Final Examination
Dr. Shafayat Abrar
Probability and Statistics

Max Points: 200
Date: May 08, 2024
Duration: 150 min

Q1 [40 pt.]: An ECE instructor at Habib University never finishes his lecture before the end of the hour, but always within eight minutes after the hour. His CS students are now habituated to the situation 😄.

Let X be the time that elapses between the end of the hour and the end of the lecture, and assume the pdf of X is as follows:

$$f_X(x) = \begin{cases} k(x + x^2) & 0 \leqslant x \leqslant 8 \\ 0 & \text{otherwise} \end{cases}$$

(1a) Find the value of k and draw the corresponding density curve.

(1b) What is the probability that the lecture ends within 5 minutes of the end of the hour?

(1c) What is the probability that the lecture continues beyond the hour for between 3 and 5 minutes?

(1d) What is the probability that the lecture continues for at least 3 minutes beyond the end of the hour?

(1e) Obtain the cumulative distribution function $F_X(x)$, and also plot it.

(1f) Find the mean and standard deviation of X.

Q2 [30 pt.]: Royal Deluxe 200cc, the 4-stroke water-cooled Sazkar Rickshaw, is popular in Karachi because of its mobility, loading capacity, and low cost. The manufacturer claims that the petrol mileage has a normal distribution with a mean value of 28 km/L and a standard deviation of 1.75 km/L. Consider randomly selecting a single such Rickshaw.

(2a) What is the probability that the fuel mileage is at most 32 km/L?
[Hint: find $P(X \leqslant 32)$] [10 pt.]

(2b) What is the probability that the fuel mileage is at least 31 km/L? [10 pt.]

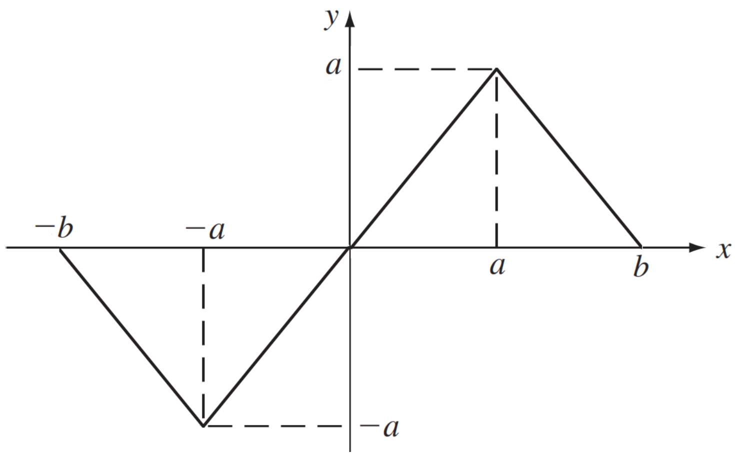
(2c) What is the probability that the fuel mileage differs from the mean value by at most 1.0 standard deviations? [10 pt.]

Q3 [30 pt.]: Vehicle speed on a Pakistani motorway can be represented as normally distributed.

(3a) What is the mean and standard deviation of vehicle speed if 5% of vehicles travel less than 80 km/h and 10% travel more than 110 km/h? [20 pt.]

(3b) What is the probability that a randomly selected vehicle's speed is between 90 and 100 km/h? [10 pt.]

Q4 [60 pt.]: A nonlinear processing element is shown below whose purpose is to ignore large values of the input data (or signal):



(4a) Find an expression for the mean and variance of $Y = g(X)$ for an arbitrary continuous random variable X. Let $a = 1$ and $b = 2$. [20 pt.]

(4b) Evaluate the mean and variance of Y if X is Laplacian with the following PDF: [20 pt.]

$$f_X(x) = \frac{1}{2}\exp(-|x|)$$

(4c) Find the CDF and PDF of the output Y. [20 pt.]

Q5 [40 pt.]: A restaurant serves three fixed-price dinners costing \$12, \$15, and \$20. For a randomly selected couple dining at this restaurant, let X = the cost of the man's dinner and Y = the cost of the woman's dinner. The joint pmf of X and Y is given in the following table:

$p_{X,Y}(x,y)$		y = 12	15	20
	12	0.05	0.05	0.10
x	15	0.05	0.10	0.35
	20	0	0.20	0.10

(5a) Compute the marginal PMF's of X and Y.

(5b) What is the probability that the man's and the woman's dinner cost at most \$15 each?

(5c) Are X and Y independent? Justify your answer.

(5d) What is the expected total cost of the dinner for the two people?

Q1. Solution -

$$f_X(x) = k(x+x^2), \quad 0 \leq x \leq 8$$

1a)
$$\int_0^8 k(x+x^2)\,dx = k\left(\frac{x^2}{2} + \frac{x^3}{3}\right)\Big|_0^8$$

$$= k\left(32 + 170.67\right) = 1$$

$$k = \frac{1}{202.67} = 0.004934$$

1b)
$$P[X \leq 5] = \int_0^5 k(x+x^2)\,dx$$

$$= k\left(\frac{x^2}{2} + \frac{x^3}{3}\right)\Big|_0^5$$

$$= 0.267$$

1c)
$$P[3 \leq X \leq 5] = \int_{\cancel{0}3}^5 k(x+x^2)\,dx$$

$$= k\left(\frac{x^2}{2} + \frac{x^3}{3}\right)\Big|_3^5$$

$$= 0.201$$

1d)
$$P[X \geq 3] = \int_3^8 k(x+x^2)\,dx$$

$$= k\left(\frac{x^2}{2} + \frac{x^3}{3}\right)\Big|_3^8$$

$$= 0.933$$

1e)
$$F_X(x) = \int_{-\infty}^x f_X(\lambda)\,d\lambda = \int_0^x f_X(\lambda)\,d\lambda$$

$$= \begin{cases} k\left(\frac{x^2}{2} + \frac{x^3}{2}\right) & 0 \leq x < 8. \\ 1 & x \geq 8 \end{cases}$$

②

1f) $E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$

$= \int_0^8 x\, k(x + x^2)\, dx$

$= k\left(\frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_0^8 = 5.894$

$E(X^2) = \int_0^8 x^2 k(x + x^2) dx = k\left(\frac{x^4}{4} + \frac{x^5}{5}\right)$

$= 37.388$

$Var(X) = E(X^2) - E(X)^2$

$= 37.388 - 5.894^2$

$= 2.648$

$std(X) = \sqrt{var(X)}$

$= \sqrt{2.648} = 1.627.$

Q2. $\mu = \text{mean} = 28\,\text{km/L}$ (3)

$\sigma = \text{std.dev} = 1.75\,\text{Km/L}.$

(2a) $P(X \leq 32) = P\left(\frac{X-\mu}{\sigma} \leq \frac{32-28}{1.75}\right)$

$= P(Z \leq 2.286) = \Phi(2.286)$

$\approx \Phi(2.29) = 0.9890$ (from table)

(2b) $P(X \geq 31) = 1 - P(X \leq 31)$

$= 1 - P\left(Z \leq \frac{31-28}{1.75}\right)$

$= 1 - P(Z \leq 1.714)$

$\approx 1 - P(Z \leq 1.71)$

$= 1 - 0.9564 = 0.0436.$

(2c) $P[\mu-\sigma \leq X \leq \mu+\sigma] = P[26.25 \leq X \leq 29.75]$

$= P\left[\frac{26.25-28}{1.75} \leq Z \leq \frac{29.75-28}{1.75}\right]$

$= P[-1 \leq Z \leq 1]$

$= \Phi(1) - \Phi(-1)$

$= 2\Phi(1) - 1$

$= 2 * 0.8413 - 1 = 0.6826.$

Q3. 3a) $X \sim N(\mu, \sigma^2)$

$P[X \leqslant 80] = 5\%$ (given)

$P[X \geqslant 110] = 10\%$ (given)

$$P[X \leqslant 80] = P\left[Z \leqslant \frac{80-\mu}{\sigma}\right] = 5\% = 0.05.$$

$$\Phi(z) = 0.05 \Rightarrow Z = \Phi^{-1}(0.05) = -\Phi^{-1}(1-0.05)$$
$$= -\Phi^{-1}(0.95)$$
$$= -[1.64, 1.65]$$
$$\approx -1.645$$

$$\Rightarrow \boxed{80-\mu = -1.645\sigma} \quad (1)$$

$$P[X \geqslant 110] = 1 - P[X \leqslant 110] = 0.1$$

$$\Rightarrow P[X \leqslant 110] = 0.9$$

$$\Rightarrow P\left[Z \leqslant \frac{110-\mu}{\sigma}\right] = 0.9 = \Phi(z)$$

$$Z = \Phi^{-1}(0.9) \approx \Phi^{-1}([0.8997, 0.9015])$$
$$\approx \frac{1.28 + 1.29}{2} \approx 1.285$$

$$\Rightarrow \boxed{110-\mu = 1.285\sigma} \quad (2)$$

Subtracting the eqs. (1) & (2), we get

$$110 - 80 = 1.285\sigma + 1.645\sigma$$
$$\Rightarrow \sigma = 10.2389 \approx 10.24.$$

This implies $\mu = 110 - 1.285\sigma$

$$= 110 - 1.285(10.24)$$

$$= 96.8416 = E[X]$$

(3b) $P[90 \leq X \leq 100] = P\left[\frac{90-\mu}{\sigma} \leq Z \leq \frac{100-\mu}{\sigma}\right]$

$$= P\left[\frac{90-96.8416}{10.24} \leq Z \leq \frac{100-96.8416}{10.24}\right]$$

$$= P[-0.668 \leq Z \leq 0.30847]$$

$$\simeq P[-0.67 \leq Z \leq 0.31]$$

$$= \Phi(0.31) - \Phi(-0.67)$$

$$= \Phi(0.31) - 1 + \Phi(0.67)$$

$$= 0.6217 + 0.7486 - 1 \simeq 0.3703.$$

$$= 37\%.$$

Q5. 5a) marginal PMF of X ⑥

x	12	15	20
$p_X(k)$	0.20	0.50	0.30

marginal PMF of Y

y	12	15	20
$p_Y(k)$	0.10	0.35	0.55

5b) $P[X \leq 15, Y \leq 15]$

$= P[12,12] + P[12,15] + P[15,12] + P[15,15]$

$= 0.05 + 0.05 + 0.05 + 0.10$

$= 0.25$

5c) $P[12,12] = 0.05$

$P[X=12] \cdot P[Y=12] = 0.20 * 0.10 = 0.02$

So $P[X=12, Y=12] \neq P[X=12]\,P[Y=12]$

X & Y are not independent.

5d) For two people (we assume one is X (man) and the other is Y (woman))

$g(x,y) = X + Y$

$E[X+Y] = E[X] + E[Y] =$

$E[X] = 12(0.2) + 15(0.5) + 20(0.3) = 15.9$

$E[Y] = 12(0.1) + 15(0.35) + 20(0.55) = 17.45$

$\Rightarrow E[X+Y] = 15.9 + 17.45 = \$33.35.$

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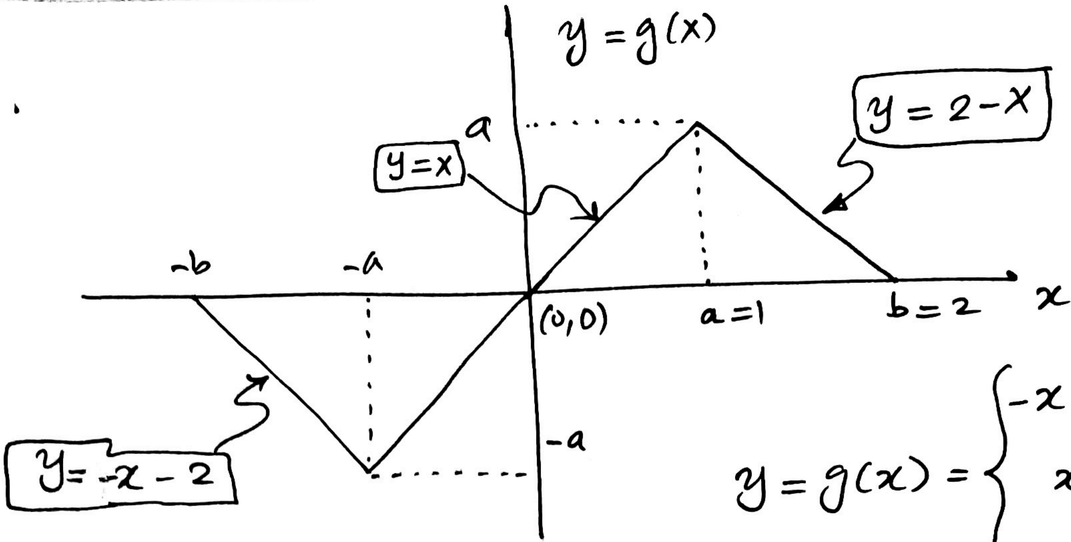
Q4.



$$y = g(x) = \begin{cases} -x-2 & -2<x<-1 \\ x & -1<x<1 \\ 2-x & 1<x<2 \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y] = E[g(X)] = \int_{-2}^{2} g(x) f_X(x)\,dx$$

$$= \int_{-2}^{-1} (-x-2) f_X(x)\,dx + \int_{-1}^{1} x f_X(x)\,dx$$

$$+ \int_{1}^{2} (2-x) f_X(x)\,dx.$$

$$E[Y^2] = E[g^2(X)] = \int_{-2}^{2} g^2(x) f_X(x)\,dx$$

$$= \int_{-2}^{-1} (-x-2)^2 f_X(x)\,dx + \int_{-1}^{1} x^2 f_X(x)\,dx$$

$$+ \int_{1}^{2} (2-x)^2 f_X(x)\,dx$$

mean of $Y = E(Y)$

variance of $Y = E(Y^2) - E(Y)^2$

Q4 (b). X is Laplacian

$$f_X(x) = \frac{1}{2}\exp(-|x|)$$

$$-\infty < x < \infty$$

X has an infinite support & symmetrical density so, its mean value is zero.

Y = g(X) is also symmetrical about y-axis, so its mean value is also zero.

But all steps are required to be shown.

(steps are easy, and are skipped here).

$\boxed{E[Y] = 0}$ Next, we need to compute $E[Y^2]$.

$$E[Y^2] = \int_{-2}^{-1} (-x-2)^2 \frac{1}{2} e^{x} dx + \int_{-1}^{0} x^2 \frac{1}{2} e^{x} dx + \int_{0}^{1} x^2 \cdot \frac{1}{2} e^{-x} dx + \int_{1}^{2} (2-x)^2 \cdot \frac{1}{2} e^{-x} dx$$

Note that

$$\frac{1}{2}\int_{-2}^{-1} (x+2)^2 e^{x} dx = \frac{e-2}{2e^2}$$

$$\frac{1}{2}\int_{-1}^{0} x^2 e^{x} dx = 1 - \frac{5}{2e}$$

$$\frac{1}{2}\int_{0}^{1} x^2 e^{-x} dx = 1 - \frac{5}{2e}$$

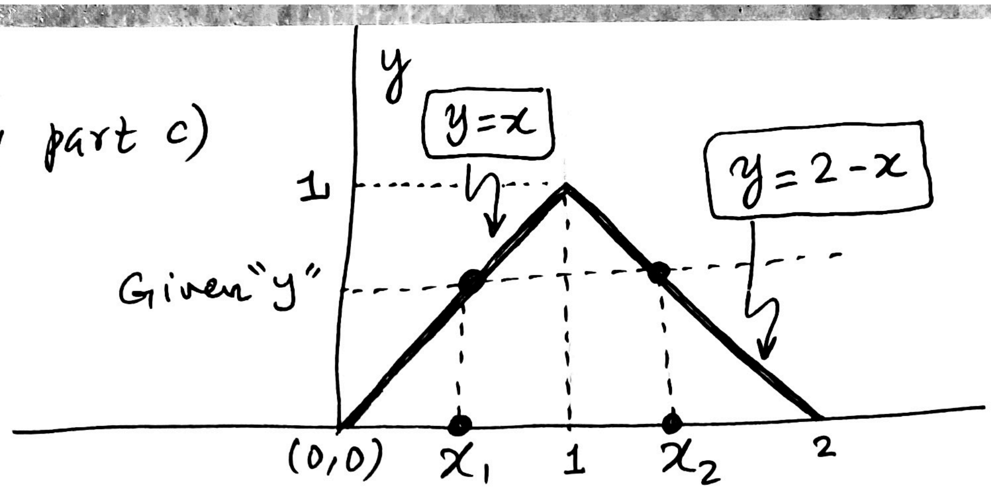
$$\frac{1}{2}\int_{1}^{2} (2-x)^2 e^{-x} dx = \frac{e-2}{2e^2}$$

$\boxed{\text{var}(Y) = 0.2578}$

$$\text{Variance}(Y) = E(Y^2) - \underbrace{E(Y)^2}_{=0} = \frac{e-2}{e^2} + 2\left(1 - \frac{5}{2e}\right)$$

$$= 0.2578.$$

Q4 part c)



$\boxed{\text{Note that } |y| \leq 1}$

Given $y > 0$, there are two possible values of x, x_1 & x_2.

$$x_1 = y \quad (y>0)$$
$$x_2 = 2-y \quad (y>0)$$

$$f_Y(y) = \sum_{i=1}^{2} f_X(x_i) \left|\frac{dx_i}{dy}\right|_{x_i = g^{-1}(y)}$$

Since $f_X(x) = \frac{1}{2} e^{-|x|} = \frac{1}{2} e^{-x}$ (for $x>0$, $\boxed{|x| = +x}$)

Therefore,

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right|$$

$$= \frac{1}{2} e^{-y} \cdot \left|\frac{dy}{dy}\right| + \frac{1}{2} e^{-(2-y)} \cdot \left|\frac{d(2-y)}{dy}\right|$$

$$\boxed{f_Y(y) = \frac{1}{2} e^{-y} + \frac{1}{2} e^{y-2} \quad \text{for } 0 < y \leq 1}$$

Q4 part c)

Let us find the area under the curve $f_Y(y)$ for $0 \leq y \leq 1$

$$\text{Area}_1 = \frac{1}{2}\int_0^1 \left(e^{-y} + e^{y-2}\right) dy = \frac{1}{2}\left(e^{y-2} - e^{-y}\right)\Big|_0^1$$

$$= \frac{1}{2}\left(e^{1-2} - e^{-1}\right) - \frac{1}{2}\left(e^{0-2} - e^{-0}\right)$$

$$= \frac{1}{2}\left(\cancel{e^{-1}} - \cancel{e^{-1}}\right) - \frac{1}{2}\left(e^{-2} - 1\right)$$

$$\boxed{\text{Area}_1 = \frac{1}{2}(1 - e^{-2}).}$$

Whenever is $x > 2$ or $x < -2$, the output $y = 0$. This is the discrete value of Y.

So, $f_Y(y) = P[Y=0]\,\delta(y)$

$$P[Y=0] = P[X<-2] + P[X>2]$$

$$= \int_{-\infty}^{-2} \frac{1}{2} e^{+x} dx + \int_{+2}^{+\infty} \frac{1}{2} e^{-x} dx$$

$$= \frac{1}{2} e^{x}\Big|_{-\infty}^{-2} + \frac{1}{2} e^{-x}\Big|_{+\infty}^{+2}$$

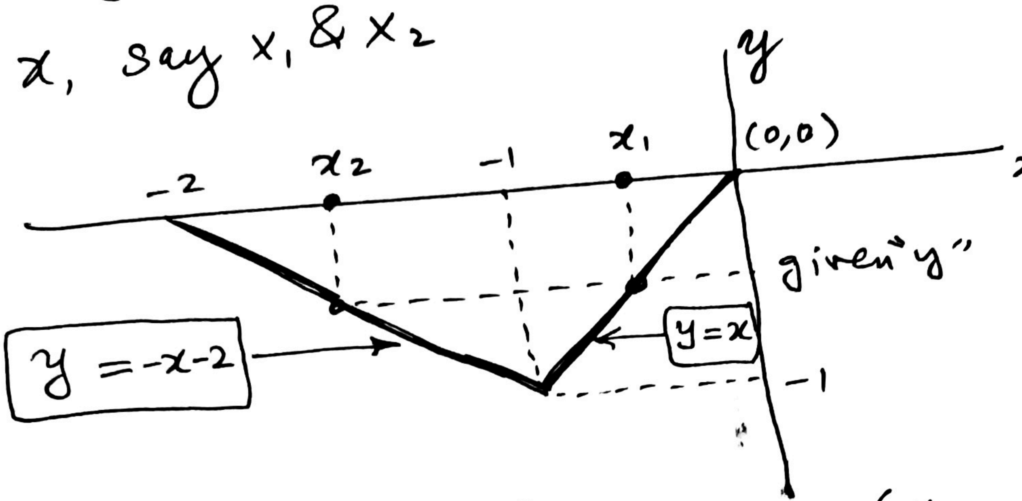
$$= \frac{1}{2}e^{-2} + \underbrace{\frac{1}{2}e^{-\infty}}_{=0} + \frac{1}{2}e^{-2} + \underbrace{\frac{1}{2}e^{-\infty}}_{=0}$$

$$= e^{-2}.$$

$$\boxed{\text{Area}_2 = e^{-2}}.$$

Q4 part c)

Given $y<0$, there are two possible values of x, say x_1 & x_2



$$x = g^{-1}(y) = \begin{cases} x_1 = y & (y<0) \\ x_2 = -y-2 & (y<0) \end{cases}$$

Since $f_X(x) = \frac{1}{2} e^{-|x|} = \frac{1}{2} e^{+x}$ (for $x<0$) $\boxed{|x| = -x}$

Therefore

$$f_Y(y) = f_X(x_1) \left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right|$$

$$= \frac{1}{2} e^{y} \left|\frac{dy}{dy}\right| + \frac{1}{2} e^{-(2+y)} \left|\frac{d(-y-2)}{dy}\right|$$

$$= \frac{1}{2}\left(e^{y} + e^{-(y+2)}\right), \text{ for}$$

$$-1 \leq y \leq 0.$$

$\text{Area}_3 = \int_{-1}^{0} f_Y(y)\,dy = \frac{1}{2}(1-e^{-2})$ ⟸ this is left for you to prove.

Finally, we get

$$f_Y(y) = \begin{cases} \frac{1}{2}\left(e^{-y} + e^{y-2}\right) & \text{for } 0 \leqslant y \leqslant +1 \\ e^{-2}\,\delta(y) & @\, y = 0 \text{ (discrete component)} \\ \frac{1}{2}\left(e^{+y} + e^{-y-2}\right) & \text{for } -1 \leqslant y \leqslant 0 \\ 0, & \text{otherwise.} \end{cases}$$