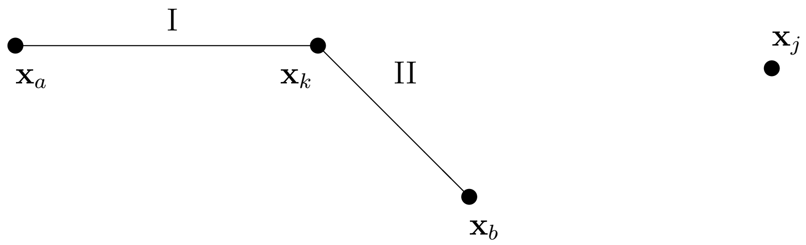
For Bree

The effect on the velocity $\mathbf{u}_j$ at $\mathbf{x}_j$ due to $\mathbf{f}_k$.



The velocity $\mathbf{u}_j$ is given by

$$\begin{aligned}\mathbf{u}_j &= \int_I H_1(r)\mathbf{f}(s) + (\mathbf{f}(s)\cdot(\mathbf{x}_j - \mathbf{x}(s))(\mathbf{x}_j - \mathbf{x}(s))H_2(r)ds \\ &+ \int_{II} H_1(r)\mathbf{f}(z) + (\mathbf{f}(z)\cdot(\mathbf{x}_j - \mathbf{x}(z))(\mathbf{x}_j - \mathbf{x}(z))H_2(r)dz\end{aligned}$$

Since we want the effect of $\mathbf{f}_k$ only, we write

$$\begin{aligned}\text{In } I: \quad \mathbf{f}(s) &= \mathbf{f}_a(1-s) + \mathbf{f}_k s \\ \text{In } II: \quad \mathbf{f}(z) &= \mathbf{f}_k(1-z) + \mathbf{f}_b z\end{aligned}$$

So the contribution to $\mathbf{u}_j$ from $\mathbf{f}_k$ is given by

$$\mathbf{u} = \int_0^1 sH_1(r)\mathbf{f}_k + s(\mathbf{f}_k\cdot(\mathbf{x}_j - \mathbf{x}(s))(\mathbf{x}_j - \mathbf{x}(s))H_2(r)ds \tag{1}$$

$$+ \int_0^1 (1-z)H_1(r)\mathbf{f}_k + (1-z)(\mathbf{f}_k\cdot(\mathbf{x}_j - \mathbf{x}(z))(\mathbf{x}_j - \mathbf{x}(z))H_2(r)dz \tag{2}$$

1 The contribution from I

$$\mathbf{x}(s) = \mathbf{x}_a(1-s) + \mathbf{x}_k s = \mathbf{x}_a + s(\mathbf{x}_k - \mathbf{x}_a) \tag{3}$$

$$\Rightarrow \quad \mathbf{x}_j - \mathbf{x}(s) = (\mathbf{x}_j - \mathbf{x}_a) - s(\mathbf{x}_k - \mathbf{x}_a)$$

$$\Rightarrow \quad r^2 = \|\mathbf{x}_j - \mathbf{x}(s)\|^2 = \|\mathbf{x}_j - \mathbf{x}_a\|^2 - 2s(\mathbf{x}_j - \mathbf{x}_a)\cdot(\mathbf{x}_k - \mathbf{x}_a) + s^2\|\mathbf{x}_k - \mathbf{x}_a\|^2$$

Let's define the matrix $M(a,k;j)$ with entries

$$\begin{aligned}M_{11} &= (x_j - x_a)^2 - 2s(x_j - x_a)(x_k - x_a) + s^2(x_k - x_a)^2 \\ M_{12} &= (x_j - x_a)(y_j - y_a) - s\left[(x_j - x_a)(y_k - y_a) + (y_j - y_a)(x_k - x_a)\right] + s^2(x_k - x_a)(y_k - y_a) \\ M_{21} &= M_{12} \\ M_{22} &= (y_j - y_a)^2 - 2s(y_j - y_a)(y_k - y_a) + s^2(y_k - y_a)^2\end{aligned}$$

Then we can write

$$(\mathbf{f}_k \cdot (\mathbf{x}_j - \mathbf{x}(s))(\mathbf{x}_j - \mathbf{x}(s)) = M(a,k;j)\mathbf{f}_k$$

where

$$M(a,k;j) = A(a,k;j) + B(a,k;j)s + C(a,k;j)s^2 \tag{4}$$

and

$$A(a,k;j) = \begin{pmatrix} (x_j - x_a)^2 & (x_j - x_a)(y_j - y_a) \\ (x_j - x_a)(y_j - y_a) & (y_j - y_a)^2 \end{pmatrix}$$

$$B(a,k;j) = -\begin{pmatrix} 2(x_j - x_a)(x_k - x_a & (x_j - x_a)(y_k - y_a) + (y_j - y_a)(x_k - x_a) \\ (x_j - x_a)(y_k - y_a) + (y_j - y_a)(x_k - x_a) & 2(y_j - y_a)(y_k - y_a) \end{pmatrix}$$

$$C(a,k;j) = \begin{pmatrix} (x_k - x_a)^2 & (x_k - x_a)(y_k - y_a) \\ (x_k - x_a)(y_k - y_a) & (y_k - y_a)^2 \end{pmatrix}$$

We can finally write the integral in (1) as

$$\int_0^1 [sH_1(r)I + sH_2(r)M(a,k;j)]\mathbf{f}_k ds$$

where

$$r^2 = \|\mathbf{x}_j - \mathbf{x}(s)\|^2 = \|\mathbf{x}_j - \mathbf{x}_a\|^2 - 2s(\mathbf{x}_j - \mathbf{x}_a)\cdot(\mathbf{x}_k - \mathbf{x}_a) + s^2\|\mathbf{x}_k - \mathbf{x}_a\|^2$$

2 The contribution from II

$$\mathbf{x}(z) = \mathbf{x}_k(1-z) + \mathbf{x}_b z = \mathbf{x}_k + z(\mathbf{x}_b - \mathbf{x}_k)$$

$$\Rightarrow \quad \mathbf{x}_j - \mathbf{x}(z) = (\mathbf{x}_j - \mathbf{x}_k) - z(\mathbf{x}_b - \mathbf{x}_k)$$

but by making the substitution $s = 1 - z$, we get

$$\int_0^1 ds = \int_0^1 dz$$

and

$$\mathbf{x}_j - \mathbf{x}(s) = (\mathbf{x}_j - \mathbf{x}_k) - (1-s)(\mathbf{x}_b - \mathbf{x}_k) = (\mathbf{x}_j - \mathbf{x}_b) - s(\mathbf{x}_k - \mathbf{x}_b)$$

which is exactly (3) with $\mathbf{x}_b$ replacing $\mathbf{x}_a$. Therefore, the integral in (2) as

$$\int_0^1 [sH_1(r)I + sH_2(r)M(b,k;j)]\mathbf{f}_k ds$$

where

$$r^2 = \|\mathbf{x}_j - \mathbf{x}(s)\|^2 = \|\mathbf{x}_j - \mathbf{x}_b\|^2 - 2s(\mathbf{x}_j - \mathbf{x}_b)\cdot(\mathbf{x}_k - \mathbf{x}_b) + s^2\|\mathbf{x}_k - \mathbf{x}_b\|^2$$

3 The necessary integrals for I

Since

$$\begin{aligned}
8\pi H_1(r) &= \frac{2\epsilon^2}{(r^2+\epsilon^2)} - \log(r^2+\epsilon^2) \\
8\pi H_2(r) &= \frac{2}{(r^2+\epsilon^2)}
\end{aligned}$$

we need

$$\begin{aligned}
I_1 &= \int_0^1 s\log(a^2 - 2sb + s^2c^2)\ ds \\
I_2 &= \int_0^1 \frac{s}{a^2 - 2sb + s^2c^2}\ ds \\
I_3 &= \int_0^1 \frac{s^2}{a^2 - 2sb + s^2c^2}\ ds \\
I_4 &= \int_0^1 \frac{s^3}{a^2 - 2sb + s^2c^2}\ ds
\end{aligned}$$

where

$$a^2 = \|\mathbf{x}_j - \mathbf{x}_a\|^2 + \epsilon^2, \qquad b = (\mathbf{x}_j - \mathbf{x}_a)\cdot(\mathbf{x}_k - \mathbf{x}_a), \qquad c^2 = \|\mathbf{x}_k - \mathbf{x}_a\|^2$$

Then the velocity contribution from I is

$$8\pi\mathbf{u} = 2\epsilon^2 I_2 - I_1 + 2A(a,k;j)I_2 + 2B(a,k;j)I_3 + 2C(a,k;j)I_4$$

and the contribution from II uses exactly the same integrals except that the coefficients (a, b, c, $A(a,k;j)$, $B(a,k;j)$ and $C(a,k;j)$) change by replacing $\mathbf{x}_a$ with $\mathbf{x}_b$.