Energy

November 10, 2010

In 2D Navier-Stokes

$$\rho(u_t + (u \cdot \nabla)u) = -\nabla p + \mu\Delta u + F$$

the variable F is force per unit area (in 2D).

Then you can write

$$F(x) = \int_\Gamma g(s)\phi_\delta(x - X(s))ds$$

where $g(s)$ is a force per unit length and s is the arclength parameter.

You can define an *Energy* as

$$\mathcal{E}(X,Y) = \int_\Gamma \frac{1}{2}S_1\left(\|\vec{X}'(s)\| - 1\right)^2 ds \equiv \int_\Gamma \epsilon(X(s),Y(s))ds \tag{1}$$

where S_1 has units of energy per unit length and $\epsilon(X(s),Y(s))$ is an energy density.

Method 1:

We can write the *net force* in the problem as

$$-\mathcal{F}_1 = \frac{\partial\mathcal{E}}{\partial X} = \int_\Gamma \frac{\partial\epsilon}{\partial X}ds, \qquad -\mathcal{F}_2 = \frac{\partial\mathcal{E}}{\partial Y} = \int_\Gamma \frac{\partial\epsilon}{\partial Y}ds$$

This should be a variational form (see Peskin's notes). There is an easy way to extract $\frac{\partial\epsilon}{\partial X}$ and $\frac{\partial\epsilon}{\partial Y}$ since

$$\frac{d\epsilon}{ds}(X(s),Y(s)) = \frac{\partial\epsilon}{\partial X}X'(s) + \frac{\partial\epsilon}{\partial Y}Y'(s)$$

all we have to do is find $d\epsilon/ds$ and identify the pieces. From Eq. (1) we have that

$$\frac{d\epsilon}{ds}(X(s),Y(s)) = S_1\left(\|\vec{X}'(s)\| - 1\right)(X''(s)X'(s) + Y''(s)Y'(s))\frac{1}{\|\vec{X}'(s)\|}$$

so it is clear that the net force is

$$\mathcal{F}_1 = -\int_\Gamma S_1\left(\|\vec{X}'(s)\| - 1\right)\frac{X''(s)}{\|\vec{X}'(s)\|}\,ds \tag{2}$$

$$\mathcal{F}_2 = -\int_\Gamma S_1\left(\|\vec{X}'(s)\| - 1\right)\frac{Y''(s)}{\|\vec{X}'(s)\|}\,ds \tag{3}$$

This shows that the force per unit length is defined as

$$g(s) = -S_1 \left(\|\vec{X}'(s)\| - 1 \right) \frac{\vec{X}''(s)}{\|\vec{X}'(s)\|}$$

and the force per unit area in the Navier-Stokes equation is

$$F(\vec{x}) = \int_\Gamma g(s)\phi_\delta(\vec{x} - \vec{X}(s))ds \tag{4}$$

Method 2:

Take Eq. (1) and discretize it as

$$\mathcal{E}_h(\ldots, X_k, Y_k, \ldots) = \sum_{n=1}^{N} \frac{1}{2} S_1 \left(\frac{\|\vec{X}_{n+1} - \vec{X}_n\|}{h} - 1 \right)^2 h$$

and differentiate it with respect to X_k and Y_k. The result is the force at position (X_k, Y_k)

$$-g_1(k)h = \frac{\partial \mathcal{E}_h}{\partial X_k} = S_1 \left(\frac{\|\vec{X}_k - \vec{X}_{k-1}\|}{h} - 1 \right) \frac{X_k - X_{k-1}}{\|\vec{X}_k - \vec{X}_{k-1}\|} - S_1 \left(\frac{\|\vec{X}_{k+1} - \vec{X}_k\|}{h} - 1 \right) \frac{X_{k+1} - X_k}{\|\vec{X}_{k+1} - \vec{X}_k\|}$$

$$-g_2(k)h = \frac{\partial \mathcal{E}_h}{\partial Y_k} = S_1 \left(\frac{\|\vec{X}_k - \vec{X}_{k-1}\|}{h} - 1 \right) \frac{Y_k - Y_{k-1}}{\|\vec{X}_k - \vec{X}_{k-1}\|} - S_1 \left(\frac{\|\vec{X}_{k+1} - \vec{X}_k\|}{h} - 1 \right) \frac{Y_{k+1} - Y_k}{\|\vec{X}_{k+1} - \vec{X}_k\|}$$

If we expand $g_1(k)$ and $g_2(k)$ in a Taylor series about $(X_k, Y_k) = (X(s_k), Y(s_k))$ we find that

$$\begin{aligned}
-g_1(k)h &= S_1 \left(\|\vec{X}'(s_k)\| - 1 \right) \frac{X''(s_k)}{\|\vec{X}'(s_k)\|} \; h + O(h^2) \\
-g_2(k)h &= S_1 \left(\|\vec{X}'(s_k)\| - 1 \right) \frac{Y''(s_k)}{\|\vec{X}'(s_k)\|} \; h + O(h^2)
\end{aligned}$$

(using the fact that $\vec{X}' \cdot \vec{X}'' = 0$). This shows that the (discrete) net force is

$$\begin{aligned}
\mathcal{F}_{1h} = \sum_k g_1(k)h &= -\sum_k S_1 \left(\|\vec{X}'(s_k)\| - 1 \right) \frac{X''(s_k)}{\|\vec{X}'(s_k)\|} \; h \\
\mathcal{F}_{2h} = \sum_k g_2(k)h &= -\sum_k S_1 \left(\|\vec{X}'(s_k)\| - 1 \right) \frac{Y''(s_k)}{\|\vec{X}'(s_k)\|} \; h
\end{aligned}$$

which are discretizations of Eq. (2)-(3).

At this point we can define the force per unit area at any location (x, y) as

$$F(\vec{x}) = \sum_k g_1(k)\phi_\delta(\vec{x} - \vec{X}_k)h$$

which is a discretization of Eq. (4).