\Delta\to 0) \\ \ln * \Rightarrow \ln(1+\Delta)\sim\Delta \quad (\Delta\to 0) \\ x、\sin x、\tan x、\arcsin x、\arctan x \text{ 任两个之差为三阶无穷小} \end{cases}$$

② 误区.

如：$\lim\limits_{x\to 0}\frac{e^{2x}-1+\sin 3x}{x}\overset{?}{=}\lim\limits_{x\to 0}\frac{2x+3x}{x}=5$ ✓

又如：$\lim\limits_{x\to 0}\frac{e^{x^2}-\cos x}{x^2}=\lim\limits_{x\to 0}\frac{(e^{x^2}-1)-(1-\cos x)}{x^2}\overset{?}{=}\lim\limits_{x\to 0}\frac{x^2+\frac{1}{2}x^2}{x^2}=\frac{3}{2}$ ✓

$\lim\limits_{x\to 0}\frac{x-\sin x}{x^3}\overset{?}{=}\lim\limits_{x\to 0}\frac{x-x}{x^3}=0$ ✗

例1. $\lim\limits_{x\to 0}\left[\left(\frac{2+\cos x}{3}\right)^{x}-1\right]/x^3=\lim\limits_{x\to 0}\frac{x\ln\frac{2+\cos x}{3}}{x^3}=\lim\limits_{x\to 0}\frac{\cos x-1}{3x^2}=-\frac{1}{6}$

例2. $\lim\limits_{x\to 0}(e^{\tan x}-e^{\sin x})/[x^2\ln(1+x)]$

$=\lim\limits_{x\to 0}\frac{e^{\sin x}}{x^3}(e^{\tan x-\sin x}-1)=\lim\limits_{x\to 0}\frac{\tan x-\sin x}{x^3}$

$=\lim\limits_{x\to 0}\frac{\tan x\cdot(1-\cos x)}{x^3}=\lim\limits_{x\to 0}\frac{\tan x}{x}\cdot\frac{1-\cos x}{x^2}=\frac{1}{2}$

例3. $\lim\limits_{x\to 0}\frac{1}{x^4}\{[\sin x-\sin(\sin x)]\cdot\sin x\}$

$=\lim\limits_{x\to 0}\frac{\sin x}{x}\times\frac{\sin x-\sin(\sin x)}{\sin^3 x}\times\left(\frac{\sin x}{x}\right)^3$

$=\lim\limits_{t\to 0}\frac{t-\sin t}{t^3}=\lim\limits_{t\to 0}\frac{1-\cos t}{3t^2}=\frac{1}{6}$

1^{∞} 型 $\begin{cases} 凑\ (1+\Delta)^{\frac{1}{\Delta}} \\ 恒等变形 \end{cases}$

例 1. $\lim\limits_{x\to 0}(1-\sin x^2)^{\frac{3}{x^2}} = \lim\limits_{x\to 0} e^{-\frac{3\sin x^2}{x^2}} = e^{-3}$

例 2、$\lim\limits_{x\to 0}\left(\frac{x}{\sin x}\right)^{\frac{1}{x^2}} = \lim\limits_{x\to 0} e^{\frac{x-\sin x}{x^2\sin x}} = \lim\limits_{x\to 0} e^{\frac{x-\sin x}{x^3}} = e^{\lim\limits_{x\to 0}\frac{1-\cos x}{3x^2}} = e^{\frac{1}{6}}$

例 3、$\lim\limits_{x\to\infty}(\cos\frac{1}{x})^{x^2} = \lim\limits_{x\to\infty} e^{(\cos\frac{1}{x}-1)\cdot x^2} = \lim\limits_{t\to 0} e^{\frac{\cos t-1}{t^2}} = e^{-\frac{1}{2}}$

$\frac{\infty}{\infty}$ 型

$\begin{cases} 洛必达法则 \\ \lim\limits_{x\to\infty}\dfrac{a_m x^m+\cdots+a_0}{b_n x^n+\cdots+b_0} & \begin{cases} 0, & m<n \\ \dfrac{a_m}{b_n}, & m=n \\ \infty, & m>n \end{cases} \\ \dfrac{0}{0} \end{cases}$

如：$\lim\limits_{x\to 0}\dfrac{2x^2+x\cos x}{x^2-x\sin\frac{1}{x}} = \lim\limits_{x\to 0}\dfrac{2+\frac{1}{x}\cos x}{1-\frac{1}{x}\sin\frac{1}{x}} = 2$

$0\times\infty$ 型

$\begin{cases} \dfrac{0}{\frac{1}{\infty}}, 即 \dfrac{0}{0} \\ \dfrac{\infty}{\frac{1}{0}}, 即 \dfrac{\infty}{\infty} \end{cases}$

例 1. $\lim\limits_{x\to+\infty}\left[x-x^2\ln(1+\frac{1}{x})\right] = \lim\limits_{x\to+\infty} x^2\left[\frac{1}{x}-\ln(1+\frac{1}{x})\right]$

$= \lim\limits_{x\to+\infty}\dfrac{\frac{1}{x}-\ln(1+\frac{1}{x})}{\frac{1}{x^2}} = \lim\limits_{t\to 0}\dfrac{t-\ln(1+t)}{t^2} = \lim\limits_{t\to 0}\dfrac{1-\frac{1}{1+t}}{2t} = \lim\limits_{t\to 0}\dfrac{\frac{1}{1+t}}{2} = \frac{1}{2}$

$\infty-\infty$ 型　　$\lim f\pm g \overset{?}{=} \lim f \pm \lim g$ ×

若 $\lim f$、$\lim g$ 存在，则 $\lim(f\pm g) = \lim f \pm \lim g$

例 1. $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right) = \lim\limits_{x\to 0}\dfrac{\sin^2 x-x^2}{x^2\sin^2 x} = \lim\limits_{x\to 0}\dfrac{\sin^2 x-x^2}{x^4}$

$= \lim\limits_{x\to 0}\dfrac{\sin x+x}{x}\times\dfrac{\sin x-x}{x^3} = 2\lim\limits_{x\to 0}\dfrac{\cos x-1}{3x^2} = -\frac{1}{3}$

例 2、$\lim\limits_{x\to+\infty}\left(\sqrt{x^2+4x}-x\right)$

法一：$I = \lim\limits_{x\to+\infty}\dfrac{4x}{\sqrt{x^2+4x}+x} = \lim\limits_{x\to+\infty}\dfrac{4}{\sqrt{1+\frac{4}{x}}+1} = 2$

法二：$I = \lim\limits_{x\to+\infty} x\left(\sqrt{1+\frac{4}{x}}-1\right) = \lim\limits_{x\to+\infty} x\cdot\frac{1}{2}\cdot\frac{4}{x} = 2$

$$\left.\begin{array}{l}\infty^0\\0^0\end{array}\right\}\Rightarrow e^{\ln}$$

例1. $\lim\limits_{x\to0^+}\left(\frac{1}{x}\right)^{\sin x} = e^{-\lim\limits_{x\to0^+}\sin x\ln x} = e^{-\lim\limits_{x\to0^+}\frac{\sin x}{x}\times\frac{\ln x}{\frac{1}{x}}}$

$= e^{-\lim\limits_{x\to0^+}\frac{1}{x}\times(-x^2)} = e^{\lim\limits_{x\to0^+}x} = e^0=1$

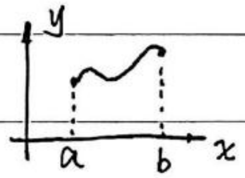
例2. $\lim\limits_{x\to0^+}x^x = e^{\lim\limits_{x\to0^+}x\ln x} = e^{\lim\limits_{x\to0^+}\frac{\ln x}{\frac{1}{x}}} = e^{\lim\limits_{x\to0^+}\frac{1}{x}\times(-x^2)} = e^0=1$

Part Ⅱ 连续与间断

一、defs.

1、连续 ① 若 $\lim\limits_{x\to a}f(x)=f(a) \Leftrightarrow f(a-0)=f(a+0)=f(a)$

称 $f(x)$ 在 $x=a$ 处连续.



② $\begin{cases} f(x)\text{ 在 }(a,b)\text{ 内点点连续} \\ f(a)=f(a+0)、f(b)=f(b-0) \end{cases}$

称 $f(x)$ 在 $[a,b]$ 上连续，记 $f(x)\in C[a,b]$

2、间断 — $\lim\limits_{x\to a}f(x)\neq f(a)$.

第一类间断点：$f(a-0)$、$f(a+0)$ 存在

$\begin{cases} f(a-0)=f(a+0)\ (\neq f(a)) & \text{，a为 可去间断点,} \\ f(a-0)\neq f(a+0) & \text{，a为 跳跃间断点.} \end{cases}$

第二类间断点：$f(a-0)$、$f(a+0)$ 至少一个不存在

例1. $f(x)=\frac{x^2-3x+2}{x^2-1}e^{\frac{1}{x}}$

解：1° $x=-1,0,1$ 为间断点,

2° $\lim\limits_{x\to-1}f(x)=\infty$，$\Rightarrow x=1$ 为第二类间断点,

$\because f(0+0)=-\infty$，$\Rightarrow x=0$ 为第二类间断点,

$\lim\limits_{x\to1}f(x)=\lim\limits_{x\to1}\frac{x-2}{x+1}e^{\frac{1}{x}}=-\frac{1}{2}e$，$\Rightarrow x=1$ 为可去间断点.

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例2. $f(x)=\frac{x}{\tan x}$

解: 1° $x=k\pi$, $x=k\pi+\frac{\pi}{2}$ $(k\in Z)$ 为间断点,

2° $\because \lim\limits_{x\to 0} f(x)=1$, $\therefore$ $x=0$ 为可去间断点,

$\lim\limits_{x\to k\pi(k\neq 0)} f(x)=\infty$, $\therefore$ $x=k\pi$ $(k\in Z$ 且 $k\neq 0)$ 为第二类间断点,

$\lim\limits_{x\to k\pi+\frac{\pi}{2}} f(x)=0$, $\therefore$ $x=k\pi+\frac{\pi}{2}$ $(k\in Z)$ 为可去间断点,

例3. $f(x)=\dfrac{1-2^{\frac{1}{x}}}{1+2^{\frac{1}{x}}}$

解: $x=0$ 为间断点.

$\lim\limits_{x\to 0^-} f(x)=1$; $\lim\limits_{x\to 0^+} f(x)=-1$. $\because f(0-0)\neq f(0+0)$, $\therefore x=0$ 为跳跃间断点

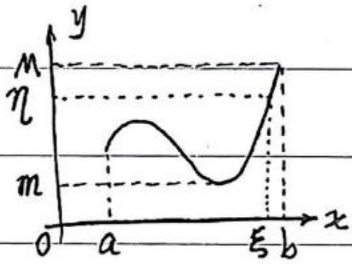
二、$f(x)\in C[a,b]$

1、$f(x)\in C[a,b] \Rightarrow \exists\, m, M$

2、$f(x)\in C[a,b] \Rightarrow \exists\, k>0$, 使 $|f(x)|\leq k$.

3、$f(x)\in C[a,b]$ 且 $f(a)f(b)<0$, 则 $\exists\, \xi\in(a,b)$, 使 $f(\xi)=0$.

☆ 4、介值定理



$f(x)\in C[a,b]$

$\forall\, \eta\in[m,M]$, $\exists\, \xi\in[a,b]$,

使 $f(\xi)=\eta$

Note:

① $f(x)\in C[a,b]$, 若 $\exists\, c\in(a,b) \Rightarrow$ 零点定理

② $f(x)\in C[a,b]$, 若 $\exists\, \xi\in[a,b] \Rightarrow$ 介值定理

例1. $f(x)\in C[a,b]$. $\forall\, p>0, q>0$. 证: $\exists\, \xi\in[a,b]$, 使 $pf(a)+qf(b)$

$=(p+q)f(\xi)$

证: $f(x)\in C[a,b] \Rightarrow \exists\, m, M$.

$(p+q)m\leq pf(a)+qf(b)\leq(p+q)M$

$m \le \frac{pf(a)+qf(b)}{p+q} \le M$ $\therefore \exists \xi \in [a,b]$，使 $f(\xi) = \frac{pf(a)+qf(b)}{p+q}$

例2. $f(x) \in C[a,b]$，证：$\exists \xi \in [a,b]$，使 $\int_a^b f(x)dx = f(\xi)(b-a)$.

1° $f(x) \in C[a,b] \Rightarrow \exists m, M$，$m \le f(x) \le M \Rightarrow m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

$\Rightarrow m \le \frac{1}{b-a}\int_a^b f(x)dx \le M$

2° $\exists \xi \in [a,b]$，使 $f(\xi) = \frac{1}{b-a}\int_a^b f(x)dx$

③ $f(x) \in C[a,b]$，出现函数值之和，用介值定理

例3. $f(x) \in C[0,2]$，且 $f(0)+2f(1)+3f(2)=6$. 证：$\exists c \in [0,2]$，使 $f(c)=1$.

证：$f(x) \in C[0,2] \Rightarrow \exists m, M$. $6m \le f(0)+2f(1)+3f(2) \le 6M$

$\Rightarrow m \le 1 \le M \Rightarrow \exists c \in [0,2]$，使 $f(c)=1$

第二章 导数与微分

一、defs.

1、导数 — $y=f(x)$ $(x \in D)$. $x_0 \in D$

$$\Delta y = f(x_0+\Delta x) - f(x_0)$$

若 $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ 存在，称 $y=f(x)$ 在 x_0 处可导

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \triangleq f'(x_0)、\left.\frac{dy}{dx}\right|_{x=x_0}$$

Notes:

① $f'(x_0) = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

$f'(x_0) = \lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$

② $f(x)$ 在 x_0 处可导 $\Rightarrow f(x)$ 在 x_0 处连续.

$\because \lim\limits_{x \to x_0}[f(x)-f(x_0)]/(x-x_0)$ 存在，$\therefore \lim\limits_{x \to x_0} f(x) = f(x_0)$

③ $\Delta x \to 0 \begin{cases} \Delta x \to 0^- \\ \Delta x \to 0^+ \end{cases}$ 或 $x \to a \begin{cases} x \to a^- \\ x \to a^+ \end{cases}$

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$$\lim_{\Delta x\to 0^-}\frac{\Delta y}{\Delta x}\left(=\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}\right)\triangleq f'_-(a)$$

$$\lim_{\Delta x\to 0^+}\frac{\Delta y}{\Delta x}\left(=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}\right)\triangleq f'_+(a)$$

☆ $f'(a)$存在 $\Leftrightarrow$ $f'_-(a)$、$f'_+(a)$ 存在且相等

例1. $f(x)=|x|$ 在 $x=0$处连续

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x}=\lim_{x\to 0^-}\frac{|x|}{x}=-1$$

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x}=\lim_{x\to 0^+}\frac{|x|}{x}=1$$

$\because f'_-(0)\neq f'_+(0)$，$\therefore f'(0)$ 不存在

④ $f(x)$ 在 $x=a$ 处连续，如 $\lim_{x\to a}\frac{f(x)-b}{x-a}=A$

则 $\begin{cases} b=f(a) \\ A=f'(a) \end{cases}$

2、可微 — $y=f(x)$ $(x\in D)$，$x_0\in D$

$$\Delta y=f(x_0+\Delta x)-f(x_0)$$

若 $\Delta y=A\Delta x+o(\Delta x)$，称 $f(x)$ 在 x_0处可微

$$\underset{\displaystyle Adx}{\overset{\displaystyle A\Delta x}{\|}}\triangleq dy\Big|_{x=x_0}$$ —— $f(x)$ 在 x_0处的微分

Note: ①可导 $\Leftrightarrow$ 可微

证明："$\Rightarrow$" 设 $f(x)$在x_0处可导，即 $\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=f'(x_0)$

$$\Rightarrow \frac{\Delta y}{\Delta x}=f'(x_0)+\alpha,\ \alpha\to 0\ (\Delta x\to 0)$$

$$\Rightarrow \Delta y=f'(x_0)\Delta x+\alpha\cdot\Delta x$$

$\because \lim_{\Delta x\to 0}\frac{\alpha\Delta x}{\Delta x}=0$，$\therefore \alpha\Delta x=o(\Delta x)$

$\therefore \Delta y=f'(x_0)\Delta x+o(\Delta x)$

"$\Leftarrow$" 设 $\Delta y=A\Delta x+o(\Delta x)$

$$\Rightarrow \frac{\Delta y}{\Delta x}=A+\frac{o(\Delta x)}{\Delta x}\Rightarrow \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=A$$

② $\Delta y = A\Delta x + o(\Delta x) \Rightarrow A = f'(x_0)$

③ $y=f(x)$ 可微. $dy = f'(x)dx$

如 $d(x^2) = 2x\,dx$

二、求导工具:

(一) 基本公式.

1. $(C)'=0$;

2. $(x^a)' = ax^{a-1}$ $\begin{cases} (\sqrt{x})' = \frac{1}{2\sqrt{x}} \\ (\frac{1}{x})' = -\frac{1}{x^2} \end{cases}$;

3. $(a^x)' = a^x\ln a$, $(e^x)' = e^x$;

4. $(\log_a x)' = \frac{1}{x\ln a}$, $(\ln x)' = \frac{1}{x}$;

5. ① $(\sin x)' = \cos x$

② $(\cos x)' = -\sin x$

③ $(\tan x)' = \sec^2 x$

④ $(\cot x)' = -\csc^2 x$

⑤ $(\sec x)' = \sec x\tan x$

⑥ $(\csc x)' = -\csc x\cot x$

6. ① $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

② $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

③ $(\arctan x)' = \frac{1}{1+x^2}$

④ $(\text{arccot}\,x)' = -\frac{1}{1+x^2}$

(二) 四则

1. $(u\pm v)' = u'\pm v'$

2. ① $(uv)' = u'v + uv'$ ② $(uvw)' = u'vw + uv'w + uvw'$

③ $(ku)' = ku'$

3. $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

(三) 复合函数求导的链式运算

$y=f(u)$ 可导. $u=\varphi(x)$ 可导且 $\varphi'(x)\neq 0$

则 $y=f[\varphi(x)]$ 可导. 且 $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = f'(u)\cdot\varphi'(x)$

$= f'[\varphi(x)]\varphi'(x)$

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$\varphi'(x)=\lim\limits_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\neq 0 \Rightarrow \Delta u = O(\Delta x)$

$\frac{dy}{dx}=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}=\lim\limits_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\cdot\lim\limits_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=f'(u)\cdot\varphi'(x)$

$=f'[\varphi(x)]\varphi'(x)$

(四) 反函数的导数

中学：$y=f(x)\Rightarrow x=\varphi(y)(\Rightarrow \underset{\times}{y=\varphi(x)})$

例1. $y=\ln(x+\sqrt{1+x^2})$. 求反函数

解：$\begin{cases} x+\sqrt{1+x^2}=e^y \\ -x+\sqrt{1+x^2}=e^{-y}\end{cases} \Rightarrow x=\frac{e^y-e^{-y}}{2}$

Th1: $y=f(x)$ 可导且 $f'(x)\neq 0$. $x=\varphi(y)$ 为其反函数.

则 $\varphi'(y)=\frac{1}{f'(x)}$

$\left.\begin{aligned} &f'(x)=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\neq 0, \quad \Delta y = O(\Delta x) \\ &\varphi'(y)=\lim\limits_{\Delta y\to 0}\frac{\Delta x}{\Delta y}\neq 0, \quad \Delta x = O(\Delta y)\end{aligned}\right\} \varphi'(y)=\frac{1}{f'(x)}$

三、求导类型

Case 1. 定义.

例1. 已知 $f'(a)$ 存在. $\lim\limits_{h\to 0}\frac{f(a+2h)-f(a-h)}{h}$?

解: $\lim\limits_{h\to 0}\frac{f(a+2h)-f(a-h)}{h}=\lim\limits_{h\to 0}\left[2\cdot\frac{f(a+2h)-f(a)}{2h}+\frac{f(a-h)-f(a)}{-h}\right]$

$=2f'(a)+f'(a)=3f'(a)$

例2. $f(x)=\begin{cases}\frac{x\cdot 2^{\frac{1}{x}}}{1+2^{\frac{1}{x}}}, & x\neq 0 \\ 0, & x=0\end{cases}$. $f'(0)$?

解: $\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x}=\lim\limits_{x\to 0}\frac{2^{\frac{1}{x}}}{1+2^{\frac{1}{x}}}$

$f'_-(0)=0$, $f'_+(0)=1$. $\because f'_-(0)\neq f'_+(0)$, $\therefore f'(0)$ 不存在

Case2. 显函数求导. $y=f(x)$

例1. $y=x^2\ln(x+\arctan x)$

解: $y' = 2x\ln(x+\arctan x) + x^2 \cdot \frac{1}{x+\arctan x} \cdot (1+\frac{1}{1+x^2})$

例2. $y = e^{\sin^2\frac{1}{x}}$

解: $y' = e^{\sin^2\frac{1}{x}} \cdot (2\sin\frac{1}{x}) \cdot (\cos\frac{1}{x}) \cdot (-\frac{1}{x^2})$

例3. $y = (1+x^2)^{\sin 2x}$

解: $y = e^{\sin 2x \ln(1+x^2)} \Rightarrow y' = e^{\sin 2x \ln(1+x^2)} \cdot [(2\cos 2x)\ln(1+x^2) + (\sin 2x)\cdot\frac{2x}{1+x^2}]$

Case 3. 隐函数求导 $F(x,y)=0$

例1. $e^{x+y} = \tan xy + 1$, 求 $\frac{dy}{dx}$.

解: $e^{x+y} \cdot (1+\frac{dy}{dx}) = \sec^2(xy) \cdot (y + x\frac{dy}{dx})$

例2. $\ln\sqrt{x^2+y^2} = \arctan\frac{y}{x}$, 求 $\frac{dy}{dx}$

解: $\frac{1}{\sqrt{x^2+y^2}} \cdot \frac{2x+2y\frac{dy}{dx}}{2\sqrt{x^2+y^2}} = \frac{1}{1+(\frac{y}{x})^2} \cdot \frac{x\frac{dy}{dx} - y\times 1}{x^2}$

Case 4. $\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$. ~~$\varphi(t)$、$\psi(t)$可导,且$\varphi'(t)\neq 0$~~. $\frac{dy}{dx}$

$\varphi'(t) = \lim\limits_{\Delta t\to 0}\frac{\Delta x}{\Delta t} \neq 0$, $\Delta x = O(\Delta t)$

$\frac{dy}{dx} = \lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{\Delta y/\Delta t}{\Delta x/\Delta t} = \lim\limits_{\Delta t\to 0}\frac{\Delta y/\Delta t}{\Delta x/\Delta t} = \frac{\psi'(t)}{\varphi'(t)}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\psi'(t)}{\varphi'(t)}$

$\frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dx} = \frac{d[\frac{\psi'(t)}{\varphi'(t)}]/dt}{dx\ /dt} = \frac{[\frac{\psi'(t)}{\varphi'(t)}]'}{\varphi'(t)}$

例1. $\begin{cases} x=\arctan t \\ y=\ln(1+t^2) \end{cases}$, $\frac{d^2y}{dx^2}=?$

解: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1+t^2}\Big/\frac{1}{1+t^2} = 2t$

$\frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})/dt}{dx/dt} = \frac{2}{\frac{1}{1+t^2}} = 2(t^2+1)$

Case 5. 分段函数求导

例1. $f(x) = \begin{cases} \frac{\sin x}{x}, & x\neq 0 \\ 1, & x=0 \end{cases}$, $f'(x)$.

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解：1°. $\lim\limits_{x\to0} f(x)=1=f(0) \Rightarrow f(x)$ 在 $x=0$ 处连续.

2°. $x\neq0$ 时，$f'(x)=\dfrac{x\cos x-\sin x}{x^2}$ ；

$x=0$ 时，$\lim\limits_{x\to0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to0}\dfrac{\frac{\sin x}{x}-1}{x}=\lim\limits_{x\to0}\dfrac{\sin x-x}{x^2}=0$

$\Rightarrow f'(0)=0 \quad \therefore f(x)=\begin{cases}0, & x=0\\ (x\cos x-\sin x)/x^2, & x\neq0\end{cases}$

例2. $f(x)=\begin{cases}e^x, & x<0\\ ax+b, & x\geq0\end{cases}$ 且 $f'(0)$ 存在，求 a,b.

解：1°. $f(0-0)=1$ ，$f(0)=f(0+0)=b$ $\because f(x)$ 在 $x=0$ 处连续，$\therefore b=1$

2° $f'_-(0)=\lim\limits_{x\to0^-}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to0^-}\dfrac{e^x-1}{x}=1$

$f'_+(0)=\lim\limits_{x\to0^+}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to0^+}\dfrac{ax+1-1}{x}=a$

$\because f'_-(0)=f'_+(0)$ ，$\therefore a=1$

Case 6. 高阶导数 $\begin{cases}\text{归纳}\\ \text{公式}\end{cases}$

例1. $y=e^x\sin x$. $y^{(n)}$?

解：$y'=e^x\sin x+e^x\cos x=e^x(\sin x+\cos x)=\sqrt2\,e^x\sin(x+\frac{\pi}{4})$

$y^{(n)}=(\sqrt2)^n e^x\sin(x+\frac{n\pi}{4})$

例2. $y=\dfrac{1}{ax+b}$ ，$y^{(n)}$?

解：$y=(ax+b)^{-1}$ ；$y'=(-1)(ax+b)^{-2}\cdot a$ ；$y''=(-1)(-2)(ax+b)^{-3}\cdot a^2$

$y^{(n)}=(-1)^n\cdot n!\,a^n/(ax+b)^{n+1}$

记：$\left(\dfrac{1}{ax+b}\right)^{(n)}=\dfrac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}$

例3. $f(x)=\dfrac{1}{x^2-3x+2}$ ，$f^{(n)}(x)$?

解：$f(x)=\dfrac{1}{(x-1)(x-2)}=\dfrac{1}{x-1}-\dfrac{1}{x-2}$

$f^{(n)}(x)=\dfrac{(-1)^n n!}{(x-2)^{n+1}}-\dfrac{(-1)^n n!}{(x-1)^{n+1}}$

记：$\begin{cases}(uv)^{(n)}=C_n^0u^{(n)}v+C_n^1u^{(n-1)}v'+\cdots+C_n^nuv^{(n)}\\ (\sin x)^{(n)}=\sin(x+\frac{n\pi}{2})\\ (\cos x)^{(n)}=\cos(x+\frac{n\pi}{2})\end{cases}$

例4. $f(x)=x^2\sin 3x$. $f^{(5)}(x)$

解: $f^{(5)}(x)=C_5^0 x^2\cdot\sin(3x+\frac{5\pi}{2})\cdot 3^5+C_5^1 2x\cdot\sin(3x+\frac{4\pi}{2})\cdot 3^4$

$+C_5^2 2\cdot\sin(3x+\frac{3\pi}{2})\cdot 3^3$

第三章 一元微分学的应用

Part I 中值定理

引言: $f'(a)\begin{cases}>0 \Rightarrow \text{左低右高}\\ <0 \Rightarrow \text{左高右低}\\ \left.\begin{array}{l}=0\\ \text{不存在}\end{array}\right\}\Rightarrow \text{可能为极值点，但不一定}\end{cases}$

① $f'(a)>0$. $f'(a)=\lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a}>0$

$\exists\delta>0$, 当 $0<|x-a|<\delta$ 时, $\frac{f(x)-f(a)}{x-a}>0$

$\begin{cases}f(x)<f(a), x\in(a-\delta,a)\\ f(x)>f(a), x\in(a,a+\delta)\end{cases}$ $\Rightarrow$ 左低右高.

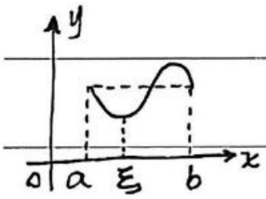
② $f'(a)<0$ $\Rightarrow$ 左高右低.

$f'(a)\neq 0 \Rightarrow x=a$ 一定不是极值点

若 $x=a$ 为极值点, $\underset{\not\Leftarrow}{\Rightarrow}$ $f'(a)=0$ 或 $f'(a)$ 不存在

如: $y=x^3$, $y'=3x^2$, $y'(0)=0$

Th1. (Rolle) $\begin{cases}① f(x)\in C[a,b]\\ ② \text{~~f(x)在(a,b)内可导~~}\\ ③ f(a)=f(b)\end{cases}$, 则 $\exists\xi\in(a,b)$, 使 $f'(\xi)=0$.



证: $f(x)\in C[a,b]\Rightarrow\exists m,M$

① $m=M$. $f(x)\equiv C_0$. $\forall\xi\in(a,b)$, $f'(\xi)=0$;

② $m<M$. $\because f(a)=f(b)$ $\therefore m,M$ 至少一个在 (a,b) 内达到.

设 $\exists\xi\in(a,b)$, 使 $f(\xi)=M\Rightarrow f'(\xi)=0$ 或 $f'(\xi)$ 不存在. $\because f(x)$

在 (a,b) 内可导, $\therefore f'(\xi)=0$

☆ Th2. (Lagrange) $\begin{cases}① f(x)\in C[a,b]\\ ② \text{~~f(x)在(a,b)内可导~~}\end{cases}$ 则 $\exists\xi\in(a,b)$ 使

$f'(\xi)=\frac{f(b)-f(a)}{b-a}$

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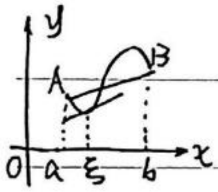
Note: ① 若 $f(a)=f(b)$，则 Lagrange $\Rightarrow$ Rolle

② 等价形式：

$$f'(\xi)=\frac{f(b)-f(a)}{b-a} \Leftrightarrow f(b)-f(a)=f'(\xi)(b-a)$$

$$\Leftrightarrow f(b)-f(a)=f'[a+\theta(b-a)](b-a) \quad (0<\theta<1)$$



③ ξ 对 a,b 有依赖性，且 a,b 可以为变量

如：$f(x)-f(a)=f'(\xi)(x-a)$

分析：过 AB： $L: y=f(x)$ 和 L_{AB}： $y-f(a)=\frac{f(b)-f(a)}{b-a}(x-a)$

即 L_{AB}： $y=f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$

证：令 $\varphi(x)=$ 曲 $-$ 直 $=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$

$\varphi(x)\in C[a,b]$，$\varphi(x)$ 在 (a,b) 内可导，$\varphi(a)=0$，$\varphi(b)=0$

$\exists\,\xi\in(a,b)$，使 $\varphi'(\xi)=0$. 而 $\varphi'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}$

$\therefore f'(\xi)=\frac{f(b)-f(a)}{b-a}$

型一：$f^{(n)}(\xi)=0$ (Rolle)

例1、$f(x)\in C[0,2]$，在 $(0,2)$ 内可导.

$3f(0)=f(1)+2f(2)$. 证：$\exists\,\xi\in(0,2)$，使 $f'(\xi)=0$

证：1° $f(x)\in[1,2] \Rightarrow \exists\, m, M$，使 $3m\le f(1)+2f(2)\le 3M$

$m\le\frac{f(1)+2f(2)}{3}\le M$. $\therefore \exists\, c\in[1,2]$，使 $f(c)=\frac{f(1)+2f(2)}{3}$

$\Rightarrow f(1)+2f(2)=3f(c)$. 2° $\because f(0)=f(c)$，$\therefore \exists\,\xi\in(0,c)\subset$ $(0,2)$，使 $f'(\xi)=0$.

例2. $f(x)\in C[a,b]$，(a,b) 内可导. $f(a)f(\frac{a+b}{2})<0$，$f(a)f(b)>0$.

证：$\exists\,\xi\in(a,b)$ 内，使 $f'(\xi)=0$.

证：1° $\exists\, x_1\in(a,\frac{a+b}{2})$，$f(x_1)=0$；$\because f(a)f(b)>0$，$\therefore f(\frac{a+b}{2})f(b)<0$，$\therefore \exists\, x_2\in(\frac{a+b}{2},b)$，使 $f(x_2)=0$； 2° $f(x_1)=f(x_2)=0$；

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$\exists\,\xi\in(x_1,x_2)\subset(a,b)$，使 $f'(\xi)=0$.

例3. $f(x)$在$[0,1]$上三阶可导，$f(1)=0$. $F(x)=x^3f(x)$. 证：$\exists\,\xi\in(0,1)$内，使 $f'''(\xi)=0$.

证：1° $F(0)=F(1)=0$. $\exists\,\xi_1\in(0,1)$，使 $f'(\xi_1)=0$.

2°. $F'(x)=3x^2f(x)+x^3f'(x)$. $\because F'(0)=0=F'(\xi_1)$

$\therefore\ \exists\,\xi_2\in(0,\xi_1)$，使 $F''(\xi_2)=0$.

3° $F''(x)=6xf(x)+3x^2f'(x)+3x^2f'(x)+x^3f''(x)$

$\because F''(0)=0$，$F''(\xi_2)=0$. $\therefore\ \exists\,\xi\in(0,\xi_2)\subset(0,1)$，使 $F'''(\xi)=0$

型二：仅有ξ、无其他字母.

① 还原法. $[\ln f(x)]'=\dfrac{f'(x)}{f(x)}$

例1. 设$f(x)\in C[0,1]$，$(0,1)$内可导. $f(1)=0$. 证：$\exists\,\xi\in(0,1)$，使 $\xi f'(\xi)+3f(\xi)=0$.

分析：$xf'(x)+3f(x)=0\Rightarrow\dfrac{f'(x)}{f(x)}+\dfrac{3}{x}=0\Rightarrow[\ln f(x)]'+(\ln x^3)'=0$

证：令 $\varphi(x)=x^3f(x)$ $\because f(1)=0$，$\therefore\ \varphi(0)=\varphi(1)=0$

$\exists\,\xi\in(0,1)$，使 $\varphi'(\xi)=0$. 而 $\varphi'(x)=3x^2f(x)+x^3f'(x)$

$\therefore\ 3\xi^2f(\xi)+\xi^3f'(\xi)=0$

$\because \xi\neq0$，$\therefore\ 3f(\xi)+\xi f'(\xi)=0$

例2. $f(x)\in C[a,b]$，(a,b)内可导. $f(a)=f(b)=0$. 证：$\exists\,\xi\in(a,b)$，使 $f'(\xi)-2f(\xi)=0$. 分析：$f'(x)-2f(x)=0\Rightarrow\dfrac{f'(x)}{f(x)}-2=0$

$\Rightarrow[\ln f(x)]'+(\ln e^{-2x})'=0$

证：令 $\varphi(x)=e^{-2x}f(x)$. $\because f(a)=f(b)=0$ $\therefore\ \varphi(a)=\varphi(b)=0$.

$\therefore\ \exists\,\xi\in(a,b)$，使 $\varphi'(\xi)=0$. 而 $\varphi'(x)=-2e^{-2x}f(x)+e^{-2x}f'(x)$

$=e^{-2x}[f'(x)-2f(x)]$. $\therefore\ e^{-2\xi}[f'(\xi)-2f(\xi)]=0$，而 $e^{-2\xi}\neq0$，$\therefore$ 得证.

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Th3. (Cauchy) $\begin{cases} ① f(x), g(x) \in C[a,b] \\ ② f(x), g(x) 在 (a,b) 内可导 \\ ③ g'(x) \neq 0 \ (a<x<b) \end{cases}$，则 $\exists \xi \in (a,b)$，使 $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$

Notes: ① $g'(x) \neq 0 \Rightarrow \begin{cases} g(b)-g(a) \neq 0 \\ g'(\xi) \neq 0 \end{cases}$

② $g'(x) \neq 0 \quad (a<x<b)$，若 $g'(a)=0$ 或 $g'(b)=0$ 不影响.

③ L: $\varphi(x) = 曲 - 直 = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$

C: $\varphi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)]$

$\varphi(a) = \varphi(b) = 0$，Rolle

④ 若 $g(x) = x$，则 $C \Rightarrow L$

②分组法.

例1. $f(x) \in C[0,1]$，在 $(0,1)$ 内可导. $f(0)=0$，$f(\frac{1}{2})=1$，$f(1)=\frac{1}{2}$.

证: ① $\exists c \in (0,1)$，使 $f(c)=c$. ② $\exists \xi \in (0,1)$，使 $f'(\xi)-2f(\xi)+2\xi=1$.

证: ① $h(x) = f(x) - x$，$h(\frac{1}{2}) = \frac{1}{2} > 0$，$h(1) = -\frac{1}{2} < 0$. $\therefore \exists c \in (\frac{1}{2}, 1) \subset (0,1)$，

使 $h(c)=0 \Rightarrow f(c)=c$

② $[f(x)-x]' - 2[f(x)-x] = 0 \Rightarrow h' - 2h = 0 \Rightarrow \frac{h'}{h} - 2 = 0$

$\Rightarrow (\ln h)' + (\ln e^{-2x})' = 0 \Rightarrow \varphi(x) = e^{-2x}[f(x)-x]$

$\because f(0)=0$，$f(c)=c$，$\therefore \varphi(0) = \varphi(c) = 0$，再用 Rolle.

型三: 有 ξ，有 a、b.

① ξ 与 a、b 可分离: ξ 与 a、b 分离 $\Rightarrow$ a、b 侧 $\begin{cases} \frac{f(b)-f(a)}{b-a} - L \\ \frac{f(b)-f(a)}{g(b)-g(a)} - C \end{cases}$

例1. $f(x) \in C[a,b]$，(a,b) 内可导 $(a>0)$.

证: $\exists \xi \in (a,b)$，使 $2\xi[f(b)-f(a)] = (b^2-a^2)f'(\xi)$

分析: $\frac{f'(\xi)}{2\xi} = \frac{f(b)-f(a)}{b^2-a^2}$

证: 令 $g(x) = x^2$，$g'(x) = 2x \neq 0$，$\exists \xi \in (a,b)$，使 $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$

即 $\frac{f(b)-f(a)}{b^2-a^2} = \frac{f'(\xi)}{2\xi}$

例2. $0<a<b$. 证: $\exists\xi\in(a,b)$, 使 $ae^b-be^a=(a-b)(1-\xi)e^\xi$

分析: $\frac{ae^b-be^a}{a-b}=(1-\xi)e^\xi \Rightarrow (\frac{e^b}{b}-\frac{e^a}{a})/(\frac{1}{b}-\frac{1}{a})$

证: $f(x)=\frac{e^x}{x}$, $g(x)=\frac{1}{x}$, $g'(x)=\frac{1}{x}\neq0$, $\exists\xi\in(a,b)$, 使

$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}$

② ξ 与 a,b 不可分 —— 凑微法

$\left.\begin{array}{l}\xi\to x\\ \text{去分母,移项}\end{array}\right\} \Rightarrow$ 式子 $=0 \Rightarrow (\underbrace{\quad ? \quad}_{\text{辅助函数}})'=0$

例1. $f(x)$、$g(x)\in C[a,b]$, (a,b) 内可导. $g'(x)\neq0$ $(a<x<b)$. 证:

$\exists\xi\in(a,b)$. 使 $\frac{f(\xi)-f(a)}{g(b)-g(\xi)}=\frac{f'(\xi)}{g'(\xi)}$

分析: $\frac{f(x)-f(a)}{g(b)-g(x)}=\frac{f'(x)}{g'(x)} \Rightarrow f(x)g'(x)-f(a)g'(x)-f'(x)g(b)+f'(x)g(x)=0$

$\Rightarrow [f(x)g(x)-f(a)g(x)-f(x)g(b)]'=0$

证: 令 $\varphi(x)=f(x)g(x)-f(a)g(x)-f(x)g(b)$

$\varphi(a)=-f(a)g(b)$, $\varphi(b)=-f(a)g(b)$

$\because\varphi(a)=\varphi(b)$, $\therefore\exists\xi\in(a,b)$, 使 $\varphi'(\xi)=0$ 而 $\varphi'(x)=\cdots$

型四、有 ξ, η.

① 仅有 $f'(\xi)$、$f'(\eta)$. $\left\{\begin{array}{l}\text{找三点,}\\ \text{两次 L.}\end{array}\right.$

例1. $f(x)\in C[0,1]$, $(0,1)$ 内可导. $f(0)=0$, $f(1)=1$, 证:

① $\exists C\in(0,1)$, 使 $f(c)=1-c$. ② $\exists\xi$、$\eta\in(0,1)$, 使 $f'(\xi)f'(\eta)=1$

证: ① 令 $\varphi(x)=f(x)-1+x$. $\varphi(0)=-1$, $\varphi(1)=1$. $\because\varphi(0)\varphi(1)<0$

$\therefore\exists C\in(0,1)$, 使 $\varphi(c)=0 \Rightarrow f(c)=1-c$

② $\exists\xi\in(0,c)$, $\eta\in(c,1)$, 使 $f'(\xi)=\frac{f(c)-f(0)}{c-0}=\frac{1-c}{c}$

$f'(\eta)=\frac{f(1)-f(c)}{1-c}=\frac{c}{1-c}$, $\therefore f'(\xi)\cdot f'(\eta)=1$

例2. $f(x)\in C[0,1]$, $(0,1)$ 内可导. $f(0)=0$, $f(1)=1$. 证: ① $\exists C\in(0,1)$,

使 $f(c)=\frac{1}{2}$. ② $\exists\,\xi、\eta\in(0,1)$，使 $\frac{1}{f'(\xi)}+\frac{1}{f'(\eta)}=2$

证：① $\varphi(x)=f(x)-\frac{1}{2}$，$\varphi(0)=-\frac{1}{2}$，$\varphi(1)=\frac{1}{2}>0$，$\exists\, c\in(0,1)$，$\varphi(c)=0$

$\Rightarrow f(c)=\frac{1}{2}$　② $\exists\,\xi\in(0,c)$，$\eta\in(c,1)$，使 $f'(\xi)=[f(c)-f(0)]/(c-0)$

$=\frac{1}{2c}$；$f'(\eta)=\frac{f(1)-f(c)}{1-c}=\frac{1}{2(1-c)}$；$\therefore \frac{1}{f'(\xi)}=2c$，$\frac{1}{f'(\eta)}=2(1-c)$

② $\xi、\eta$ 复杂度不同

留复杂 $\Rightarrow$ $\begin{cases} (\quad)' — L \\ \frac{(\quad)'}{(\quad)'} — C \end{cases}$

例1. $f(x)\in C[a,b]$，(a,b) 内可导，且 $f(a)=f(b)=1$. 证：$\exists\,\xi、\eta\in(a,b)$，

使 $e^{\eta-\xi}[f'(\eta)+f(\eta)]=1$.

分析：$e^{\eta}[f'(\eta)+f(\eta)]=e^{\xi}\Rightarrow e^{x}f(x)$

证：令 $\varphi(x)=e^{x}f(x)$. $\exists\,\eta\in(a,b)$，使 $[\varphi(b)-\varphi(a)]/(b-a)=\varphi'(\eta)$

$(e^{b}-e^{a})/(b-a)=e^{\eta}[f'(\eta)+f(\eta)]$

令 $h(x)=e^{x}$，$\exists\,\xi\in(a,b)$，$h'(\xi)=\frac{h(b)-h(a)}{b-a}\Rightarrow e^{\xi}=\frac{e^{b}-e^{a}}{b-a}$

$\Rightarrow e^{\xi}=e^{\eta}(f'(\eta)+f(\eta))$

例2. $f(x)\in C[a,b]$，(a,b) 内可导 $(a>0)$，证：$\exists\,\xi,\eta\in(a,b)$，使

$f'(\xi)=(a+b)\frac{f'(\eta)}{2\eta}$

证：令 $g(x)=x^2$，$g'(x)=2x\neq0$ $(a<x<b)$. $\exists\,\eta\in(a,b)$，使

$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\eta)}{g'(\eta)}$. 即 $\frac{f(b)-f(a)}{b^2-a^2}=\frac{f'(\eta)}{2\eta}$，$\frac{f(b)-f(a)}{b-a}=(a+b)\frac{f'(\eta)}{2\eta}$.

$\exists\,\xi\in(a,b)$，$f'(\xi)=\frac{f(b)-f(a)}{b-a}$　$\therefore f'(\xi)=(a+b)\frac{f'(\eta)}{2\eta}$.

背景：$\lim\limits_{x\to0}\frac{x-\sin x}{x^3}\overset{?}{=}\lim\limits_{x\to0}\frac{x-x}{x^3}=0$ ×

$\sin x=x+?x^3+o(x^3)$

Th4. (Taylor) 设 $f(x)$ 在 $x=x_0$ 邻域内 $n+1$ 阶可导.

则 $f(x)=P_n(x)+R_n(x)$

$P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

$$R_n(x)=\begin{cases}\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} & \text{—— L型}\\ o\left((x-x_0)^n\right) & \text{—— 佩亚诺型}\end{cases}$$

若 $x_0=0$，则 $f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+R_n(x)$

称 $f(x)$ 的麦克劳林公式.

记：

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\cdots$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\cdots$$

$$\frac{1}{1-x}=1+x+x^2+x^3\cdots$$

$$\frac{1}{1+x}=1-x+x^2-x^3\cdots$$

$$(1+x)^a=1+ax+\frac{a(a-1)}{2!}x^2+\cdots+\frac{a(a-1)\cdots(a-n+1)}{n!}x^n\cdots$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\cdots$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}\cdots$$

$$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}\cdots$$

$$\sin x=x-\frac{1}{6}x^3+o(x^3)$$

$$\arcsin x=x+\frac{1}{6}x^3+o(x^3)$$

$$\tan x=x+\frac{1}{3}x^3+o(x^3)$$

$$\arctan x=x-\frac{1}{3}x^3+o(x^3)$$

型五、求极限

例1. $\lim\limits_{x\to0}\frac{x-\sin x}{x^3}$.

解：$\sin x=x-\frac{1}{6}x^3+o(x^3)\Rightarrow x-\sin x\sim\frac{1}{6}x^3$

$\therefore\lim\limits_{x\to0}\frac{x-\sin x}{x^3}=\frac{1}{6}$

例2. $\lim\limits_{x\to0}\frac{e^{-\frac{x^2}{2}}-1+\frac{x^2}{2}}{x^3\arctan x}$

解：$\lim\limits_{x\to0}\frac{1+(-\frac{x^2}{2})+\frac{1}{2}(-\frac{x^2}{2})^2-1+\frac{x^2}{2}}{x^4}=\frac{1}{8}$

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例3. $\lim\limits_{x\to0}\dfrac{\sqrt{1+x}+\sqrt{1-x}-2}{x^2}=\lim\limits_{x\to0}\dfrac{1+\frac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2}x^2+1-\frac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2}x^2+1-2}{x^2}$

$=-\dfrac{1}{4}$

洛必达法则.

1° $\lim\dfrac{f(x)}{F(x)}$ $\begin{cases}\frac{0}{0}\\ \frac{\infty}{\infty}\end{cases}$

2° $\lim\dfrac{f'(x)}{F'(x)}$ 存在

$\Big\}$ $\lim\dfrac{f(x)}{F(x)}=\lim\dfrac{f'(x)}{F'(x)}$

如: $\lim\limits_{x\to0}\dfrac{x-\sin x}{x^3}=\lim\limits_{x\to0}\dfrac{1-\cos x}{3x^2}=\dfrac{1}{6}$

又如: $\lim\limits_{x\to\infty}\dfrac{2x+\sin x}{x}=\lim\limits_{x\to\infty}(2+\cos x)$ 不存在【洛必达失败，不代表极限无】

而 $\lim\limits_{x\to\infty}\dfrac{2x+\sin x}{x}=\lim\limits_{x\to\infty}(2+\frac{1}{x}\sin x)=2$

型一 $f^{(n)}(\xi)=0$ $\begin{cases}①还原法\\ ②分组法\end{cases}$

型二 仅有 ξ

型三 有 ξ，有 $a.b$

Case 1. ξ与$a.b$可分离：

ξ与$a.b$分离 $\Rightarrow$ $a.b$一侧 $\Rightarrow\begin{cases}\dfrac{f(b)-f(a)}{b-a} — L\\ \dfrac{f(b)-f(a)}{F(b)-F(a)} — C\end{cases}$

Case 2. ξ与$a.b$不可分 —— 凑微法

$\xi\to x$ $\Rightarrow$ 去分母、移项 $\Rightarrow$ 函数 $=0\Rightarrow(\varphi(x))'=0$

Rolle

型四、有 ξ、η

①仅有 $f'(\xi)$、$f'(\eta)$ $\Rightarrow$ 找三点，用两次 L

例1. $f(x)\in C[a,b]$. (a,b)内可导，$f(a)=f(b)$. $f'_+(a)>0$.

证：$\exists\,\xi、\eta\in(a,b)$，使 $f'(\xi)>0$，$f'(\eta)<0$.

1° $f'_+(a)>0\Rightarrow\exists\,C\in(a,b)$，使 $f(C)>f(a)$

2° $\exists\,\xi\in(a,C)$，$\eta\in(C,b)$，使 $f'(\xi)=\dfrac{f(C)-f(a)}{C-a}>0$，

$f'(\eta)=\dfrac{f(b)-f(C)}{b-C}<0$.

②有ξ、η 对应项复杂度不同.

留复杂 $\Rightarrow \begin{cases} (\quad)' & — L \\ \dfrac{(\quad)'}{(\quad)'} & — C \end{cases}$

例2. $f(x)\in C[a,b]$, (a,b)内可导$(a>0)$. 证: $\exists \xi、\eta\in(a,b)$,

使 $ab f'(\xi)=\eta^2 f'(\eta)$ 　　分析: $\dfrac{f'(\eta)}{1/\eta^2}=\dfrac{(\quad)'}{(\quad)'}$

证: 1° $F(x)=-\frac{1}{x}$, $F'(x)=\frac{1}{x^2}\neq 0$, $\exists\eta\in(a,b)$, $\dfrac{f(b)-f(a)}{-\frac{1}{b}+\frac{1}{a}}=\dfrac{f'(\eta)}{\frac{1}{\eta^2}}$

2° $ab\cdot\dfrac{f(b)-f(a)}{b-a}=\eta^2 f'(\eta)$

③ ξ、η复杂度对等

如: $f'(\xi)+\xi^2=f'(\eta)+\eta^2$

$\varphi(x)=f(x)+\frac{1}{3}x^3$, 找3点,用2次L

型五: $f''(x)>0\ (<0)$

① $f''(x)>0 \Rightarrow f'(x)\nearrow$

例1. $f'(x)>0$, $f''(x)>0$, $\Delta x>0$, $\Delta y=f(x_0+\Delta x)-f(x)$,

$dy=f'(x_0)\Delta x$. 问 0、Δy、dy大小?

解: $\Delta y=f'(\xi)\Delta x\ (x_0<\xi<x_0+\Delta x)$

$\because f''>0$, $\therefore f'\nearrow$, 又$\because x_0<\xi$, $\therefore f'(x_0)<f'(\xi)$

$\because \Delta x>0$ $\therefore 0<f'(x_0)\Delta x<f'(\xi)\Delta x$, 即 $0<dy<\Delta y$

例2. $f'(1)=0$, $\lim\limits_{x\to1}\dfrac{f''(x)}{(x-1)^2}=3$. $x=1$?

解: $\lim\limits_{x\to1}\dfrac{f''(x)}{(x-1)^2}=3>0 \Rightarrow \exists\ \delta>0$, 当 $0<|x-1|<\delta$ 时,

$\dfrac{f''(x)}{(x-1)^2}>0 \Rightarrow f''(x)>0$ 　$\because f'(1)=0$, $\therefore \begin{cases} f'(x)<0, & x\in(1-\delta,1) \\ f'(x)>0, & x\in(1,1+\delta) \end{cases}$

$\therefore x=1$为极小点

例3. $f''>0$, $f(0)=0$, $0<a<b$. 证: $f(a)+f(b)<f(a+b)$

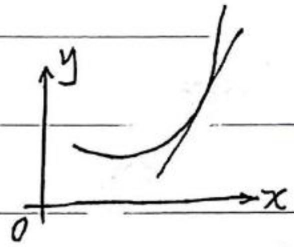
证: 1° $f(a)-f(0)=f'(\xi_1)a\quad(0<\xi_1<a)$

$f(a+b)-f(b)=f'(\xi_2)\cdot a \quad (b<\xi_2<a+b)$

2° $f''>0 \Rightarrow f'\uparrow$ $\because \xi_1<\xi_2$, $\therefore f'(\xi_1)\cdot a<f'(\xi_2)\cdot a$

$\Rightarrow f(a)-f(0)<f(a+b)-f(b)$

② $f''(x)>0 \Rightarrow$ 凹



切线 $y-f(x_0)=f'(x_0)(x-x_0)$

即 $y=f(x_0)+f'(x_0)(x-x_0)$

$\therefore \underline{f(x)\geq f(x_0)+f'(x_0)(x-x_0)}$, "$=$" $\Leftrightarrow$ "$x=x_0$"

证: $f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(\xi)}{2!}(x-x_0)^2$

$\because f''(x)>0$ $\therefore f(x)\geq f(x_0)+f'(x_0)(x-x_0)$, "$=$" $\Leftrightarrow$ "$x=x_0$"

例1. $f''>0$, $\lim\limits_{x\to0}\frac{f(x)}{x}=1$. 证: $f(x)\geq x$

1° $\lim\limits_{x\to0}\frac{f(x)}{x}=1 \Rightarrow f(0)=0,\ f'(0)=1$.

2° $f(x)=f(0)+f'(0)x+\frac{f''(\xi)}{2!}x^2$,

$\because f''(x)>0$, $\therefore f(x)\geq f(0)+f'(0)x=x$

例2. $f''>0 \ (a\leq x\leq b)$. 证: $\frac{f(a)+f(b)}{2}>f(\frac{a+b}{2})$

证明: $\frac{a+b}{2}=x_0$ $\because f''>0$ $\therefore f(x)\geq f(x_0)+f'(x_0)(x-x_0)$

"$=$" $\Leftrightarrow$ "$x=x_0$". $f(a)>f(x_0)+f'(x_0)(a-x_0)$

$f(b)>f(x_0)+f'(x_0)(b-x_0) \Rightarrow \frac{f(a)+f(b)}{2}>f(x_0)+f'(x_0)(x_0-x_0)$

$\therefore \frac{f(a)+f(b)}{2}>f(\frac{a+b}{2})$

型六 Taylor 常规

$f(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1}$

$x_0\begin{cases} ① f(c) \Rightarrow x_0=c \\ ② x=\frac{a+b}{2} \\ ? \end{cases}$ $\quad x\begin{cases} ① f(c)=?\ (无 f'(c)) \Rightarrow x=c \\ ② 端点 \\ ? \end{cases}$

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习惯: $\left.\begin{matrix} f(a)、f(b)、f(c) \\ f'(a)、f'(b)、f'(c) \end{matrix}\right\}$ 2L $\left.\begin{matrix} f^{(n)}(x)\ (n\geq 2) \\ f(a)、f'(c)、f(b) \end{matrix}\right\}$ T

例1、$f(x)$ 在 $[0,a]$ 上二阶可导，$|f''(x)|\leq M$，$f(x)$ 在 $(0,a)$ 内取最大值。证：$|f'(0)|+|f'(a)|\leq Ma$

1° $\exists c\in(0,a)$，$f(c)$ 最大 $\Rightarrow f'(c)=0$

2° $\begin{cases} f'(c)-f'(0)=f''(\xi_1)c & (0<\xi_1<c) \\ f'(a)-f'(c)=f''(\xi_2)(a-c) & (c<\xi_2<a) \end{cases}$

3° $\begin{cases} |f'(0)|\leq Mc \\ |f'(a)|\leq M(a-c) \end{cases}$

例2、$f''(x)\in C[0,1]$，$f(0)=f(1)=0$，$\min\limits_{0\leq x\leq 1} f(x)=-1$，证：

$\exists\xi\in(0,1)$，$f''(\xi)\geq 8$.

1° $\exists c\in(0,1)$，$f(c)=-1$，$f'(c)=0$

2° $\begin{cases} f(0)=f(c)+\dfrac{f''(\xi_1)}{2!}(0-c^2) & (\xi_1\in(0,c)) \\ f(1)=f(c)+\dfrac{f''(\xi_2)}{2!}(1-c^2) & (\xi_2\in(c,1)) \end{cases}$

$\Rightarrow \begin{cases} 1=\dfrac{c^2}{2}f''(\xi_1) \\ 1=\dfrac{(1-c)^2}{2}f''(\xi_2) \end{cases} \Rightarrow \begin{cases} f''(\xi_1)=\dfrac{2}{c^2} \\ f''(\xi_2)=\dfrac{2}{(1-c)^2} \end{cases}$

3° ① $c\in(0,\frac{1}{2}] \Rightarrow f''(\xi_1)\geq 8$，取 $\xi=\xi_1$

② $c\in(\frac{1}{2},1) \Rightarrow f''(\xi_2)\geq 8$，取 $\xi=\xi_2$

例3、$f'''(x)\in C[-1,1]$，$f(-1)=0$，$f'(0)=0$，$f(1)=1$

证：$\exists\xi\in(-1,1)$，$f'''(\xi)=3$.

1° $f(-1)=f(0)+\dfrac{f''(0)}{2!}(-1-0)^2+\dfrac{f'''(\xi_1)}{3!}(-1-0)^3$，$\xi_1\in(-1,0)$

$f(1)=f(0)+\dfrac{f''(0)}{2!}(1-0)^2+\dfrac{f'''(\xi_2)}{3!}(1-0)^3$，$\xi_2\in(0,1)$

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$$\Rightarrow \begin{cases} 0 = f(0) + \frac{1}{2}f''(0) - \frac{1}{6}f'''(\xi_1) \\ 1 = f(0) + \frac{1}{2}f''(0) + \frac{1}{6}f'''(\xi_2) \end{cases}$$

2° $f'''(\xi_1) + f'''(\xi_2) = 6$

3° $f'''(x) \in C[\xi_1, \xi_2] \Rightarrow \exists m, M, m \le \frac{f'''(\xi_1)+f'''(\xi_2)}{2} \le M$

$\Rightarrow m \le 3 \le M \Rightarrow \exists \xi \in [\xi_1, \xi_2] \subset (-1,1)$，使 $f'''(\xi) = 3$.

型七 L常规

① $f(b) - f(a) \Rightarrow L$

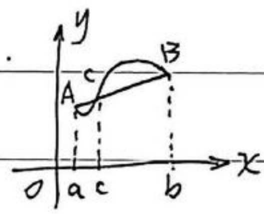
例1. $\lim\limits_{x\to\infty} f'(x) = e$. $\lim\limits_{x\to\infty}[f(x+1) - f(x)] = \lim\limits_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x$. $a=?$

解: 1° $f(x+1) - f(x) = f'(\xi)$ $(x < \xi < x+1)$

$$\left.\begin{aligned} 左 &= \lim_{x\to\infty} f'(\xi) = e \\ 右 &= e^{\lim\limits_{x\to\infty}\frac{2ax}{x-a}} = e^{2a} \end{aligned}\right\} \therefore a = \frac{1}{2}$$

② 遇到 $f(a)$、$f(c)$、$f(b) \Rightarrow 2L$

例2. $f(x)$ 在 $[a,b]$ 上二阶可导. 连 $A(a, f(a))$、$B(b, f(b))$ 的直线交 $y = f(x)$ 于 $C(c, f(c))$ $(a<c<b)$.



证: $\exists \xi \in (a,b)$, $f''(\xi) = 0$

1° $\exists \xi_1 \in (a,c)$, $\xi_2 \in (c,b)$, 使

$$f'(\xi_1) = \frac{f(c) - f(a)}{c-a}, \quad f'(\xi_2) = \frac{f(c) - f(b)}{c-b}$$

2° $\because k_{AC} = k_{BC}$, $\therefore f'(\xi_1) = f'(\xi_2)$

例3. $f(x)$ 在 $[a,b]$ 上可导, $|f'(x)| \le M$. $f(x)$ 在 (a,b) 内至少一个零点. 证: $|f(a)| + |f(b)| \le M(b-a)$

1° $\exists c \in (a,b)$, 使 $f(c) = 0$

2° $\begin{cases} f(c) - f(a) \\ f(b) - f(c) \end{cases}$

③ 见到 $f'、f \Rightarrow \{$ N-L

Part Ⅱ 单调性与极值

$y=f(x)$.

1°. $x\in D$;

2°. $f'(x)\begin{cases}=0 & \text{(驻点)} \\ \text{不可导点}\end{cases}$ (不一定)

3°. 判别法

方法一:

① $\begin{cases}x<x_0, f'<0 \\ x>x_0, f'>0\end{cases} \Rightarrow x_0$ 为极小点

② $\begin{cases}x_0<x_0, f'>0 \\ x>x_0, f'<0\end{cases} \Rightarrow x_0$ 为极大点

方法二:

$f'(x_0)=0.\ f''(x_0)\begin{cases}>0 \Rightarrow \text{小} \\ <0 \Rightarrow \text{大}\end{cases}$

$f''(x_0)=\lim\limits_{x\to x_0}\frac{f'(x)}{x-x_0}>0$

$\Rightarrow \exists\,\delta>0$, 当 $0<|x-x_0|<\delta$ 时, $\frac{f'(x)}{x-x_0}>0$

$\begin{cases}f'(x)<0, & x\in(x_0-\delta, x_0) \\ f'(x)>0, & x\in(x_0, x_0+\delta)\end{cases} \Rightarrow x_0$ 为极小点

型一 极值点判断

例1、$f'(1)=0$. $\lim\limits_{x\to 1}\frac{f'(x)}{\sin^3\pi x}=-1$. $x=1$?

解: $\exists\,\delta>0$, 当 $0<|x-1|<\delta$ 时, $\frac{f'(x)}{\sin^3\pi x}<0$

$\begin{cases}f'(x)<0, & x\in(1-\delta, 1) \\ f'(x)>0, & x\in(1, 1+\delta)\end{cases} \Rightarrow x=1$ 为极小点

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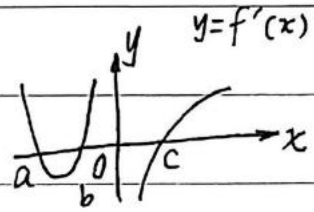
例2、$f(x)\in C(-\infty,+\infty)$.



1° $x=a、b、0、c$

2° $\begin{cases} x<a, f'>0 \\ x>a, f'<0 \end{cases}$ 大 $\begin{cases} x<b, f'<0 \\ x>b\ f'>0 \end{cases}$ 小

$\begin{cases} x<0\ , f'>0 \\ x>0\ , f'<0 \end{cases}$ 大 $\begin{cases} x<c\ , f'<0 \\ x>c\ , f'>0 \end{cases}$ 小

例3、$f(x)$ 连续可导. $f(0)=0$. $\lim\limits_{x\to 0}\dfrac{f(x)+f'(x)}{x}=2$. $x=0$?

1° $\lim\limits_{x\to 0}\dfrac{f(x)+f'(x)}{x}=2 \Rightarrow \lim\limits_{x\to 0}[f(x)+f'(x)]=0$

$\Rightarrow f(0)+f'(0)=0 \Rightarrow f'(0)=0$

2° $2=\lim\limits_{x\to 0}\dfrac{f(x)-f(x)}{x}+\lim\limits_{x\to 0}\dfrac{f'(x)-f(0)}{x}$

$=f'(0)+f''(0) \Rightarrow f''(0)=2>0$ 极小

例4、$f(x)$：$xf''(x)-3xf'^2(x)=1-e^{-x}$

① $x=a\ (\neq 0)$ 为极值点. 大、小?

② $f''(x)$ 连续. $x=0$ 为极值点. 大.小?

解：① $f'(a)=0$. $af''(a)=1-e^{-a} \Rightarrow f''(a)=\dfrac{1-e^{-a}}{a}$

$\left.\begin{array}{l} a<0 \Rightarrow -a>0 \Rightarrow e^{-a}>1 \Rightarrow f''(a)>0 \\ a>0 \Rightarrow -a<0 \Rightarrow e^{-a}<1 \Rightarrow f''(a)>0 \end{array}\right\} \Rightarrow$ 极小点

② $f'(0)=0$. $x\neq 0$ 时, $f''(x)=\dfrac{1-e^{-x}}{x}+3f'^2(x)$

$\because f''(x)$ 连续 $\therefore f''(0)=\lim\limits_{x\to 0}f''(x)=1+3f'^2(0)=1>0$

$\therefore$ 极小

型二：不等式证明：

① 单调性

例. $e<a<b$，证：$a^b>b^a$

证：$a^b > b^a \Leftrightarrow b\ln a - a\ln b > 0$

令 $\varphi(x) = x\ln a - a\ln x$，$\varphi(a)=0$

$\varphi'(x) = \ln a - \frac{a}{x} > 0\ (x>a)$，$\varphi'(x)>0\ (x>a)$ $\Big\} \Rightarrow \varphi(x)>0\ (x>a)$

$\because b>a$，$\therefore \varphi(b)>0$

例2. $x>0$，证：$\frac{x}{1+x} < \ln(1+x) < x$

证：$f(x) = x-\ln(1+x)$，$f(0)=0$，$f'(x)=1-\frac{1}{1+x}\ (x>0) > 0$

$\begin{cases} f(0)=0 \\ f'(x)>0\ (x>0) \end{cases} \Rightarrow f(x)>0\ (x>0)$

$g(x) = \ln(1+x) - \frac{x}{1+x}$，$g(0)=0$. $g'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2}\ (x>0) > 0$

$\begin{cases} g(0)=0 \\ g'(x)>0 \end{cases} \Rightarrow g(x)>0\ (x>0)$

例3、$f(a)=g(a)$，$f'(a)=g'(a)$，$f''(x)>g''(x)\ (x>a)$. 证：$f(x)>g(x)\ (x>a)$

证：令 $\varphi(x) = f(x)-g(x)$，$\varphi(a)=0$，$\varphi'(a)=0$，$\varphi''(x)>0\ (x>a)$

$\begin{cases} \varphi'(a)=0 \\ \varphi''(x)>0\ (x>a) \end{cases} \Rightarrow \varphi'(x)>0\ (x>a)$；$\begin{cases} \varphi(a)=0 \\ \varphi'(x)>0\ (x>a) \end{cases} \Rightarrow \varphi(x)>0\ (x>a)$

例4. 证：$1+x\ln(x+\sqrt{1+x^2}) \geq \sqrt{1+x^2}$

证：$f(x) = 1+x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2}$，$f(0)=0$

$f'(x) = \ln(x+\sqrt{1+x^2}) + x\cdot\frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{x}{\sqrt{1+x^2}}\right) - \frac{x}{\sqrt{1+x^2}}$

$= \ln(x+\sqrt{1+x^2})$. $f'(0)=0$

$f''(x) = \frac{1}{x+\sqrt{1+x^2}}\cdot\left(1+\frac{x}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}} > 0$ $\Big\}$ 极小

$\Rightarrow x=0$ 为 $f(x)$ 最小值点，而 $f(0)=0$，$\therefore f(x)\geq 0$

例5、$x>0$，证：$(x^2-1)\ln x \geq (x-1)^2$

证：$f(x) = (x^2-1)\ln x - (x-1)^2$，$f(1)=0$

$f'(x) = 2x\ln x - x - \frac{1}{x} - 2(x-1) = 2x\ln x - x - \frac{1}{x} + 2$

$f''(x) = 2\ln x + 2 - 1 + \frac{1}{x^2} = 2\ln x + 1 + \frac{1}{x^2}$. $f''(1)=2$

$f'''(x) = \frac{2}{x} - \frac{2}{x^3} = \frac{2(x^2-1)}{x^3}$ $\begin{cases} <0, & 0<x<1 \\ >0, & x>1 \end{cases}$

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$\Rightarrow x=1$ 为 $f''(x)$ 最小点，而 $f''(1)=2$，$\therefore f''(x)\geq 2>0$

$\begin{cases} f'(1)=0 \\ f''(x)>0 \end{cases} \Rightarrow \begin{cases} f'(x)<0, & 0<x<1 \\ f'(x)>0, & x>1 \end{cases} \Rightarrow x=1$ 为 $f(x)$ 最小点，

而 $f(1)=0$，$\therefore f(x)\geq 0$

② 中值定理证明不等式 $\begin{cases} f(b)-f(a) \text{ 或 } \frac{f(b)-f(a)}{b-a} & \text{— L} \\ [f(b)-f(a)]/[F(b)-F(a)] & \text{— C} \end{cases}$

例1. $0<a<b$. 证：$\frac{\ln b-\ln a}{b-a}>\frac{2a}{a^2+b^2}$

$(\Leftrightarrow (a^2+b^2)(\ln b-\ln a)-2a(b-a)>0$.

$\varphi(x)=(a^2+x^2)(\ln x-\ln a)-2a(x-a)$

$\varphi(a)=0$，$\varphi'(x)>0\ (x>a) \Rightarrow \varphi(x)>0$

$\because b>a$，$\therefore \varphi(b)>0$)

证：令 $f(x)=\ln x$，$f'(x)=\frac{1}{x}$，左 $=f'(\xi)=\frac{1}{\xi}\ (a<\xi<b)$

$\because \frac{1}{\xi}>\frac{1}{b}>\frac{2a}{a^2+b^2}$ $\therefore$ 左 > 右

例2. $a<b$，证 $\arctan b-\arctan a\leq b-a$

证：令 $f(x)=\arctan x$，$f'(x)=\frac{1}{1+x^2}$

左 $=f(b)-f(a)=f'(\xi)(b-a)\ (a<\xi<b)$

$=\frac{b-a}{1+\xi^2}\leq b-a$

例3. $x>0$. 证：$\frac{x}{1+x}<\ln(1+x)<x$

证：令 $\varphi(t)=\ln(1+t)$，$\varphi'(t)=\frac{1}{1+t}$

$\ln(1+x)=\varphi(x)-\varphi(0)=\varphi'(\xi)x\quad (0<\xi<x)$

$=\frac{x}{1+\xi}$ $\quad \because \frac{x}{1+x}<\frac{x}{1+\xi}<x$

$\therefore \frac{x}{1+x}<\ln(1+x)<x$

型三 函数零点或方程的根

① $f(x)\in C[a,b]$，$f(a)f(b)<0$

② 单调性

③ Rolle. $f(x) \Rightarrow F(x)$, $F'(x)=f(x)$

若 $F(a)=F(b)$. $\exists\, c\in(a,b)$, $F'(c)=0 \Rightarrow f(c)=0$

例1. $a_0+\frac{a_1}{2}+\cdots+\frac{a_n}{n+1}=0$. 证明: $a_0+a_1x+\cdots+a_nx^n=0$ 至少一正根.

证: 令 $f(x)=a_0+a_1x+\cdots+a_nx^n$, $F(x)=a_0x+\frac{a_1}{2}x^2+\cdots+\frac{a_n}{n+1}x^{n+1}$

$F'(x)=f(x)$. $F(0)=0$, $F(1)=0$, $\exists\,\xi\in(0,1)$, 使 $F'(\xi)=0$,

$\Rightarrow f(\xi)=0$.

例2. 证: $e^x=-x^2+2x$ 不可能有3个不同根.

证: $f(x)=e^x+x^2-2x$, 设 $\exists\, c_1<c_2<c_3$, 使 $f(c_1)=f(c_2)=f(c_3)=0$.

$\exists\,\xi_1\in(c_1,c_2)$, $\xi_2\in(c_2,c_3)$, 使 $f'(\xi_1)=f'(\xi_2)=0$.

$\exists\,\xi\in(\xi_1,\xi_2)$, $f''(\xi)=0$. 而 $f''(\xi)=e^\xi+2\neq 0$

∴ 不可能.

例3. $a>0$, 讨论 $xe^{-x}=a$ 几根?

解: 1° $f(x)=xe^{-x}-a$ $(-\infty<x<+\infty)$

2° $f'(x)=(1-x)e^{-x}=0 \Rightarrow x=1$

$\begin{cases} f'>0, & x<1 \\ f'<0, & x>1 \end{cases} \Rightarrow x=1$ 为最大点, $M=f(1)=\frac{1}{e}-a$

3° ① $M<0$, 即 $a>\frac{1}{e}$, 无零点,

② $M=0$, 即 $a=\frac{1}{e}$, 一个零点,

③ $M>0$, 即 $0<a<\frac{1}{e}$

$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}(xe^{-x}-a)=-\infty$

$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\left(\frac{x}{e^x}-a\right)=-a<0$

例4. 讨论 $\ln x=\frac{x}{e}-2\sqrt{2}$ 几根?

解: 1° $f(x)=\ln x-\frac{x}{e}+2\sqrt{2}$ $(x>0)$

2° $f'(x)=\frac{1}{x}-\frac{1}{e}=0 \Rightarrow x=e$

$f''(x)=-\frac{1}{x^2}$，$\because f''(e)=-\frac{1}{e^2}<0$，$\therefore$ $x=e$为 $f(x)$ 的最大点，

$M=F(e)=2\sqrt{2}>0$

$3°.\ \lim\limits_{x\to 0^+} f(x)=-\infty$. $\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}(\ln x-\frac{x}{e}+2\sqrt{2})=-\infty$

Part Ⅲ 几个小知识

一、凹凸性.

1、def — $y=f(x)$ $(x\in D)$

① 若 $\forall x_1, x_2\in D$ 且 $x_1\neq x_2$，有 $f(\frac{x_1+x_2}{2})<\frac{f(x_1)+f(x_2)}{2}$

称 $f(x)$ 在 D 内为凹函数

② 若 $\forall x_1, x_2\in D$ 且 $x_1\neq x_2$，有 $f(\frac{x_1+x_2}{2})>\frac{f(x_1)+f(x_2)}{2}$

称 $f(x)$ 在 D 内为凸函数.

2、判别法：

Th: $f''>0(<0)\Rightarrow f(x)$ 凹（凸）

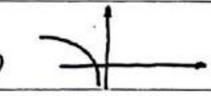
证：设 $f''>0$，$\forall x_1, x_2\in D$ 且 $x_1\neq x_2$. 令 $x_0=\frac{x_1+x_2}{2}$

$\because f''(>0)$，$\therefore f(x)\geqslant f(x_0)+f'(x_0)(x-x_0)$

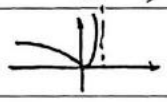
"$=$" $\Leftrightarrow$ "$x=x_0$" $\begin{cases} f(x_1)>f(x_0)+f'(x_0)(x_1-x_0) \\ f(x_2)>f(x_0)+f'(x_0)(x_2-x_0)\end{cases}$

$\Rightarrow \frac{f(x_1)+f(x_2)}{2}>f(x_0)=f(\frac{x_1+x_2}{2})$

例1、$y=|\ln(1-x)|$

解：$y=\ln x$  ，$y=\ln(-x)$ 

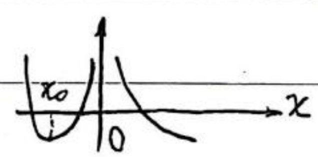
$y=\ln(1-x)=\ln[-(x-1)]$ ，

$y=|\ln(1-x)|$ ，

$\therefore$ $x=0$ 为极小点，$(0,0)$ 为拐点，

例2. $f(x)\in C(-\infty,+\infty)$，

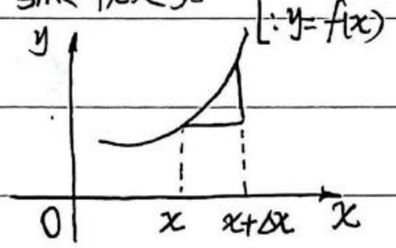
拐点？

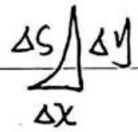


$\begin{cases} x<x_0, & f''<0 \\ x>x_0, & f''>0 \end{cases} \Rightarrow (x_0, f(x_0))$为拐点

$\begin{cases} x<0, & f''>0 \\ x>0, & f''<0 \end{cases} \Rightarrow (0, f(0))$为拐点

二、弧微分





$(\Delta s)^2 \approx (\Delta x)^2 + (\Delta y)^2$

$(ds)^2 = (dx)^2 + (dy)^2$

1、弧微分

① $L: y=f(x) \qquad ds=\sqrt{1+(\frac{dy}{dx})^2}\,dx = \sqrt{1+f'^2(x)}\,dx$

② $L: \begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases} \qquad ds=\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt = \sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt$

③ $L: r=r(\theta) \Rightarrow L: \begin{cases} x=r(\theta)\cos\theta \\ y=r(\theta)\sin\theta \end{cases} \qquad ds=\sqrt{(\frac{dx}{d\theta})^2+(\frac{dy}{d\theta})^2}\,d\theta$

$= \sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta$

2、曲率、曲率半径.

$$K=\frac{|y''|}{(1+y'^2)^{\frac{3}{2}}}, \quad R=\frac{1}{K}$$

三、渐近线

1、水平 —— $\lim\limits_{x\to\infty} f(x)=A \Rightarrow y=A$ 为水平渐近线

2、铅直渐近线 —— $\begin{cases} f(a-0)=\infty \\ f(a+0)=\infty \\ \lim\limits_{x\to a} f(x)=\infty \end{cases}$

$x=a$为$f(x)$的铅直渐近线

3、斜渐近线 — $L: y=f(x)$

$\lim\limits_{x\to\infty}\frac{f(x)}{x}=a\,(\neq 0,\infty)$

$\lim\limits_{x\to\infty}[f(x)-ax]=b$

$\Rightarrow y=ax+b$为斜渐近线.

例1、$f(x)=\frac{x^3-3x+2}{x^2-1}e^{\frac{1}{x}}$，求渐近线.

解：$\lim\limits_{x\to\infty}f(x)=\infty$，无水平渐近线

$\lim\limits_{x\to-1}f(x)=\infty$，$x=-1$ √

$\lim\limits_{x\to1}f(x)=e^{\lim\limits_{x\to1}\frac{x^3-3x+2}{x^2-1}}=e^{\lim\limits_{x\to1}\frac{3x^2-3}{2x}}=0$，$x=1$ ×

$\lim\limits_{x\to0^+}f(x)=-\infty$，$x=0$ √

$\lim\limits_{x\to\infty}\frac{f(x)}{x}=1$　$\lim\limits_{x\to\infty}[f(x)-x]=?$

第四章 不定积分

一、defs.

1、原函数 — $f(x)$、$F(x)$. 若 $F'(x)=f(x)$，称 $F(x)$ 为 $f(x)$ 的原函数.

Notes: ① 连续函数一定有原函数

② 有第一类间断点的函数无原函数

③ 若 $f(x)$ 有原函数，则一定有无数个，且任两个原函数相差常数.

④ $F(x)$　$f(x)$　$f'(x)$

偶 ⟵ 奇 ⟶ 偶

$f(x)=3x^2$，$F(x)=x^3+C$　奇 $\not\leftarrow$ 偶 ⟶ 奇

$f(x)=1-\cos x$，$F(x)=x\sin x+C$　周期 $\not\leftarrow$ 周期 ⟶ 周期

2、不定积分 — $\int f(x)\,dx=F(x)+C$

二、不定积分的两大工具.

(一) 基本公式

1、$\int k\,dx=kx+C$

2、① $\int x^a\,dx=\frac{1}{a+1}x^{a+1}+C\ (a\neq-1)$　② $\int x^{-1}\,dx=\ln|x|+C$

3. ① $\int a^x dx = \frac{a^x}{\ln a} + C$　　② $\int e^x dx = e^x + C$

4、① $\int \sin x \, dx = -\cos x + C$

② $\int \cos x \, dx = \sin x + C$

③ $\int \tan x \, dx = -\ln|\cos x| + C$

④ $\int \cot x \, dx = \ln|\sin x| + C$

⑤ $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

⑥ $\int \csc x \, dx = \ln|\csc x - \cot x| + C$

⑦ $\int \sec^2 x \, dx = \tan x + C$

⑧ $\int \csc^2 x \, dx = -\cot x + C$

⑨ $\int \sec x \tan x \, dx = \sec x + C$

⑩ $\int \csc x \cot x \, dx = -\csc x + C$

5、平方和、平方差

① $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

② $\int \frac{1}{\sqrt{a^2-x^2}} dx = \frac{1}{a} \arcsin x + C$

③ $\int \frac{1}{1+x^2} dx = \arctan x + C$

④ $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$

⑤ $\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln(x + \sqrt{x^2+a^2}) + C$

⑥ $\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln|x + \sqrt{x^2-a^2}| + C$

⑦ $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$

⑧ $\int \sqrt{a^2-x^2} \, dx = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$

(二) 积分法

方法一：换元积分法

① 第一类换元积分法

例1、$\int \frac{1}{x\ln^2 x}dx = \int \frac{1}{\ln^2 x} d\ln x = -\frac{1}{\ln x} + C$

例2、$\int \frac{1}{\sqrt{x(1-x)}}dx = 2\int \frac{dx}{2\sqrt{x}\sqrt{1-x}} = 2\int \frac{d\sqrt{x}}{\sqrt{1-(\sqrt{x})^2}} = 2\arcsin\sqrt{x} + C$

例3、$\int \frac{1}{x^2+x+1}dx = 1/[(\frac{\sqrt{3}}{2})^2 + (x+\frac{1}{2})^2]\, d(x+\frac{1}{2}) = \frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C$

思路：$\int f[\varphi(x)]\varphi'(x)dx = \int f[\varphi(x)]d\varphi(x) \xlongequal{\varphi(x)=t} \int f(t)dt = F(t)+C$

$= F(\varphi(x)) + C$

② 第二类换元积分法

(1) 无理 $\Longrightarrow$ 有理 (不一定)

例1、$\int \frac{dx}{\sqrt{x}(1+x)} = 2\int \frac{1}{2\sqrt{x}}\cdot\frac{1}{1+x}dx = 2\int \frac{d(\sqrt{x})}{1+(\sqrt{x})^2}$

$= 2\arctan\sqrt{x} + C$

例2、$\int \sqrt{\frac{1+x}{1-x}}dx = \int \frac{1+x}{\sqrt{1-x^2}}dx = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{-2x}{2\sqrt{1-x^2}}dx$

$= \arcsin x - \int \frac{d(1-x^2)}{2\sqrt{1-x^2}} = \arcsin x - \sqrt{1-x^2} + C$

例3、$\int \frac{dx}{1+\sqrt{x}} \xlongequal{x=t^2} 2\int \frac{t}{1+t}dt = 2\int (1-\frac{1}{1+t})dt = 2t - 2\ln(1+t) + C$

$= 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C$

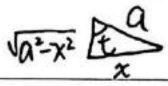
例4、$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} \xlongequal{x=t^6} \int \frac{6t^5}{t^3+t^2}dt = 6\int \frac{t^3+1-1}{t+1}dt = 6\int t^2 - t + \frac{1}{t+1}dt$

$= 2t^3 - 3t^2 + 6\ln(t+1) + C = 2\sqrt{x} - 3\sqrt[3]{x} + 6\ln(\sqrt[6]{x}+1) + C$

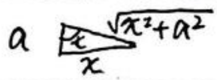
思想：$\int f(x)dx \xlongequal{x=\varphi(t)} \int f[\varphi(t)]\varphi'(t)dt = \int g(t)dt = G(t) + C$

$= G(\varphi^{-1}(x)) + C$

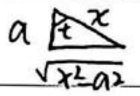
(2) $\sqrt{a^2-x^2} \xlongequal{x=a\sin t} a\cos t$



$\sqrt{x^2+a^2} \xlongequal{x=a\tan t} a\sec t$



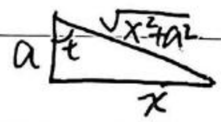
$\sqrt{x^2-a^2} \xlongequal{x=a\sec t} a\tan t$



例1、$\int \frac{dx}{\sqrt{x^2+a^2}} \xlongequal{x=a\tan t} \int \frac{a\sec^2 t}{a\sec t}dt = \int \sec t\,dt = \ln|\sec t + \tan t| + C$

$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right| + C$

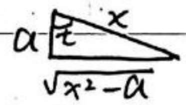
$= \ln(x + \sqrt{x^2+a^2}) + C$



例2、$\int\frac{dx}{\sqrt{x^2-a^2}} \xlongequal{x=a\sec t} \int\frac{a\sec t\tan t}{a\tan t}dt = \int \sec t\,dt = \ln|\sec t+\tan t|+C$

$=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C$

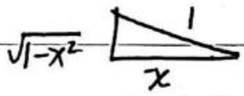
$=\ln|x+\sqrt{x^2-a^2}|+C$



例3、$\int\frac{dx}{x^2\sqrt{1-x^2}} \xlongequal{x=\sin t} \int \frac{\cos t}{\sin^2 t\cdot\cos t}dt = \int\csc^2 t\,dt$

$=-\cot t+C$

$=-\frac{\sqrt{1-x^2}}{x}+C$



方法二：分部积分法

$(uv)'=u'v+uv' \Rightarrow uv'=(uv)'-u'v$

$\int u\,dv = uv-\int v\,du$

case1. $\int x^n \cdot \begin{cases} e^x\\ \ln x \\ 三角 \\ 反三角\end{cases} dx$

例1、$\int x^2e^x dx = \int x^2 de^x = x^2e^x - \int e^x dx^2$

$=x^2e^x-2\int xe^xdx = x^2e^x-2\int x\,de^x$

$=x^2e^x-2[xe^x-\int e^xdx] = (x^2-2x+2)e^x+C$

例2、$\int x^2\ln x\,dx = \int \ln x\, d(\frac13 x^3) = \frac13x^3\ln x - \frac13\int x^3\frac1x dx$

$=\frac13 x^3\ln x - \frac19 x^3 + C$

例3、$\int x\cos 2x\,dx = 2\int x\,d(\sin 2x) = \frac12x\sin2x-\frac12\int\sin2x\,dx$

$=\frac12x\sin 2x+\frac14\cos2x+C$

例4、$\int x\arctan x\,dx = \int\arctan x\, d(\frac12x^2) = \frac12x^2\arctan x$

$- \frac12\int\frac{x^2}{1+x^2}dx = \frac12x^2\arctan x-\frac12\int(1-\frac{1}{1+x^2})dx$

Case2. $\int e^{ax}\cdot\begin{cases}\sin bx\\ \cos bx\end{cases}dx$

例5、$\int e^{2x}\cos3x\,dx$ 解：$I=\int e^{2x}\cos 3x\,dx = \frac13\int e^{2x}\,d\sin3x$

$= \frac{1}{3}[e^{2x}\sin 3x - \int 2e^{2x}\sin 3x\,dx] = \frac{1}{3}[e^{2x}\sin 3x - 2\int e^{2x}\sin 3x\,dx]$

$= \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}\int e^{2x}d(\cos 3x) = \frac{1}{3}e^{3x}\sin 3x + \frac{2}{9}[e^{2x}\cos 3x - 2\int e^{x}$

$\cos 3x\,dx]$

$I = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}I = \frac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x) + C$

~~Case 3.~~ $\begin{cases}\int \sec^n x\,dx \\ \int \csc^n x\,dx\end{cases}$，n为奇数

例 $\int \tan^4 x\,dx = \int(\sec^2 x - 1)^2 dx = \int \sec^4 x\,dx - 2\int \sec^2 x\,dx + \int dx$

$= \int \tan^2 x + 1\ d(\tan x) - 2\tan x + x$

$= \frac{1}{3}\tan^3 x - \tan x + x + C$

$n=1$: $\begin{cases}\int \sec x\,dx = \ln|\sec x + \tan x| + C \\ \int \csc x\,dx = \ln|\csc x - \cot x| + C\end{cases}$

例1. $\int \sec^3 x\,dx$

解: $I_3 = \int \sec^3 x\,dx = \int \sec x\,d\tan x = \sec x\tan x - \int \sec x\tan^2 x\,dx$

$= \sec x\tan x - \int \sec x(\sec^2 x - 1)dx = \sec x\tan x - I_3 + \int \sec x\,dx$

$= \sec x\tan x - I_3 + \ln|\sec x + \tan x|$

$I_3 = \frac{1}{2}(\sec x\tan x + \ln|\sec x + \tan x|) + C$

~~Case4、见到~~ $\begin{cases}\ln \\ e^x \\ \text{arc}\end{cases}$ ~~，用分部积分~~

例1. $\int \frac{1+\sin x}{1+\cos x}e^x dx = \int \frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}e^x dx = \int(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2})e^x dx$

$= \int e^x d(\tan\frac{x}{2}) + \int e^x \tan\frac{x}{2}dx = e^x\tan\frac{x}{2} - \int e^x\tan\frac{x}{2}dx$

$+ e^x\tan\frac{x}{2}dx = e^x\tan\frac{x}{2} + C$

例2、$\int \frac{xe^x}{(x+1)^2}dx = \int \frac{(x+1)e^x - e^x}{(x+1)^2}dx = \int \frac{e^x}{x+1}dx + \int e^x d(\frac{1}{x+1})$

例3、$\int \ln(1+\sqrt{x})dx \xlongequal{\sqrt{x}=t} \int \ln(1+t)\,d(t^2) = t^2\ln(1+t) - \int \frac{t^2}{1+t}dt$

例4. $\int \arcsin x \arccos x \, dx = x\arcsin x \arccos x - \int \frac{x(\arccos x - \arcsin x)}{\sqrt{1-x^2}} dx$

$= x\arcsin x \arccos x + \int (\arccos x - \arcsin x) \, d\sqrt{1-x^2}$

例5. $\int \frac{e^{\arctan x}}{(1+x^2)^{\frac{3}{2}}} dx \overset{x=\tan t}{=\!=} \int \frac{e^t}{\sec^3 t} d\tan t = \int \frac{e^t}{\sec t} dt$

$= \int e^t \cos t \, dt = \int e^t d\sin t = e^t \sin t - \int \sin t \, e^t dt = e^t \sin t + \int e^t d\cos t$

$= e^t \sin t + e^t \cos t - \int e^t \cos t \, dt = \frac{1}{2} e^t (\sin t + \cos t) + C$

$= \frac{1}{2} e^{\arctan t} \frac{x+1}{\sqrt{x^2+1}} + C$

(三) 特例 —— 有理函数不定积分

$\int R(x) dx$

$R(x) = \frac{P(x)}{Q(x)} \begin{cases} 真分式, \deg P < \deg Q \\ 假分式, \deg P \geq \deg Q \end{cases}$

Case 1. $R(x)$ 为真分式

$R(x) = \frac{不变}{因式分解} \Rightarrow$ 部分和

① $\frac{x+5}{x^3-4x} = \frac{x+5}{x(x-2)(x+2)} = \frac{\frac{5}{4}}{x} + \frac{\frac{7}{8}}{x-2} + \frac{\frac{3}{8}}{x+2}$

② $\frac{2x-1}{(x-1)(x+1)^2} = \frac{\frac{1}{4}}{x-1} + \frac{\frac{3}{2}}{(x+1)^2} + \frac{-\frac{1}{4}}{x+1}$

③ $\frac{x^2-3x}{(x-2)(x^2+1)} = \frac{-\frac{2}{5}}{x-2} + \frac{Bx+C}{x^2+1} \Rightarrow B = \frac{7}{5}, C = -\frac{1}{5}$

Case 2. $R(x)$ 为假分式

$R(x) =$ 多项式 + 真分式

例1. ① $\int \frac{dx}{x^2-x-2} = \int \frac{dx}{(x+1)(x-2)} = \frac{1}{3} \int (\frac{1}{x-2} - \frac{1}{x+1}) dx$

$= \frac{1}{3} \ln \left| \frac{x-2}{x+1} \right| + C$

② $\int \frac{dx}{x^2+2x+2} = \int \frac{d(x+1)}{1+(x+1)^2} = \arctan(x+1) + C$

例2. ① $\int \frac{5x-1}{x^2-x-2} dx = \int \frac{5x-1}{(x+1)(x-2)} dx = 2\int \frac{1}{x+1} dx + 3\int \frac{1}{x-2} dx$

$= 2\ln|x+1| + 3\ln|x-2| + C$

② $\int \frac{x-2}{x^2+x+1} dx = \frac{1}{2} \int \frac{2x+1-5}{x^2+x+1} dx = \frac{1}{2} \int \frac{d(x^2+x+1)}{x^2+x+1} - \frac{5}{2} \int \frac{d(x+\frac{1}{2})}{(x+\frac{1}{2})^2+\frac{3}{4}}$

$= \frac{1}{2} \ln|x^2+x+1| - \frac{5}{2} \cdot \sqrt{\frac{4}{3}} \arctan[(x+\frac{1}{2}) \cdot \sqrt{\frac{4}{3}}] + C$

第五章 定积分

一、实际应用及定积分的定义

例1. $v=v(t)$ ，$t\in[a,b]$

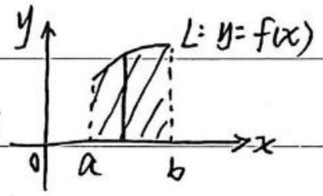
1° $t_0=a<t_1<\cdots<t_n=b$

$[a,b]=[t_0,t_1]\cup[t_1,t_2]\cup\cdots\cup[t_{n-1},t_n]$

2° $\forall\ \xi_i\in[t_{i-1},t_i]$ ，$\sum_{i=1}^{n}v(\xi_i)\Delta t_i\approx S$ ；

3° $\lambda=\max\limits_{1\le i\le n}\{\Delta t_i\}$ ，$\lim\limits_{\lambda\to 0}\sum_{i=1}^{n}v(\xi_i)\Delta t_i=S$

例2、1° $a=x_0<x_1<\cdots<x_n=b$



2° $\forall\ \xi_i\in[x_{i-1},x_i]$ ，$\Delta A_i\approx f(\xi_i)\Delta x_i$

$A\approx\sum_{i=1}^{n}f(\xi_i)\Delta x_i$ ；

3° $\lambda=\max\limits_{1\le i\le n}\{\Delta x_i\}$ ，$A=\lim\limits_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i$

定积分的定义 —— $y=f(x)$ 在 $[a,b]$ 上有界

1° $a=x_0<x_1<\cdots<x_n=b$

2° $\forall\ \xi_i\in[x_{i-1},x_i]$ ，作 $\sum_{i=1}^{n}f(\xi_i)\Delta x_i$ ；

3° $\lambda=\max\limits_{1\le i\le n}\{\Delta x_i\}$ ，则如果 $\lim\limits_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i$ 存在，称 $f(x)$ 在 $[a,b]$ 上可积。

Note: ① $\lim\limits_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i$ 与 $\begin{cases}[a,b]\text{划分方法}\\ \xi_i\text{的取法}\end{cases}$ 无关。

② $\lambda\to 0 \underset{\not\Leftarrow}{\Rightarrow} n\to\infty$

"$\Rightarrow$" $b-a=\Delta x_1+\cdots+\Delta x_n\le n\lambda$ ，$n\ge\frac{b-a}{\lambda}\to+\infty\ (\lambda\to 0)$

"$\not\Leftarrow$" $\underset{x_0}{a}$ —— x_1 ······ b ，$n\to\infty$，但 $\lambda=\frac{b-a}{2}>0$

③ $f(x)$ 在 $[a,b]$ 上有界是可积的必要条件

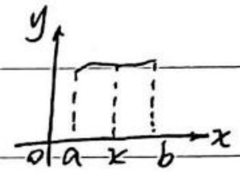
如：$f(x)=\begin{cases}1, & x\in Q\\ 0, & x\in R/Q\end{cases}$ 在 $[a,b]$ 上有界，但不可积.

Case 1. $\xi_i\in Q$ ，$\lim\limits_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i=b-a$ ；

Case 2. $\xi_i \in R \setminus Q$. $\lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i = 0$. $\therefore \lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i$ 不存在。

④ $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n} f(\frac{i}{n}) = \int_0^1 f(x)dx$

二、定积分理论



Th1 . $f(x)\in C[a,b]$, $\Phi(x) = \int_a^x f(t)dt$

则 $\Phi'(x) = f(x)$

证: $\Delta\Phi = \Phi(x+\Delta x) - \Phi(x) = \int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt$

$= \int_x^{x+\Delta x} f(t)dt = f(\xi)\Delta x$ (ξ介于 x与 $x+\Delta x$之间)

$\frac{\Delta\Phi}{\Delta x} = f(\xi)$, $\lim\limits_{\Delta x\to 0}\frac{\Delta\Phi}{\Delta x} = \lim\limits_{\Delta x\to 0} f(\xi) \Rightarrow \Phi'(x) = f(x)$

Q1. $\int_a^x \overset{\text{不同}}{\overline{f(x)\,dx}} = \int_a^x f(t)dt$

Q2. $\int_a^x \overset{\text{同}}{\overline{f(x,t)}}\,dt$

Notes:

① $\frac{d}{dx}\int_a^{\varphi(x)} f(t)dt = f[\varphi(x)]\cdot\varphi'(x)$

② $\frac{d}{dx}\int_{\varphi_1(x)}^{\varphi_2(x)} f(t)dt = f[\varphi_2(x)]\cdot\varphi_2'(x) - f[\varphi_1(x)]\cdot\varphi_1'(x)$

例1. $f(x)$ 连续. $f(0)=0$, $f'(0)=2$. $F(x) = \int_0^x t f(x^2-t^2)dt$.

求 $\lim\limits_{x\to 0}\frac{F(x)}{x^4}$?

解: 1° $F(x) = -\frac{1}{2}\int_0^x f(x^2-t^2)d(x^2-t^2)$

$\xlongequal{x^2-t^2=u} -\frac{1}{2}\int_{x^2}^0 f(u)du = \frac{1}{2}\int_0^{x^2} f(u)du$

2° $I = \lim\limits_{x\to 0}\frac{\frac{1}{2}f(x^2)\cdot 2x}{4x^3} = \frac{1}{4}\lim\limits_{x\to 0}\frac{f(x^2)}{x^2} = \frac{1}{4}\lim\limits_{t\to 0}\frac{f(t)-f(0)}{t-0} = \frac{1}{4}f'(0)$

例2. $f(x)$ 连续. $f(0)=0$, $f'(0)=1$. $\lim\limits_{x\to 0}\frac{\int_0^x t f(x-t)dt}{x-\arctan x} = ?$

解: 1° $\arctan x = x - \frac{1}{3}x^3 + o(x^3)$, $x - \arctan x \sim \frac{1}{3}x^3$

2° $\int_0^x t f(x-t)dt \xlongequal{x-t=u} \int_x^0 (x-u)f(u)(-du) = \int_0^x (x-u)f(u)du$

$= x\int_0^x f(u)du - \int_0^x uf(u)du$

3° $I = \lim\limits_{x\to 0}\frac{\int_0^x f(u)du + x\cdot f(x) - xf(x)}{x^2} = \frac{1}{2}\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x} = \frac{1}{2}f'(0) = \frac{1}{2}$

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3、$\frac{d^2}{dx^2}\int_0^x (x-t)f(t)dt = ?$

解：1° $\int_0^x (x-t)f(t)dt = x\int_0^x f(t)dt - \int_0^x tf(t)dt$

2° $\frac{d}{dx}\int_0^x (x-t)f(t)dt = \int_0^x f(t)dt$

3° $\frac{d^2}{dx^2}\int_0^x (x-t)f(t)dt = f(x)$.

Th2. (N-L) $\int_a^b f(x)dx = F(b)-F(a)$

证：令 $\Phi(x) = \int_a^x f(t)dt$ $\because \Phi'(x) = f(x)$，$F'(x) = f(x)$

$\therefore \Phi(x) - F(x) \equiv C_0$，$\Phi(b) - F(b) = \Phi(a) - F(a)$

$\because \Phi(a) = 0$，$\therefore \Phi(b) = F(b) - F(a)$，即 $\int_a^b f(x)dx = F(b) - F(a)$

三、定积分性质

(一) 一般性质

5. ① $f(x) \geq 0\ (a \leq x \leq b) \Rightarrow \int_a^b f(x)dx \geq 0$

② $f(x) \geq g(x)\ (a \leq x \leq b) \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$

☆ ③ $f(x) \in C[a,b]$，则 $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

证：$-|f(x)| \leq f(x) \leq |f(x)|$

$-\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$

$\Rightarrow \left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

6. ①（积分中值定理）$f(x) \in C[a,b]$，则 $\exists \xi \in [a,b]$，

使 $\int_a^b f(x)dx = f(\xi)(b-a)$

证：$f(x) \in C[a,b] \Rightarrow \exists m, M$，$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

$\therefore m \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M$

$\therefore \exists \xi \in [a,b]$，使 $f(\xi) = \frac{\int_a^b f(x)dx}{b-a}$，$\int_a^b f(x)dx = f(\xi)(b-a)$

②（推广）$f(x) \in C[a,b]$，则 $\exists \xi \in (a,b)$，使 $\int_a^b f(x)dx = f(\xi)(b-a)$

证：令 $F(x) = \int_a^x f(t)dt$，$F'(x) = f(x)$

$\int_a^b f(x)dx = F(b) - F(a) = F'(\xi)(b-a) \quad (a<\xi<b)$

$= f(\xi)(b-a)$

(二) 特殊性质

1、$f(x) \in [-a,a]$，则 $\int_{-a}^{a} f(x)dx = \int_0^a [f(x)+f(-x)]dx$

证：左 $= \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$ 而 $\int_{-a}^0 f(x)dx \xlongequal{x=-t} \int_a^0 f(-t)(-dt)$

$= \int_0^a f(-x)dx$ ，∴ 左 = 右

2、① $\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$

证：左 $\xlongequal{x+t=\frac{\pi}{2}} \int_{\frac{\pi}{2}}^0 f(\cos t)(-dt) = \int_0^{\frac{\pi}{2}} f(\cos x)dx =$ 右

记：$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = I_n$

$$\begin{cases} I_n = \frac{n-1}{n} I_{n-2} \\ I_0 = \frac{\pi}{2} \\ I_1 = 1 \end{cases}$$

如：$\int_0^{\frac{\pi}{2}} \sin^8 x\,dx = I_8 = \frac{7}{8} I_6 = \frac{7}{8}\cdot\frac{5}{6} I_4 = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}} \cos^{11} x\,dx = I_{11} = \frac{10}{11}\cdot\frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1$

② $\int_0^{\pi} f(\sin x)dx = 2\int_0^{\frac{\pi}{2}} f(\sin x)dx$

例1. $\int_0^1 x^2\sqrt{1-x^2}\,dx \xlongequal{x=\sin t} \int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t\,dt$

$= \int_0^{\frac{\pi}{2}} \sin^2 t(1-\sin^2 t)dt = I_2 - I_4 = \frac{1}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$

例2. $\int_{-\pi}^{\pi} \frac{\sin^2 x}{1+e^x}dx = \int_0^{\pi} \frac{\sin^2 x}{1+e^x} + \frac{\sin^2 x}{1+e^{-x}}dx$

$= \int_0^{\pi}\left(\frac{1}{1+e^x} + \frac{1}{1+e^{-x}}\right)\sin^2 x\,dx = \int_0^{\pi}\sin^2 x\,dx = 2I_2 = 2\times\frac{1}{2}\times\frac{\pi}{2}$

例3. ① $f(x)$、$g(x) \in C[-a,a]$

$f(x)+f(-x) \equiv A$，$g(-x)=g(x)$. 证：$\int_{-a}^{a} f(x)g(x)dx = A\int_0^a g(x)dx$

证：左 $= \int_0^a [f(x)g(x)+f(-x)g(-x)]dx = \int_0^a [f(x)+f(-x)]g(x)dx =$ 右

② 求 $\int_{-\pi}^{\pi} \arctan e^x \cdot \sin^2 x\,dx = ?$

$=\int_0^{\pi}(\arctan e^{x}+\arctan e^{-x})\sin^2x\,dx$

$\because (\arctan e^{x}+\arctan e^{-x})'=\frac{e^{x}}{1+e^{2x}}+\frac{-e^{-x}}{1+e^{-2x}}=0$

$\therefore \arctan e^{x}+\arctan e^{-x}\equiv A$. 取 $x=0$, $A=\frac{\pi}{2}$

$I=\frac{\pi}{2}\int_0^{\pi}\sin^2x\,dx=\frac{\pi}{2}\cdot 2\cdot I_2=\frac{\pi}{2}\cdot 2\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi^2}{4}$

3. $f(x)$ 以 T 为周期.

☆ ① $\int_a^{a+T}f(x)\,dx=\int_0^{T}f(x)\,dx$

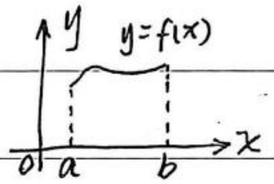
证: 左 $=\int_a^{0}f(x)\,dx+\int_0^{T}f(x)\,dx+\int_T^{a+T}f(x)\,dt$.

$\int_T^{a+T}f(x)\,dx\ \underline{\underline{x-T=t}}\ \int_0^{a}f(t+T)\,dt=\int_0^{a}f(x)\,dx$

$\therefore$ 左 $=\int_0^{T}f(x)\,dx$

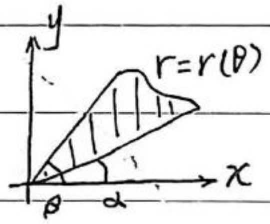
② $\int_0^{nT}f(x)\,dx=n\int_0^{T}f(x)\,dx$

四、定积分的应用 — 几何



(一) 面积

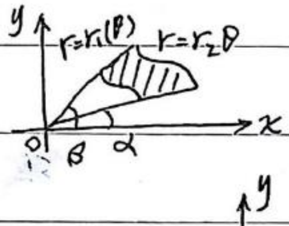
1. $A=\int_a^{b}f(x)\,dx$



2、1° $[\theta,\theta+d\theta]\subset[\alpha,\beta]$

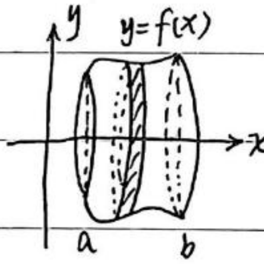
2° $dA=\frac{1}{2}r^2(\theta)\,d\theta$

3° $A=\frac{1}{2}\int_\alpha^{\beta}r^2(\theta)\,d\theta$



3、$A=\frac{1}{2}\int_\alpha^{\beta}[r_2^2(\theta)-r_1^2(\theta)]\,d\theta$

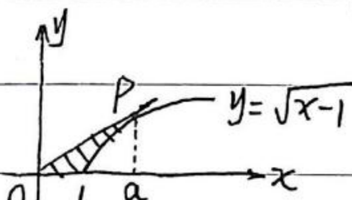
4、1° $[x,x+dx]\subset[a,b]$



2° $dA=2\pi|f(x)|\cdot ds=2\pi|f(x)|\sqrt{1+f'^2(x)}\,dx$

3° $A=2\pi\int_a^{b}|f(x)|\sqrt{1+f'^2(x)}\,dx$

例1、① 求切线方程



② 求阴影面积

③ 求旋转表面积(绕 x)

解：① 令 $P(a, \sqrt{a-1})$，由 $\frac{1}{2\sqrt{a-1}} = \frac{\sqrt{a-1}}{a} \Rightarrow a=2$，$k=\frac{1}{2}$

切线 $y=\frac{1}{2}x$.

② $A = 1 - \int_1^2 \sqrt{x-1}\,dx = 1 - \int_1^2 (x-1)^{\frac{1}{2}}\,d(x-1) = \frac{1}{3}$

③ $[x, x+dx] \subset [1,2]$，$dA = 2\pi\sqrt{x-1}\cdot ds = 2\pi\sqrt{x-1}\sqrt{1+\frac{1}{4(x-1)}}\,dx$

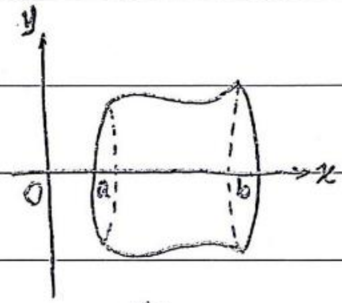
$= \pi\sqrt{4x-3}\,dx$，$S_内 = \pi\int_1^2 (4x-3)^{\frac{1}{2}}\,d(4x-3) = \frac{\pi}{4} \times \frac{2}{3}(4x-3)^{\frac{3}{2}}\Big|_1^2 = \frac{\pi}{6}(5\sqrt{5}-1)$；

$[x, x+dx] \subset [0,2]$，$dA = 2\pi\cdot\frac{x}{2}\cdot ds = \pi x\sqrt{1+\frac{1}{4}}\,dx = \frac{\sqrt{5}}{2}\pi x\,dx$

$S_外 = \frac{\sqrt{5}}{2}\pi\int_0^2 x\,dx = \sqrt{5}\pi$

(二) 体积

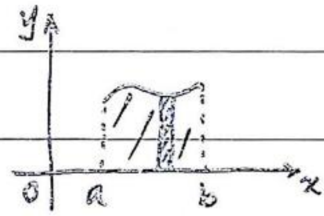
1. 1° $[x, x+dx] \subset [a,b]$



2° $dV = \pi f^2(x)\,dx$

3° $V_x = \pi\int_a^b f^2(x)\,dx$

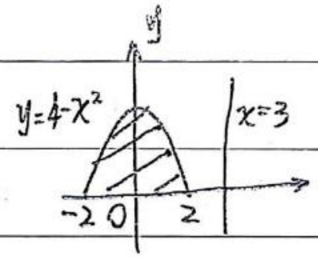
2、1° $[x, x+dx] \subset [a,b]$



2° $dV = 2\pi|x|\cdot|f(x)|\cdot dx$

3° $V_y = 2\pi\int_a^b |x|\cdot|f(x)|\,dx$

例2.



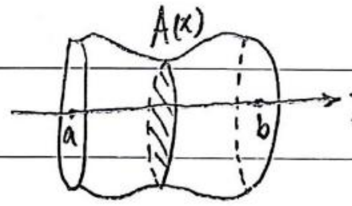
解：$[x, x+dx] \subset [-2,2]$

$dV = 2\pi(3-x)(4-x^2)\,dx$

$V = 2\pi\int_{-2}^2 (3-x)(4-x^2)\,dx$

$= 12\pi\int_0^2 (4-x^2)\,dx$

3、$V = \int_a^b A(x)\,dx$



(三) 弧长

1. $L: y=f(x) \quad (a \le x \le b)$

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1° $[x, x+dx] \subset [a,b]$

2° $ds = \sqrt{1+f'^2(x)}\ dx$

3° $l = \int_a^b \sqrt{1+f'^2(x)}\ dx$

2、$L: \begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases} \quad (\alpha \le t \le \beta)$

1° $[t, t+dt] \subset [\alpha, \beta]$

2° $ds = \sqrt{\varphi'^2(t) + \psi'^2(t)}\ dt$

3° $l = \int_\alpha^\beta \sqrt{\varphi'^2(t)+\psi'^2(t)}\ dt$

五、广义积分

正常积分 $\int_a^b f(x)dx$ $\begin{cases} \text{①区间有限} \\ \text{② } f(x) \text{ 在 } [a,b] \text{ 连续或有限个第一类间断点} \end{cases}$

(一)区间无限

1、$f(x) \in [a, +\infty)$. $\int_a^{+\infty} f(x)dx$

def - $\int_a^b f(x)dx = F(b) - F(a)$

$\lim\limits_{b\to+\infty} [F(b)-F(a)] \begin{cases} =A. & \int_a^{+\infty} f(x)dx = A \\ \text{无}. & \text{发散} \end{cases}$

判别法:

$\lim\limits_{x\to+\infty} x^n f(x) = C_0\ (\neq 0) \begin{cases} \text{收敛}, & a>1 \\ \text{发散}, & a\le 1 \end{cases}$

2、$f(x) \in C(-\infty, a]$. $\int_{-\infty}^a f(x)dx$

def - $\int_b^a f(x)dx = F(a) - F(b)$

$\lim\limits_{b\to-\infty} [F(a)-F(b)] \begin{cases} =A. & \int_{-\infty}^a f(x)dx = A \\ \text{无}. & \text{发散} \end{cases}$

判别法:

$\lim\limits_{x\to-\infty} x^\alpha f(x) = C_0\ (\neq 0) \begin{cases} \text{收敛}, & \alpha>1 \\ \text{发散}, & \alpha\le 1 \end{cases}$

3、$f(x) \in C(-\infty, +\infty)$. $\int_{-\infty}^{+\infty} f(x)dx$

$\int_{-\infty}^{+\infty} f(x)dx$ 收敛 $\Leftrightarrow$ $\int_{-\infty}^a f(x)dx$ 与 $\int_a^{+\infty} f(x)dx$ 皆收敛

Q: $\int_{-\infty}^{+\infty} \frac{x}{1+2x^2} dx \overset{?}{=} 0$

对 $\int_0^{+\infty} \frac{x}{1+2x^2} dx$，$\because \lim_{x\to+\infty} x\cdot\frac{x}{1+2x^2}=\frac{1}{2}$ 且 $\alpha=1\le 1$

$\therefore \int_0^{+\infty} \frac{x}{1+2x^2} dx$ 发散.

(二) 区间有限. 函数有无穷间断点.

1、$f(x)\in C(a,b]$ 且 $f(a+0)=\infty$. $\int_a^b f(x)dx$

def- $\forall \varepsilon>0$. $\int_{a+\varepsilon}^b f(x)\,dx = F(b)-F(a+\varepsilon)$

$\lim_{\varepsilon\to0^+}[F(b)-F(a+\varepsilon)]\begin{cases}=A. & \int_a^b f(x)dx=A\\ \text{无}. & \text{发散}\end{cases}$

例1. $\int_1^2\frac{dx}{\sqrt{x-1}}$?

解：$\forall\varepsilon>0$. $\int_{1+\varepsilon}^2\frac{dx}{\sqrt{x-1}} = 2\sqrt{x-1}\Big|_{1+\varepsilon}^2 = 2(1-\sqrt{\varepsilon})$

$\because \lim_{\varepsilon\to0^+}2(1-\sqrt{\varepsilon})=2$ $\therefore \int_1^2\frac{dx}{\sqrt{x-1}}=2$

例2. $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{2}{n}\cdots\frac{n}{n}\right)^{\frac{1}{n}} = e^{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln\frac{i}{n}} = e^{\int_0^1\ln x\,dx}$

$\forall\varepsilon>0$, $\int_\varepsilon^1\ln x\,dx = (x\ln x-x)\Big|_\varepsilon^1 = 0-\varepsilon\ln\varepsilon-(1-\varepsilon)$

$\because \lim_{\varepsilon\to0^+}\varepsilon\ln\varepsilon = \lim_{\varepsilon\to0^+}\frac{\ln\varepsilon}{\frac{1}{\varepsilon}} = \lim_{\varepsilon\to0^+}\frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}}=0$

$\therefore \int_0^1\ln x\,dx=-1$, $\therefore I=e^{-1}$

判别法：

$\lim_{x\to a^+}(x-a)^{\alpha}\cdot f(x)=C_0(\neq0)\begin{cases}\text{收敛}, & \alpha<1\\ \text{发散}, & \alpha\geqslant1\end{cases}$

2、$f(x)\in C[a,b)$. $f(b-0)=\infty$. $\int_a^b f(x)dx$

def- $\forall\varepsilon>0$. $\int_a^{b-\varepsilon}f(x)dx=F(b-\varepsilon)-F(a)$

$\lim_{\varepsilon\to0^+}[F(b-\varepsilon)-F(a)]\begin{cases}=A. & \int_a^b f(x)dx=A\\ \text{无}. & \text{发散}\end{cases}$

判别法：

$\lim_{x\to b^-}(b-x)^{\alpha}\cdot f(x)=C_0(\neq0)\begin{cases}\text{收敛}, & \alpha<1\\ \text{发散}, & \alpha\geqslant1\end{cases}$

3、$f(x)\in C[a,c)\cup(c,b]$. $\lim_{x\to c}f(x)=\infty$

$\int_a^b f(x)dx$ 收敛 $\Leftrightarrow$ $\int_a^c f(x)dx$ 与 $\int_c^b f(x)dx$ 皆收敛.

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☆ Γ 函数

$$\Gamma(\alpha) \triangleq \int_0^{+\infty} x^{\alpha-1} e^{-x} dx$$

如：$\int_0^{+\infty} x^5 e^{-x} dx = \Gamma(6)$

$\int_0^{+\infty} \sqrt{x}\, e^{-x} dx = \Gamma(\frac{3}{2})$

$$\begin{cases} \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \\ \Gamma(n+1) = n! \\ \Gamma(\frac{1}{2}) = \sqrt{\pi} \end{cases}$$

例1. $\int_0^{+\infty} x^5 e^{-x} dx = \Gamma(6) = 5!$

例2. $\int_0^{+\infty} \sqrt{x}\, e^{-x} dx = \Gamma(\frac{1}{2}+1) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$

例3. $\int_0^{+\infty} x^3 e^{-x^2} dx = \frac{1}{2}\int_0^{+\infty} x^2 e^{-x^2} d(x^2) = \frac{1}{2}\int_0^{+\infty} x e^{-x} dx = \frac{1}{2}\Gamma(2) = \frac{1}{2}$

例4. $\int_0^{+\infty} x^4 e^{-x^2} dx \xlongequal{x^2=t} \int_0^{+\infty} t^2 e^{-t} \times \frac{1}{2\sqrt{t}} dt$

$= \frac{1}{2}\int_0^{+\infty} t^{\frac{3}{2}} e^{-t} dt = \frac{1}{2}\Gamma(\frac{3}{2}+1) = \frac{1}{2}\cdot\frac{3}{2}\Gamma(\frac{1}{2}+1) = \frac{1}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma(\frac{1}{2})$

$= \frac{3}{8}\sqrt{\pi}$

例1、$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dx}{\sqrt{|x-x^2|}}$?

解：1° $I = \int_{\frac{1}{2}}^{1} \frac{dx}{\sqrt{x-x^2}} + \int_1^{\frac{3}{2}} \frac{dx}{\sqrt{x^2-x}} \triangleq I_1 + I_2$

2° 对 I_1

$\lim\limits_{x\to1^-} (1-x)^{\frac{1}{2}} \cdot \frac{1}{\sqrt{x-x^2}} = 1$ 且 $\alpha = \frac{1}{2} < 1$，$\therefore I_1$ 收敛

$\lim\limits_{x\to1^+} (x-1)^{\frac{1}{2}} \cdot \frac{1}{\sqrt{x^2-x}} = 1$ 且 $\alpha = \frac{1}{2} < 1$，$\therefore I_2$ 收敛

3° 方法一：

$I_1 = 2\int_{\frac{1}{2}}^{1} \frac{d\sqrt{x}}{\sqrt{1-\sqrt{x}^2}} = 2\arcsin\sqrt{x}\,\Big|_{\frac{1}{2}}^{1}$

方法二：

$I_1 = \int_{\frac{1}{2}}^{1} \frac{d(x-\frac{1}{2})}{\sqrt{(\frac{1}{2})^2-(x-\frac{1}{2})^2}} = \arcsin(2x-1)\Big|_{\frac{1}{2}}^{1}$

例2、$\int_2^{+\infty} \frac{dx}{(x-1)^5\sqrt{x^2-2x}}$?

解：1° $\lim\limits_{x\to2^+} (x-2)^{\frac{1}{2}} \frac{1}{(x-1)^5\sqrt{x^2-2x}} = \lim\limits_{x\to2^+} \frac{1}{(x-1)^5\sqrt{x}} = \frac{1}{\sqrt{2}}$ 且 $\alpha = \frac{1}{2} < 1$

$\lim\limits_{x\to+\infty} x^6 \frac{1}{(x-1)^5\sqrt{x^2-2x}} = 1$ 且 $\alpha = 6 > 1$，$\therefore$ 收敛

2° $I=\int_2^{+\infty}\frac{d(x-1)}{(x-1)^5\sqrt{(x-1)^2-1}} = \int_1^{+\infty}\frac{dx}{x^5\sqrt{x^2-1}} \xlongequal{x=\sec t} \int_0^{\frac{\pi}{2}}\frac{\sec t\tan t}{\sec^5 t\tan t}dt$

$=\int_0^{\frac{\pi}{2}}\cos^4 t\,dt = \frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$

型一 计算

例1. $\int_0^2 x^2\sqrt{2x-x^2}\,dx = \int_0^2[(x-1)+1]^2\sqrt{1-(x-1)^2}\,d(x-1)$

$=\int_{-1}^1 (x+1)^2\sqrt{1-x^2}\,dx = 2\int_0^1 (1+x^2)\sqrt{1-x^2}\,dx \xlongequal{x=\sin t} 2\int_0^{\frac{\pi}{2}}(1+\sin^2 t)\cos^2 t\,dt$

$=2\int_0^{\frac{\pi}{2}}(1-\sin^4 t)\,dt = 2\left(\frac{\pi}{2}-\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}\right)$

2. $f(x)>0$, $f(x)\cdot f(-x)\equiv 1$, 计算 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2 x}{1+f(x)}dx$

$=\int_0^{\frac{\pi}{2}}\left[\frac{\sin x^2}{1+f(x)}+\frac{\sin^2 x}{1+f(-x)}\right]dx = \int_0^{\frac{\pi}{2}}\left[\frac{1}{1+f(x)}+\frac{1}{1+1/f(x)}\right]\sin^2 x\,dx$

$=\int_0^{\frac{\pi}{2}}\sin^2 x\,dx = \frac{1}{2}\times\frac{\pi}{2}$

3. ① 证 $\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$.

证：左 $=I \xlongequal{x+t=\pi+0} \int_{\pi}^{0}(\pi-t)f(\sin t)(-dt) = \int_0^{\pi}(\pi-t)f(\sin t)\,dt$

$=\int_0^{\pi}(\pi-x)f(\sin x)\,dx = \pi\int_0^{\pi}f(\sin x)\,dx - \int_0^{\pi}xf(\sin x)\,dx$

$=\frac{\pi}{2}\int_0^{\pi}f(\sin x)\,dx$

② $f(x)=\frac{x\sin x}{1+\cos^2 x}+\int_{-\pi}^{\pi}f(x)\,dx$ (太简单,换一题)

$f(x)=\frac{x}{1+\cos^2 x}+\int_{-\pi}^{\pi}f(x)\sin x\,dx$, 求 $\int_0^{\pi}f(x)dx$

解: 令 $\int_{-\pi}^{\pi}f(x)\sin x\,dx=A$, $f(x)=\frac{x}{1+\cos^2 x}+A$

$f(x)\sin x = \frac{x\sin x}{1+\cos^3 x}+A\sin x$

$A=\int_{-\pi}^{\pi}\frac{x\sin x}{1+\cos^2 x}dx = 2\int_0^{\pi}x\cdot\frac{\sin x}{1+\cos^2 x}dx$

$=-\pi\int_0^{\pi}\frac{d\cos x}{1+\cos^2 x} = -\pi\arctan\cos x\Big|_0^{\pi} = \frac{\pi^2}{2}$

$f(x)=\frac{x}{1+\cos^2 x}+\frac{\pi^2}{2}$

$\int_0^{\pi}f(x)=\int_0^{\pi}x\cdot\frac{1}{1+\cos^2 x}+\frac{\pi^2}{2}$

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4、$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos^2 x} = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\sec^2 x+1}dx = \int_0^{\frac{\pi}{2}} \frac{d(\tan x)}{(\sqrt{2})^2+\tan^2 x}$

$= \frac{1}{\sqrt{2}}\arctan\frac{\tan x}{\sqrt{2}}\Big|_0^{\frac{\pi}{2}} = \frac{1}{\sqrt{2}}(\frac{\pi}{2}-0)$

变积分限函数的积分—分部积分

1、$f(x) = \int_1^x e^{-t^2}dt$. 求 $\int_0^1 x^2 f(x)dx$?

解: $\int_0^1 x^2 f(x)dx = \int_0^1 f(x)\, d(\frac{1}{3}x^3) = \frac{1}{3}x^3 f(x)\Big|_0^1 - \int_0^1 \frac{1}{3}x^3 f'(x)dx$

$= -\frac{1}{3}\int_0^1 x^3 \times e^{-x^2}dx = -\frac{1}{6}\int_0^1 x^2 e^{-x^2}d(x^2) = -\frac{1}{6}\int_0^1 xe^{-x}dx$

$= \frac{1}{6}\int_0^1 x\,d(e^{-x}) = \frac{1}{6}xe^{-x}\Big|_0^1 - \frac{1}{6}\int_0^1 e^{-x}dx = \frac{1}{6}e^{-1} + \frac{1}{6}e^{-x}\Big|_0^1$

$= \frac{1}{6e} + \frac{1}{6}(e^{-1}-1)$

2、$f(x) = \int_0^x \frac{\sin t}{\pi - t}dt$. 求 $\int_0^\pi f(x)dx$

解: $\int_0^\pi f(x)dx = xf(x)\Big|_0^\pi - \int_0^\pi xf'(x)dx$

$= \pi f(\pi) - \int_0^\pi x\cdot\frac{\sin x}{\pi - x}dx = \int_0^\pi \frac{\pi\sin x}{\pi - x}dx - \int_0^\pi \frac{x\sin x}{\pi - x}dx$

$= \int_0^\pi \sin x\, dx = 2\int_0^{\frac{\pi}{2}}\sin x\, dx = 2$

对等

1、$\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x+\cos^3 x}dx \triangleq I \xlongequal{x+t=\frac{\pi}{2}+0} \int_{\frac{\pi}{2}}^0 \frac{\cos^3 t}{\cos^3 t+\sin^3 t}(-dt)$

$= \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos^3 x+\sin^3 x}dx$, $2I = \frac{\pi}{2}$, $I = \frac{\pi}{4}$

型二 证明

(一) f(x) 连续

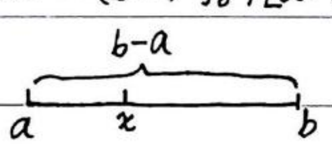
1、f(x) 连续，证 $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

证: 左 $\xlongequal{x+t=a+b} \int_b^a f(a+b-t)(-dt)$

$= \int_a^b f(a+b-x)dx$

2、$f(x) \in C[a,b]$. 证 $\int_a^b f(x)dx = (b-a)\int_0^1 f[a+(b-a)x]dx$

证: 左 $\xlongequal{x=a+(b-a)t} \int_0^1 f[a+(b-a)]\cdot$



$(b-a)dt = (b-a)\int_0^1 f[a+(b-a)x]dx$

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3、$f(x)\in C[a,b]$. $\forall x,y\in[a,b]$ 有 $|f(x)-f(y)|\le 2|x-y|$

证：$\left|\int_a^b f(x)-f(a)(b-a)\right|\le(b-a)^2$

证：$f(a)(b-a)=\int_a^b f(a)\,dx$

左$=\left|\int_a^b[f(x)-f(a)]dx\right|\le\int_a^b|f(x)-f(a)|\,dx\le\int_a^b 2(x-a)\,d(x-a)$

$=(x-a)^2\Big|_a^b=(b-a)^2$

Case1. $f(x)$ 连续 + 单调

1、$f(x)\in C[a,b]$，$\nearrow$，证：$\int_a^b xf(x)\,dx\ge\frac{a+b}{2}\int_a^b f(x)\,dx$

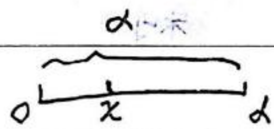
证：令 $\varphi(x)=\int_a^x tf(t)\,dt-\frac{a+x}{2}\int_a^x f(t)\,dt$

$\varphi(a)=0$. $\varphi'(x)=xf(x)-\frac{1}{2}\int_a^x f(t)\,dt-\frac{a+x}{2}f(x)$

$=\frac{x-a}{2}f(x)-\frac{1}{2}\int_a^x f(t)\,dt$

$=\frac{x-a}{2}[f(x)-f(\xi)]\ge 0\ (a\le\xi\le x)(x>a)$

$\begin{cases}\varphi(a)=0\\ \varphi'(x)\ge 0\ (x>a)\end{cases}\Rightarrow\varphi(x)\ge 0\ (a<x\le b)$



2、$f(x)\in C[0,1]\searrow$，$0<\alpha<1$，证：$\int_0^\alpha f(x)\,dx\ge\alpha\int_0^1 f(x)\,dx$

方法一：左 $\underline{\underline{x=\alpha t}}$ $\int_0^1 f(\alpha t)\cdot\alpha\,dt=\alpha\int_0^1 f(\alpha x)\,dx$

$\because f\searrow$ 且 $\alpha x\le x$，$\therefore f(\alpha x)\ge f(x)$. $\therefore$ 左 $\ge\alpha\int_0^1 f(x)\,dx=$ 右

方法二：左－右 $=\int_0^\alpha f(x)\,dx-\alpha\left(\int_0^\alpha f(x)\,dx+\int_\alpha^1 f(x)\,dx\right)$

$=(1-\alpha)\int_0^\alpha f(x)\,dx-\alpha\int_\alpha^1 f(x)\,dx=(1-\alpha)f(\xi_1)\alpha-\alpha f(\xi_2)(1-\alpha)$

$=\alpha(1-\alpha)[f(\xi_1)-f(\xi_2)]\ge 0\ (0\le\xi_1\le\alpha,\ \alpha\le\xi_2\le 1)$

Case2. 连续 + 周期

(二) $f(x)$ 可导

工具：$\begin{cases}f(x)-f(a)=f'(\xi)(x-a) & (积分中无f')\\ f(x)-f(a)=\int_a^x f'(t)\,dt & (积分中有f')\end{cases}$

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手法：$\begin{cases} 含|\cdot| \Rightarrow 内移 \\ 含(\ )^2 \Longrightarrow 柯西不等式 \\ 两边积分 \end{cases}$

1、$f'(x)\in C[0,a]$，$|f'(x)|\le M$，$f(0)=0$. 证：$|\int_0^a f(x)|\le \frac{M}{2}a^2$

1° $f(x)=f(x)-f(0)=f'(\xi)x\ (0<\xi<x)$

2° $|f(x)|=|f'(\xi)|\cdot x\le Mx$

3° 左 $\le \int_0^a|f(x)|dx\le M\int_0^a x\,dx=\frac{M}{2}a^2$

2、$f'(x)\in C[a,b]$，$f(a)=f(b)=0$，$a<c<b$

证：$|f(c)|\le\frac{1}{2}\int_a^b|f'(x)|dx$

1° $f(c)-f(a)=\int_a^c f'(x)dx$

$f(b)-f(c)=\int_c^b f'(x)dx$

2° $\begin{cases}|f(c)|\le\int_a^c|f'(x)|dx \\ |f(c)|\le\int_c^b|f'(x)|dx\end{cases}$

柯西不等式

$f(x)\cdot g(x)\in C[a,b]$，则

$$\left(\int_a^b f(x)g(x)dx\right)^2\le\int_a^b f^2(x)dx\cdot\int_a^b g^2(x)dx$$

3、$f'(x)\in C[0,1]$. $f(1)-f(0)=1$. 证：$\int_0^1 f'^2(x)dx\ge1$

1° $1=f(1)-f(0)=\int_0^1 f'(x)dx$

2° $1=1^2=\left(\int_0^1 1\times f'(x)dx\right)^2\le\int_0^1 1^2dx\int_0^1 f'^2(x)dx$

4、$f'(x)\in C[a,b]$，$f(a)=0$，证：$\int_a^b f^2(x)dx\le\frac{(b-a)^2}{2}\int_a^b f'^2(x)dx$

1° $f(x)=f(x)-f(a)=\int_a^x f'(t)dt$

2° $f^2(x)=\left(\int_a^x 1\times f'(t)dt\right)^2\le\int_a^x 1^2dt\cdot\int_a^x f'^2(t)dt$

$\le(x-a)\int_a^b f'^2(x)dx$

3° $\int_a^b f^2(x)dx\le\int_a^b(x-a)d(x-a)\int_a^b f'^2(x)dx$

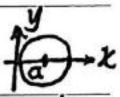
(三) 高阶可导 — Taylor $\begin{cases} f(x) \\ F(x)=\int_a^x f(t)dt\end{cases}$

型三 应用

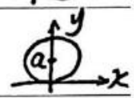
$\begin{cases} 几何 (一、二、三) \\ 物理 (一、二) ☆ \end{cases}$

记：1.圆：① $x^2+y^2=a^2 \Leftrightarrow r=a$ $\quad \begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$

② $x^2+y^2=2ax \Leftrightarrow (x-a)^2+y^2=a^2 \Leftrightarrow r=2a\cos\theta$



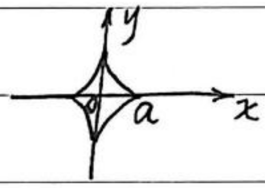
③ $x^2+y^2=2ay \Leftrightarrow x^2+(y-a)^2=a^2 \Leftrightarrow r=2a\sin\theta$



2、心脏线：$r=a(1+\cos\theta)$

3、星形线：

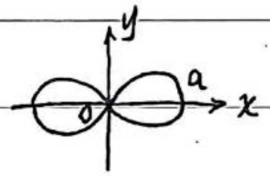
$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ $\quad \begin{cases} x=a\cos^3 t \\ y=a\sin^3 t \end{cases}$



4、双纽线

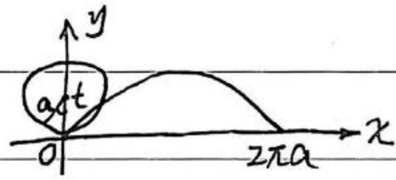
$(x^2+y^2)^2=a^2(x^2-y^2)$

$r^2=a^2\cos2\theta$



5、摆线

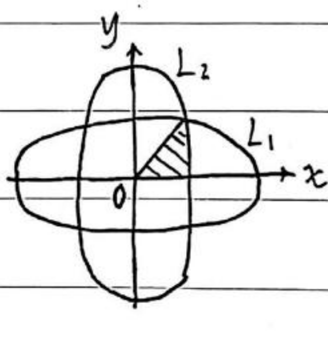
$L: \begin{cases} x=a(t-\sin t) \\ y=a(1-\cos t) \end{cases}$



几何应用

1、$L_1: \frac{x^2}{9}+\frac{y^2}{4}=1$，$L_2: \frac{x^2}{4}+\frac{y^2}{9}=1$

求公共部分面积.



1° L_1: $r^2=\frac{36}{4\cos^2\theta+9\sin^2\theta}$

L_2: $r^2=\frac{36}{4\sin^2\theta+9\cos\theta}$

2° $A_1=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{36}{4\sin^2\theta+9\cos^2\theta}d\theta=18\int_0^{\frac{\pi}{4}}\frac{\sec^2\theta}{4\tan^2\theta+9}d\theta$

$=9\int_0^{\frac{\pi}{4}}\frac{d(2\tan\theta)}{3^2+(2\tan\theta)^2}=3\arctan\frac{2\tan\theta}{3}\Big|_0^{\frac{\pi}{4}}=3\arctan\frac{2}{3}$

3° $A=8A_1=24\arctan\frac{2}{3}$

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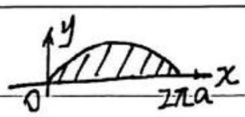
2. $L: \begin{cases} x = a(t-\sin t) \\ y = a(1-\cos t) \end{cases}$ $(0 \le t \le 2\pi)$



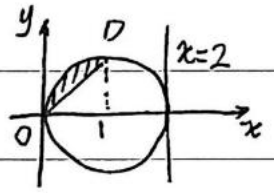
① 求A ② 求V_x

解: ① $A = \int_0^{2\pi a} f(x)dx = \int_0^{2\pi a} y\,dx = \int_0^{2\pi} a^2(1-\cos t)^2 dt$

$= a^2 \int_0^{2\pi} (2\sin^2\frac{t}{2})^2 dt = 8a^2 \int_0^{\pi} \sin^4 t\,dt = 16a^2 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$

② $V_x = \pi \int_0^{2\pi a} f^2(x)dx = \pi \int_0^{2\pi a} y^2 dx = \pi \int_0^{2\pi} a^3(1-\cos t)^3 dt$

3、求旋转体积



解: 1° $[x, x+dx] \subset [0,1]$

2° $dV = 2\pi(2-x)(\sqrt{2x-x^2}-x)$

3° $V = 2\pi \int_0^1 (2-x)(\sqrt{2x-x^2}-x)dx$

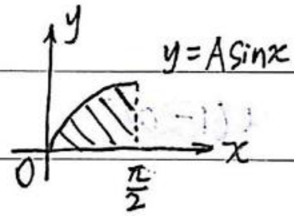
$= 2\pi \int_0^1 [1-(x-1)]\sqrt{1-(x-1)^2}\,d(x-1) - 2\pi \int_0^1 (2x-x^2)dx$

$= 2\pi \int_{-1}^0 (1-x)\sqrt{1-x^2}\,dx - 2\pi(1-\frac{1}{3})$

$\underline{\underline{x=-t}}\ 2\pi \int_1^0 (1+t)\sqrt{1-t^2}(-dt) - 2\pi(1-\frac{1}{3}) = 2\pi \int_0^1 (1+x)\sqrt{1-x^2}\,dx - \frac{4}{3}\pi$

$\underline{\underline{x=\sin t}}\ 2\pi \int_0^{\frac{\pi}{2}} (1+\sin t)(1-\sin^2 t)dt - \frac{4}{3}\pi$

4、 $V_x = V_y$, $A = ?$



解: $V_x = \pi \int_0^{\frac{\pi}{2}} y^2 dx = \pi A^2 \int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

$= \frac{\pi^2}{4}A^2$

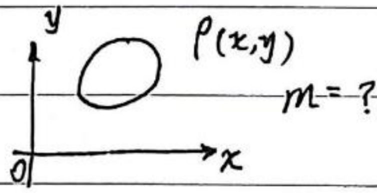
1° $[x, x+dx] \subset [0, \frac{\pi}{2}]$

2° $dV_y = 2\pi x(A\sin x)dx$

3° $V_y = 2\pi A \int_0^{\frac{\pi}{2}} x\sin x\,dx = 2\pi A(\sin x - x\cos x)\Big|_0^{\frac{\pi}{2}}$

第六章 二重积分

一、实例



1°、$D \Rightarrow \Delta\sigma_1, \Delta\sigma_2, \cdots, \Delta\sigma_n$;

2°、$\forall(\xi_i, \eta_i) \in \Delta\sigma_i$ $(1 \le i \le n)$

$\Delta m_i \approx \rho(\xi_i, \eta_i)\Delta\sigma_i \qquad m \approx \sum_{i=1}^{n}\rho(\xi_i,\eta_i)\Delta\sigma_i$

3° λ为 $\Delta\sigma_1, \Delta\sigma_2, \cdots, \Delta\sigma_n$ 直径最大者

$m = \lim_{\lambda\to 0}\sum_{i=1}^{n}\rho(\xi_i,\eta_i)\Delta\sigma_i$

二、def — $f(x,y)$ 在D上有界

1° $D \Rightarrow \Delta\sigma_1, \Delta\sigma_2, \cdots, \Delta\sigma_n$；

2° $\forall(\xi_i,\eta_i)\in\Delta\sigma_i$ 作 $\sum_{i=1}^{n}f(\xi_i,\eta_i)\Delta\sigma_i$；

3° λ为 $\Delta\sigma_1, \Delta\sigma_2, \cdots, \Delta\sigma_n$ 直径最大者

若 $\lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i,\eta_i)\Delta\sigma_i$ 存在，称此极限为 $f(x,y)$ 在D上的二重积分

记 $\iint_D f(x,y)d\sigma$

即 $\iint_D f(x,y)d\sigma \triangleq \lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i,\eta_i)\Delta\sigma_i$

Note: $D=\{(x,y)\,|\,0\le x\le 1, 0\le y\le 1\}$. 则

$$\lim_{\substack{m\to\infty\\ n\to\infty}}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}f\left(\frac{i}{m},\frac{j}{n}\right)=\iint_D f(x,y)d\sigma$$

如：$\lim_{\substack{m\to\infty\\ n\to\infty}}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{m+i}\cdot\frac{1}{n+2j}=\lim_{\substack{m\to\infty\\ n\to\infty}}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left(1+\frac{i}{m}\right)\left(1+2\cdot\frac{j}{n}\right)\right]^{-1}$

$=\iint_D\frac{d\sigma}{(1+x)(1+2y)}$

三、性质

1、$\iint_D 1\,d\sigma = A$

2、D关于y轴对称，右D_1

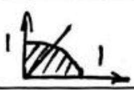
① $f(-x,y)=-f(x,y) \Rightarrow \iint_D f(x,y)d\sigma = 0$

② $f(-x,y)=f(x,y) \Rightarrow \iint_D f(x,y)d\sigma = 2\iint_{D_1}f(x,y)d\sigma$

3. D关于$y=x$对称

$\iint_D f(x,y)d\sigma = \iint_D f(y,x)d\sigma$

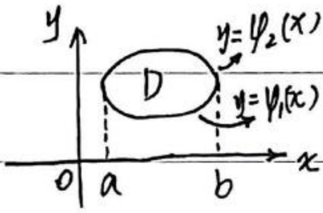
例1、$f(u)$连续. $f(u)>0$. $\forall a>0, b>0$. $I=\iint_D\frac{a\sqrt{f(x)}+b\sqrt{f(y)}}{\sqrt{f(x)}+\sqrt{f(y)}}d\sigma=?$



解：$I=\iint_D\frac{a\sqrt{f(y)}+b\sqrt{f(x)}}{\sqrt{f(x)}+\sqrt{f(y)}}d\sigma$，$2I=\iint_D(a+b)d\sigma=\frac{\pi}{4}(a+b)$

四、积分法

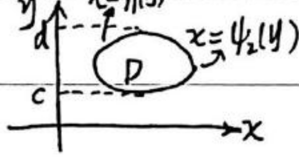
方法一：直角坐标法



X-型区域

$D=\{(x,y)\mid a\le x\le b,\ \varphi_1(x)\le y\le \varphi_2(x)\}$

$$\iint\limits_D f(x,y)\,d\sigma=\int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$$



Y-型区域

$D=\{(x,y)\mid \psi_1(y)\le x\le \psi_2(y),\ c\le y\le d\}$

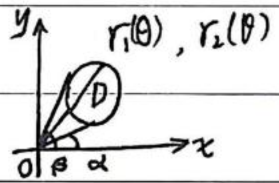
$$\iint\limits_D f(x,y)\,d\sigma=\int_c^d dy\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx$$

方法二：极坐标法

特征：① D的边界含 x^2+y^2

② $f(x,y)$ 中含 x^2+y^2

变换：$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$

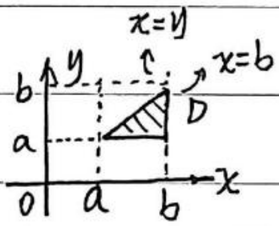


$(\alpha\le\theta\le\beta.\ r_1(\theta)\le r\le r_2(\theta))$ $\quad d\sigma=dx\,dy=r\,dr\,d\theta$

$$\iint\limits_D f(x,y)\,d\sigma=\int_\alpha^\beta d\theta\int_{r_1(\theta)}^{r_2(\theta)} r\cdot f(r\cos\theta,r\sin\theta)\,dr$$

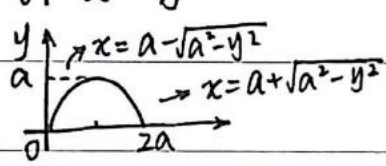
型一 改变积分次序

1、$\int_a^b dx\int_a^x f(x,y)\,dy$



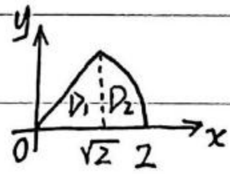
解：$I=\int_a^b dy\int_y^b f(x,y)\,dx$

2、$\int_0^{2a} dx\int_0^{\sqrt{2ax-x^2}} f(x,y)\,dy$



解：$I=\int_0^a dy\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx$

3、$\int_0^{\frac{\pi}{4}} d\theta\int_0^2 r f(r^2)\,dr$ 改为先y后x



解：$D_1=\{(x,y)\mid 0\le x\le\sqrt2,\ 0\le y\le x\}$

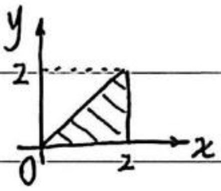
$D_2=\{(x,y)\mid \sqrt2\le x\le 2,\ 0\le y\le\sqrt{4-x^2}\}$

$$I=\int_0^{\sqrt2}dx\int_0^x f(x^2+y^2)\,dy+\int_{\sqrt2}^2 dx\int_0^{\sqrt{4-x^2}} f(x^2+y^2)\,dy$$

4、$\int_0^2 dy\int_y^2 x^2e^{x^2}dx$

不可积	可积
$x^{2n}e^{\pm x^2}$	$x^{2n+1}e^{\pm x^2}$
$e^{\frac{1}{x}}$	$\frac{1}{x^2}e^{\frac{1}{x}}$
$\sin\frac{1}{x}$	$\frac{1}{x^2}\sin\frac{1}{x}$
$\cos\frac{1}{x}$	$\frac{1}{x^2}\cos\frac{1}{x}$

解：



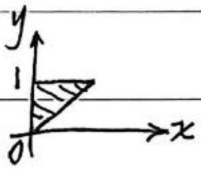
$I=\int_0^2 dx\int_0^x x^2e^{x^2}dy$

$=\int_0^2 x^2e^{x^2}dx$

$=\frac{1}{2}\int_0^2 x^2e^{x^2}dx^2=\frac{1}{2}\int_0^4 xe^xdx$

型二　计算

例1. $\iint_D\sqrt{y^2-xy}\,d\sigma$



解：1° $D=\{(x,y)\mid 0\le x\le y, 0\le y\le 1\}$

2° $I=\int_0^1 dy\int_0^y\sqrt{y^2-xy}\,dx$

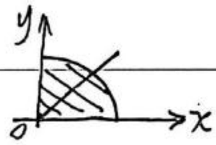
$\int_0^y\sqrt{y^2-xy}\,dx=-\frac{1}{y}\int_0^y(y^2-xy)^{\frac{1}{2}}d(y^2-xy)$

$=\frac{1}{y}\times\frac{2}{3}(y^2-xy)^{\frac{3}{2}}\Big|_0^y$

$=-\frac{2}{3y}(0-|y|^3)=\frac{2}{3}y^2$

3° $I=\frac{2}{3}\int_0^1 y^2dy=\frac{2}{9}$

例2 $\iint_D\sin x^2\cos y^2\,d\sigma$

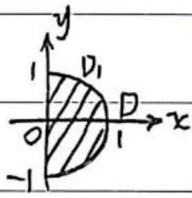


解：1° $I=\iint_D\cos x^2\sin y^2\,d\sigma$

2° $2I=\iint_D\sin(x^2+y^2)d\sigma=\int_0^{\frac{\pi}{2}}d\theta\int_0^1 r\sin r^2dr$

$=\frac{\pi}{4}\int_0^1\sin r^2d(r^2)\;=\frac{\pi}{4}\int_0^1\sin r\,dr$

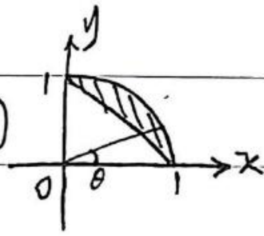
3、$\iint_D\frac{1+xy}{1+x^2+y^2}d\sigma$



1° $I=\iint_D\frac{1}{1+x^2+y^2}d\sigma=2\iint_{D_1}\frac{1}{1+x^2+y^2}d\sigma$

2° $I=2\int_0^{\frac{\pi}{2}}d\theta\int_0^1\frac{r}{1+r^2}dr=\frac{\pi}{2}\ln(1+r^2)\Big|_0^1=\frac{\pi}{2}\ln 2$

4、$\iint_D\frac{d\sigma}{\sqrt{x^2+y^2}}$



1° 令 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$ $\left(0\le\theta\le\frac{\pi}{2},\ \frac{1}{\sin\theta+\cos\theta}\le r\le 1\right)$

2° $I=\int_0^{\frac{\pi}{2}}d\theta\int_{\frac{1}{\sin\theta+\cos\theta}}^1 1\,dr$

$=\int_0^{\frac{\pi}{2}}\left(1-\frac{1}{\sin\theta+\cos\theta}\right)d\theta=\frac{\pi}{2}-\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\csc\left(\theta+\frac{\pi}{4}\right)d\left(\theta+\frac{\pi}{4}\right)$

$=\frac{\pi}{2}-\frac{1}{\sqrt{2}}\ln\left|\csc\left(\theta+\frac{\pi}{4}\right)-\cot\left(\theta+\frac{\pi}{4}\right)\right|\Big|_0^{\frac{\pi}{2}}$

第七章 多元函数微分学

一、defs.

1、极限 — $f(x,y)$ $((x,y)\in D)$

若 $\forall \varepsilon>0, \exists \delta>0$，当 $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$ 时

$$|f(x,y)-A|<\varepsilon$$

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=A$$

例1、 $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$，$\lim\limits_{\substack{x\to0\\ y\to0}} f(x,y)$？

解：$\lim\limits_{\substack{x\to0\\ y=x}} f(x,y)=\lim\limits_{x\to0}\dfrac{x^2}{x^2+x^2}=\dfrac{1}{2}$

$\lim\limits_{\substack{x\to0\\ y=-x}} f(x,y)=\lim\limits_{x\to0}\dfrac{-x^2}{x^2+x^2}=-\dfrac{1}{2}$

$\therefore \lim\limits_{\substack{x\to0\\ y\to0}}$ 不存在

2、连续 — $f(x,y)$ $((x,y)\in D)$，$(x_0,y_0)\in D$

若 $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=f(x_0,y_0)$. 称 $f(x,y)$ 在 (x_0,y_0) 处连续.

3、偏导数 — $z=f(x,y)$ $((x,y)\in D)$，$(x_0,y_0)\in D$

关于x的偏增量 $\Delta z_x=f(x_0+\Delta x, y_0)-f(x_0,y_0)=f(x,y_0)-f(x_0,y_0)$

关于y的偏增量 $\Delta z_y=f(x_0, y_0+\Delta y)-f(x_0,y_0)=f(x_0,y)-f(x_0,y_0)$

全增量 $\Delta z=f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)=f(x,y)-f(x_0,y_0)$

$$\lim_{\Delta x\to0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}=\lim_{\Delta x\to0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}\triangleq f'_x(x_0,y_0)\triangleq\frac{\partial z}{\partial x}\Big|_{(x_0,y_0)}$$

$$\lim_{\Delta y\to0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}=\lim_{\Delta y\to0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}\triangleq f'_y(x_0,y_0)\triangleq\frac{\partial z}{\partial y}\Big|_{(x_0,y_0)}$$

4、可(全)微

一元：$y=f(x)$，$x_0\in D$，$\Delta y=f(x_0+\Delta x)-f(x_0)\overset{\text{if}}{=\!=}A\Delta x+o(\Delta x)$

①可导 $\Leftrightarrow$ 可微 ② $A=f'(x_0)$ $A\Delta x\triangleq dy|_{x=x_0}=Adx$

二元：$z=f(x,y)$ $(x_0,y_0)\in D$ $\Delta z=f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)$

$\overset{\text{if}}{=\!=} A\Delta x+B\Delta y+o(\rho)$ $(=f(x,y)-f(x_0,y_0))$

$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$ $\left(\overset{\text{if}}{=\!=} A(x-x_0)+B(y-y_0)+o(\rho),\ \rho=\sqrt{(x-x_0)^2+(y-y_0)^2}\right)$

称 $f(x,y)$ 在 (x_0, y_0) 处可(全)微

$$A\Delta x + B\Delta y \triangleq dz\big|_{(x_0,y_0)} = Adx + Bdy$$

注：① 可微与可偏导不等价

② $A = f'_x(x_0, y_0)$，$B = f'_y(x_0, y_0)$

二、理论

1、

$$\begin{tikzcd}
\text{连续} \arrow[rr, shift left, "/"] & & \text{可偏导} \arrow[ll, shift left, "/"] \arrow[dl, shift left, "/"] \\
& \text{可微} \arrow[ul, shift left] \arrow[ur, shift left] & \\
& \text{连续可偏导} \arrow[u] &
\end{tikzcd}$$

$\downarrow$ 强

连续：$\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y) \overset{?}{=} f(x_0, y_0)$

可偏导：$\begin{cases} \lim\limits_{\Delta x\to 0} \dfrac{\Delta z_x}{\Delta x} \ ? \\ \lim\limits_{\Delta y\to 0} \dfrac{\Delta z_y}{\Delta y} \ ? \end{cases}$

可微：$\Delta z \overset{?}{=} A\Delta x + B\Delta y + o(\rho)$

证明：①"可微"$\Rightarrow$"连续"

$$\Delta z = f(x,y) - f(x_0,y_0) = A(x-x_0) + B(y-y_0) + o(\rho)$$

$$\rho = \sqrt{(x-x_0)^2+(y-y_0)^2}$$

$\because \lim\limits_{\rho\to 0} \Delta z = 0$ $\therefore \lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = f(x_0,y_0)$，即连续

②"可微"$\Rightarrow$"可偏导"

$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$$

$$= A\Delta x + B\Delta y + o(\rho)$$

取 $\Delta y = 0$. $\Delta z_x = A\Delta x + o(\Delta x) \Rightarrow \dfrac{\Delta z_x}{\Delta x} = A + \dfrac{o(\Delta x)}{\Delta x}$

$\because \lim\limits_{\Delta x\to 0} \dfrac{\Delta z_x}{\Delta x} = A$，即可偏导，且 $A = f'_x(x_0, y_0)$

同理，$B = f'_y(x_0, y_0)$

反例：①"连续"$\not\Rightarrow$"可偏导"

例1：$f(x,y) = \sqrt{x^2+y^2}$ 在 $(0,0)$ 处连续

$\lim\limits_{x\to 0} \dfrac{f(x,0)-f(0,0)}{x-0} = \lim\limits_{x\to 0} \dfrac{|x|}{x}$ 不存在，$\therefore f(x,y)$ 在 $(0,0)$ 处对 x 不可偏导.

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同理，$f(x,y)$ 在 $(0,0)$ 处对 y 不可偏导.

② "可偏导" $\nRightarrow$ "连续"

例 2. $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

$$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{0}{x^3}=0 \Rightarrow f'_x(0,0)=0$$

同理，$f'_y(0,0)=0$，即 $f(x,y)$ 在 $(0,0)$ 处可偏导

$\because \lim\limits_{\substack{x\to 0\\ y=x}} f(x,y)=\frac{1}{2},\ \lim\limits_{\substack{x\to 0\\ y=-x}} f(x,y)=-\frac{1}{2}$ $\therefore \lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ 不存在

而 $f(0,0)=0$，$\therefore f(x,y)$ 在 $(0,0)$ 处不连续

Notes：① 可偏导是可微的必要不充分条件.

② 若 $\Delta z=A\Delta x+B\Delta y+o(\rho)$

则 $A=f'_x(x_0,y_0)$，$B=f'_y(x_0,y_0)$

③ $\Delta z=A\Delta x+B\Delta y+o(\rho)\Leftrightarrow$

$\Delta z-A\Delta x-B\Delta y=o(\rho)\Leftrightarrow$

$$\lim_{\rho\to 0}\frac{\Delta z-A\Delta x-B\Delta y}{\rho}=0$$

☆ 若 $f(x,y)$ 在 (x_0,y_0) 处可偏导，则可微 $\Leftrightarrow \lim\limits_{\rho\to 0}\dfrac{\Delta z-A\Delta x-B\Delta y}{\rho}=0$

③ "连续" $\nRightarrow$ "可微" $\nLeftarrow$ "可偏导"

例 3. $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

解：$\because 0\leq|f(x,y)|=|x|\cdot\dfrac{|y|}{\sqrt{x^2+y^2}}\leq|x|$

又 $\because \lim\limits_{\substack{x\to 0\\ y\to 0}}|x|=0$ $\therefore \lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=0=f(0,0)$

$$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{0}{x|x|}=0\Rightarrow f'_x(0,0)=0$$

同理 $f'_y(0,0)=0$

$\Delta z=f(x,y)-f(0,0)=\dfrac{xy}{\sqrt{x^2+y^2}}$，$\rho=\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}$

$A=0$，$B=0$. $\lim\limits_{\rho\to 0}\dfrac{\Delta z-A(x-0)-B(y-0)}{\rho}=\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{xy}{x^2+y^2}$ 不存在

$\therefore f(x,y)$ 在 $(0,0)$ 处不可微

例 $z=x^3+2x^2y-xy^2+2y^3$

$\frac{\partial z}{\partial x}=3x^2+4xy-y^2$, $\frac{\partial z}{\partial y}=2x^2-2xy+6y^2$

$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=4x-2y$, $\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=4x-2y$

2、$z=f(x,y)$ 二阶连续可偏导

$$\not\Rightarrow \frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$$

三. 求偏导

(一) 显函数求偏导

1、$z=x^y$

$\frac{\partial z}{\partial x}=yx^{y-1}$, $\frac{\partial z}{\partial y}=x^y\ln x$

2、$z=\arctan^2\frac{x+y}{1-xy}$

解: $\frac{\partial z}{\partial x}=2\arctan\frac{x+y}{1-xy}\cdot\frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2}\cdot\frac{1-xy+(x+y)y}{(1-xy)^2}$

☆ (二) 复合函数求偏导

例1、$z=f(x^2+y^2)$, f二阶可导，求$\frac{\partial^2 z}{\partial x\partial y}$

解: $\frac{\partial z}{\partial x}=2xf'(x^2+y^2)$

$\frac{\partial^2 z}{\partial x\partial y}=2x\cdot 2y\cdot f''(x^2+y^2)$

例2、$z=f(x^2+y^2, e^x\sin y)$. f二阶连续可偏导，求$\frac{\partial^2 z}{\partial x\partial y}$

解: $\frac{\partial z}{\partial x}=2xf_1'+e^x\sin y\cdot f_2'$

$\frac{\partial^2 z}{\partial x\partial y}=2x(2yf_{11}''+e^x\cos y f_{12}'')+e^x\cos y\cdot f_2'+e^x\sin y(2yf_{21}''+e^x\cos y f_{12}'')$

例3. $z=f(x+y, xy, x^2)$. f二阶连续可偏导，求$\frac{\partial^2 z}{\partial x\partial y}$

解: $\frac{\partial z}{\partial x}=f_1'+yf_2'+2xf_3'$

$\frac{\partial^2 z}{\partial x\partial y}=f_{11}''+xf_{12}''+f_2'+y(f_{21}''+xf_{22}'')+2x(f_{31}''+xf_{32}'')$

(三) 隐函数(组)求偏导

例1. $\begin{cases}x+2y+3z=0\\ x^2+y^2+z^2=21\end{cases}$ ，求$\frac{dz}{dx}$

解：$\begin{cases} x+2y+3z=0 \\ x^2+y^2+z^2=21 \end{cases} \Rightarrow \begin{cases} y=y(x) \\ z=z(x) \end{cases}$

$\therefore \begin{cases} 1+2\frac{dy}{dx}+3\frac{dz}{dx}=0 \\ 2x+2y\frac{dy}{dx}+2z\frac{dz}{dx}=0 \end{cases}$

$F(x,y,z)=0 \Rightarrow z=\varphi(x,y)$

$\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases} \Rightarrow \begin{cases} y=y(x) \\ z=z(x) \end{cases}$

$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \Rightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$

例2. $\begin{cases} F(x,y,t)=0 \\ t=f(x,y) \end{cases}$，求 $\frac{dt}{dx}$

解：1° (三个字母，两个约束条件 ⇒ 两个一元)

$\begin{cases} F(x,y,t)=0 \\ t=f(x,y) \end{cases} \Rightarrow \begin{cases} y=y(x) \\ t=t(x) \end{cases}$

2° $\begin{cases} F_1'+F_2'\frac{dy}{dx}+F_3'\frac{dt}{dx}=0 \\ \frac{dt}{dx}=f_1'+f_2'\frac{dy}{dx} \end{cases}$

例3. $\begin{cases} xu+yv=1 \\ xv-yu=3 \end{cases}$，求 $\frac{\partial u}{\partial x}$，$\frac{\partial v}{\partial x}$

解：1° (四个字母，两个约束条件 ⇒ 两个二元)

$\begin{cases} xu+yv=1 \\ xv-yu=3 \end{cases} \Rightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$

2° $\begin{cases} u+x\frac{\partial u}{\partial x}+y\frac{\partial v}{\partial x}=0 \\ v+x\frac{\partial v}{\partial x}-y\frac{\partial u}{\partial x}=0 \end{cases}$

(四) 反问题

$f_x'(x,0)=x^2-1$

例1. $\frac{\partial^2 z}{\partial x\partial y}=e^x+2\sin y$，$z=f(x,y)$，$f(0,0)=3$，求 $f(x,y)$.

解：$\frac{\partial^2 z}{\partial x\partial y}=e^x+2\sin y \Rightarrow \frac{\partial z}{\partial x}=e^x y-2\cos y+\varphi(x)$

$\because f_x'(x,0)=x^2-1$，$\therefore \varphi(x)-2=x^2-1 \Rightarrow \varphi(x)=x^2+1$

$\therefore \frac{\partial z}{\partial x}=e^x y-2\cos y+x^2+1$

$z=e^x y-2x\cos y+\frac{1}{3}x^3+x+\psi(y)$

四、应用 (一、二、三) 极值

(一) 无条件极值

一元：$y=f(x)$

1° $x\in D$　　2° $f'(x)\begin{cases} =0 \\ 不存在 \end{cases}$　　3° 判别法

方法一：$\begin{cases} x<x_0, f'>0 \\ x>x_0, f'<0 \end{cases} \Rightarrow$ 大；$\begin{cases} x<x_0, f'<0 \\ x>x_0, f'>0 \end{cases} \Rightarrow$ 小

方法二：$f'(x_0)=0$，$f''(x_0)\begin{cases} >0, & 小 \\ <0, & 大 \end{cases}$

二元：$z=f(x,y)$，$(x,y)\in D$（开）

1° $\begin{cases} \frac{\partial z}{\partial x}=\cdots=0 \\ \frac{\partial z}{\partial y}=\cdots=0 \end{cases} \Rightarrow \begin{cases} x=? \\ y=? \end{cases}$（驻点）

2° 判别法：

设(x_0,y_0)为驻点. $A=f''_{xx}(x_0,y_0)$，$B=f''_{xy}(x_0,y_0)$，$C=f''_{yy}(x_0,y_0)$

$$AC-B^2\begin{cases} >0 \Rightarrow \checkmark \begin{cases} A>0 & 小 \\ A<0 & 大 \end{cases} \\ <0 \Rightarrow \times \end{cases}$$

例1. $f(x,y)=xe^{-x^2+y^2}$，求极值点与极值.

解：1° $\begin{cases} \frac{\partial f}{\partial x}=(1-2x^2)e^{-x^2+y^2}=0 \\ \frac{\partial f}{\partial y}=2xye^{-x^2+y^2}=0 \end{cases} \Rightarrow \begin{cases} x=-\frac{1}{\sqrt{2}} \\ y=0 \end{cases}$ $\begin{cases} x=\frac{1}{\sqrt{2}} \\ y=0 \end{cases}$

2° $\frac{\partial^2 f}{\partial x^2}=?$ $\frac{\partial^2 f}{\partial x\partial y}=?$ $\frac{\partial^2 f}{\partial y^2}=?$

当$x=-\frac{1}{\sqrt{2}}$，$y=0$时，$A=?$ $B=?$ $C=?$ $AC-B^2$?

（二）条件极值

Case 1. $z=f(x,y)$ s.t. $\varphi(x,y)=0$（等式）

方法一：L一法

1° $F=f(x,y)+\lambda\varphi(x,y)$

2° $\begin{cases} F'_x=f'_x+\lambda\varphi'_x=0 \\ F'_y=f'_y+\lambda\varphi'_y=0 \\ F'_\lambda=\varphi(x,y)=0 \end{cases} \Rightarrow \begin{cases} x=? \\ y=? \end{cases}$

方法二：$\varphi(x,y)=0 \Rightarrow \begin{cases} x=x(t) \\ y=y(t) \end{cases}$ $(\alpha\le t\le\beta)$

代入 $z=f[x(t),y(t)]$ $(\alpha\le t\le\beta)$

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例1、(P124. 例3) 求函数 $z=x^2+12xy+2y^2$ 在区域 $D: 4x^2+y^2\le 25$ 上的最大和最小值.

解：① 当 $4x^2+y^2<25$ 时，

由 $\begin{cases} z_x' = 2x+12y=0 \\ z_y' = 12x+4y=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases} \Rightarrow z(0,0)=0$；

② $4x^2+y^2=25$ 时,

方法一：L-法 ：1° $F=x^2+12xy+2y^2+\lambda(4x^2+y^2-25)$

2° $\begin{cases} F_x' = 2x+12y+8\lambda x=0 \\ F_y' = 12x+4y+2\lambda y=0 \\ F_\lambda' = 4x^2+y^2-25=0 \end{cases}$ $\begin{cases} (1+4\lambda)x+6y=0 & ① \\ 6x+(2+\lambda)y=0 & ② \\ 4x^2+y^2=25 & ③ \end{cases}$

3° (0,0)不是③的解，

所以①②有非零解，

$\because D=\begin{vmatrix} 1+4\lambda & 6 \\ 6 & 2+\lambda \end{vmatrix}=0$

$\therefore \lambda_1=2$ 或 $\lambda_2=-\frac{17}{4}$

$\begin{cases} x=-2 \\ y=3 \\ \lambda=2 \end{cases}$ $\begin{cases} x=2 \\ y=-3 \\ \lambda=2 \end{cases}$ $\begin{cases} x=-\frac{3}{2} \\ y=-4 \\ \lambda=-\frac{17}{4} \end{cases}$ $\begin{cases} x=\frac{3}{2} \\ y=4 \\ \lambda=-\frac{17}{4} \end{cases}$

③ 内部1个，边上4个怀疑对象，分别算出即可.

方法二：令 $\begin{cases} x=\frac{5}{2}\cos t \\ y=5\sin t \end{cases}$ $(0\le t\le 2\pi)$，$z=?$

第七章 微分方程

一、defs. (两个字母)

1、微分方程 —— 含导数或微分的方程

2、微分方程的解 — 使微分方程成立的函数

- Part Ⅰ 一阶微分方程
- Part Ⅱ 可降阶的微分方程 (一、二)
- Part Ⅲ 高阶线性微分方程

Part Ⅰ 一阶微分方程

一、可分离变量的D.E.

def — 设 $\frac{dy}{dx}=f(x,y)$

若 $f(x,y)=\varphi_1(x)\varphi_2(y)$，称 $\frac{dy}{dx}=f(x,y)$ 为可分离变量的D.E.

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解法：$\frac{dy}{dx}=f(x,y) \Rightarrow \frac{dy}{dx}=\varphi_1(x)\varphi_2(y)$

$\Rightarrow \int \frac{dy}{\varphi_2(y)} = \int \varphi_1(x)dx + C$

例1、$\frac{dy}{dx}=1+x+y^2+xy^2$

解：$\frac{dy}{dx}=(1+x)(1+y^2) \Rightarrow \int \frac{dy}{1+y^2}=\int(1+x)dx+C$

$\arctan y = x+\frac{1}{2}x^2+C$

例2、$\frac{dy}{dx}-2xy=0$

解：$\frac{dy}{dx}=2xy$，① $y=0$ 为方程的解.

② $y\neq 0$时，$\int \frac{dy}{y}=\int 2x\,dx+C_0 \Rightarrow \ln|y|=x^2+C_0$

$\Rightarrow |y|=e^{C_0}\cdot e^{x^2} \Rightarrow y=\pm e^{C_0}e^{x^2} \xRightarrow{\pm e^{C_0}=C} y=Ce^{x^2}\ (C\neq 0)$

∴通解：$y=Ce^{x^2}$（C为任意常数）

二、齐次D.E.

def — 设$\frac{dy}{dx}=f(x,y)$，若$f(x,y)=\varphi(\frac{y}{x})$，称齐次D.E.

如：$\frac{dy}{dx}=\frac{x-2y}{2x+y} \Rightarrow \frac{dy}{dx}=\frac{1-2y/x}{2+2y/x}$

$\frac{dy}{dx}=\frac{x^2-2xy}{xy+y^2} \Rightarrow \frac{dy}{dx}=\frac{1-2y/x}{y/x+(y/x)^2}$

解法：$\frac{dy}{dx}=\varphi(\frac{y}{x})$，令$\frac{y}{x}=u \Rightarrow y=xu \Rightarrow \frac{dy}{dx}=u+x\frac{du}{dx}$

∴ $u+x\frac{du}{dx}=\varphi(u) \Rightarrow x\frac{du}{dx}=\varphi(u)-u \Rightarrow \int\frac{du}{\varphi(u)-u}=\int\frac{dx}{x}+C$

例2、$xdy-(y+\sqrt{x^2+y^2})dx=0\ (x>0)$

解：$\frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x} \Rightarrow \frac{dy}{dx}=\frac{y}{x}+\sqrt{(\frac{y}{x})^2+1}$

令$\frac{y}{x}=u$，$\frac{dy}{dx}=u+x\frac{du}{dx}$ ∴ $u+x\frac{du}{dx}=u+\sqrt{u^2+1}$

$\int\frac{du}{\sqrt{u^2+1}}=\int\frac{dx}{x}+\ln C \Rightarrow \ln(u+\sqrt{u^2+1})=\ln x+\ln C$

$\begin{cases} u+\sqrt{u^2+1}=Cx \\ -u+\sqrt{u^2+1}=\frac{1}{Cx}\end{cases} \Rightarrow u=\frac{1}{2}(Cx-\frac{1}{Cx}) \Rightarrow y=\frac{1}{2}(Cx^2-\frac{1}{C})$

三、一阶齐次线性D.E.

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def - $\frac{dy}{dx} + P(x)\, y = 0$　(*)

当 $y=0$ 时，为方程的解；

当 $y\neq 0$ 时，$\frac{dy}{y} = -P(x)\,dx \Rightarrow \ln|y| = -\int P(x)dx + C_0$

$\Rightarrow |y| = e^{C_0}\cdot e^{-\int P(x)dx} \Rightarrow y = \pm e^{C_0} e^{-\int P(x)dx} \xrightarrow{令\pm e^{C_0}=C} y = Ce^{-\int P(x)dx}\ (C\neq 0)$

∴ 通解：　$y = Ce^{-\int P(x)dx}$，（C为任意常数）

如：$y'+2xy=0$，$y = Ce^{-\int 2x dx} = Ce^{-x^2}$

四、一阶非齐次线性 D.E.

def - $y' + P(x)\, y = Q(x)$　(**)

解法：常数变易法

$y'+P(x)y=0$ (*) $\Rightarrow y = Ce^{-\int P(x)dx}$

令(**)解为 $y = C(x)e^{-\int P(x)dx}$ 代入(**)，得

$C'(x)e^{-\int P(x)dx} - P(x)C(x)e^{-\int P(x)dx} + P(x)C(x)e^{-\int P(x)dx} = Q(x)$

$C'(x) = Q(x)e^{\int P(x)dx} \Rightarrow C(x) = \int Q(x)e^{\int P(x)dx}dx + C$

∴ $y = \left[\int Q(x)e^{\int P(x)dx}dx + C\right]e^{-\int P(x)dx}$

例：$y=f(x)$ 满足 $\frac{dy}{dx} - \frac{2}{x}y = -1$，$y=f(x)$ $(0\leq x\leq 1)$（绕x轴旋转体积最小，求 $f(x)$

1° $\frac{dy}{dx} - \frac{2}{x}y = -1$

法一：$\frac{dy}{dx} - \frac{2}{x}y = 0 \Rightarrow y = Ce^{-\int -\frac{2}{x}dx} = Cx^2$

令原方程解为 $y = C(x)\cdot x^2$.

由 $C'(x)\cdot x^2 + C(x)\cdot 2x - \frac{2}{x}\cdot C(x)\cdot x^2 = -1$

$C'(x) = -\frac{1}{x^2} \Rightarrow C(x) = \frac{1}{x} + C \Rightarrow y = (\frac{1}{x}+C)x^2 = Cx^2 + x$

法二：$y = \left[\int Q(x)e^{\int P(x)dx}dx + C\right]e^{-\int P(x)dx}$

$= e^{-\int -\frac{2}{x}dx}\left[\int (-1)e^{\int -\frac{2}{x}dx}dx + C\right] = Cx^2 + x$

2° $V(c)=\pi\int_0^1(c^2x^4+2cx^3+x^2)dx$

$=\pi(\frac{c^2}{5}+\frac{c}{2}+\frac{1}{3})$

3° $V'(c)=\pi(\frac{2c}{5}+\frac{1}{2})=0\Rightarrow c=-\frac{5}{4}$

$V''(c)=\frac{2\pi}{5}>0$，$\therefore c=-\frac{5}{4}$时 $V(c)$ 最小

$\therefore f(x)=-\frac{5}{4}x^2+x$

Part Ⅱ 可降阶的高阶 D.E.

一、$y^{(n)}=f(x)$ 如 $y''=3x^2\Rightarrow y'=x^3+C_1$，$y=\frac{1}{4}x^4+C_1x+C_2$

二、$f(x,y',y'')=0$ (缺 y)

令 $y'=p$，$y''=\frac{dp}{dx}$ $\Rightarrow f(x,p,\frac{dp}{dx})=0$

例1、$xy''+2y'=0$

法一：$xy''+2y'=0\Rightarrow x^2y''+2xy'=0\Rightarrow(x^2y')'=0\Rightarrow x^2y'=C_1$

$\Rightarrow y'=\frac{C_1}{x^2}\Rightarrow y=-\frac{C_1}{x}+C_2$

法二：令 $y'=p$，$y''=\frac{dp}{dx}$. $x\frac{dp}{dx}+2p=0$. $\frac{dp}{dx}+\frac{2}{x}p=0$

$p=C_1e^{-\int\frac{2}{x}dx}=\frac{C_1}{x^2}$

三、$f(y,y',y'')=0$ (缺 x)

令 $y'=p$，$y''=\frac{dp}{dx}=\frac{dy}{dx}\cdot\frac{dp}{dy}=p\frac{dp}{dy}$

$\Rightarrow f(y,p,p\frac{dp}{dy})=0$

例2. $yy''=y'^2$. $y(0)=1$. $y'(0)=1$.

法一：$yy''-y'^2=0\Rightarrow\frac{yy''-y'^2}{y^2}=0\Rightarrow(\frac{y'}{y})'=0\Rightarrow\frac{y'}{y}=C_1$

$\because y(0)=1$，$y'(0)=1$，$\therefore C_1=1$ $\therefore y'-y=0$ $\therefore y=C_2e^x$

$\because y(0)=1$，$\therefore C_2=1$，$\therefore y=e^x$

法二：令 $y'=p$，$y''=p\frac{dp}{dy}$，$\therefore yp\frac{dp}{dy}=p^2$. $\because p\neq0$，$\therefore y\frac{dp}{dy}-p=0$.

$\frac{dp}{dy}-\frac{1}{y}p=0\Rightarrow p=C_1e^{-\int-\frac{1}{y}dy}=C_1y$，即 $y'=C_1y$

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Part Ⅲ 高阶线性微分方程

一、defs.

$$y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_{n-1}(x) y' + a_n(x) y = 0 \quad (*)$$

(n阶齐次线性微分方程)

$\cdots = f(x) \quad (**)$ (n阶非齐次线性微分方程)

若 $f(x) = f_1(x) + f_2(x)$，则 $\begin{cases} \cdots = f_1(x) & (**)' \\ \cdots = f_2(x) & (**)'' \end{cases}$

二、结构

1、$\varphi_1(x), \cdots, \varphi_s(x)$ 为 $(*)$ 的解，则

$k_1\varphi_1(x) + \cdots + k_s\varphi_s(x)$ 为 $(*)$ 的解

2、$\varphi_1(x), \cdots, \varphi_s(x)$ 为 $(**)$ 的解

① $k_1\varphi_1(x) + \cdots + k_s\varphi_s(x)$ 为 $(**)$ 的解 $\Leftrightarrow k_1 + \cdots + k_s = 1$

② $k_1\varphi_1(x) + \cdots + k_s\varphi_s(x)$ 为 $(*)$ 的解 $\Leftrightarrow k_1 + \cdots + k_s = 0$

3、$\varphi_1(x), \varphi_2(x)$ 为 $(*)$、$(**)$ 的解，则

$\varphi_1(x) + \varphi_2(x)$ 为 $(**)$ 的解.

4、$\varphi_1(x)$、$\varphi_2(x)$ 为 $(**)'$、$(**)''$ 的解，则

$\varphi_1(x) + \varphi_2(x)$ 为 $(**)$ 的解

三、Special Cases.（特例）

(一) $y'' + py' + qy = 0 \quad (*)$

while p、q arc constants

1° $\lambda^2 + p\lambda + q = 0$

2° ① $\Delta > 0 \Rightarrow \lambda_1 \neq \lambda_2 \Rightarrow y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

如 $y'' - y' - 2y = 0 \Rightarrow \lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = 2$

$\Rightarrow y = C_1 e^{-x} + C_2 e^{2x}$

② $\Delta=0 \Rightarrow \lambda_1=\lambda_2 \Rightarrow y=(C_1+C_2x)e^{\lambda_1 x}$

如：$y=(x+3)e^{2x}$ 为 $y''+py'+qy=0$ 的解，$p=?$，$q=?$

解：$\lambda_1=\lambda_2=2 \Rightarrow \lambda^2+p\lambda+q=0 \Rightarrow p=-4, q=4$

③ $\Delta<0 \Rightarrow \lambda_{1,2}=\alpha\pm i\beta \Rightarrow y=e^{\alpha x}\cdot(C_1\cos\beta x+C_2\sin\beta x)$

如：$y''-6y'-9y=0 \Rightarrow \lambda^2-6\lambda+9=0 \Rightarrow \lambda_1=\lambda_2=3$

$y=(C_1+C_2x)e^{3x}$

又如：$y''-2y'+2y=0 \Rightarrow \lambda^2-2\lambda+2=0 \Rightarrow \lambda_{1,2}=\frac{2\pm\sqrt{-4}}{2}=1\pm i$

$y=e^{x}(C_1\cos x+C_2\sin x)$

(二) $y'''+py''+qy'+ry=0$

$\lambda^3+p\lambda^2+q\lambda+r=0$

① λ_1、λ_2、$\lambda_3\in R$ 且两两不等.

$y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}+C_3e^{\lambda_3 x}$

② λ_1、λ_2、$\lambda_3\in R$，$\lambda_1=\lambda_2\neq\lambda_3$

$y=(C_1+C_2x)e^{\lambda_1 x}+C_3e^{\lambda_3 x}$

③ $\lambda_1=\lambda_2=\lambda_3\in R$

$y=(C_1+C_2x+C_3x^2)e^{\lambda_1 x}$

④ $\lambda_1\in R$，$\lambda_{2,3}=\alpha\pm i\beta$

$y=C_1e^{\lambda_1 x}+e^{\alpha x}(C_2\cos\beta x+C_3\sin\beta x)$

(三) $y''+py'+qy=f(x)$ ** (二阶常系数非齐次线性微分方程)

通解 = 齐通解 + 特解

型一：$f(x)=e^{kx}P_n(x)$ 特解：按照右边的样子假设；k 和 λ 比较。

例1、$y''-y'-2y=(2x+1)e^{x}$ (**)

1° $\lambda^2-\lambda-2=0 \Rightarrow \lambda_1=-1, \lambda_2=2$

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$y''-y'-2y=0 \Rightarrow y=C_1e^{-x}+C_2e^{2x}$

2° $y_0(x)=(ax+b)e^{x}$ 代入(**)

例2、$y''-3y'+2y=(2x+1)e^{x}$

1° $\lambda^2-3\lambda+2=0 \Rightarrow \lambda_1=1, \lambda_2=2$,

$y''-3y'+2y=0 \Rightarrow y=C_1e^{x}+C_2e^{2x}$

2° $y_0(x)=(ax^2+bx)e^{x}$

例3、$y''-2y'+y=xe^{x}$

1° $\lambda^2-2\lambda+1=0 \Rightarrow \lambda_1=\lambda_2=1$

$y''-2y'+y=0 \Rightarrow y=(C_1+C_2x)e^{x}$

2° $y_0(x)=(ax^3+bx^2)e^{x}$

第八章 级数

①常数项级数 ②幂级数 ③傅里叶级数

Part 1 常数项级数

一、defs.

1、$\{a_n\}$. $\sum_{n=1}^{\infty}a_n$ 称常数项级数

2、$S_n=a_1+a_2+\cdots+a_n$ — 部分和

① S_n 与 $\sum_{n=1}^{\infty}a_n$ 不同

② $\lim_{n\to\infty}S_n$ 与 $\sum_{n=1}^{\infty}a_n$ 无区别

$$\lim_{n\to\infty}S_n \begin{cases} =S. & \sum_{n=1}^{\infty}a_n=S \\ 不存在. & 发散 \end{cases}$$

例1. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$?

解: 1° $S_n=\frac{1}{1\times 2}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$

2° $\lim\limits_{n\to\infty} S_n=1$ $\therefore \sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}=1$.

二、性质

1、$\sum\limits_{n=1}^{\infty}a_n=A$，$\sum\limits_{n=1}^{\infty}b_n=B \Rightarrow \sum\limits_{n=1}^{\infty}(a_n\pm b_n)=A\pm B$

2、$\sum\limits_{n=1}^{\infty}a_n=S \Rightarrow \sum\limits_{n=1}^{\infty}ka_n=kS$

☆ $k\neq0$时，$\sum\limits_{n=1}^{\infty}a_n$与$\sum\limits_{n=1}^{\infty}ka_n$敛散性相同.

3、增加、减少、改变有限项，级数敛散性不变

4、添括号提高收敛性

5、$\sum\limits_{n=1}^{\infty}a_n$收敛 $\overset{\nLeftarrow}{\Rightarrow}$ $\lim\limits_{n\to\infty}a_n=0$

证：$S_n=a_1+\cdots+a_n$

$\lim\limits_{n\to\infty}S_n=S$，$a_n=S_n-S_{n-1}$，$\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}S_n-\lim\limits_{n\to\infty}S_{n-1}=0$

三、两个重要级数

1、P-级数

$\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ 称为p级数 $\begin{cases} p>1，收敛 \\ p\leq1，发散 \end{cases}$

$p=1$ — 调和级数 $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ 发散

2、几何级数

$\sum\limits_{n=1}^{\infty}aq^n\ (a\neq0)$ $\begin{cases} |q|<1，\dfrac{第一项}{1-q} \\ |q|\geq1，发散 \end{cases}$

如 $\sum\limits_{n=1}^{\infty}2(\frac{2}{3})^n=\dfrac{2\times\frac{2}{3}}{1-\frac{2}{3}}=4$

又如 $\sum\limits_{n=0}^{\infty}x^n=\dfrac{x^0}{1-x}=\dfrac{1}{1-x}$ $(|x|<1)$

四、正项级数

(一) def — $\sum\limits_{n=1}^{\infty}a_n$ $(a_n\geq0)$ 称正项级数

注：$\sum\limits_{n=1}^{\infty}a_n\ (a_n\geq0)$，$S_n=a_1+a_2+\cdots+a_n$

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$S_1 \le S_2 \le \cdots$ 即 $\{S_n\}\uparrow$

$\begin{cases} ① \{S_n\} 无上界, \lim\limits_{n\to\infty} S_n = +\infty \\ ② \{S_n\} 有上界, \lim\limits_{n\to\infty} S_n 存在 \end{cases}$

(二) 判别法

方法一：比较审敛法

Th1（基本形式）$\sum\limits_{n=1}^{\infty} a_n$、$\sum\limits_{n=1}^{\infty} b_n$ $(a_n \geq 0, b_n \geq 0)$

① $\begin{cases} a_n \le b_n \\ \sum\limits_{n=1}^{\infty} b_n 收敛 \end{cases} \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ 收敛

② $\begin{cases} a_n \geq b_n \\ \sum\limits_{n=1}^{\infty} b_n 发散 \end{cases} \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ 发散

例1. $a_n \le b_n \le c_n$，$\sum\limits_{n=1}^{\infty} a_n$、$\sum\limits_{n=1}^{\infty} c_n$ 收敛，证：$\sum\limits_{n=1}^{\infty} b_n$ 收敛

1° $a_n \le b_n \le c_n \Rightarrow 0 \le b_n - a_n \le c_n - a_n$

2° $\sum\limits_{n=1}^{\infty}(c_n - a_n)$ 收敛 $\Rightarrow \sum\limits_{n=1}^{\infty} b_n - a_n$ 收敛

3° $\sum\limits_{n=1}^{\infty} a_n$、$\sum\limits_{n=1}^{\infty}(b_n - a_n)$ 收敛 $\Rightarrow \sum\limits_{n=1}^{\infty} b_n$ 收敛

Th1'（极限形式）$\sum\limits_{n=1}^{\infty} a_n$、$\sum\limits_{n=1}^{\infty} b_n$ $(a_n \geq 0, b_n \geq 0)$

$$\lim_{n\to\infty} \frac{b_n}{a_n} = l \quad (0 < l < +\infty)$$

则两个级数敛散性相同.

例2. $\sum\limits_{n=1}^{\infty}\left[\frac{1}{n} - \ln(1+\frac{1}{n})\right]$？

解：1° $\because x > 0$ 时，$\ln(1+x) < x$

$\therefore \frac{1}{n} - \ln(1+\frac{1}{n}) > 0$

2° $\because \lim\limits_{x\to 0}[x - \ln(1+x)]/x^2 = \lim\limits_{x\to 0} \frac{1-\frac{1}{1+x}}{2x} = \frac{1}{2}$

$\therefore \lim\limits_{n\to\infty} \frac{1}{n} - \ln(1+\frac{1}{n}) / \frac{1}{n^2} = \frac{1}{2}$

$\therefore \sum\limits_{n=1}^{\infty}\left[\frac{1}{n} - \ln(1+\frac{1}{n})\right]$ 与 $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ 敛散性相同.

方法二：比值审敛法

Th2: $\sum_{n=1}^{\infty} a_n \ (a_n \geq 0)$ $\qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho \begin{cases} 收敛, & \rho<1 \\ 发散, & \rho>1 \\ 不用, & \rho=1 \end{cases}$

例1: $\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{n^n}$?

解: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}\cdot(n+1)!}{(n+1)^{n+1}} \Big/ \frac{2^n \cdot n!}{n^n} = 2\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$

$= 2\lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{2}{e} < 1 \quad \therefore \sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$ 收敛

方法三：根值审敛法

Th3. $\sum_{n=1}^{\infty} a_n \ (a_n \geq 0)$. $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho \begin{cases} <1, & 收敛 \\ >1, & 发散 \\ =1, & 不用 \end{cases}$

五、交错级数

(一) def - $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$ $\ (a_n > 0)$

或 $\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \cdots$

(二) 判别法（莱布尼茨法）

Th. $\sum_{n=1}^{\infty} (-1)^{n-1} a_n \ (a_n > 0)$ 若 ① $\{a_n\}\downarrow$；② $\lim_{n\to\infty} a_n = 0$

则 $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - \cdots$ 收敛，且 $S \leq a$

例4 $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

解: $a_n = \frac{1}{\sqrt{n}}$. $\because \{a_n\}\downarrow$ 且 $\lim_{n\to\infty} a_n = 0$. $\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ 收敛

Q1: $\sum_{n=1}^{\infty} a_n$ 收敛 $\overset{?}{\Longrightarrow} \sum_{n=1}^{\infty} a_n^2$ 收敛 ? ×

如 $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ 收敛，而 $\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{\sqrt{n}}\right)^2$ 发散

Q2: $\sum_{n=1}^{\infty} a_n (a_n \geq 0)$ 收敛 $\overset{?}{\Longrightarrow} \sum_{n=1}^{\infty} a_n^2$ 收敛 √

证: $\lim_{n\to\infty} a_n = 0$，取 $\varepsilon = 1$，$\exists N > 0$，当 $n > N$ 时，$|a_n - 0| < 1$

$\Rightarrow 0 \leq a_n < 1 \Rightarrow 0 \leq a_n^2 \leq a_n < 1$

$\because \sum_{n=1}^{\infty} a_n$ 收敛 $\quad \therefore \sum_{n=1}^{\infty} a_n^2$ 收敛

六、绝对收敛、条件收敛

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ 收敛而 $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right|$ 发散

1、$\sum_{n=1}^{\infty} a_n$ 收敛，$\sum_{n=1}^{\infty} |a_n|$ 发散，称 $\sum_{n=1}^{\infty} a_n$ 条件收敛

2、$\sum_{n=1}^{\infty} |a_n|$ 收敛，称 $\sum_{n=1}^{\infty} a_n$ 绝对收敛

1、$\sum_{n=1}^{\infty} a_n$ 敛. ① $\sum_{n=1}^{\infty} (a_{2n-1}+a_{2n})$? ② $\sum_{n=1}^{\infty} (a_n+a_{n+1})$? ③ $\sum_{n=1}^{\infty} a_n a_{n+1}$?

解: ① $\sum_{n=1}^{\infty} (a_{2n-1}+a_{2n}) = (a_1+a_2)+(a_3+a_4)+\cdots$ 收敛

② $\sum_{n=1}^{\infty} (a_n+a_{n+1}) = (a_1+a_2)+(a_2+a_3)+\cdots$

$S_n = a_1+a_2+\cdots+a_n$

$S_n^{(1)} = (a_1+a_2)+(a_2+a_3)+\cdots+(a_n+a_{n+1})$

$= 2S_n - a_1 + a_{n+1}$

$\because \lim_{n\to\infty} S_n^{(1)} = 2S - a_1$，$\therefore \sum_{n=1}^{\infty} (a_n+a_{n+1})$ 收敛

③ 不一定：$a_n = \frac{(-1)^n}{\sqrt{n}}$，$\sum_{n=1}^{\infty} a_n$ 收敛，$\sum_{n=1}^{\infty} a_n a_{n+1} = -1\cdot\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$

$\because \lim_{n\to\infty} \frac{1}{\sqrt{n(n+1)}} / \frac{1}{n} = 1$ 且 $\sum_{n=1}^{\infty} \frac{1}{n}$ 发散，$\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$ 发散

2、$a_n>0$, $b_n>0$. $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$. $\sum_{n=1}^{\infty} a_n$ 收敛. 证 $\sum_{n=1}^{\infty} b_n$ 收敛

证: 取 $\varepsilon=1$，$\exists N>0$，当 $n>N$ 时，$\left|\frac{b_n}{a_n}-0\right|<1 \Rightarrow 0<b_n<a_n$

$\because \sum_{n=1}^{\infty} a_n$ 收敛 $\therefore \sum_{n=1}^{\infty} b_n$ 收敛

3、$a_1=2$，$a_{n+1} = \frac{1}{2}(a_n+\frac{1}{a_n})$. ① 证 $\lim_{n\to\infty} a_n$ 存在；② 证 $\sum_{n=1}^{\infty} (1-\frac{a_{n+1}}{a_n})$ 收敛

证: ① $a_{n+1} \geq 1$，$a_{n+1}-a_n = \frac{1}{2}(a_n+\frac{1}{a_n}) - a_n = \frac{1-a_n^2}{2a_n} \leq 0$

$\Rightarrow \{a_n\}\downarrow$ $\therefore \lim_{n\to\infty} a_n$ 存在

② $\because \{a_n\}\downarrow$，$\therefore 1-\frac{a_{n+1}}{a_n} \geq 0$

$0 \leq 1-\frac{a_{n+1}}{a_n} = \frac{a_n-a_{n+1}}{a_n} \leq a_n - a_{n+1}$

对 $\sum_{n=1}^{\infty} (a_n-a_{n+1})$，用定义. $S_n = (a_1-a_2)+\cdots+(a_n-a_{n+1}) = 2-a_{n+1}$

$\because \lim_{n\to\infty} S_n$ 存在 $\therefore \sum_{n=1}^{\infty} (a_n-a_{n+1})$ 收敛.

4、$\{a_n\}$: $\begin{cases} a_n>0 \\ \{a_n\}\downarrow \end{cases}$，$\sum_{n=1}^{\infty} (-1)^n a_n$ 发散，问 $\sum_{n=1}^{\infty} (\frac{1}{1+a_n})^n$?

1° $\begin{cases} a_n>0 \\ \{a_n\}\downarrow \end{cases} \Rightarrow \lim_{n\to\infty} a_n = A \geq 0$

2° $\sum_{n=1}^{\infty}(-1)^n a_n$ 发散 $\Rightarrow \lim_{n\to\infty} a_n = A > 0$

3° $\lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{1+a_n}\right)^n} = \lim_{n\to\infty}\frac{1}{1+a_n} = \frac{1}{1+A} = \rho < 1$

Part Ⅱ 幂级数

一、def — $\begin{cases}\sum_{n=0}^{\infty}a_n x^n = a_0 + a_1 x + a_2 x^2 \\ \sum_{n=0}^{\infty}a_n(x-x_0)^n = a_0 + a_1(x-x_0)+\cdots\end{cases}$

二、阿贝尔定理

$\sum_{n=0}^{\infty}a_n x^n$，$\exists R \geq 0$ $\begin{cases}|x|<R，绝对收敛 \\ |x|>R，发散 \\ |x|=R，一切皆有可能\end{cases}$

R— 收敛半径

Th1. 对 $\sum_{n=0}^{\infty}a_n x^n$，

① $x=x_0$ 时，$\sum_{n=0}^{\infty}a_n x_0^n$ 收敛 $\Rightarrow |x|<|x_0|$ 时，级数绝对收敛

② $x=x_1$ 时，$\sum_{n=0}^{\infty}a_n x_1^n$ 发散 $\Rightarrow |x|>|x_1|$ 时，级数发散

Th2. 对 $\sum_{n=0}^{\infty}a_n x^n$，若 $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho$ 或 $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \rho \Rightarrow R = \frac{1}{\rho}$

三、分析性质

$S(x) = \sum_{n=0}^{\infty}a_n x^n \quad x\in(-R,R)$ 和函数

Th1. $x\in(-R,R)$ 时，$\left(\sum_{n=0}^{\infty}a_n x^n\right)' = \sum_{n=0}^{\infty}(a_n x^n)' = \sum_{n=1}^{\infty}n a_n x^{n-1}$

且 $\sum_{n=1}^{\infty}n a_n x^{n-1}$ 收敛半径为R —— 逐项可导性

Th2. $x\in(-R,R)$ 时，$\int_0^x\left(\sum_{n=0}^{\infty}a_n x^n\right)dx = \sum_{n=0}^{\infty}\int_0^x a_n x^n dx$

$= \sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$ 且收敛半径为R —— 逐项可积性

四、任务：①收敛半径、收敛域 ②求 $S(x)$

③展开 ④特殊常数项级数之和

(一) R、域 $\begin{cases}收敛区间：(-R,R) \\ 收敛域：讨论 \pm R\end{cases}$

1. $\sum_{n=1}^{\infty}\frac{x^n}{n}$？

解：1° $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{n+1}\Big/\frac{1}{n} = 1 \Rightarrow R=1$

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2° $x=-1$，$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ 收敛；$x=1$，$\sum_{n=1}^{\infty}\frac{1}{n}$ 发散；收敛域 $[-1,1)$

2、$\sum_{n=1}^{\infty}\frac{(2x-1)^n}{2^n n^2}$?

解：1° $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{2} \Rightarrow R=2$

2° $2x-1=\pm 2$ 时，$\because \sum_{n=1}^{\infty}\left|\frac{(\pm 2)^n}{2^n n^2}\right|=\sum_{n=1}^{\infty}\frac{1}{n^2}$ 收敛

$\therefore$ $2x-1=\pm 2$ 时 级数收敛，$\therefore -2\le 2x-1\le 2$

$\Rightarrow -\frac{1}{2}\le x\le\frac{3}{2}$，收域域 $[-\frac{1}{2},\frac{3}{2}]$

（二）展开 $f(x)\Rightarrow\sum_{n=0}^{\infty}a_n(x-x_0)^n$

方法一：直接法

$f(x)$ 在 $x=x_0$ 邻域内任意阶可导，则

$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ —泰勒级数

$x_0=0$ 时，$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ —麦克劳林级数

记：① $e^x=1+x+\cdots+\frac{x^n}{n!}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!}\ (-\infty<x<+\infty)$

② $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}\ (-\infty<x<+\infty)$

③ $\cos x=1-\frac{x^2}{2!}+\frac{x}{4!}-\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^n\ (-\infty<x<+\infty)$

④ $\frac{1}{1-x}=1+x+\cdots+x^n+\cdots=\sum_{n=0}^{\infty}x^n\ (-1<x<1)$

⑤ $\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n\ (-1<x<1)$

⑥ $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n\ (-1<x\le 1)$

☆ ⑦ $-\ln(1-x)=\sum_{n=1}^{\infty}\frac{x^n}{n}\ (-1\le x<1)$

方法二：

间接法 $\begin{cases}①\sim⑦\\ \text{逐项可导、逐项可积}\end{cases}$

例1. $f(x)=\frac{5x-1}{x^2-x-2}$ 展成 $(x-1)$ 的幂级数

1° $f(x)=\frac{5x-1}{(x+1)(x-2)}=\frac{2}{x+1}+\frac{3}{x-2}$

2° $\frac{1}{x+1}=\frac{1}{2+(x-1)}=\frac{1}{2}\cdot\frac{1}{1+\frac{x-1}{2}}=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-1}{2}\right)^n$

$=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-1)^n \quad (-1<x<3)$

$\frac{1}{x-2}=\frac{1}{-1+(x-1)}=-\frac{1}{1-(x-1)}=-\sum_{n=0}^{\infty}(x-1)^n \quad (0<x<2)$

$3^\circ\ f(x)=\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2}-3\right](x-1)^n \quad (0<x<2)$

例2. $f(x)=\arctan x$ 展成 x 的幂级数

解: $1^\circ\ f'(x)=\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n} \quad (-1<x<1)$

$2^\circ\ f(0)=0$. $f(x)=f(x)-f(0)=\int_0^x f'(x)dx=\sum_{n=0}^{\infty}\int_0^x(-1)^n x^{2n}dx$

$=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} \quad (-1\leq x\leq 1)$

☆(三) 求 $S(x)$ $\begin{cases} ①\sim⑦ \\ \text{逐项可导、逐项可积} \\ \text{微分方程} \end{cases}$

Case 1. $\sum P(n)x^n <$ ④ ⑤

例1. $\sum_{n=0}^{\infty}nx^{n+1}$, 求 $S(x)$

$1^\circ\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1 \Rightarrow R=1$, ∵ $x=\pm1$ 时, $n\cdot(\pm1)^{n+1}\not\to 0\ (n\to\infty)$

∴收敛域为 $(-1,1)$

$2^\circ\ S(x)=\sum_{n=0}^{\infty}nx^{n+1}=\sum_{n=1}^{\infty}nx^{n+1}=x^2\sum_{n=1}^{\infty}nx^{n-1}=x^2\sum_{n=1}^{\infty}(x^n)'$

$=x^2\left(\sum_{n=1}^{\infty}x^n\right)'=x^2\left(\frac{x}{1-x}\right)'$

例2. $\sum_{n=1}^{\infty}n^2x^n$, 求 $S(x)$

$1^\circ\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1 \Rightarrow R=1$, $x=\pm1$ 时, $n^2(\pm1)^n\not\to 0\ (n\to\infty)$

收敛域 $(-1,1)$

$2^\circ\ S(x)=\sum_{n=1}^{\infty}n^2x^n=\sum_{n=1}^{\infty}[n(n-1)+n]x^n$

$=x^2\sum_{n=2}^{\infty}n(n-1)x^{n-2}+x\sum_{n=1}^{\infty}nx^{n-1}=x^2\left(\sum_{n=2}^{\infty}x^n\right)''+x\left(\sum_{n=1}^{\infty}x^n\right)'$

$=x^2\left(\frac{x^2}{1-x}\right)''+x\left(\frac{x}{1-x}\right)'$

Case 2. $\sum\frac{x^n}{P(n)} <$ ⑥⑦ 灭分母

例1. $\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}$

Date . .

1° $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1 \Rightarrow R=1$，$x=\pm1$时，$\sum_{n=1}^{\infty}\left|\frac{(\pm1)^n}{n(n+1)}\right|=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$

收敛域 $[-1,1]$

2° $S(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}-\sum_{n=1}^{\infty}\frac{x^n}{n+1}$

① $S(0)=0$；② $x\neq0$时，$S(x)=-\ln(1-x)-\frac{1}{x}\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1}=-\ln(1-x)-\frac{1}{x}\sum_{n=2}^{\infty}\frac{x^n}{n}$

$=-\ln(1-x)-\frac{1}{x}\left(\sum_{n=1}^{\infty}\frac{x^n}{n}-x\right)=\left(\frac{1}{x}-1\right)\ln(1-x)+1 \quad (-1\le x<1)(x\neq0)$

③ $S(1)=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$

$$S_x=\begin{cases}0, & x=0\\ \cancel{1, \quad x=1} \\ \left(\frac{1}{x}-1\right)\ln(1-x)+1, & -1\le x<1 \text{ 且 } x\neq0\end{cases}$$

例4. $\sum_{n=0}^{\infty}\frac{x^{2n}}{2n+1}$，$S(x)$

解：1° $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1\Rightarrow R=1$，$x=\pm1$时，$\sum_{n=0}^{\infty}\frac{1}{2n+1}$.

$\because \lim_{n\to\infty}\frac{\frac{1}{2n+1}}{\frac{1}{n}}=\frac{1}{2}$ 且 $\sum_{n=1}^{\infty}\frac{1}{n}$ 发散，$\therefore \sum_{n=0}^{\infty}\frac{1}{2n+1}$ 发散，$\therefore$ 收敛域 $(-1,1)$

2° $S(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{2n+1}$，① $S(0)=1$；② $x\neq0$时．$xS(x)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$

$\Rightarrow [xS(x)]'=\sum_{n=0}^{\infty}x^{2n}=\frac{1}{1-x^2}$

$xS(x)=xS(x)-0S(0)=\int_0^x[xS(x)]'dx=-\int_0^x\frac{1}{x^2-1}dx$

$=-\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\Big|_0^x=-\frac{1}{2}\ln\frac{1-x}{1+x}$

$S(x)=-\frac{1}{2x}\ln\frac{1-x}{1+x}$，$\therefore S(x)=\begin{cases}1, & x=0\\ -\frac{1}{2x}\ln\frac{1-x}{1+x}, & -1<x<1 \text{ 且 } x\neq0\end{cases}$

Case 3. $\sum\frac{x^n}{\text{含阶乘}}$ < ①②③ / 微分方程

例5. $\sum_{n=0}^{\infty}\frac{n^2+1}{2^n n!}(3x-2)^n$，$S(x)$

1° $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=0\Rightarrow R=+\infty$，收敛域 $(-\infty,+\infty)$

2° $\sum_{n=0}^{\infty}\frac{n^2+1}{2^n n!}(3x-2)^n=\sum_{n=0}^{\infty}\frac{n^2+1}{n!}\left(\frac{3x-2}{2}\right)^n \xlongequal{\frac{3x-2}{2}=t} \sum_{n=0}^{\infty}\frac{n^2+1}{n!}t^n=S(t)$

$=\sum_{n=1}^{\infty}\frac{n^2}{n!}t^n+\sum_{n=0}^{\infty}\frac{t^n}{n!}=\sum_{n=1}^{\infty}\frac{(n-1)+1}{(n-1)!}t^n+e^t=\sum_{n=2}^{\infty}\frac{n-1}{(n-1)!}t^n+t\sum_{n=1}^{\infty}\frac{t^{n-1}}{(n-1)!}+e^t$

$=t^2\sum_{n=2}^{\infty}\frac{t^{n-2}}{(n-2)!}+te^t+e^t=(t^2+t+1)e^t$

第九章 重积分（二重积分、三重积分）

三重积分

一、def — Ω 有界闭区域，$f(x,y,z)$ 在 Ω 上有界

1° $\Omega \Rightarrow \Delta V_1, \Delta V_2, \cdots, \Delta V_n$；

2° $\forall(\xi_i, \eta_i, \zeta_i) \in \Delta V_i$

作 $\sum\limits_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i)\Delta V_i$

3° λ 为 $\Delta V_1, \Delta V_2, \Delta V_n$ 的最大直径

若 $\lim\limits_{\lambda \to 0} \sum\limits_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i)\Delta V_i$ 存在，称此极限为

$f(x,y,z)$ 在 Ω 上的三重积分，记 $\iiint\limits_{\Omega} f(x,y,z)\,dv$

$$\iiint\limits_{\Omega} f\,dv \triangleq \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i)\Delta V_i$$

二、性质：

1、$\iiint\limits_{\Omega} dv = V$

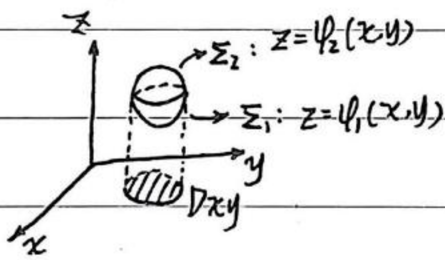
2、① Ω 关于 xoy 面对称，$f(x,y,-z) = -f(x,y,z)$，$\iiint\limits_{\Omega} f\,dv = 0$

② Ω 关于 xoy 面对称，$f(x,y,-z) = f(x,y,z)$，$\iiint\limits_{\Omega} f\,dv = 2\iiint\limits_{\Omega_1} f\,dv$

三、积分法

方法一：直角坐标法

①铅直投影法



$\iiint\limits_{\Omega} f\,dv$

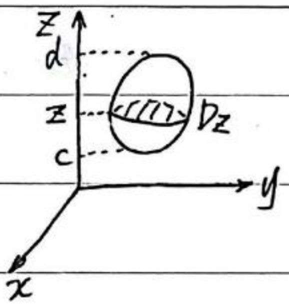
$$\Omega: \begin{cases} (x,y) \in D_{xy} \\ \varphi_1(x,y) \le z \le \varphi_2(x,y) \end{cases}$$

$$\iiint\limits_{\Omega} f\,dv = \iint\limits_{D_{xy}} dx\,dy \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\,dz$$

②切片法

$\iiint\limits_{\Omega} f\,dv$

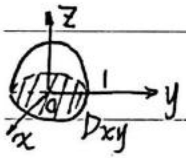
$$\Omega: \begin{cases} (x,y) \subset D_z \\ c \le z \le d \end{cases}$$



$$\iiint_\Omega f\,dv = \int_c^d dz \iint_{D_z} f(x,y,z)\,dxdy$$

例1. $\iiint_\Omega \sqrt{x^2+y^2}\,dv$

① $\Omega=\{(x,y,z)\mid (x,y)\in D_{xy},\ 0\le z\le \sqrt{1-x^2-y^2}\}$, $D_{xy}: x^2+y^2\le 1$

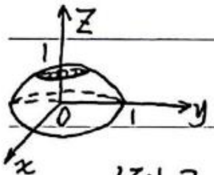


$$I=\iint_{D_{xy}}\sqrt{x^2+y^2}\,dxdy\int_0^{\sqrt{1-x^2-y^2}}dz=\iint_{D_{xy}}\sqrt{x^2+y^2}\cdot\sqrt{1-x^2-y^2}\,dxdy$$

$$=\int_0^{2\pi}d\theta\int_0^1 r^2\sqrt{1-r^2}\,dr=2\pi\int_0^{\frac{\pi}{2}}\sin^2t\cos^2t\,dt$$

$$=2\pi\int_0^{\frac{\pi}{2}}\sin^2t-\sin^4t\,dt=2\pi\left(\frac12\times\frac{\pi}{2}-\frac34\times\frac12\times\frac12\right)=\frac{\pi^2}{8}$$

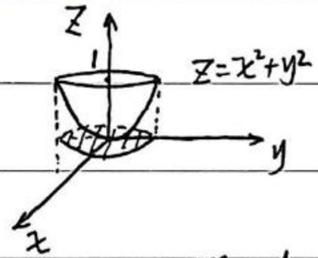
② $\Omega=\{(x,y,z)\mid(x,y)\in D_z,\ 0\le z\le 1\}$ $D_z: x^2+y^2\le 1-z^2$



$$I=\int_0^1 dz\iint_{D_z}\sqrt{x^2+y^2}\,dxdy=\int_0^1 dz\int_0^{2\pi}d\theta\int_0^{\sqrt{1-z^2}}r^2\,dr$$

$$=\frac{2\pi}{3}\int_0^1(1-z^2)^{\frac32}dz=\frac{2\pi}{3}\int_0^{\frac{\pi}{2}}\cos^4t\,dt=\frac{2\pi}{3}\frac34\frac12\frac{\pi}{2}=\frac{\pi^2}{8}$$

例2、 算质心坐标.



解: $\bar{x}=0,\ \bar{y}=0$. $\bar{z}=\dfrac{\iiint_\Omega z\,dv}{\iiint_\Omega dv}$

① $\Omega=\{(x,y,z)\mid(x,y)\in D_{xy},\ x^2+y^2\le z\le 1\}$

$$\iiint_\Omega dv=\iint_{D_{xy}}dxdy\int_{x^2+y^2}^1 1\,dz=\iint_{D_{xy}}(1-x^2-y^2)\,dxdy$$

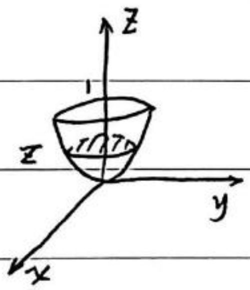
重心: $\bar{x}=\dfrac{\iiint_\Omega x\rho\,dv}{\iiint_\Omega \rho\,dv}$

$$=\int_0^{2\pi}d\theta\int_0^1(r-r^3)\,dr=2\pi\left(\frac12-\frac14\right)=\frac{\pi}{2}$$

$$\iiint_\Omega z\,dv=\iint_{D_{xy}}dxdy\int_{x^2+y^2}^1 z\,dz=\frac12\iint_{D_{xy}}[1-(x^2+y^2)^2]\,dxdy$$

$$=\frac12\int_0^{2\pi}\int_0^1(r-r^5)\,dr=\frac12\cdot2\pi\cdot\left(\frac12-\frac16\right)=\frac{\pi}{3}$$

$$\bar{z}=\frac{\frac{\pi}{3}}{\frac{\pi}{2}}=\frac23$$ ∴ 质心 $(0,0,\frac23)$



② $\Omega=\{(x,y,z)\mid(x,y)\in D_z,\ 0\le z\le 1\}$ $D_z: x^2+y^2\le z$

$$\iiint_\Omega dv=\int_0^1 dz\iint_{D_z}1\,dxdy=\pi\int_0^1 z\,dz=\frac{\pi}{2}$$

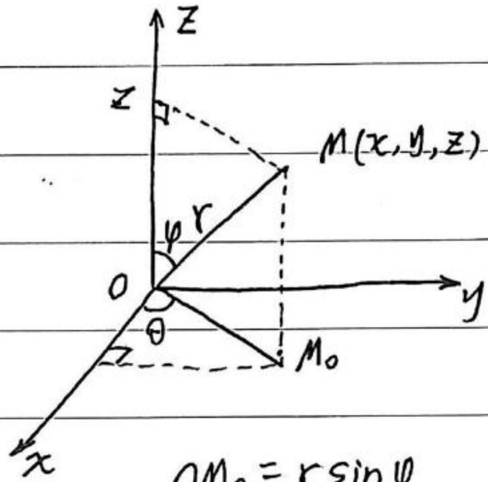
$$\iiint_\Omega z\,dv=\int_0^1 z\,dz\iint_{D_z}1\,dxdy=\pi\int_0^1 z^2\,dz=\frac{\pi}{3}$$

方法三：球面坐标法

① 特征：$\begin{cases} \Omega\text{的边界含 } x^2+y^2+z^2 \\ f\text{中含 } x^2+y^2+z^2 \end{cases}$

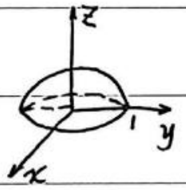
② 变换：$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$

③ $dV = r^2\sin\varphi\, dr\, d\theta\, d\varphi$



$OM_0 = r\sin\varphi$

例3、$\iiint_\Omega \sqrt{x^2+y^2}\, dV$



解：令 $\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$ $\quad 0\le r\le 1,\ 0\le\theta\le 2\pi,\ 0\le\varphi\le\frac{\pi}{2}$

$$I = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 r^3\sin^2\varphi\, dr$$

$$= 2\pi\int_0^{\frac{\pi}{2}}\sin^2\varphi\, d\varphi \int_0^1 r^3 dr = 2\pi\times\frac{1}{2}\times\frac{\pi}{2}\times\frac{1}{4} = \frac{\pi^2}{8}$$

例4. $f(u)$ 连续. $f(0)=0$, $f'(0)=2$. $\Omega: x^2+y^2+z^2\le t^2\ (t>0)$

$\lim\limits_{t\to 0}\dfrac{\iiint_\Omega f(\sqrt{x^2+y^2+z^2})\,dV}{t^4}$?

1° $\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$ $\quad 0\le\theta\le 2\pi,\ 0\le\varphi\le\pi,\ 0\le r\le t$

$$\iiint_\Omega f(\sqrt{x^2+y^2+z^2}) = \int_0^{2\pi} d\theta\int_0^\pi d\varphi\int_0^t f(r)\, r^2\sin\varphi\, dr$$

$$= 2\pi\int_0^\pi \sin\varphi\, d\varphi\int_0^t r^2 f(r)\, dr = 4\pi\int_0^t r^2 f(r)\, dr$$

2° $I = 4\pi\lim\limits_{t\to 0}\dfrac{\int_0^t r^2 f(r)dr}{t^4} = 4\pi\lim\limits_{t\to 0}\dfrac{t^2 f(t)}{4t^3} = \pi\lim\limits_{t\to 0}\dfrac{f(t)-f(0)}{t} = \pi f'(0) = 2\pi$

第十章 空间解析几何

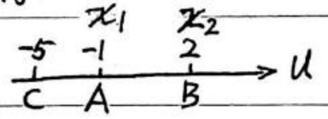
Part I 理论 — 向量

一、def

1、向量 —— 既有大小，又有方向. $\vec{a}$、$\vec{b}$、$\overrightarrow{AB}$

2、向量的坐标

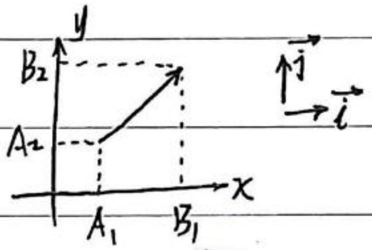
1-dim：



$\vec{e}$

$$\overrightarrow{AB}=3\vec{e} \qquad \overrightarrow{AC}=-4\vec{e} \qquad \overrightarrow{AB}=(x_2-x_1)\vec{e}$$

2-dim：



$A(x_1,y_1)$、$B(x_2,y_2)$

$$\overrightarrow{AB}=\overrightarrow{A_1B_1}+\overrightarrow{A_2B_2}$$
$$=(x_2-x_1)\vec{i}+(y_2-y_1)\vec{j}$$
$$\triangleq\{x_2-x_1,\ y_2-y_1\}$$

3-dim：

$A(x_1,y_1,z_1)$、$B(x_2,y_2,z_2)$

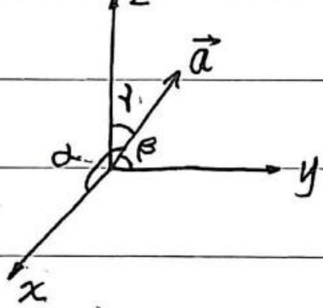
$$\overrightarrow{AB}=(x_2-x_1)\vec{i}+(y_2-y_1)\vec{j}+(z_2-z_1)\vec{k}$$
$$\triangleq\{x_2-x_1,\ y_2-y_1,\ z_2-z_1\}$$

3、向量的模、向量角、向量余弦

设 $\vec{a}=\{a_1,b_1,c_1\}$

① $|\vec{a}|=\sqrt{a_1^2+b_1^2+c_1^2}$

② $\vec{a}^0=\frac{1}{|\vec{a}|}\vec{a}=\{\frac{a_1}{|\vec{a}|},\frac{b_1}{|\vec{a}|},\frac{c_1}{|\vec{a}|}\}$



③ α、β、γ —— 方向角

④ 方向余弦：

$$\cos\alpha=\frac{a_1}{|\vec{a}|}=\frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$$

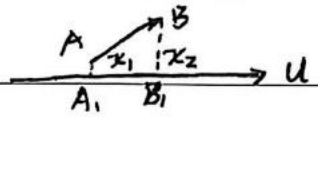
$$\cos\beta=\frac{b_1}{|\vec{a}|}=\frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$$

$$\cos\gamma=\frac{c_1}{|\vec{a}|}=\frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$$

记：$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$\{\cos\alpha,\cos\beta,\cos\gamma\}=\frac{1}{|\vec{a}|}\vec{a}=\vec{a}^0$

4、投影 ——



$\vec{e}$

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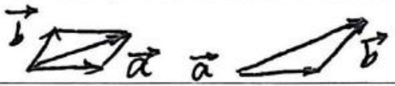
$\overrightarrow{A_1B_1} = (x_2 - x_1)\vec{e}$

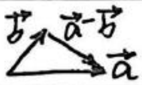
$A_1B_1 = x_2 - x_1$ —— $\overrightarrow{AB}$在u轴上的投影

$A_1B_1 \triangleq Prj_u \overrightarrow{AB} = |\overrightarrow{AB}| \cdot \cos(\widehat{\overrightarrow{AB}, u})$

二、向量的运算

(一) 几何刻画

1、$\vec{a} + \vec{b}$: 

2、$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$ 

3、$k\vec{a}$ $\begin{cases} k>0，方向同\vec{a}，长k倍 \\ k=0，k\vec{a} = \vec{0} \\ k<0，方向与\vec{a}反，长|k|倍 \end{cases}$

4、$\vec{a} \cdot \vec{b} \triangleq |\vec{a}| \cdot |\vec{b}| \cdot \cos(\widehat{\vec{a}, \vec{b}})$

5、$\vec{a} \times \vec{b} = \begin{cases} 方向：右手定则 \\ 大小：|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \cdot \sin(\widehat{\vec{a}, \vec{b}}) \end{cases}$

(二) 代数刻画

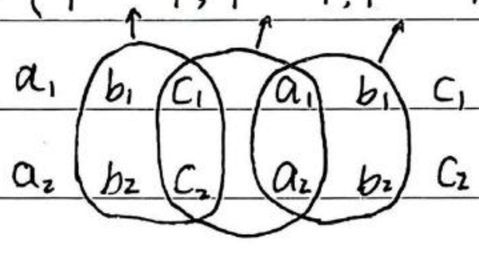
设 $\vec{a} = \{a_1, b_1, c_1\}$，$\vec{b} = \{a_2, b_2, c_2\}$

1、$\vec{a} \pm \vec{b} = \{a_1 \pm a_2, b_1 \pm b_2, c_1 \pm c_2\}$

2、$k\vec{a} = \{ka_1, kb_1, kc_1\}$

3、$\vec{a} \cdot \vec{b} = a_1a_2 + b_1b_2 + c_1c_2$

4、$\vec{a} \times \vec{b} = \{|\quad|, |\quad|, |\quad|\}$



Notes:

1、$\vec{a} \cdot \vec{b}$

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① $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

② $\vec{a}\cdot\vec{a}=|\vec{a}|=a_1^2+b_1^2+c_1^2$

$\vec{a}\cdot\vec{a}=0 \Leftrightarrow \vec{a}=\vec{0}$

③ $\vec{a}\perp\vec{b} \Leftrightarrow \vec{a}\cdot\vec{b}=0 \Leftrightarrow a_1a_2+b_1b_2+c_1c_2=0$

2、$\vec{a}\times\vec{b}$:

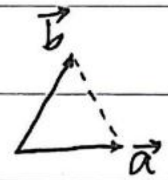
① $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$

② $\vec{a}\times\vec{b}\perp\vec{a}$, $\vec{a}\times\vec{b}\perp\vec{b}$

③ $\vec{a}\times\vec{b}=\vec{0} \Leftrightarrow \vec{a}/\!/\vec{b} \Leftrightarrow \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

☆ ④ $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin(\widehat{\vec{a},\vec{b}})=2S_\Delta$

即 $|\vec{a}\times\vec{b}|=2S_\Delta$



Part Ⅱ 应用

一、空间曲面

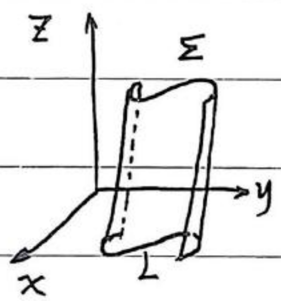
$\Sigma: F(x,y,z)=0$

二、特殊曲面

(一) 柱面

1、$\Sigma: F(x,y)=0$

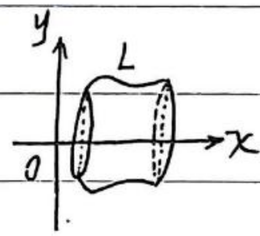
$L: \begin{cases} F(x,y)=0 \\ z=0 \end{cases}$



2、旋转曲面

Case1: (2-dim)

$L: \begin{cases} f(x,y)=0 \\ z=0 \end{cases}$



$\Sigma_x: f(x, \pm\sqrt{y^2+z^2})=0$

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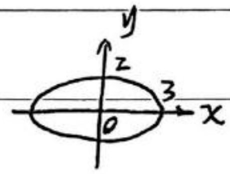
$\Sigma_y: f(\pm\sqrt{x^2+z^2}, y)=0$

如：$L: \begin{cases} \frac{x^2}{9}+\frac{y^2}{4}=1 \\ z=0 \end{cases}$



$\Sigma_x: \frac{x^2}{9}+\frac{y^2}{4}+\frac{z^2}{4}=1$

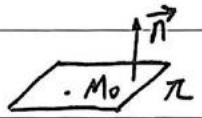
$\Sigma_y: \frac{x^2}{9}+\frac{y^2}{4}+\frac{z^2}{9}=1$

Case 2: (3-dim) (?)

★(二) 退化 — 平面

1、点法式

$M_0(x_0, y_0, z_0)\in\pi$



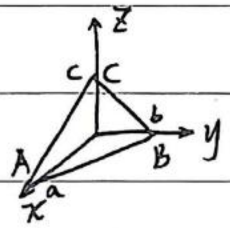
$\vec{n}=\{A,B,C\}\perp\pi$

$\forall M(x,y,z)\in\pi \Leftrightarrow \overrightarrow{M_0M}\perp\vec{n} \Leftrightarrow \vec{n}\cdot\overrightarrow{M_0M}=0$

$\pi: A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

2、截距式

$\overrightarrow{AB}=\{-a, b, 0\}$



$\overrightarrow{AC}=\{-a, 0, c\}$

$\vec{n}=\overrightarrow{AB}\times\overrightarrow{AC}=\{bc, ac, ab\}$

$\pi: bc(x-a)+ac(y-0)+ab(z-0)=0$

$\pi: \frac{x-a}{a}+\frac{y}{b}+\frac{z}{c}=0$

即：$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

3、一般式

$\pi: Ax+By+Cz+D=0$

(三) 空间曲面 $\begin{cases} \text{切平面} \\ \text{法线} \end{cases}$

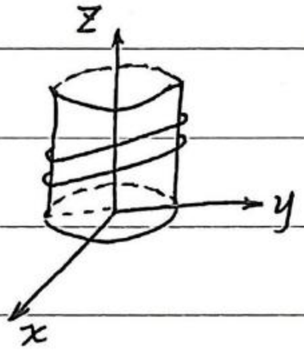
$M_0(x_0, y_0, z_0) \in \Sigma$

$\vec{n} = \{F'_x, F'_y, F'_z\}_{M_0}$

二、空间曲线

(一) 形式

1、一般形式



$$L: \begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$$

例 $L: \begin{cases} x = R\cos\omega t \\ y = R\sin\omega t \\ z = v_0 t \end{cases}$

2、参数式

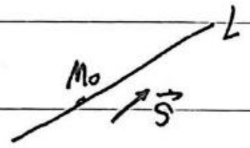
$$L: \begin{cases} x = \varphi(t) \\ y = \psi(t) \\ z = \omega(t) \end{cases}$$

(二) 退化一直线

1、一般式

$$L: \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

2、点向式



$M_0(x_0, y_0, z_0) \in L$

$\vec{s} = \{m, n, p\} \parallel L$

$M(x, y, z) \in L \Leftrightarrow \overrightarrow{M_0M} \parallel \vec{s}$

$$L: \frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p}$$

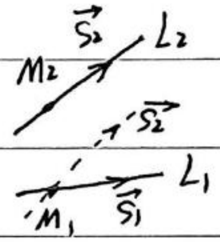
3、参数式 $L: \begin{cases} x = x_0 + mt \\ y = y_0 + nt \\ z = z_0 + pt \end{cases}$

$\frac{x-1}{2} = \frac{2y+3}{4} = \frac{z}{1}$

$M_0(1, -\frac{3}{2}, 0) \in L$

$\vec{s} = \{2, 2, 1\}$

5、异面直线之距



判别：① L_1, L_2 共面 $\Leftrightarrow (\vec{s_1}\times\vec{s_2})\cdot\overrightarrow{M_1M_2}=0$

② L_1, L_2 异面 $\Leftrightarrow (\vec{s_1}\times\vec{s_2})\cdot\overrightarrow{M_1M_2}\neq 0$

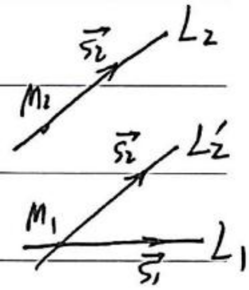
若 L_1, L_2 异面，过 M_1 作 $\vec{s_2}$，与 $\vec{s_1}$ 形成新平面（点法式），求 M_2 到新平面距离即可.

例2. L_1: $\frac{x}{1}=\frac{y-1}{-1}=\frac{z+1}{2}$

L_2: $\frac{x-1}{-1}=\frac{y}{3}=\frac{z-2}{1}$

解: 1° $M_1(0,1,-1)$，$\vec{s_1}=\{1,-1,2\}$

$M_2(1,0,2)$，$\vec{s_2}=\{-1,3,1\}$



$\overrightarrow{M_1M_2}=\{1,-1,3\}$，$\vec{s_1}\times\vec{s_2}=\{-7,-3,2\}$

2° $(\vec{s_1}\times\vec{s_2})\cdot\overrightarrow{M_1M_2}=-7+3+6=2\neq0$，$L_1L_2$ 异面

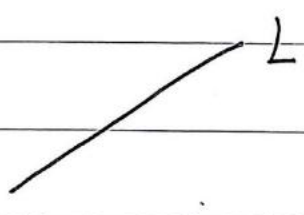
3° π: $-7(x-0)-3(y-1)+2(z+1)=0$

π: $-7x-3y+2z+5=0$

4° $d=M_2$ 到 π 的距离.

平面束：

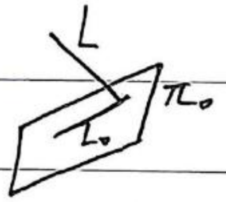
$$L:\begin{cases}A_1x+B_1y+C_1z+D_1=0\\A_2x+B_2y+C_2z+D_2=0\end{cases}$$



π: $A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0$

例3、L: $\begin{cases}x-y-z-2=0\\2x+y-z-1=0\end{cases}$. π_0: $x+y+z-2=0$

求投影 L_0



1° π: $(x-y-z-2)+\lambda(2x+y-z-1)=0$

π: $(2\lambda+1)x+(\lambda-1)y-(\lambda+1)z-2-\lambda=0$

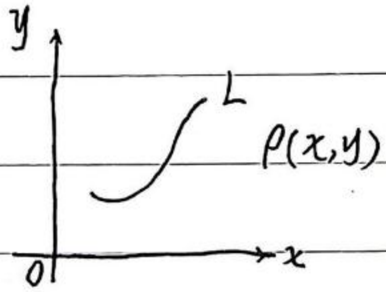
2°. $\{1,1,1\}\cdot\{2\lambda+1,\lambda-1,-\lambda-1\}=0 \Rightarrow \lambda=\frac{1}{2}$

第十一章 曲线曲面积分

Part I 曲线积分

一、对弧长的曲线积分(第一类)

(一) 背景：求 m



1° $\forall ds \subset L$

2° $dm = \rho(x,y)ds$

3° $m = \int_L dm = \int_L \rho(x,y)ds$

(二) def - $\int_L f(x,y)ds$ - $f(x,y)$在L上对弧长的曲线积分

(三) 性质

1、$\int_L 1\,ds = l$

2、① L关于y轴对称，右L_1

若$f(-x,y)=-f(x,y)$，$\int_L f(x,y)ds=0$；

若$f(-x,y)=f(x,y)$，$\int_L f(x,y)ds = 2\int_{L_1} f(x,y)ds$.

② L关于$y=x$对称

$$\int_L f(x,y)ds = \int_L f(y,x)ds$$

(四) 积分法

法一：替代法

1、$L: \frac{x^2}{4}+y^2=1$. L之长为a. $\int_L (x-2y)^2 ds = \int_L (x^2+4y^2-4xy)ds$

$= \int_L (x^2+4y^2)ds = 4\int_L(\frac{x^2}{4}+y^2)ds = 4\int_L ds = 4a$

法二：定积分法

Case 1. $L: y=\varphi(x)\ (a\le x\le b)$

$$\int_L f(x,y)ds = \int_a^b f[x,\varphi(x)]\sqrt{1+\varphi'^2(x)}\,dx$$

Case 2. $L: x=\varphi(t),\ y=\psi(t),\ (\alpha\le t\le\beta)$

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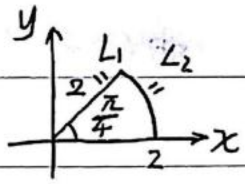
$$\int_L f(x,y)ds = \int_\alpha^\beta f[\varphi(t),\psi(t)]\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt$$

例2. $\int_L x e^{x^2+y^2} ds$.



① $L_1: y=x \quad (0\le x\le \sqrt{2})$

$$I_1 = \int_0^{\sqrt2} x e^{2x^2}\cdot\sqrt{1+\quad 1}\ dx$$

$$=\frac{1}{\sqrt2}\int_0^{\sqrt2}\sqrt2 x\, e^{(\sqrt2 x)^2} d(\sqrt2 x) = \frac{1}{\sqrt2}\int_0^2 x e^{x^2} dx$$

$$=\frac{1}{\sqrt2}\times\frac12 e^{x^2}\Big|_0^2$$

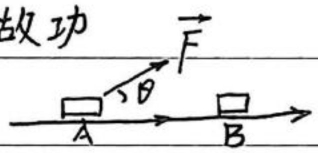
② $L_2: \begin{cases} x=2\cos t \\ y=2\sin t\end{cases} \quad (0\le t\le\frac{\pi}{4})$

$$I_2 = \int_0^{\frac{\pi}{4}} 2\cos t\times e^4\cdot\sqrt{4\sin^2 t+4\cos^2 t}\ dt$$

二、对坐标的曲线积分(第二类)

(一) 背景: 做功

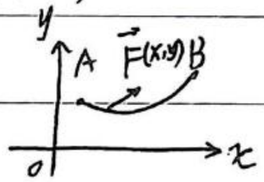
Case 1.



$$W = |\vec F|\cdot\cos\theta\cdot|\vec{AB}| = |\vec F||\vec{AB}|\cos(\vec F \hat{,} \vec{AB})$$

$$\triangleq \vec F\cdot\vec{AB}$$

Case 2: L为xOy面内有向曲线 (2-dim)



$$\vec F(x,y)\triangleq\{P(x,y),Q(x,y)\}$$

1° $\forall\,\vec{ds}\subset L$, $\vec{ds}=\{dx,dy\}$

2° $dW = \vec F(x,y)\cdot\vec{ds} = P(x,y)dx+Q(x,y)dy$

3° $W=\int_L P(x,y)dx+Q(x,y)dy$

Case 3: (3-dim) L为三维空间有向曲线段

$$\vec F = \{P, Q, R\}$$

1° $\forall\,\vec{ds}\subset L$, $\vec{ds}=\{dx,dy,dz\}$;

2° $dw=\vec F\cdot\vec{ds} = Pdx+Qdy+Rdz$

3° $w=\int_L Pdx+Qdy+Rdz$.

(二) defs:

1. (2-dim): $\int_L P dx + Q dy$

2. (3-dim): $\int_L P dx + Q dy + R dz$

(三) 性质

1. $\int_{L^-} = -\int_L$

2. ?

(四) 积分法

Case 1. 2-dim: $\int_L P dx + Q dy$

1. 定积分法

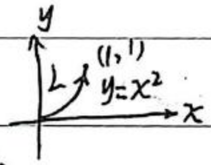
① $L: y=\varphi(x)$ (起: $x=a$; 终: $x=b$)

$\int_L P dx + Q dy = \int_a^b P[x,\varphi(x)]dx + Q[x,\varphi(x)]\varphi'(x)dx$

② $L: \begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$ (起: $t=\alpha$; 终: $t=\beta$)

$\int_L P dx + Q dy = \int_\alpha^\beta P[\varphi(t),\psi(t)]\varphi'(t)dt + Q[\varphi(t),\psi(t)]\psi'(t)dt$

例1. $\int_L (x+2y)dx + (2x-y^2)dy$

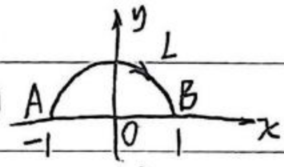


解: 1° $L: y=x^2$ (起 $x=0$, 终 $x=1$)

2° $I = \int_0^1 (x+2x^2)dx + (2x-x^4)\cdot 2x\,dx$

$= \int_0^1 (x+6x^2-2x^5)dx = \frac{1}{2}+2-\frac{1}{3}$

例2. $\int_L (x^3+1)dy - (2x+y^2)dx$



解: 1° $L: \begin{cases} x=\cos t \\ y=\sin t \end{cases}$ (起 $t=\pi$, 终 $t=0$)

2° $I = \int_\pi^0 (\cos^3 t+1)\cdot\cos t\,dt - (2\cos t+\sin^2 t)(-\sin t)dt$

$= \int_\pi^0 (\cos^4 t+\cos t+2\sin t\cos t+\sin^3 t)dt$

$= -\int_0^\pi (\cos^4 t+\sin^3 t)dt = -2(I_4+I_3)$

$= -2\left[\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)+\left(\frac{2}{3}\times 1\right)\right]$

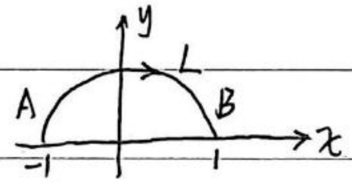
方法二：二重积分法（格林公式）

Th. ① D为连通区域，L为D正向边界

☆ ② $P(x,y)$, $Q(x,y)$在D上连续可偏导

则 $\oint_L Pdx+Qdy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$

例1. $\int_L (x^3+1)dy-(2x+3y)dx$



解：$I=\oint_{L+BA}+\int_{\overline{AB}}=I_1+I_2$

$I_1=-\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma=-\iint_D(3x^2+3)d\sigma$

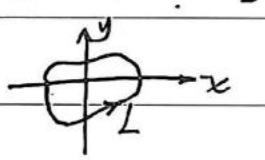
$=-3\iint_D(x^2+1)d\sigma=-3\int_0^\pi d\theta\int_0^1(r^2\cos^2\theta+1)rdr$

$=-3\int_0^\pi\cos^2\theta d\theta\int_0^1 r^3dr-3\int_0^\pi d\theta\int_0^1 rdr$

$=-\frac{6}{4}\times\frac{1}{2}\times\frac{\pi}{2}-3\times\pi\times\frac{1}{2}$

AB: $y=0$ 起，终 $x=1$. $I_2=\int_{-1}^1 -2xdx=0$. $I=I_1+I_2$

例2: $\oint_L\frac{xdy-ydx}{x^2+y^2}$



解: 1° $P=-\frac{y}{x^2+y^2}$, $Q=\frac{x}{x^2+y^2}$

$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}=\frac{x^2-y^2}{(x^2+y^2)^2}$ $((x,y)\neq(0,0))$

2° $L_0: x^2+y^2=r^2$ $(r>0)$ (L_0在L内) (L_0逆)

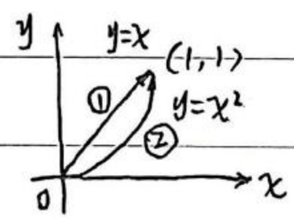
3° $\oint_{L+L_0^-}=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma=0$

$\Rightarrow\oint_L+\oint_{L_0^-}=0\Rightarrow\oint_L=\oint_{L_0}\frac{xdy-ydx}{x^2+y^2}=\frac{1}{r^2}\oint_{L_0}xdy-ydx$

$=\frac{2}{r^2}\iint_{D_2}d\sigma=\frac{2}{r^2}\times\pi r^2=2\pi$

☆ 3. 曲线与路径无关的条件.

例1. $\int_L x^2ydx+(x^2y+2y)dy$



① L: $y=x$ (起 $x=0$, 终 $x=1$)

$I=\int_0^1 x^3dx+(x^3+2x)dx$

$=\int_0^1(2x^3+2x)dx=\frac{1}{2}+1=\frac{3}{2}$

② L: $y=x^2$ (起 $x=0$, 终 $x=1$)

$$I=\int_0^1 x^5dx+(x^4+2x^2)\cdot 2x\,dx$$

$$=\int_0^1(3x^5+4x^3)=\frac{1}{2}+1=\frac{3}{2}$$

Th. D为单连通区域, P、Q在D上连续可偏导, 以下4个等价:

① $\int_L Pdx+Qdy$ 与路径无关

② $\forall C\subset D$ (闭), 有 $\oint_C Pdx+Qdy=0$

③ $\frac{\partial Q}{\partial x}\equiv\frac{\partial P}{\partial y}$ (柯西黎曼条件)

④ $\exists u(x,y)$, $Pdx+Qdy=du$

Note: ① 若 $\frac{\partial Q}{\partial x}\equiv\frac{\partial P}{\partial y}$, 则 $\int_L=\int_{(x_0,y_0)}^{(x_1,y_1)}=\int_{x_0}^{x_1}P(x,y_0)dx+\int_{y_0}^{y_1}Q(x_1,y)dy$

② 若 $\frac{\partial Q}{\partial x}\equiv\frac{\partial P}{\partial y}$, 则 $u(x,y)=\int_{(x_0,y_0)}^{(x,y)}Pdx+Qdy$

$$=\int_{x_0}^{x}P(x,y_0)dx+\int_{y_0}^{y}Q(x,y)dy$$

③ 若 $\frac{\partial Q}{\partial x}\equiv\frac{\partial P}{\partial y}$, 且 $Pdx+Qdy=du$, 则

$$\int_L=\int_{(x_0,y_0)}^{(x_1,y_1)}Pdx+Qdy=u(x_1,y_1)-u(x_0,y_0)$$

例1. $\int_{(0,1)}^{(1,2)}(xy^2+y)dx+(x^2y+x)dy=\int_{(0,1)}^{(1,2)}d(\frac{1}{2}x^2y^2+xy)$

$$=(\frac{1}{2}x^2y^2+xy)\Big|_{(0,1)}^{(1,2)}$$

例2、证明: $x>0$时, $\frac{xdy-ydx}{x^2+y^2}$ 为一个$u(x,y)$的全微分, 求$u(x,y)$.

证: 1° $P=-\frac{y}{x^2+y^2}$, $Q=\frac{x}{x^2+y^2}$

$\because \frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}=\frac{y^2-x^2}{(x^2+y^2)^2}$ $\therefore \exists u(x,y)$, 使 $\frac{xdy-ydx}{x^2+y^2}=du$

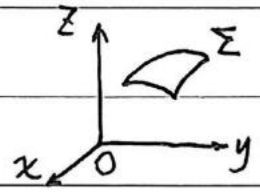
2° $u(x,y)=\int_{(1,0)}^{(x,y)}\frac{xdy-ydx}{x^2+y^2}=\int_1^x 0\,dx+\int_0^y\frac{xdy}{x^2+y^2}=\arctan\frac{y}{x}$.

Case 2. 3-dim: $\int_L Pdx+Qdy+Rdz$ (不讲)

Part Ⅱ 曲面积分

一、对面积的曲面积分. (第一类)

(一) 背景: 求m



$1^\circ\ \forall ds \subset \Sigma$; $2^\circ\ dm=\rho(x,y,z)ds$ $3^\circ\ m=\iint\limits_\Sigma \rho(x,y,z)ds$

(二) def — $\iint\limits_\Sigma f(x,y,z)\,ds$

(三) 性质：

1、$\iint\limits_\Sigma 1\,ds = A$

2、① Σ 关于 xoy 面对称，上 Σ_1，

若 $f(x,y,-z)=-f(x,y,z)$，则 $\iint\limits_\Sigma f\,ds=0$

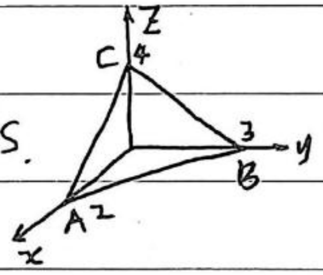
若 $f(x,y,-z)=f(x,y,z)$，则 $\iint\limits_\Sigma = 2\iint\limits_{\Sigma_1}$

(四) 积分法

方法一：替代法

例1、Σ: $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$

$(x\geq 0, y\geq 0, z\geq 0)$，求 $\iint\limits_\Sigma (2x+\frac{4}{3}y+z)\,ds$.



解：$4\iint\limits_\Sigma (\frac{x}{2}+\frac{y}{3}+\frac{z}{4})\,ds$

$=4\iint\limits_\Sigma 1\,ds = 4S$

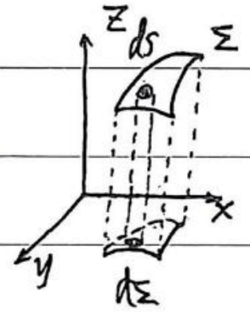
$\vec{AB}=\{-2,3,0\}$，$\vec{AC}=\{-2,0,4\}$，$\vec{AB}\times\vec{AC}=\{12,8,6\}$

$|\vec{AB}\times\vec{AC}|=\sqrt{144+64+36}=\sqrt{244}=2\sqrt{61}=2S$

$\therefore I=4\sqrt{61}$

方法二：二重积分法

$\iint\limits_\Sigma f(x,y,z)\,ds$

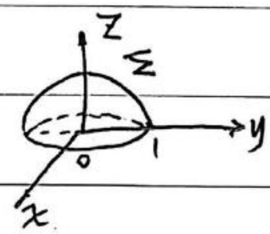


1° $\Sigma: z=\varphi(x,y)$，$(x,y)\in D_{xy}$；

2° $ds=\sqrt{1+z_x'^2+z_y'^2}\,d\sigma$

3° $\iint\limits_\Sigma f\,ds=\iint\limits_{D_{xy}} f[x,y,\varphi(x,y)]\sqrt{1+z_x'^2+z_y'^2}\,d\sigma$

例1、$\iint\limits_\Sigma \sqrt{x^2+y^2}\,ds$



1° $\Sigma: z=\sqrt{1-x^2-y^2}$，$D_{xy}: x^2+y^2\leq 1$

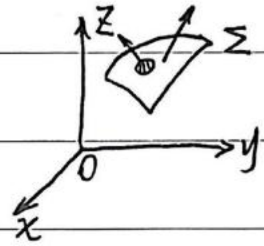
2° $dS=\sqrt{1+z_x'^2+z_y'^2}\,d\sigma=\frac{1}{\sqrt{1-x^2-y^2}}\,d\sigma$

3° $I=\iint_{D_{xy}}\sqrt{x^2+y^2}\,(\sqrt{1-x^2-y^2})^{-1}\,d\sigma$

$=\int_0^{2\pi}d\theta\int_0^1\frac{r^2}{\sqrt{1-r^2}}\,dr=2\pi\int_0^{\frac{\pi}{2}}\frac{\sin^2 t}{\cos t}\cos t\,dt=\frac{\pi^2}{2}$

二、对坐标的曲面积分（第二类）

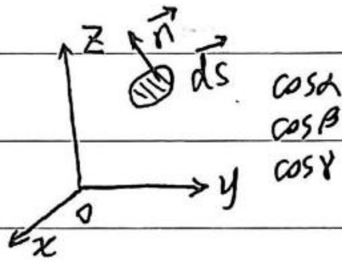
(一) 背景：流量



$\vec{v}=\{P,Q,R\}$，求Φ？

1° $\forall\, d\vec{S}\subset\Sigma$

$d\vec{S}=\{dydz,\ dzdx,\ dxdy\}$



1′ 从x正轴将$d\vec{S}$向yoz面投影.

投影记为dydz

$\begin{cases} dydz>0, & \cos\alpha>0 \\ dydz<0, & \cos\alpha<0 \end{cases}$

2′ 从y正轴将$d\vec{S}$向xoz面投影，投影记为dzdx

$\begin{cases} dzdx>0, & \cos\beta>0 \\ dzdx<0, & \cos\beta<0 \end{cases}$

3′ 从z正轴将$d\vec{S}$向xoy面投影，投影记为dxdy

$\begin{cases} dxdy>0, & \cos\gamma>0 \\ dxdy<0, & \cos\gamma<0 \end{cases}$

($dydz=dS\cos\alpha$, $dzdx=dS\cos\beta$, $dxdy=dS\cos\gamma$)

2° $d\Phi=\vec{v}\,d\vec{S}=P\,dydz+Q\,dzdx+R\,dxdy$

3° $\Phi=\iint_\Sigma P\,dydz+Q\,dzdx+R\,dxdy$

(二) def- $\iint_\Sigma P\,dydz+Q\,dzdx+R\,dxdy$

(三) 性质：

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1、$\iint\limits_{\Sigma^-} = -\iint\limits_{\Sigma}$

2、$\iint\limits_{\Sigma} P\,dydz + Q\,dzdx + R\,dxdy$

$= \iint\limits_{\Sigma} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,dS$

(四)计算方法

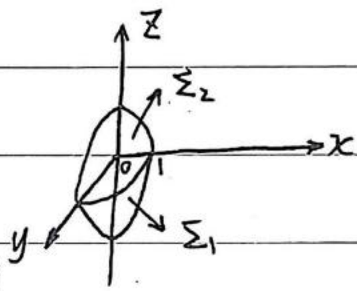
方法一：二重积分法

$\iint\limits_{\Sigma} R\,dxdy \quad \begin{cases} dxdy > 0, & \cos\gamma > 0 \\ dxdy < 0, & \cos\gamma < 0 \end{cases}$

1° $\Sigma: z = \varphi(x,y)$, $(x,y) \in D_{xy}$;

2° $\iint\limits_{\Sigma} R\,dxdy = \pm \iint\limits_{D_{xy}} R[x, y, \varphi(x,y)]\,dxdy$

例1. $\iint\limits_{\Sigma} \sqrt{x^2+y^2} \cdot z\,dxdy$



1° $\iint\limits_{\Sigma} = \iint\limits_{\Sigma_1} + \iint\limits_{\Sigma_2} = I_1 + I_2$

2° ① $\Sigma_1: z = -\sqrt{1-x^2-y^2}$, $(x,y) \in D_{xy}$

② $I_1 = -\iint\limits_{D_{xy}} \sqrt{x^2+y^2} \cdot (-\sqrt{1-x^2-y^2})\,dxdy$

$= \iint\limits_{D_{xy}} \sqrt{x^2+y^2} \cdot \sqrt{1-x^2-y^2}\,dxdy$

3° ① $\Sigma_2: z = \sqrt{1-x^2-y^2}$, $(x,y) \in D_{xy}$

② $I_2 = \iint\limits_{D_{xy}} \sqrt{x^2+y^2}\sqrt{1-x^2-y^2}\,dxdy$

4° $I = 2\iint\limits_{D_{xy}} \sqrt{x^2+y^2}\sqrt{1-x^2-y^2}\,d\sigma$

$= 2\int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^2\sqrt{1-r^2}\,dr$

$= \pi \int_0^{\frac{\pi}{2}} \sin^2 t\,(1-\sin^2 t)\,dt$

$= \pi\left(\frac{1}{2} \times \frac{\pi}{2} - \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}\right)$

$\iint\limits_{\Sigma} P\,dydz$

1° $\Sigma: x = \varphi(y,z)$, $(y,z) \in D_{yz}$

2° $\iint\limits_{\Sigma} P\,dydz = \pm \iint\limits_{D_{yz}} P[\varphi(y,z), y, z]\,dydz$

$\cos\alpha > 0$ 取 +，$\cos\alpha < 0$ 取 −

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方法二：三重积分法

Th. Ω为几何体，Σ为Ω的外表面

P、Q、R在Ω上连续可偏导，则

$$\oiint_{\Sigma} P\,dydz + Q\,dzdx + R\,dxdy = \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV.$$

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