Stoke’s Theorem is the Generalization of Many Calculus Theorems

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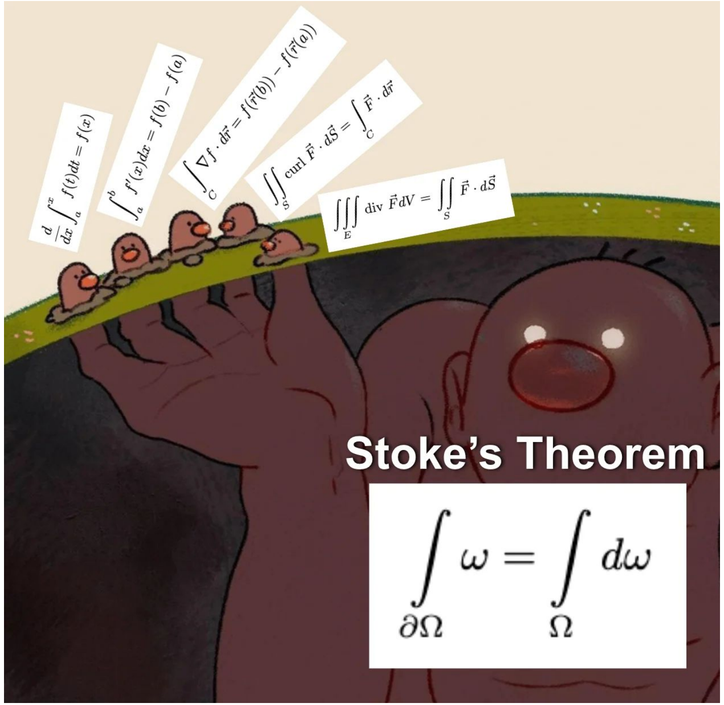
Introduction

- Stokes' Theorem generalizes calculus theorems, providing a unified approach in vector calculus.
- It links with fundamental concepts like Green's Theorem, curl, and conservative fields for a cohesive understanding.
- Similar to the Divergence Theorem, it relates surface and line integrals, emphasizing the vector field curl.

Looking Under the Hood of Stokes Theorem

- Curl Vector and Circulation
- Fundamental Theorem of Calculus
- Green's Theorem
- Divergence Theorem



Relationship between Curl Vector, Circulation and Stokes' Theorem

Circulation measures the tendency of a vector field to rotate around a curve.

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{Counterclockwise circulation}} = \underbrace{\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{Curl integral}}$$

Curl is a measure of rotating effect of the fluid about axis **n**.

- Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C.
- Let **F = Mi + Nj + Pk** be a vector field whose components have continuous first partial derivatives on an open region containing **S**.
- Then the circulation of F around C in the direction counterclockwise with respect to the surface unit normal vector n equals the integral of the curl vector field **∇ x F over S.**

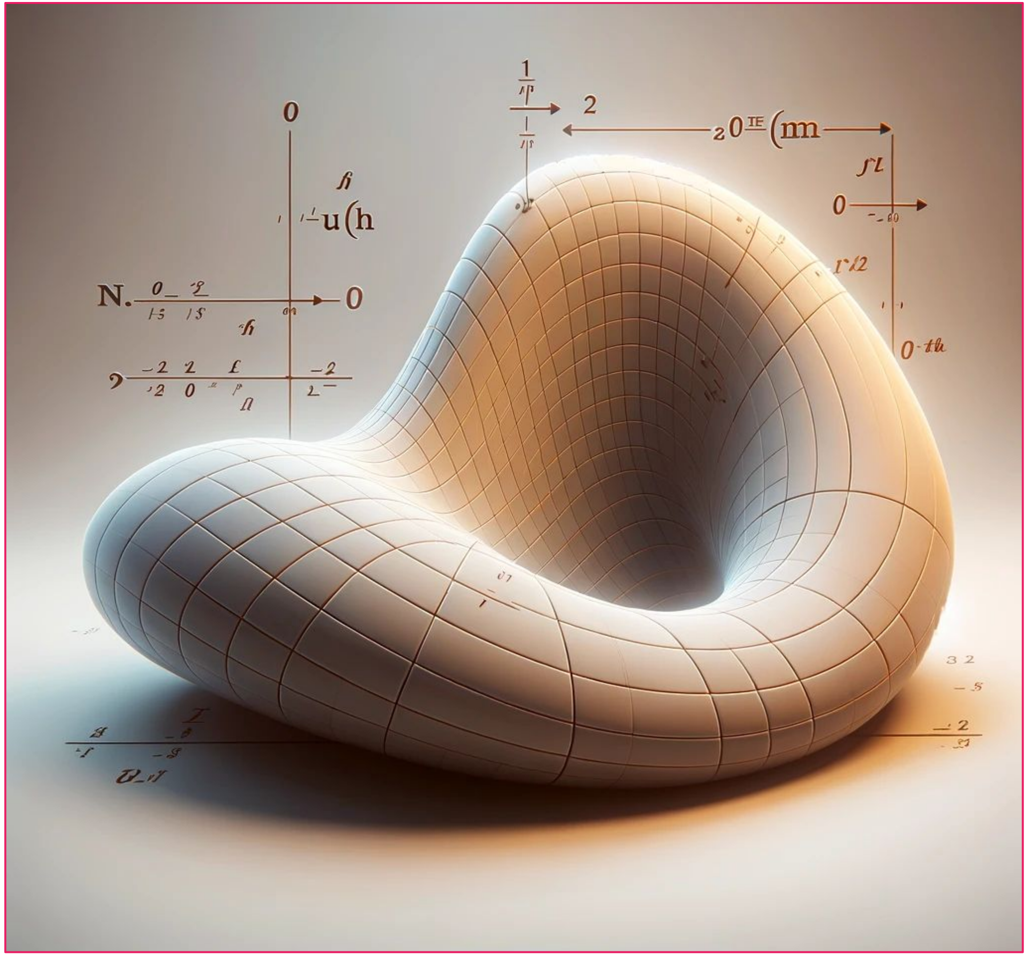
Second Fundamental Theorem of Calculus

Second Fundamental Theorem of Calculus states that:

If there exist a function f and F(x) is the antiderivative of function f, then definite integral of f(x) over interval [a,b] is the difference of antiderivative at b and a.

But How is Stoke’s theorem is the generalization of second fundamental theorem of calculus?

- With the equation $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$ we know that dF/dx = f(x). Hence f(x)dx is an exterior derivative of 0-th form but general stoke's theorem applies on higher differential forms.
- The closed interval [a,b] can be considered one-dimensional manifold with boundary on the points of a and b. But since Stoke’s theorem deals with differential forms on manifolds instead of functions with intervals, Integrating f over the interval may be generalized to integrating forms on a higher-dimensional manifold.
- Hence second fundamental theorem of calculus can be seen as a special case of Stoke's theorem for 0th-form functions with one-dimensional manifolds.



Visualization of one-dimensional manifold with 0th form

We have the equation:

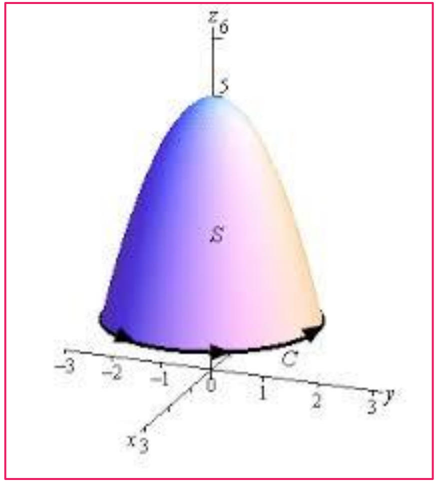
$$\int_{[a,b]} f(x)\,\mathrm{d}x = \int_{[a,b]} \mathrm{d}F = \int_{\{a\}^{-}\cup\{b\}^{+}} F = F(b) - F(a).$$

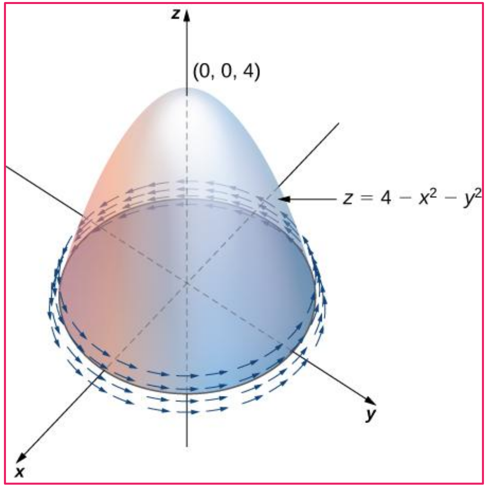
Stokes theorem would state this in higher dimension as:

$$\int_M d\omega = \int_{\partial M} \omega.$$

Where:

- M = manifold
- dω = exterior derivative of ω
- ∂M = boundary of manifold





Relationship between Green's Theorem and Stokes' Theorem

Green's theorem applies only to two-dimensional vector fields and to regions in the two-dimensional plane. Stokes' theorem generalized Green's theorem to three dimensions.

We write **Green's Theorem** as:

$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

Now converting this to 3D. **According to Stoke Theorem definition**, we take curl F with respect to normal n.

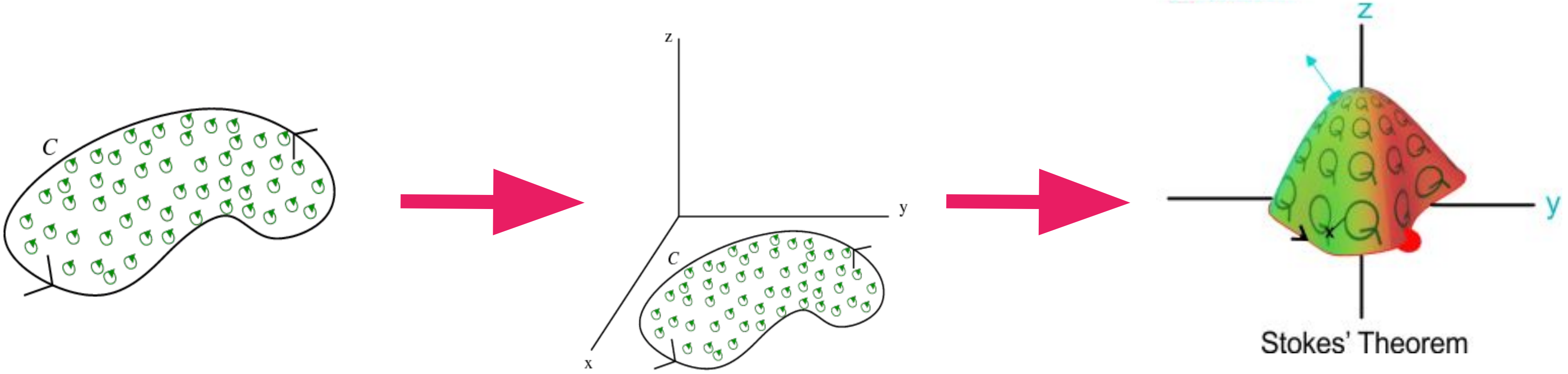
$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}}$$

where, $\nabla \times F = \cancel{\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}}} + \cancel{\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$

now multiplying k-th component with normal n gives us **Stokes Theorem**

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \\ = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \end{aligned}$$

Converting Green's Theorem to Stokes Theorem



https://mathinsight.org/stokes_theorem_idea (for visualisation)

Relationship between Divergence Theorem and Stokes' Theorem

The usual (3-dimensional) Stokes' and Divergence theorems **both involve a surface integral**, but they are in rather different circumstances.

In the **Divergence theorem**, the surface S is the boundary of a bounded region R of space, and you're taking the flux through this surface of a vector field F defined in R and on its boundary

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_R \operatorname{div} \mathbf{F} \, dV}$$

In **Stokes' theorem**, the surface is generally not the boundary of a region: instead it has a boundary which is a curve C; you're taking the flux, not of an arbitrary vector field, but of the curl of some other field:

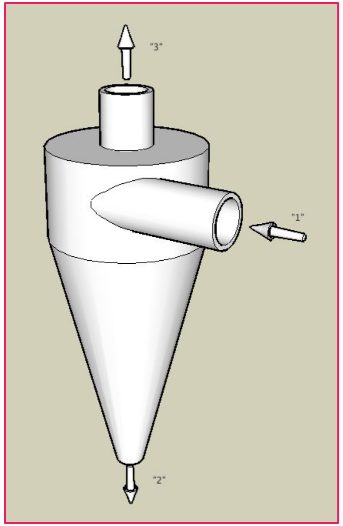
$$\boxed{\iint_S \operatorname{curl} \mathbf{G} \cdot d\mathbf{S} = \oint_C \mathbf{G} \cdot d\mathbf{r}}$$

Relationship between Divergence Theorem and Stokes' Theorem

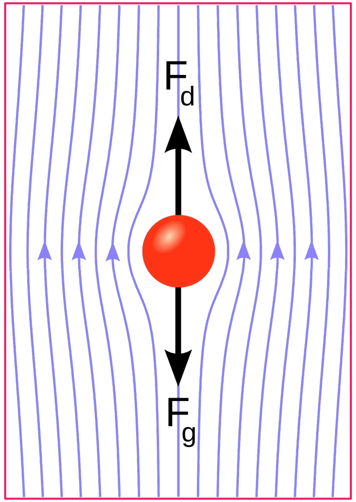
In scenarios where the vector field F is the curl of another field G, and the surface S is the boundary of bounded region R, both Divergence and Stoke Theorem yield a result of 0 due to cancellation of divergence and circulation respectively.

Long story short, Stokes' Theorem evaluates the flux going through a single surface, while the Divergence Theorem evaluates the flux going in and out of a solid through its surface(s).

Think of Stokes' Theorem as "air passing through your window", and of the Divergence Theorem as "air going in and out of your room". Clearly, if you only had the window, both results should coincide, but what if you also count the door in your room?



Hydrocyclone

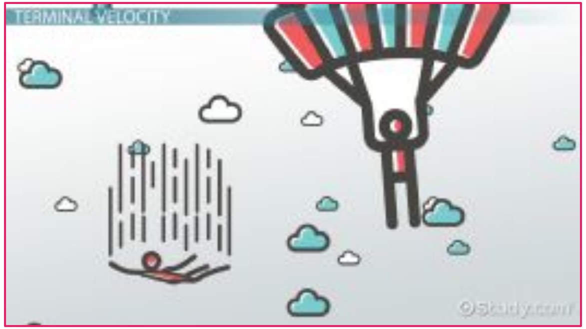


Fluid Dynamics

Centrifuge

Practical Applications

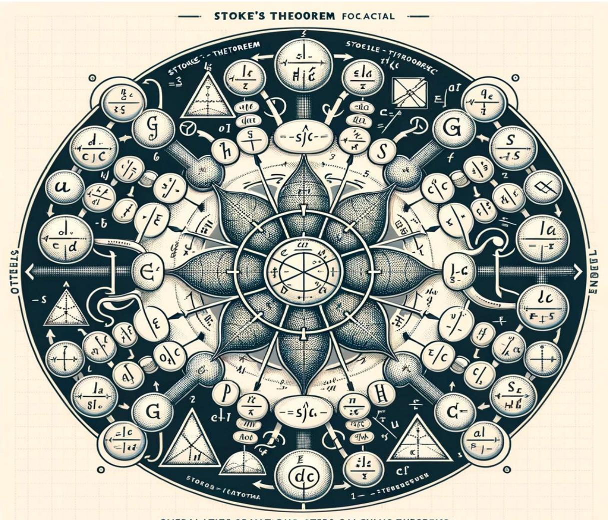
- Fluid Dynamics (flow etc)
- Field Stimulation (computer graphics)
- Study of Cloud formation
- Magnetic and Electric Fields calculation (Electromagnetism)
- Parachuting (drag force)



Parachuting

To Conclude:

The examples we discussed are **some** of the theorems of calculus encompassed by Stoke’s Theorem. Stoke's theorem is a fundamental result in calculus, particularly in the context of vector fields and the study of circulation and flux. Outside of calculus and mathematics, Stoke's theorem is not directly applicable. However, its principles and concepts can be used in various fields such as physics and engineering to analyze and solve problems related to fluid dynamics, electromagnetism, and more.



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THANK YOU


WHEN THEY ASK IF
YOU UNDERSTOOD THE PROOF
BUT YOU DIDN'T EVEN
UNDERSTAND THE THEOREM
imgflip.com