Midterm-II RETAKE
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Probability and Statistics

Max Points: 100
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Duration: 70 min

Q1 [10 pt]: If $\mathrm{E}[X] = 1$ and $\mathrm{Var}(X) = 5$, find

(1a) $\mathrm{E}\left[(2+X)^2\right]$

(1b) $\mathrm{Var}(4+3X)$.

Q2 [15 pt]: Let X be a Bernoulli random variable such that

$$p_X(-1) = P[X = -1] = 1 - p$$
$$p_X(1) = P[X = 1] = p$$

Find $A \neq 1$, where A is a real number not equal to unity, such that $\mathrm{E}\left[A^X\right] = 1$.

Q3 [15 pt]: Let X be a Poisson random variable with parameter λ.
Prove that the value of λ that maximizes $P[X = k]$ for $k \geq 0$ is $\lambda = k$.

Q4 [15 pt]: Suppose that X is a random variable that takes on one of the values $\mathbf{0}, \mathbf{1}, \mathbf{2}$. If for some constant $c > 0$,

$$p_X(i-1) = P[X = i-1]$$
$$p_X(i) = P[X = i] = c \cdot p_X(i-1), \quad i = 1, 2.$$

find $\mathrm{E}[X]$.

Q5 [10 pt]: A card is drawn at random from a deck consisting of cards numbered 2 through 10. A player wins 1 dollar if the number on the card is odd, and loses 1 dollar if the number is even. What is the expected value of his/her winnings?

Q6 [15 pt]: Exactly one of five similar **bolts** (externally helical threaded fasteners) fits into a certain **nut**. Let X be the number of attempts you will have to try, one after another, to get the bolt inserted into the nut. Find the PMF of X, and obtain its mean value, $\mathrm{E}[X]$.

Q7 [20 pt]: In a certain manufacturing process, the (Fahrenheit) temperature never varies by more than 2° from 62°. Assume that the temperature is a discrete random variable F with a distribution

$$\begin{pmatrix} k \\ p_F(k) \end{pmatrix} = \begin{pmatrix} 60 & 61 & 62 & 63 & 64 \\ \frac{1}{10} & \frac{2}{10} & \frac{4}{10} & \frac{2}{10} & \frac{1}{10} \end{pmatrix}.$$

(a) Find $\mathrm{E}[F]$ and $\mathrm{var}(F)$.
(b) Define $T = 2F - 62$. Find $\mathrm{E}[T]$ and $\mathrm{var}(T)$.

Midterm II Retake

Fact: $var(X) = E(X^2) - (E(X))^2$

Q1. Solution

$E(X) = 1$ (given)

$var(X) = 5$ (given)

(a) $E(2+X)^2 = E(4 + 4X + X^2)$

$= 4 + 4E(X) + E(X^2)$

$= 4 + 4E(X) + var(X) + [E(X)]^2$

$= 4 + 4 \times 1 + 5 + (1)^2$

$= 4 + 4 + 5 + 1 = 14.$

Fact: $var(aX+b) = a^2 Var(X)$

(b) $Var(4+3X)$

$= 9\, var(X) = 5 * 9 = 45.$

Q2 solution

Find $E[A^X]$

$$p_X(k) = \begin{cases} 1-p & k=-1 \\ p & k=+1 \end{cases}$$ (given)

$$= \sum_k A^k p_X(k)$$

$$= A^{(-1)} \cdot (1-p) + A^{(+1)} \cdot p$$

$$= \frac{1}{A}(1-p) + Ap = 1 \quad (E(A^X) = 1, \text{given})$$

$$\Rightarrow (1-p) + A^2 p = A$$

$$\boxed{pA^2 - A + (1-p) = 0}$$ quadratic equation.

$$A = \frac{-(-1) \pm \sqrt{(-1)^2 - 4p(1-p)}}{2p}$$

$$= \frac{1 \pm \sqrt{1 - 4p + 4p^2}}{2p}$$

$$= \frac{1 \pm \sqrt{(2p-1)^2}}{2p}$$

$$= \frac{1 \pm |2p-1|}{2p}$$

Either $A = \frac{2p-1+1}{2p} = 1$

or $A = \frac{-(2p-1)+1}{2p} = \frac{-p+1}{p} = \frac{1-p}{p} > 0$

Since, $A \neq 1$ (given)
Therefore,
$$A = \frac{1-p}{p}$$
is the final answer

Q3. Solution. X is Poisson R.V. (given)

$$p_X(k) = \frac{e^{-\lambda}\lambda^k}{k!} = P[X = k]$$

The value of λ that maximizes $p_X(k)$ may be obtained by taking derivative of $p_X(k)$, and setting that equal to zero.

$$\frac{d}{d\lambda}\frac{e^{-\lambda}\lambda^k}{k!} = \frac{k\lambda^{k-1}e^{-\lambda}}{k!} - \frac{e^{-\lambda}\lambda^k}{k!} = 0$$

$$\Rightarrow e^{-\lambda}(k\lambda^{k-1} - \lambda^k) = 0 \quad \text{(divide by } e^{-\lambda}\text{)}$$

$$\Rightarrow k\lambda^{k-1} - \lambda^k = 0 \quad \text{(divide by } \lambda^k\text{)}$$

$$k\lambda^{-1} - 1 = 0 \quad \text{(multiply with } \lambda\text{)}$$

$$k - \lambda = 0$$

or $\boxed{\lambda = k}$.

So $\lambda^* = k$ maximizes $p_X(k)$.

Let λ be the rate, then $X = \lambda$ is the most likely event.

Q4. Solution:

$$p_X(i-1) = P[X = i-1]$$

for $i = 1$

$$p_X(0) = p \text{ (assume). } \& \ p_X(1) = c\,p_X(0) = cp$$

for $i = 2$

$$p_X(2) = c\,p_X(1) = c^2 p.$$

$$\sum_k p_X(k) = 1. \text{ (fact).}$$

$$\Rightarrow \quad p + cp + c^2 p = 1$$

$$(c^2 + c + 1) = \frac{1}{p}, \text{ or } p = \frac{1}{c^2 + c + 1}$$

$$E[X] = \sum_k k\,p_X(k) \qquad k = 0, 1, 2. \text{(given).}$$

$$= 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2\,p_X(2)$$

$$= cp + 2c^2 p = \frac{c + 2c^2}{1 + c + c^2}.$$

Q5. Solution:

$$k = 2, 3, 4, \ldots\ldots, 10 \quad \Rightarrow S_X = \{2, \ldots\ldots, 10\}$$

All cards are equally likely.

$$|S_X| = 9$$

$$p_X(k) = \frac{1}{9} \quad \forall k.$$

$$E[\text{win}] = E[\text{win} \mid X \text{ is even}]\, P[X \text{ is even}] + E[\text{win} \mid X \text{ is odd}]\, P[X \text{ is odd}]$$

$$= (-1)\left(\frac{5}{9}\right) + (+1)\left(\frac{4}{9}\right)$$

$$= -\frac{5}{9} + \frac{4}{9} = -\frac{1}{9}.$$

Q6. No of bolts = 5

$p_X(1) = \frac{1}{5}$ (bolt fits to nut in first attempt)

$p_X(2) = \frac{4}{5} \cdot \frac{1}{4}$ (incorrect bolt could one of 4 in the first attempt, so, 4/5. The correct bolt could be one of the four remaining ones)

$= \frac{1}{5}$

$p_X(3) = \frac{\cancel{4}}{5} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot \frac{1}{\cancel{3}} = \frac{1}{5}$

$p_X(4) = \frac{\cancel{4}}{5} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{1}{\cancel{2}} = \frac{1}{5}$

$p_X(5) = \frac{\cancel{4}}{5} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{1}{\cancel{2}} \cdot 1 = \frac{1}{5}$

$E[X] = 1 \cdot \frac{1}{5} + 2 \cdot \frac{1}{5} + 3 \cdot \frac{1}{5} + 4 \cdot \frac{1}{5} + 5 \cdot \frac{1}{5}$

$= [1+2+3+4+5]/5$

$= \frac{15}{5} = 3.$

Ans.

Q7. Solution.

(a) $E(F) = \left(60 \times \frac{1}{10} + \frac{61 * 2}{10} + \frac{62 * 4}{10} + \frac{63 * 2}{10} + \frac{64 * 1}{10}\right)$

$= \frac{1}{10}(60 + 122 + 248 + 126 + 64)$

$= \frac{620}{10} = 62.$ (an obvious answer due to symmetrical distribution).

$\mathrm{Var}(F) = E(F^2) - [E(F)]^2$

$E(F^2) = \left(60^2 \times \frac{1}{10} + \frac{61^2 * 2}{10} + \frac{62^2 * 4}{10} + \frac{63^2 * 2}{10} + \frac{64^2 * 1}{10}\right)$

$= \frac{38452}{10} = 3845.2.$

$\mathrm{Var}(F) = 3845.2 - 62^2 = 1.2.$

(b) $\mathrm{Var}(T) = \mathrm{Var}(2F - 62) = 4\,\mathrm{Var}(F) = 4 * 1.2 = 4.8$

$E(T) = E(2F - 62) = 2E(F) - 62 = 2 * 62 - 62 = 62.$