Quiz 12 Solution

Sunday, 28 April 2024 6:27 pm

NAME:
HABIB ID:

LINEAR ALGEBRA

SPRING 2024 – SECTIONS L1, L3, L5

QUIZ 12 (04 April 2024)

Max Marks: 10

Time: 07 minutes

Q. Let the vector space P_2 have the inner product

$$\langle \boldsymbol{p}, \boldsymbol{q} \rangle = \int_{-1}^{1} p(x)q(x)dx$$

Apply the Gram-Schmidt process to transform the standard basis $S = \{1, x, x^2\}$ into an orthonormal basis (find the first two orthonormal vectors).

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Solution

EXAMPLE 2 Weighted Euclidean Inner Product

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors in R^2. Verify that the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1 v_1 + 2u_2 v_2$$

satisfies the four inner product axioms.

Solution

Note first that if $\mathbf{u}$ and $\mathbf{v}$ are interchanged in this equation, the right side remains the same. Therefore,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$$

If $\mathbf{z} = (z_1, z_2)$, then

$$\begin{aligned} \langle \mathbf{u} + \mathbf{v}, \mathbf{z} \rangle &= 3(u_1 + v_1)z_1 + 2(u_2 + v_2)z_2 \\ &= (3u_1 z_1 + 2u_2 z_2) + (3v_1 z_1 + 2v_2 z_2) \\ &= \langle \mathbf{u}, \mathbf{z} \rangle + \langle \mathbf{v}, \mathbf{z} \rangle \end{aligned}$$

which establishes the second axiom.

Next,

$$\langle k\mathbf{u}, \mathbf{v} \rangle = 3(ku_1)v_1 + 2(ku_2)v_2 = k(3u_1 v_1 + 2u_2 v_2) = k\langle \mathbf{u}, \mathbf{v} \rangle$$

which establishes the third axiom.

Finally,

$$\langle \mathbf{v}, \mathbf{v} \rangle = 3v_1 v_1 + 2v_2 v_2 = 3v_1^2 + 2v_2^2$$

Obviously, $\langle \mathbf{v}, \mathbf{v} \rangle = 3v_1^2 + 2v_2^2 \geq 0$. Further, $\langle \mathbf{v}, \mathbf{v} \rangle = 3v_1^2 + 2v_2^2 = 0$ if and only if $v_1 = v_2 = 0$—that is, if and only if $\mathbf{v} = (v_1, v_2) = \mathbf{0}$. Thus the fourth axiom is satisfied.

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NAME:
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LINEAR ALGEBRA

SPRING 2024 – SECTIONS L2, L4, L6

QUIZ 12 (02 April 2024)

Max Marks: 10

Time: 08 minutes

Q. Let $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ be vectors in R^2. Verify that the weighted Euclidean inner product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 3u_1v_1 + 2u_2v_2$$

satisfies the four inner product axioms.

SOLUTION: (HINTS)

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

$S = \{1, x, x^2\}$, HERE

$u_1 = 1$, $u_2 = x$, $u_3 = x^2$

① $u_1 = v_1 = 1$

② $\|v_1\| = \|1\| = \langle v_1, v_1 \rangle^{\frac{1}{2}}$

$= \langle 1, 1 \rangle^{1/2} = \left(\int_{-1}^{1} 1\,dx \right)^{\frac{1}{2}} = \sqrt{2}$

F

③ $\underline{v}_2 = \underline{u}_2 - \dfrac{\langle \underline{u}_2, \underline{v}_1 \rangle \underline{v}_1}{\|\underline{v}_1\|^2}$

$= x - \dfrac{\cancel{\langle x, 1 \rangle}^{\,0}}{\|1\|^2} = x$

④ $\|\underline{v}_2\| = \|x\| = \left(\int_{-1}^{1} x^2 \, dx \right)^{\frac{1}{2}}$

$= \sqrt{\frac{2}{3}}$

v_1

$= \left\{ \dfrac{\underline{v}_1}{\|\underline{v}_1\|}, \dfrac{\underline{v}_2}{\|\underline{v}_2\|} \right.$

$= \left\{ \dfrac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x, \right.$

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