Final Exam (Fall 2022): Solution

Intro to Probability and Statistics - EE 354 / CE 361 / MATH 310

Question 1 - (10 points)

a) A lot of 100 semiconductor chips contain 20 that are defective. Two chips are selected at random, without replacement, from the lot.
   i. What is the probability that the first one selected is defective? **[1]**
   ii. What is the probability that the second one selected is defective, given that the first one was defective? **[2]**
   iii. What is the probability that both are defective? **[2]**

b) Suppose X and Y are independent random variables, such that X has probability mass function $p_X(x) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{x-1}$ for integers $x \geq 1$ and Y has probability mass function $p_Y(y) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{y-1}$ for integers $y \geq 1$.
   i. What is $Var(X - 2Y)$? **[3]**
   ii. What is $p_{X|Y}(1|2)$? **[2]**

(a)

Let

A = { First selected chip is defective }

B = { The second selected chip is defective }

i)

$$P(A) = \frac{20}{100} = 0.2$$

ii)

$$P(B|A) = \frac{19}{99}$$

iii)

$$P(A \cap B) = P(B|A)\,P(A)$$

$$= \left(\frac{19}{99}\right)\left(\frac{20}{100}\right)$$

(b)

X and Y are Geometric random variables with parameters $p = \frac{1}{3}$ and $p = \frac{3}{4}$ respectively

i)

Since X and Y are independent

$$Var(X-2Y) = Var(X) + (-2)^2 Var(Y)$$

$$= Var(X) + 4 Var(Y)$$

$\because$ Variance of a Geometric RV is $\frac{1-p}{p^2}$

$$\Rightarrow \quad Var(X-2Y) = \frac{\left(\frac{2}{3}\right)}{\left(\frac{1}{3}\right)^2} + 4\left[\frac{\left(\frac{1}{4}\right)}{\left(\frac{3}{4}\right)^2}\right]$$

$$= 6 + \frac{4^2}{3^2}$$

$$Var(X-2Y) = 7.77 \text{ — Ans}$$

ii)

Due to independence of X and Y :-

$$p_{X|Y}(x|y) = p_X(x)$$

$$p_{X|Y}(1|2) = p_X(1) = \frac{1}{3}\left(\frac{2}{3}\right)^{1-1}$$

$$= \frac{1}{3} \text{ — Ans}$$

Question 2 - (10 points)

a) Suppose that the number of kilometers you drive before you get into a car accident is modeled as an exponential random variable with a mean of 15,000 kms. You are planning to drive in your car form Karachi to Khunjerab Pass and back. What is the probability that you will make it back to Karachi without getting into an accident? **[5]**
(Hint: According to Google Maps, the distance from Karachi to Khunjerab Pass is 2,100 km.)

b) X and Y are random variables with a joint PDF given by:
$f_{X,Y}(x,y) = x + cy^3$
for $0 \le x \le 1$ and $0 \le y \le 1$. The joint PDF is zero for all other values of x and y.
   i. What is the value of constant c? **[2]**
   ii. Are X and Y independent? Justify your answer.**[3]**

(a)

X = Number of kms you drive before getting into an accident

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\text{Mean} = \frac{1}{\lambda} = 15000$$

$$\Rightarrow \lambda = \frac{1}{15000}$$

For an Exponential RV:

$$P(X \geq a) = e^{-\lambda a}$$

$$P(X \geq 4200) = e^{-\left(\frac{1}{15000}\right)(4200)}$$

$$= e^{-0.28}$$

$$= 0.756$$ — Ans

(b)

$$f_{X,Y}(x,y) = \begin{cases} x+cy^3 & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{—(A)}$$

i)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$$

$$\int_0^1\int_0^1 (x+cy^3)\, dx\, dy = 1 \quad \text{—(1)}$$

$$\int_0^1 (x+cy^3)\, dx = \frac{x^2}{2} + cy^3x \Big|_0^1$$

$$= \frac{1}{2} + cy^3$$

From (1)

$$\int_0^1 \left(\frac{1}{2} + cy^3\right) dy = 1$$

$$\frac{1}{2}y + c\frac{y^4}{4} \Big|_0^1 = 1$$

$$\frac{1}{2} + \frac{c}{4} = 1$$

$$\frac{c}{4} = \frac{1}{2}$$

$$c = 2 \quad \text{— Ans}$$

ii)

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \cdot dy = \int_0^1 (x + 2y^3)\, dy$$

$$= xy + 2 \cdot \frac{y^4}{4} \Big|_0^1$$

$$= xy + \frac{y^4}{2} \Big|_0^1$$

$$= x + \frac{1}{2} \qquad 0 \leq x \leq 1 \qquad \text{— (2)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_0^1 (x + 2y^3)\, dx$$

$$= \frac{x^2}{2} + 2y^3 x \Big|_0^1$$

$$f_Y(y) = \frac{1}{2} + 2y^3 \qquad 0 \leq y \leq 1 \qquad (3)$$

From (2) and (3), -

$$f_X(x)\, f_Y(y) = \left(x + \frac{1}{2}\right)\left(\frac{1}{2} + 2y^3\right)$$

$$= x\left(\frac{1}{2} + 2y^3\right) + \frac{1}{2}\left(\frac{1}{2} + 2y^3\right)$$

$$f_X(x) f_Y(y) = \frac{x}{2} + 2y^3 x + \frac{1}{4} + y^3 \qquad 0 \leq x \leq 1, 0 \leq y \leq 1$$

— (B)

From (A) and (B):

$$f_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y)$$

$\Rightarrow$ X and Y are not independent. — Ans

Question 3 - (10 points)

X and Y are independent continuous random variables that are uniformly distributed with $X \sim U[0,2]$ and $Y \sim U[0,2]$. Find the following:

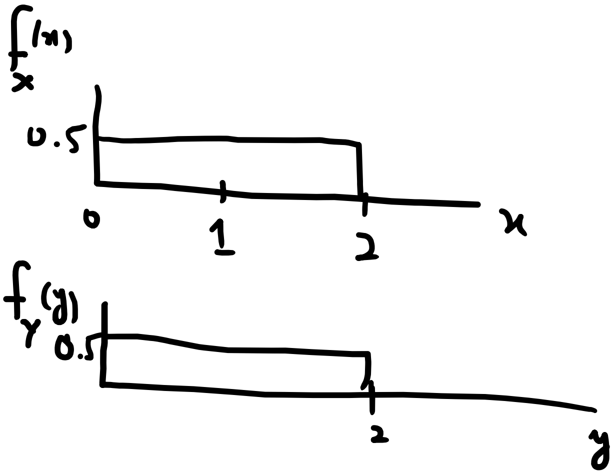
a) $f_{X,Y}(x,y)$ **[2]**

b) $F_{X,Y}(1.5,1.5)$ **[2]**

c) $f_{Y|X}(y|0.5)$ **[2]**

d) $f_{X|Y}(0.5|1)$ **[2]**

e) $E[X|Y > 1.5]$ **[2]**



(a)

Due to independence,

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

$$f_{X,Y}(x,y) = \begin{cases} 0.25 & 0 \leq x \leq 2,\ 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases} \quad \text{— Ans}$$

(b)

$F_{X,Y}(1.5,1.5) = P(X \leq 1.5, Y \leq 1.5)$

= Volume under the $f_{X,Y}(x,y)$ over the region $\{X \leq 1.5, Y \leq 1.5\}$

$= (1.5)(1.5)(0.25)$

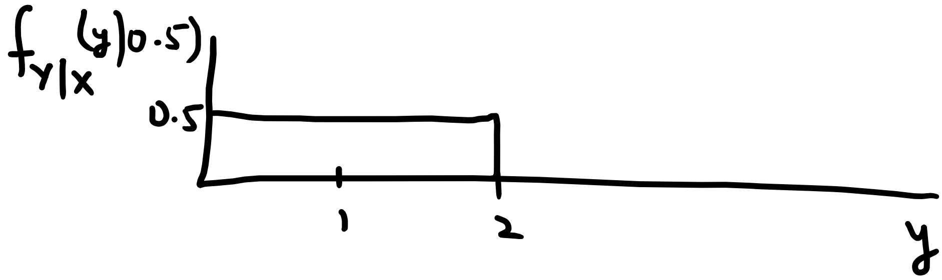
$= 0.5625$ — Ans

(c)

Due to Independence:

$$f_{Y|X}(y|x) = f_Y(y)$$

$$\Rightarrow \quad f_{Y|X}(y|0.5) = f_Y(y)$$



(d)

Due to independence:

$$f_{X|Y}(x|y) = f_X(x)$$

$$\Rightarrow \quad f_{X|Y}(0.5|1) = f_X(0.5)$$

$$= 0.5$$

— Ans

(e)

Let A be the event $Y > 1.5$.

Due to independence of X and Y:-

$$f_{X|A}(x) = f_X(x)$$

Now,

$$E[X|A] = \int_{-\infty}^{\infty} x \, f_{X|A}(x).dx$$

$$= \int_{-\infty}^{\infty} x \, f_X(x)\, dx = E[X] = 1 \quad \text{Ans}$$

Question 4 - (10 points)

A city has two internet service providers, Charter and Xfinity, with the market share of 30% and 70% respectively. At any given time, the internet speed experienced by a customer of Charter is between 7Mbps and 15Mbps with all speeds being equally likely. At any given time, the internet speed experienced by a customer of Xfinity is between 10Mbps and 20Mbps with all speeds being equally likely. If an internet customer in the city is currently experiencing an internet speed of 12Mbps, what is the probability that the customer is served by Charter?

Let I be the random variable that takes on following values.

$I = 0$ when ISP is Charter

$I = 1$ when ISP is Xfinity

$$p_I(i) = \begin{cases} 0.3 & i = 0 \\ 0.7 & i = 1 \end{cases}$$

Let S be the speed experienced by an internet customer.

$$f_{S|I}(s|0) = \begin{cases} 1/8 & 7 \leq s \leq 15 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{S|I}(s|1) = \begin{cases} 1/10 & 10 \leq s \leq 20 \\ 0 & \text{otherwise} \end{cases}$$

Req. Prob. $= p_{I|S}(0|12) = ?$

By Total Prob. Theorem:

$$f_S(s) = \sum_i p_I(i) f_{S|I}(s|i)$$

$$f_S(s) = p_I(0)\, f_{S|I}(s|0) + p_I(1)\, f_{S|I}(s|1) \quad \text{—(1)}$$

Using Bayes' Rule:

$$p_{I|S}(i|s) = \frac{p_I(i)\, f_{S|I}(s|i)}{f_S(s)}$$

$$\Rightarrow \quad p_{I|S}(0|12) = \frac{p_I(0)\, f_{S|I}(12|0)}{f_S(12)} \quad \text{—(2)}$$

From (1)

$$f_S(12) = p_I(0)\, f_{S|I}(12|0) + p_I(1)\, f_{S|I}(12|1)$$

$$= 0.3\left(\frac{1}{8}\right) + 0.7\left(\frac{1}{10}\right)$$

$$f_S(12) = 0.1075$$

From (2) :-

$$p_{I|S}(0|12) = \frac{0.3\left(\frac{1}{8}\right)}{0.1075}$$

$$= \frac{0.0375}{0.1075}$$

$$= 0.349 \quad \text{— Ans}$$

Question 5 - (10 points)

a) Binary signal S is transmitted across a communication channel. The received signal is $Y = N + S$, where N is normal noise with zero mean and unit variance, independent of S. Let A be the event that the noise is between -0.25 and 0.25 (i.e. $N \epsilon [-0.25, 0.25]$). What is $E[N|A]$? Justify your answer. **[5]**

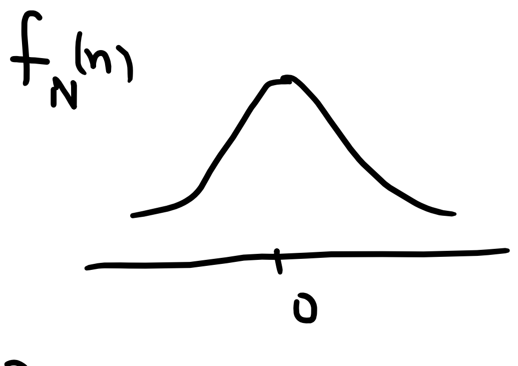
b) Suppose you were told that the average odometer reading (number of kilometers a car has traveled in its lifetime so far) of all the red cars in Karachi was 55,000 km. Express this information in terms of two random variables X and Y. Define the random variables X and Y appropriately. (You can use any of the concepts applicable to two random variables such as joint PDF, conditional PDF, etc.) **[5]**

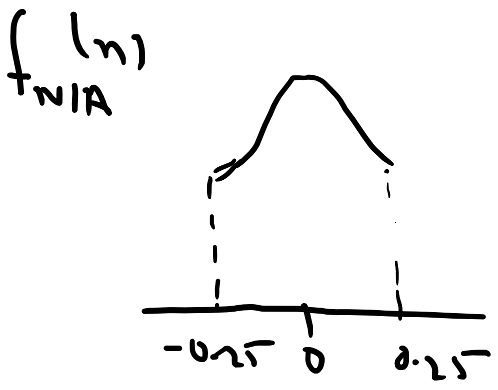
(a)

$$N \sim N(0,1)$$

$$A : N \in [-0.25, 0.25]$$

$$f_{N|A}(n) = \begin{cases} \dfrac{f_N(n)}{P(N \in [-0.25, 0.25])} & -0.25 \leq n \leq 0.25 \\ 0 & \text{otherwise} \end{cases}$$





By Symmetry

$$\Rightarrow \quad E[N|A] = 0 \quad \text{— Ans}$$

(b)

X = The odometer reading of a randomly selected car in Karachi

Y = The color of a randomly selected car in Karachi

$$E[X \mid Y = \text{Red}] = 55{,}000 \text{ km}$$

Ans