Table of Quadrature Results

We are solving the Laplacian $\Delta u(\mathbf{x}) = 0$ on the unit disk D with boundary conditions $g(\theta) = \sin(3\theta)$ on ∂D. The solution in boundary integral form is given by:

$$u(\mathbf{x}) = \int_{\partial D} f(\mathbf{y}) \frac{\partial}{\partial \mathbf{n}} G(\mathbf{y}, \mathbf{x}) \, \mathrm{d}\mathbf{y},$$

where $f(\mathbf{y}) = f(r, \theta) = 2\sin(3\theta)$ is the density function associated with the boundary conditions, $\mathbf{n}$ is the outward pointing normal, and $G(\mathbf{y}, \mathbf{x}) = \ln(|\mathbf{y} - \mathbf{x}|)/(2\pi)$ is the free-space Green's function for the Laplacian. The normal derivative of G is given by

$$\frac{\partial}{\partial \mathbf{n}} G(\mathbf{y}, \mathbf{x}) = \frac{(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}}{2\pi |\mathbf{y} - \mathbf{x}|^2}.$$

The solution within the unit disk is known to be $u(r, \theta) = r^3 \sin(3\theta)$, and the solution outside is $u(r, \theta) = -r^{-3} \sin(3\theta)$.

The unit circle is discretized into N points with chords between each pair of points on which to do quadrature. Each chord is divided into M points. The boundary elements are either constant on each chord or they are bilinear "hats" that span 2 chords, one rise and one fall. Quadrature is performed using the composite trapezoid rule on each chord.

Basis Element	N	h	Error	Error/h	Basis Element	N	h	Error	Error/h^2
Constant	4	1.4142	8.2210e-01	0.5813	Hat	4	1.4142	1.1575e+00	0.5787
	8	0.7654	7.5498e-01	0.9864		8	0.7654	2.7571e-01	0.4707
	16	0.3902	1.6655e-02	0.0427		16	0.3902	1.3790e-01	0.9058
	32	0.1960	7.5065e-02	0.3829		32	0.1960	6.4301e-02	1.6732
	64	0.0981	4.5538e-02	0.4640		64	0.0981	1.1335e-03	0.1177
	128	0.0491	2.6521e-02	0.5403		128	0.0491	2.6648e-03	1.1061
	256	0.0245	1.6822e-02	0.6854		256	0.0245	1.0026e-03	1.6644
	512	0.0123	8.8121e-03	0.7181		512	0.0123	1.0052e-04	0.6675
	1024	0.0061	4.4558e-03	0.7262		1024	0.0061	3.0173e-05	0.8014

Table 1: BEM with the trapezoid rule. N is the number of chords; M is fixed at 1000; the error is for the point $1.01 * (\cos(0.7), \sin(0.7))$. The spacing between the points h is given by the length of the chord $h = 2\sin(\pi/N)$.

When increasing the number of chords using constant functions, we achieve about $O(h)$ convergence. When using hat functions, the order is approximately h^2; when dividing by h, there is a rapidly decreasing sequence of numbers and when dividing by h^3, there is a rapidly increasing sequence. This indicates that even though the values jump around, the convergence is about h^2.