Brinkmanlets + Dipoles in 2D

1 Introduction

According to Bree's writeup, the singular Brinkmanlet in 2D is

$$\mu\mathbf{v}(\mathbf{x}) = \mathbf{f}H_1(r) + (\mathbf{f}\cdot\mathbf{x})\mathbf{x}H_2(r) \tag{1}$$

where

$$\begin{aligned} H_1(r) &= \left[\frac{K_1(\lambda r) + \lambda r K_0(\lambda r)}{2\pi\lambda r} - \frac{1}{2\pi\lambda^2 r^2}\right] = \left(\frac{1}{2\pi\lambda r}\right)\left[\lambda r K_0(\lambda r) + K_1(\lambda r) - \frac{1}{\lambda r}\right] \quad (2)\\ H_2(r) &= \left[-\frac{2K_1(\lambda r) + \lambda r K_0(\lambda r)}{2\pi\lambda r^3} + \frac{2}{2\pi\lambda^2 r^4}\right] = \left(\frac{1}{2\pi\lambda^2 r^4}\right)\left[2 - \lambda^2 r^2 K_2(\lambda r)\right] \quad (3)\end{aligned}$$

According to me, we have that

$$\mu\Delta\mathbf{v}(\mathbf{x}) = \mathbf{f}Q_1(r) + (\mathbf{f}\cdot\mathbf{x})\mathbf{x}Q_2(r)$$

where

$$\begin{aligned} Q_1(r) &= \Delta H_1(r) + 2H_2(r), \quad (4)\\ Q_2(r) &= \Delta H_2(r) + 4\frac{H_2'(r)}{r} \quad (5)\end{aligned}$$

This leads to

$$\begin{aligned} Q_1(r) &= \left(\frac{\lambda}{2\pi r}\right)\left[\lambda r K_0(\lambda r) + K_1(\lambda r)\right], \quad (6)\\ Q_2(r) &= -\left(\frac{\lambda^2}{2\pi r^2}\right)K_2(\lambda r) \quad (7)\end{aligned}$$

So, the Brinkmanlet + dipole velocity is

$$\mu\mathbf{v}(\mathbf{x}) = \mathbf{f}[H_1(r) + \alpha Q_1(r)] + (\mathbf{f}\cdot\mathbf{x})\mathbf{x}[H_2(r) + \alpha Q_2(r)]$$

If we evaluate it at $r = a$ and set the solution equal to $(U, 0)$, we have that $\mathbf{f} = (f, 0)$ and

$$\begin{aligned} U &= f[H_1(a) + \alpha Q_1(a)], \quad (8)\\ 0 &= H_2(a) + \alpha Q_2(a) \quad (9)\end{aligned}$$

The second equation gives

$$\alpha = -\frac{H_2(a)}{Q_2(a)} = \frac{1}{\lambda^2}\left(\frac{2}{\lambda^2 a^2 K_2(\lambda a)} - 1\right)$$

and the first one

$$\begin{aligned}
\frac{U}{f} &= H_1(a) + \alpha Q_1(a) \\
&= \left(\frac{1}{2\pi\lambda a}\right)\left[\lambda a K_0(\lambda a) + K_1(\lambda a) - \frac{1}{\lambda a}\right] + \alpha\left(\frac{\lambda}{2\pi a}\right)\left[\lambda a K_0(\lambda a) + K_1(\lambda a)\right] \\
&= \left(\frac{1}{2\pi\lambda a}\right)\left[\lambda a K_0(\lambda a) + K_1(\lambda a) - \frac{1}{\lambda a} - \frac{\left[\lambda a K_0(\lambda a) + 2K_1(\lambda a) - \frac{2}{\lambda a}\right]}{\left[\lambda a K_0(\lambda a) + 2K_1(\lambda a)\right]}\left[\lambda a K_0(\lambda a) + K_1(\lambda a)\right]\right] \\
&= \left(\frac{1}{2\pi\lambda a}\right)\frac{K_0(\lambda a)}{\lambda a K_0(\lambda a) + 2K_1(\lambda a)}
\end{aligned}$$

or

$$f = U\frac{2\pi\lambda^2 a^2 K_2(\lambda a)}{K_0(\lambda a)}$$

To simplify the velocity equation, we use

$$\begin{aligned}
H_1(r) + \alpha Q_1(r) &= \left(\frac{1}{2\pi\lambda r}\right)\left[\lambda r K_0(\lambda r) + K_1(\lambda r) - \frac{1}{\lambda r}\right] + \alpha\left(\frac{\lambda}{2\pi r}\right)\left[\lambda r K_0(\lambda r) + K_1(\lambda r)\right] \\
&= \left(\frac{1}{2\pi\lambda r}\right)\left[(1+\alpha\lambda^2)(\lambda r K_0(\lambda r) + K_1(\lambda r)) - \frac{1}{\lambda r}\right] \\
&= \left(\frac{1}{2\pi\lambda r}\right)\left[\frac{2(\lambda r K_0(\lambda r) + K_1(\lambda r))}{\lambda^2 a^2 K_2(\lambda a)} - \frac{1}{\lambda r}\right]
\end{aligned}$$

which leads to

$$f(H_1(r) + \alpha Q_1(r)) = \frac{U}{K_0(\lambda a)}\left(K_0(\lambda r) + K_2(\lambda r) - \frac{a^2}{r^2}K_2(\lambda a)\right)$$

Also

$$\begin{aligned}
H_2(r) + \alpha Q_2(r) &= \left(\frac{1}{2\pi\lambda^2 r^4}\right)\left[2 - \lambda^2 r^2 K_2(\lambda r)\right] - \alpha\left(\frac{\lambda^2}{2\pi r^2}\right)K_2(\lambda r) \\
&= \frac{2}{2\pi\lambda^2 r^4} - (1+\alpha\lambda^2)\frac{K_2(\lambda r)}{2\pi r^2} \\
&= \frac{2}{2\pi\lambda^2 r^4}\left(1 - \frac{r^2 K_2(\lambda r)}{a^2 K_2(\lambda a)}\right)
\end{aligned}$$

which leads to

$$fr^2(H_2(r) + \alpha Q_2(r)) = \frac{2}{K_0(\lambda a)} \left(\frac{a^2}{r^2} K_2(\lambda a) - K_2(\lambda r) \right)$$

So the velocity becomes

$$\mu u(\mathbf{x}) = \frac{U}{K_0(\lambda a)} \left(K_0(\lambda r) + K_2(\lambda r) - \frac{a^2}{r^2} K_2(\lambda a) \right) + \frac{2\cos^2\psi}{K_0(\lambda a)} \left(\frac{a^2}{r^2} K_2(\lambda a) - K_2(\lambda r) \right) \quad (10)$$

$$\mu v(\mathbf{x}) = \frac{2\sin\psi\cos\psi}{K_0(\lambda a)} \left(\frac{a^2}{r^2} K_2(\lambda a) - K_2(\lambda r) \right) \quad (11)$$