

Throughout this paper we will refer to a fixed regulatory network $\mathbf{RN} = (V, E)$ with network nodes $V = \{x_1, x_2, \dots, x_n\}$ and interactions $E \subset V \times V \times \{\rightarrow, \vdash\}$. We define the *targets* of a node as

$$\mathbf{T}(x_i) := \{x_j \mid (x_i, x_j) \in \mathbf{RN}\}$$

and the *sources* of a node as

$$\mathbf{S}(x_i) := \{x_j \mid (x_j, x_i) \in \mathbf{RN}\}$$

1 General System

Definition 1.1. We define a general system of \mathbf{RN} as

$$\dot{x}_j = -\gamma_j x_j + \sum_{I \in \mathcal{I}} \left(\prod_{i \in I} \sigma_{ji}(x_i) \right) \quad j = 1, \dots, n \quad (1)$$

where $\mathcal{I} \subseteq \mathcal{P}\{1, \dots, n\}$ and $\sigma_{ji}(x_i)$ is a linear ramp defined if $(x_i, x_j) \in E$ as

$$\sigma_{ji}(x_i) := \begin{cases} l_{ji} & \text{for } x_i \leq \theta_{ji} - \frac{\epsilon_{ji}}{2} \text{ and } x_i \rightarrow x_j, \text{ or } x_i \geq \theta_{ji} + \frac{\epsilon_{ji}}{2} \text{ and } x_i \vdash x_j \\ u_{ji} & \text{for } x_i \geq \theta_{ji} + \frac{\epsilon_{ji}}{2} \text{ and } x_i \vdash x_j, \text{ or } x_i \leq \theta_{ji} - \frac{\epsilon_{ji}}{2} \text{ and } x_i \rightarrow x_j \\ +\frac{u_{ji}-l_{ji}}{\epsilon_{ji}}(x_i - \theta_{ji}) + \frac{u_{ji}+l_{ji}}{2} & \text{for } \theta_{ji} - \frac{\epsilon_{ji}}{2} < x_i < \theta_{ji} + \frac{\epsilon_{ji}}{2} \text{ and } x_i \rightarrow x_j \\ -\frac{u_{ji}-l_{ji}}{\epsilon_{ji}}(x_i - \theta_{ji}) + \frac{u_{ji}+l_{ji}}{2} & \text{for } \theta_{ji} - \frac{\epsilon_{ji}}{2} < x_i < \theta_{ji} + \frac{\epsilon_{ji}}{2} \text{ and } x_i \vdash x_j \end{cases} \quad (2)$$

Peter: I want to add the assumptions for parameter choices here. Not really sure how to write up the formalized definition of parameter choices as well. That remains to be done. In doing so, the next paragraph probably needs to be rewritten. Should I just borrow mostly from the parameter graph paper? Also, I don't have formal definitions for \mathbf{RN} .

The constants u_{ji} and l_{ji} and the thresholds θ_{ji} are as previously defined. The constants ϵ_{ji} are introduced for all $\sigma_{ji}(x_i)$ such that $\epsilon_{ji} > 0$ and ϵ_{ji} is small enough to satisfy $\theta_{ki} - \frac{\epsilon_{ki}}{2}, \theta_{ki} + \frac{\epsilon_{ki}}{2} \notin [\theta_{ji} - \frac{\epsilon_{ji}}{2}, \theta_{ji} + \frac{\epsilon_{ji}}{2}]$ for any i, j, k with $j \neq k$. That is, no epsilon bands of any single variable overlap, and so for any particular value of x_i , at most only one $\sigma_{ji}(x_i)$, where $x_j \in \mathbf{T}(x_i)$, is in its linear region. Therefore, we can classify intervals of x_i as two distinct types.

Definition 1.2. We define a closed interval of x_i , denoted $I_{R,i}$, as *Type R*, i.e. in a particular linear ramp region, if $\exists x_j \in \mathbf{T}(x_i)$ with associated thresholds $\theta_{ji} - \frac{\epsilon_{ji}}{2}, \theta_{ji} + \frac{\epsilon_{ji}}{2}$ such that $I_{R,i} = [\theta_{ji} - \frac{\epsilon_{ji}}{2}, \theta_{ji} + \frac{\epsilon_{ji}}{2}]$ for some $x_j \in \mathbf{T}(x_i)$.

Notice that for all $x_i \in V$, the associated $\sum_{I \in \mathcal{I}} (\prod_{i \in I} \sigma_{ji}(x_i))$ is bounded above and below. Then, for each we can choose some c_{\min}

We define a closed interval of x_i , denoted $I_{C,i}$, as *Type C* if $\exists x_q, x_r \in \mathbf{T}(x_i)$ with $q \neq r$ and $\theta_{qi} + \frac{\epsilon_{qi}}{2} < \theta_{ri} - \frac{\epsilon_{ri}}{2}$ such that $I_{C,i} := [\theta_{qi} + \frac{\epsilon_{qi}}{2}, \theta_{ri} - \frac{\epsilon_{ri}}{2}]$ or $I_{C,i} := (-\infty, \theta_{qi} - \frac{\epsilon_{qi}}{2}]$ or $I_{C,i} := [\theta_{ri} + \frac{\epsilon_{ri}}{2}, \infty)$ and there is no Type R interval $I_{R,i}$ such that $I_{R,i} \subset I_{C,i}$.

Peter: I'd like to get rid of the infinities as endpoints of the Type C intervals. Bree mentioned something about compact metric spaces or something of the like. Could that be used?

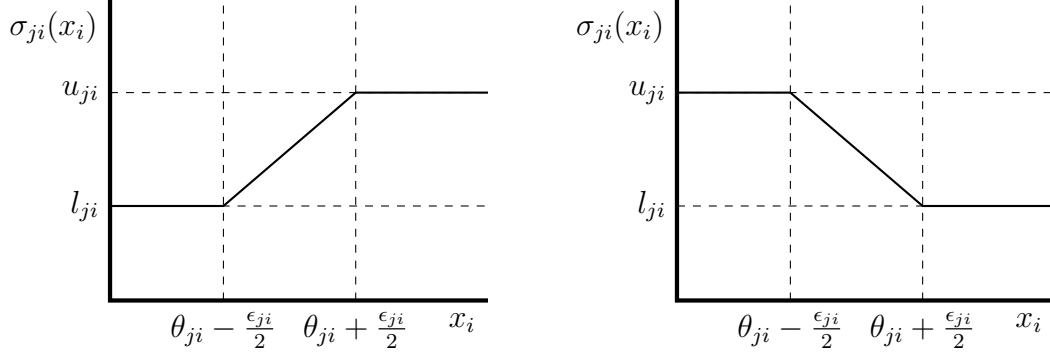


Figure 1: Left: A graph of $\sigma_{ji}(x_i)$ versus x_i if $x_i \rightarrow x_j$. Right: The graph if $x_i \dashv x_j$.

Remark 1.1. From Definition 1.2, note that every Type R interval is connected to two Type C intervals.

Definition 1.3. We divide phase space into *cells*, denoted κ . We define κ as the product of the closed intervals

$$\kappa := \prod_{i=1}^n I_{*,i} \quad \text{where} \quad * = \begin{cases} C & \text{if } i \in \alpha_C \\ R & \text{if } i \in \alpha_R \end{cases} \quad (3)$$

where α_R and α_C partition $\{1, \dots, n\}$. We label κ a m -cell, where m is the cardinality of α_R , because in κ , there are m equations of system (1) that are in their linear ramp section.

We defined two cells κ and κ' as adjacent if

$$\kappa := \prod_{i=1}^n I_{*,i} \quad \text{where} \quad * = \begin{cases} C & \text{if } i \in \alpha_C \\ R & \text{if } i \in \alpha_R \end{cases}$$

and

$$\kappa' := \prod_{i=1}^n I_{*,i} \quad \text{where} \quad *' = \begin{cases} C & \text{if } i \in \alpha'_C \\ R & \text{if } i \in \alpha'_R \end{cases}$$

where α_R and α_C partition $\{1, \dots, n\}$ and there exists some $k \in \{1, \dots, n\}$ such that $\alpha'_C = \alpha_C \setminus \{k\}$ and $\alpha'_R = \alpha_R \cup \{k\}$, and $I_{C,k}$ and $I_{R,k}$ are adjacent intervals. Then κ' is an $(m+1)$ -cell, where m is the cardinality of α_R .

We define the $n-1$ dimensional hyperplane $\kappa \cap \kappa'$ as the *wall* between cells κ and κ' as the product

$$\tau := \prod_{i=1}^{k-1} I_{*,i} \times \{\vartheta_k\} \times \prod_{i=k+1}^n I_{*,i}$$

where $\vartheta_k = \theta_{jk} \pm \frac{\epsilon_{jk}}{2}$ for some j . We refer to τ as a wall of the m -cell κ .

Remark 1.2. It is worth noting that all cells and walls are closed. Also, note that we use $\theta_{ji} + \frac{\epsilon_{ji}}{2}$, with two subscripts, to denote a threshold of system (1), and ϑ_k , with one subscript, to denote some constant that is the endpoint of some interval of x_k .

Definition 1.4. Suppose H is some subset of a co-dimension $n - 1$ hyperplane with normal vector \vec{h} and suppose a vector $\dot{x}(P)$ such that $\dot{x}(P) = \langle \dot{x}_1(P), \dots, \dot{x}_n(P) \rangle$ where $\dot{x}_j(P)$ denotes \dot{x}_j evaluated at some point P . We define the *flow* of system (1) to be *monotonic* in the direction of \vec{h} on H if \forall points P such that $P \in H$, $\vec{h} \cdot \vec{x}(P) > 0$, and monotonic in the direction of $-\vec{h}$ on H if \forall points P such that $P \in H$, $\vec{h} \cdot \vec{x}(P) < 0$. We define the flow to be *bidirectional* across H if \exists points P_1, P_2 such that $P_1, P_2 \in H$, $\vec{h} \cdot \vec{x}(P_1) > 0$ and $\vec{h} \cdot \vec{x}(P_2) < 0$.

Definition 1.5. Let A be any closed bounded region of \mathbb{R}^n , and let P and P' be points. We introduce the function $\text{sgn}(A, k)$ to classify the behavior of the solutions to system (1) in A , defined as

$$\text{sgn}(A, k) = \begin{cases} +1 & \text{if } \forall P \in A, \dot{x}_k(P) > 0 \\ -1 & \text{if } \forall P \in A, \dot{x}_k(P) < 0 \\ 0 & \text{if } \exists P, P' \in A \text{ such that } \dot{x}_k(P) > 0 \text{ and } \dot{x}_k(P') < 0 \\ \text{undefined} & \text{otherwise} \end{cases} \quad (4)$$

Remark 1.3. From Definitions 1.4 and 1.5 that if τ is some wall in \mathbb{R}^n such that $\tau \subseteq \{x_k = \vartheta_k\}$, then $\text{sgn}(\tau, k) = +1$ implies that the flow is monotonic in the \vec{e}_k direction across τ , and similarly $\text{sgn}(\tau, k) = -1$ implies that the flow is monotonic in the $-\vec{e}_k$ direction across τ . We will use this fact to construct the domain graph of system (1).

Definition 1.6. Let A be a closed region of \mathbb{R}^n such that $A := \prod_{i=1}^n a_i$ where

$$a_i = \begin{cases} I_{C,i} & \text{if } i \in \alpha_C \\ I_{R,i} & \text{if } i \in \alpha_R \\ \{f_i\} & \text{otherwise} \end{cases}$$

and α_C and α_R partition some subset of $\{1, \dots, n\}$, $I_{C,i}$ is some Type C interval of x_i , $I_{R,i}$ is some Type R interval of x_i , and f_i is any arbitrary constant. For each $i \in \alpha_C$, let c_i^+ and c_i^- be the endpoints of $I_{C,i}$ such that $I_{C,i} = [c_i^-, c_i^+]$, and for each $i \in \alpha_R$, let r_i^+ and r_i^- be the endpoints of $I_{R,i}$ such that $I_{R,i} = [r_i^-, r_i^+]$. We define the set of *corner points* of A as $\mathbb{P}(A)$, where

$$\mathbb{P}(A) := \prod_{i=1}^n P_i \quad \text{where} \quad P_i = \begin{cases} \{c_i^-, c_i^+\} & \text{if } i \in \alpha_C \\ \{r_i^-, r_i^+\} & \text{if } i \in \alpha_R \\ \{f_i\} & \text{otherwise} \end{cases}$$

Theorem 1.1. Suppose A be a closed region of \mathbb{R}^n with associated α_C and α_R , and suppose $\alpha_C \cup \alpha_R \neq \emptyset$. For some $k \in \alpha_C \cup \alpha_R$, let $I_{*,k}$ be the associated interval, and let ϑ_k and ϑ'_k be the endpoints of $I_{*,k}$ such that $I_{*,k} = [\vartheta_k, \vartheta'_k]$. Let $W := A \cap \{x_k = \vartheta_k\}$ and let $W' := A \cap \{x_k = \vartheta'_k\}$. If $\text{sgn}(W, j) = \text{sgn}(W', j) \neq 0$, then $\text{sgn}(W, j) = \text{sgn}(A, j)$.

Proof. Suppose Q is any point such that $Q \in A$. Then there exists some scalar $\beta \geq 0$ such that $Q - \beta \vec{e}_k \in W$. Let $f : [0, 1] \rightarrow A$ be a function such that $f(s) = (Q - \beta \vec{e}_k) + s(\vartheta'_k - \vartheta_k) \vec{e}_k$ for $0 \leq s \leq 1$. Note that $f(0) \in W$ and $f(1) \in W'$. Also note that from system (1), \dot{x}_j is an affine function of only x_k on $f(s)$. Recall from Definition 1.5, if $\text{sgn}(W, j) = \text{sgn}(W', j) = +1$, then $\dot{x}_j(f(0)) > 0$ and $\dot{x}_j(f(1)) > 0$. It follows that for all $s^* \in [0, 1]$, $\dot{x}_j(f(s^*)) > 0$, and so $\dot{x}_j(Q) > 0$. As Q was arbitrary, then from Definition 1.5, $\text{sgn}(A, j) = \text{sgn}(W, j)$. A similar argument is used when $\text{sgn}(W, j) = \text{sgn}(W', j) = -1$. \square

Theorem 1.2. *Let A be a closed region of \mathbb{R}^n with associated α_C and α_R . If and only if for all $p \in \mathbb{P}(A)$, $\text{sgn}(p, j)$ is defined and the same, then $\text{sgn}(A, j) = \text{sgn}(p_1, j)$.*

Proof. First we will show the converse direction. Suppose $\text{sgn}(A, j) = \text{sgn}(p_1, j)$. Note that for a single point p_1 , $\text{sgn}(p_1, j) \neq 0$, and so $\text{sgn}(A, j) \neq 0$, and the proof is immediate from Definition 1.5.

Now we will show the forward direction. Suppose $n = 1$. If $1 \notin \alpha_C \cup \alpha_R$, then $A = \{f_1\}$, and therefore $A = \mathbb{P}(A)$ and $\text{sgn}(A, 1) = \text{sgn}(p_1, 1)$. However, if $1 \in \alpha_C \cup \alpha_R$, then $A = I_{*,1} = [\vartheta_1, \vartheta'_1]$. Let $W = \{\vartheta_1\}$ and $W' = \{\vartheta'_1\}$. Note that $\mathbb{P}(A) = \{W, W'\}$, and so by Theorem 1.1 the theorem holds.

Now assume for all n -dimensional systems in the form of (1) the theorem holds. Consider any $(n+1)$ -dimensional system in the form of (1). Let A be a closed region of $\mathbb{R}^{(n+1)}$ with associated α_C and α_R , and suppose for all $p \in \mathbb{P}(A)$, $\text{sgn}(p, j)$ is defined and the same. If $\alpha_C \cup \alpha_R = \emptyset$, then A is a single point and the proof is immediate. If $\alpha_C \cup \alpha_R \neq \emptyset$, then we can find some $k \in \alpha_C \cup \alpha_R$. Let $I_{*,k}$ be the associated interval of A with endpoints ϑ_k and ϑ'_k . Let $W = A \cap \{x_k = \vartheta_k\}$ and $W' = A \cap \{x_k = \vartheta'_k\}$. Notice that on the hyperplane $\{x_k = \vartheta_k\}$, the system reduces to an n -dimensional system. Also note that $\mathbb{P}(W) \subseteq \mathbb{P}(A)$, and so by assumption $\text{sgn}(W, j) = \text{sgn}(p_1, j)$. By a similar argument $\text{sgn}(W', j) = \text{sgn}(p_1, j)$. Then by Theorem 1.1, $\text{sgn}(A, j) = \text{sgn}(W', j) = \text{sgn}(p_1, j)$ and the theorem holds. \square

Remark 1.4. We apply Theorem 1.2 to the walls and cells of **RN**. Given any wall, we can now unambiguously determine the flow through the wall if we know the sgn of all corner points of that wall. Notice that for all $i \in \alpha_R$, $I_{R,i} = [r_i^-, r_i^+]$, and recall that every Type R interval is connected to two Type C intervals and all intervals are closed, and so there must exist two Type C intervals $I_{C,i}$ and $I'_{C,i}$ such that $r_i^- \in I_{C,i}$ and $r_i^+ \in I'_{C,i}$. Then it follows that for each point $p \in \mathbb{P}(\kappa)$, there exists some 0-cell κ^* , and some wall τ^* of κ^* such that $p \in \tau^*$.

Furthermore, note that for any wall τ , if τ is a wall of some cell κ , then $\mathbb{P}(\tau) \subset \mathbb{P}(\kappa)$, and so all points in $\mathbb{P}(\tau)$ are also contained in the wall of a zero cell. By our assumptions, all walls of zero cells have defined, non-zero sgn . We use this fact to unambiguously determine the flow through all walls of **RN**, and then to construct the domain graph of **RN**.

2 Algorithm

Definition 2.1. Suppose τ is some wall of an m -cell where $m \neq 0$, with associated α_C and α_R . Let $\mathbb{P}(\tau)$ be the associated set of corner points. We define the *reduced set of corner points*, denoted $\bar{\mathbb{P}}(\tau)$ as $\bar{\mathbb{P}}(\tau) := \prod_{i=1}^n \bar{P}_i$, where \bar{P}_i is the set defined as defined as

$$\bar{P}_i := \begin{cases} \{r_i^-, r_i^+\} & \text{if } i \in \alpha_R \\ \{c_i^*\} & \text{if } i \in \alpha_C \end{cases}$$

where c_i^* is an arbitrary value chosen such that $c_i^* \in I_{C,i}$. Note that the cardinality $|\bar{\mathbb{P}}(\tau)|$ is 2^m , where m is the cardinality of α_R . Let \bar{p}_i be a component of $\bar{\mathbb{P}}(\tau)$.

Algorithm. For each wall τ of an m -cell where $m \neq 0$ and $\tau \subseteq \{x_k = \vartheta_k\}$, let $\mathbb{P}(\tau)$ be the associated set of corner points, and let $\bar{\mathbb{P}}(\tau)$ be the reduced set of corner points. For each $\bar{p}_i \in \bar{\mathbb{P}}(\tau)$, let $S_i := \text{sgn}(\bar{p}_i, k)$.

$$\text{If } \left| \sum_{i=1}^{2^m} S_i \right| < \sum_{i=1}^{2^m} |S_i|, \text{ then } \text{sgn}(\tau, k) = 0; \text{ otherwise } \text{sgn}(\tau, k) = S_1. \quad (5)$$