Regularized Stokeslets Integration

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Background

$$\phi_\delta(r) = \frac{2d^4}{\pi\left(d^2+r^2\right)^3}$$

$$G_\delta(r) = -\frac{d^2}{4\pi(d^2+r^2)} + \frac{1}{4\pi}\log\left[d^2+r^2\right]$$

$$B'_\delta(r) = -\frac{r}{8\pi} + \frac{r\log\left[d^2+r^2\right]}{8\pi}$$

Assume we are given a closed boundary in 2D parametrized by $\mathbf{X}(s) = (X(s), Y(x))$ where s is some parameter so that $0 \le s \le L$. Let $\mathbf{x} = (x, y)$ be an arbitrary point in the fluid. We want to compute

$$\mathbf{u}(\mathbf{x}) = \int_0^L \mathbf{f}(s)H_1(\mathbf{x}-\mathbf{X}(s)) + \mathbf{f}(s)\cdot(\mathbf{x}-\mathbf{X}(s))(\mathbf{x}-\mathbf{X}(s))H_2(\mathbf{x}-\mathbf{X}(s))ds$$

where

$$\begin{aligned} H_1(r) &= \frac{B'_\delta(r)}{r} - G_\delta(r) = \frac{2\delta^2}{8\pi(r^2+\delta^2)} - \frac{1}{8\pi}\log(r^2+\delta^2) \\ H_2(r) &= \frac{rB''_\delta(r) - B'_\delta(r)}{r^3} = \frac{1}{4\pi(r^2+\delta^2)} \end{aligned}$$

This integral is nearly-singular if the evaluation point $\mathbf{x}$ is close to a boundary point. Let $\mathbf{X}(s_d)$ be a boundary point that is *near* the given evaluation point $\mathbf{x}$ (not necessarily the closest but could be the closest discrete point). Then we can divide the integral above into a piece corresponding to $s \in [s_d - \ell, s_d + \ell]$ and the rest.

The piece *far* from the near singularity

$$\begin{aligned} \mathbf{u}_1(\mathbf{x}) &= \int_0^{s_d-\ell} \mathbf{f}(s)H_1(\mathbf{x}-\mathbf{X}(s)) + \mathbf{f}(s)\cdot(\mathbf{x}-\mathbf{X}(s))(\mathbf{x}-\mathbf{X}(s))H_2(\mathbf{x}-\mathbf{X}(s))ds \\ &+ \int_{s_d+\ell}^{L} \mathbf{f}(s)H_1(\mathbf{x}-\mathbf{X}(s)) + \mathbf{f}(s)\cdot(\mathbf{x}-\mathbf{X}(s))(\mathbf{x}-\mathbf{X}(s))H_2(\mathbf{x}-\mathbf{X}(s))ds \end{aligned}$$

can be computed using Gaussian quadrature. If the forces $\mathbf{f}(s)$ are known only at discrete points on the boundary, one can interpolate linearly in between.

We now concentrate on evaluating

$$\mathbf{u}_2(\mathbf{x}) = \int_{s_d-\ell}^{s_d+\ell} \mathbf{f}(s) H_1(\mathbf{x}-\mathbf{X}(s)) + \mathbf{f}(s)\cdot(\mathbf{x}-\mathbf{X}(s))(\mathbf{x}-\mathbf{X}(s)) H_2(\mathbf{x}-\mathbf{X}(s)) ds$$

Somewhere near $\mathbf{X}(s_d)$ is the boundary point that is closest to $\mathbf{x}$. Call this point $\mathbf{X}(s_0)$. Later we discuss how to find it. Now, the integral for $\mathbf{u}_2$ has a near singularity at s_0.

Let n be the largest integer that satisfies $n \leq 1 - \log(\delta)$ and make the change of variables

$$s = s_0 + \delta \sinh(n \sinh^{-1}(\alpha)) \qquad \Rightarrow \qquad ds = \frac{\delta n \cosh(n \sinh^{-1}(\alpha))}{\sqrt{1+\alpha^2}}$$

Using the notation $\mathbf{Z}(\alpha) = \mathbf{X}(s(\alpha))$ and $\mathbf{g}(\alpha) = \mathbf{f}(s(\alpha))$, we write

$$\mathbf{u}(\mathbf{x}) = \int_{-A}^{A} \mathbf{F}(\alpha) \frac{\delta n \cosh(n \sinh^{-1}(\alpha))}{\sqrt{1+\alpha^2}} d\alpha \tag{1}$$

where

$$\begin{aligned} \mathbf{F}(\alpha) &= \mathbf{g}(\alpha) H_1(\mathbf{x}-\mathbf{Z}(\alpha)) + \mathbf{g}(\alpha)\cdot(\mathbf{x}-\mathbf{Z}(\alpha))(\mathbf{x}-\mathbf{Z}(\alpha)) H_2(\mathbf{x}-\mathbf{Z}(\alpha)) \\ A &= \sinh\left(\frac{1}{n}\sinh^{-1}\left(\frac{\ell}{\delta}\right)\right) \end{aligned}$$

The near-singularity corresponding to $s = s_0$ is at $\alpha = 0$.

computing the velocity from given forces

The algorithm is:

- choose n equally spaced points in α and set $\mathbf{Z}_k = \mathbf{Z}(\alpha_k)$.
- If $\mathbf{g}(\alpha)$ is an unknown function, let $\mathbf{g}_k = \mathbf{g}(\alpha_k)$ and interpolate linearly

$$\mathbf{g}(\alpha) = \frac{\alpha_{k+1}-\alpha}{\alpha_{k+1}-\alpha_k}\mathbf{g}_k + \frac{\alpha-\alpha_k}{\alpha_{k+1}-\alpha_k}\mathbf{g}_{k+1}$$

- Now the entire function $\mathbf{F}(\alpha)$ is known and one can use Gaussian quadrature.

Finding the boundary point closest to x

In order to find the boundary point $\mathbf{X}(s_0)$ closest to $\mathbf{x}$, we assume the boundary is linear in $[s_k, s_{k+1}]$

$$\mathbf{X}(s) = \frac{s_{k+1}\mathbf{X}_k - s_k\mathbf{X}_{k+1}}{s_{k+1}-s_k} + \frac{s(\mathbf{X}_{k+1}-\mathbf{X}_k)}{s_{k+1}-s_k}$$

To minimize

$$d(s) = (x - X(s))^2 + (y - Y(s))^2$$

for $s_k \leq s \leq s_{k+1}$, we set

$$(x - X(s))\frac{X_{k+1} - X_k}{s_{k+1} - s_k} + (y - Y(s))\frac{Y_{k+1} - Y_k}{s_{k+1} - s_k} = 0$$

which is a linear function of s so it can be solved explicitly from

$$\begin{aligned} s\left(\frac{(X_{k+1} - X_k)^2 + (Y_{k+1} - Y_k)^2}{s_{k+1} - s_k}\right) &= x(X_{k+1} - X_k) + y(Y_{k+1} - Y_k) \\ &- \frac{(X_{k+1} - X_k)(s_{k+1}X_k - s_k X_{k+1})}{s_{k+1} - s_k} \\ &- \frac{(Y_{k+1} - Y_k)(s_{k+1}Y_k - s_k Y_{k+1})}{s_{k+1} - s_k} \end{aligned}$$