Showing divergence free velocity fields from a Stokeslet

Bree Cummins

November 11, 2010

We have that the velocity from a regularized Stokeslet (with fluid viscosity $\mu = 1$) is

$$\mathbf{u} = (\mathbf{f} \cdot \nabla)\nabla B_\epsilon(r) - \mathbf{f}\nabla^2 B_\epsilon(r),$$

where $r = |\mathbf{x}|$ and the radially symmetric function B_ϵ varies with the choice of blob function. The assumption of radial symmetry allows us to make some simplifications. In the following, ∂_i means take the derivative with respect to the i-th Cartesian coordinate, and n is the dimension of the domain.

$$\begin{aligned}
\nabla B_\epsilon &= \frac{B'_\epsilon(r)}{r}\mathbf{x} \\
\left(\left(\mathbf{f}\cdot\nabla\right)\nabla B_\epsilon\right)_j &= \sum_{i=1}^{n} f_i \partial_i \left(\frac{B'_\epsilon(r)}{r} x_j\right) \\
&= \sum_{i=1}^{n} \frac{B'_\epsilon(r)}{r} f_i \delta_{ij} + \sum_{i=1}^{n} x_j f_i \partial_i \frac{B'_\epsilon(r)}{r} \\
&= \frac{B'_\epsilon(r)}{r} f_j + \sum_{i=1}^{n} x_j \frac{f_i x_i}{r}\left(\frac{B''_\epsilon(r)}{r} - \frac{B'_\epsilon(r)}{r^2}\right) \\
\Rightarrow \left(\mathbf{f}\cdot\nabla\right)\nabla B_\epsilon &= \frac{B'_\epsilon(r)}{r}\mathbf{f} + \left(\frac{B''_\epsilon(r)}{r^2} - \frac{B'_\epsilon(r)}{r^3}\right)(\mathbf{f}\cdot\mathbf{x})\mathbf{x}.
\end{aligned}$$

Then we may write

$$\begin{aligned}
\mathbf{u} &= \left(\frac{B'_\epsilon(r)}{r} - \nabla^2 B_\epsilon(r)\right)\mathbf{f} + \left(\frac{B''_\epsilon(r)}{r^2} - \frac{B'_\epsilon(r)}{r^3}\right)(\mathbf{f}\cdot\mathbf{x})\mathbf{x} \\
&= H_1(r)\mathbf{f} + H_2(r)(\mathbf{f}\cdot\mathbf{x})\mathbf{x}. \qquad (1)
\end{aligned}$$

The Laplacian in H_1 introduces different dependencies on $B'_\epsilon(r)$ in two and three dimensions. In two dimensions,

$$\begin{aligned}
H_1^{2D} &= \frac{B'_\epsilon(r)}{r} - \nabla^2 B_\epsilon \\
&= \frac{B'_\epsilon(r)}{r} - \frac{1}{r}\left(rB'_\epsilon(r)\right)' \\
&= \frac{B'_\epsilon(r)}{r} - \frac{1}{r}\left(B'_\epsilon(r) + rB''_\epsilon(r)\right) \\
&= -B''_\epsilon(r). \qquad (2)
\end{aligned}$$

Additionally, in two dimensions we add a constant to H_1, to remove a constant flow induced everywhere by the Stokeslet. This is equivalent to solving for a shifted velocity field. Since the divergence of a constant times the force, $c\mathbf{f}$, is zero, we ignore it in the following work.

In three dimensions, the Laplacian has a different form, and consequently, so does H_1:

$$\begin{aligned} H_1^{3D} &= \frac{B_\epsilon'(r)}{r} - \nabla^2 B_\epsilon \\ &= \frac{B_\epsilon'(r)}{r} - \frac{1}{r^2}\left(r^2 B_\epsilon'(r)\right)' \\ &= \frac{B_\epsilon'(r)}{r} - \frac{1}{r^2}\left(2rB_\epsilon'(r) - r^2 B_\epsilon''(r)\right) \\ &= \frac{B_\epsilon'(r)}{r} - \frac{2B_\epsilon'(r)}{r} - B_\epsilon''(r) \\ &= -B_\epsilon''(r) - \frac{B_\epsilon'(r)}{r}. \end{aligned} \tag{3}$$

We take the divergence (in Cartesian coordinates) of Eq. (1) in n dimensions:

$$\begin{aligned} \nabla\cdot\mathbf{u} &= \sum_{i=1}^n \partial_i(H_1(r)f_i) + \partial_i(H_2(r)(\mathbf{f}\cdot\mathbf{x})x_i) \\ &= \sum_{i=1}^n \frac{H_1'(r)}{r} f_i x_i + \sum_{i=1}^n \frac{H_2'(r)}{r}(\mathbf{f}\cdot\mathbf{x})x_i^2 + \sum_{i=1}^n H_2(r)(\mathbf{f}\cdot\mathbf{x}) + \sum_{i=1}^n H_2(r) f_i x_i \\ &= \frac{H_1'(r)}{r}(\mathbf{f}\cdot\mathbf{x}) + \frac{H_2'(r)}{r}(\mathbf{f}\cdot\mathbf{x})r^2 + nH_2(r)(\mathbf{f}\cdot\mathbf{x}) + H_2(r)(\mathbf{f}\cdot\mathbf{x}) \\ &= (\mathbf{f}\cdot\mathbf{x})\left(\frac{H_1'(r)}{r} + rH_2'(r) + (n+1)H_2(r)\right). \end{aligned}$$

In order for $\mathbf{u}$ to be divergence free, we require that $\frac{H_1'(r)}{r} + rH_2'(r) + (n+1)H_2(r) = 0$. In both two and three dimensions, we have that

$$\begin{aligned} H_2'(r) &= \frac{\partial}{\partial r}\left(\frac{B_\epsilon''(r)}{r^2} - \frac{B_\epsilon'(r)}{r^3}\right) \\ &= \frac{B_\epsilon'''(r)}{r^2} - \frac{2B_\epsilon''(r)}{r^3} - \frac{B_\epsilon''(r)}{r^3} + \frac{3B_\epsilon'(r)}{r^4} \\ \Rightarrow rH_2'(r) &= \frac{B_\epsilon'''(r)}{r} - \frac{3B_\epsilon''(r)}{r^2} + \frac{3B_\epsilon'(r)}{r^3}. \end{aligned}$$

In two dimensions,

$$\frac{(H_1^{2D})'(r)}{r} = -\frac{B_\epsilon'''(r)}{r},$$

so that

$$\begin{aligned} \frac{H_1'(r)}{r} + rH_2'(r) + (n+1)H_2(r) &= -\frac{B_\epsilon'''(r)}{r} + \frac{B_\epsilon'''(r)}{r} - \frac{3B_\epsilon''(r)}{r^2} + \frac{3B_\epsilon'(r)}{r^3} + \frac{3B_\epsilon''(r)}{r^2} - \frac{3B_\epsilon'(r)}{r^3} \\ &= 0, \end{aligned}$$

yielding a divergence free velocity field as desired. In three dimensions,

$$\begin{aligned}\frac{(H_1^{3D})'(r)}{r} &= \frac{1}{r}\left(-B_\epsilon'''(r) - \frac{B_\epsilon''(r)}{r} + \frac{B_\epsilon'(r)}{r^2}\right)\\ &= -\frac{B_\epsilon'''(r)}{r} - \frac{B_\epsilon''(r)}{r^2} + \frac{B_\epsilon'(r)}{r^3},\end{aligned}$$

so that

$$\begin{aligned}\frac{H_1'(r)}{r} + rH_2'(r) + (n+1)H_2(r) &= -\frac{B_\epsilon'''(r)}{r} - \frac{B_\epsilon''(r)}{r^2} + \frac{B_\epsilon'(r)}{r^3} + \frac{B_\epsilon'''(r)}{r} - \frac{3B_\epsilon''(r)}{r^2} + \frac{3B_\epsilon'(r)}{r^3}\\ &\quad + \frac{4B_\epsilon''(r)}{r^2} - \frac{4B_\epsilon'(r)}{r^3}\\ &= 0,\end{aligned}$$

also yielding a divergence free velocity. This is the expected result, since the solution for $\mathbf{u}$ is constructed using the incompressibility condition.

However, instead of starting with a blob function and deriving B_ϵ, one could replace $B'(r) = -1/8\pi$ (the derivative of the biharmonic function in the singular case) with a regularizing function $B_\delta'(r) = -(1/8\pi)F_\delta(r)$. By defining H_1 and H_2 using $B_\delta'(r)$ instead of $B_\epsilon'(r)$ in Eqs. (1), (2), and (3), we have just shown that a divergence free velocity would result. One could then back-solve to discover the associated blob function. This is what Karin did in the Brinkman paper.