Habib University – City Campus
Course: CS 212 Nature of Computation
Instructors: Behroz Mirza (L1, L4), Abdullah Zafar (L2), Waqar Saleem (L3)
Term : Fall 2023
Examination: Final Term | Date: 14 Dec, 2023
Total Marks: 80 | Duration: 150 minutes

Instructions:

1. You may consult any offline resources kept with you.
2. You are allowed to keep only the following items with you on your table: writing utensils, stationery and container, your HU ID card, your reference resources, and a drinks container. Please deposit other items, e.g., bags and devices, at the front of the exam hall.
3. Make sure that your HU ID card is clearly visible on your table.
4. Please ensure sufficient writing utensils in working condition. Do not bother others during the exam.
5. Use of unfair means, including but not limited to collaboration, attempts at collaboration, and copying, violates academic honesty and will be met with disciplinary action.
6. Please submit this problem sheet with your answer book when you are done.
7. This exam consists of 16 problems for 80 points, printed on 5 sides.
8. The problem scores are set to ensure uniformity rather than to indicate their comparative difficulty.
9. Attempt all problems.

Sample Solution

viel Spaß und viel Glück!

Student ID: ____________________

Student Name: ____________________

True/False Problems

For each of the following problems, write "True" or "False" in the space provided alongside it below.

1. [2 points] In the definition of a Turing machine, the input alphabet and the tape alphabet need not be different. **False**

   Solution: They will differ in at least one symbol: the tape's blank symbol.

2. [2 points] If A is a decidable language and $B \subseteq A$, then B is necessarily decidable. **False**

   Solution: Here is an easy counterexample. Every language is a subset of Σ^* which is decidable. But there are languages which are undecidable.

3. [2 points] If A is a decidable language and $B \leq_m A$, then B is necessarily decidable. **True**

   Solution: Stated in Theorem 5.22 in the book.

4. [2 points] If A is a decidable language and $B \cup A = \emptyset$, then B is necessarily decidable. **True**

   Solution: In this case, $A = B = \emptyset$. Given that A is decidable, then so is B.

5. [2 points] If a language is unrecognizable, then it is also undecidable. **True**

   Solution: Decidable languages are a subset of recognizable languages.

6. [2 points] The complement of a recognizable language in unrecognizable. **False**

   Solution: Theorem 4.22 establishes that there are recognizable languages whose complements are also recognizable.

7. [2 points] A Turing machine can compute anything that a desktop PC can, although it might take more time. **True**

   Solution: Follows from the definition of computation as per the Church-Turing thesis.

8. [2 points] Given that $B \subseteq A$ and B is closed under some operation, f, then so is A. **False**

Sample Solution

Solution: We have a counter-example in terms of classes of regular, R, and context-free, CF, languages. We know that $R \subseteq F$ and that R is closed under intersection, but CF is not.

9. $\boxed{\text{2 points}}$ $TIME(n^5) \subseteq$ P. **True**

Solution: Follows from the definition of P.

10. $\boxed{\text{2 points}}$ The Halting Problem is in NP. **False**

Solution: The class NP only contains decidable problems, which the Halting Problem is not.

Longer Problems

Answer each of the following problems on the answer sheet. You may attempt the problems in any order provided your solution is legible and clearly indicates the problem number.

11. $\boxed{\text{10 points}}$ Prove that the language $A = \{n \in \mathbb{Z}^+ \mid n \text{ is prime and the sum of two squares}\}$ is decidable. You may do so by writing a high-level description of a Turing machine that decides A.

Solution: We show that A is decidable by building a decider for it.

We approach this step-wise. One step is to decide primality, the other is to decide if a given number is the sum of two squares. We then combine these to decide A.

Primality We showed in a WC that $PRIMES$ is decidable. Let M be a decider for P.

Sum of 2 squares We build a decider, N, to decide if a given number, n, is the sum of 2 squares. On input n:

- for i, j in $\{0, 1, \ldots, n\}^2$:
  - if $i^2 + j^2$ equals n; *accept*.
- *reject*.

Deciding A On input n:

- If P accepts n and N accepts n. *accept*; else *reject*.

12. $\boxed{\text{10 points}}$ Let $A = \{\langle M, k\rangle \mid M \text{ is a Turing machine that halts on all strings, } w, \text{ where } |w| \leq k\}$. Show that A is undecidable.

Solution: We show that A is undecidable by showing that $HALT_{TM}$ reduces to it. Recall that $HALT_{TM}$ is undecidable and defined as, $HALT_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM that halts on input } w\}$. We will show that a decider for A can be used to build a decider for $HALT_{TM}$.

We will use a similar idea as in the proof of Theorem 5.2 which states that E_{TM} is undecidable. Note that the idea in our proof is similar to the idea in that proof. There is no other relevance of E_{TM} in our proof. We will modify M to obtain M' that accepts (halts) all strings of length n or less *except* string w for which it works as usual. We will then check if $\langle M', n\rangle$ belongs to A.

Proof. Let us assume a decider, R, for A.
Then R can be used to construct S to decide $HALT_{TM}$ as follows.
On input $\langle M, w\rangle$:

Sample Solution

- Determine the length of w, call it n.
- Construct M' as follows: On input x,
  - If $|x| \leq n$ and $v \neq w$, accept.
  - If $v = w$, run M on input x and accept if M does.
- Run R on input $\langle M' \rangle$.
- If R accepts, *accept*; if R rejects, *reject*.

□

13. $\boxed{\text{10 points}}$ Show that A is decidable iff $A \leq_m 0^*1^*$.

Solution: This is similar to a WC.

The proof of the biconditional breaks down to two cases, one for each implication.

Proof. Case 1: A is decidable $\implies A \leq_m 0^*1^*$
We have to prove the existence of a reduction, f, from A to 0^*1^*.
Let S be the decider of A.
We construct f as follows:
On input u,

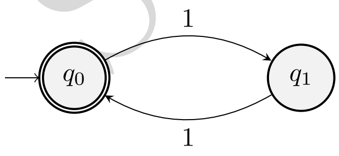
- Simulate S on u.
- If S accepts, return 01; if S rejects, return 10.

Case 2: $A \leq_m 0^*1^* \implies A$ is decidable
We have to show that 0^*1^* is decidable.
0^*1^* can be decided by checking that no 0 occurs after a 1 in an input string. □

14. $\boxed{\text{10 points}}$ Let $A = \{\langle M \rangle \mid M$ is a DFA that does not accept any string containing an even number of 1s$\}$. Show that A is decidable.

Solution: We show that A is decidable by using the results that E_{DFA} is decidable (Theorem 4.4) and that given two DFAs, another DFA can be constructed which recognizes the intersection of their languages.

Proof. Let $D1$ be the DFA equivalent to the following NFA:



$D1$ accepts strings with an even number of 1s.
Let E be the decider for E_{DFA}.
We build N to decide A as follows:
On input $\langle M \rangle$,

- Construct the DFA $D2$ such that $L(D2) = L(D1) \cap L(M)$.
- Output the result of running E on $\langle D2 \rangle$.

□

Sample Solution

15. $\boxed{\text{10 points}}$ Given a polynomial time multi-tape decider for a language, A, argue that A is in P.

Solution:

Proof. Let us denote the polynomial time multi-tape decider as M.
The running time of M is then $O(n^k)$ for some $k \in \mathbb{Z}^+$.
By Theorem 7.8, M has an equivalent $O(n^{2k})$ time single-tape Turing machine.
$\therefore A \in \text{P}$. □

16. $\boxed{\text{10 points}}$ Prove that the language, $CYCLE = \{\langle G \rangle \mid G$ is an undirected graph that contains a cycle$\}$, is in NP.

Solution:

Proof. We provide a polynomial time (deterministic) verifier, V, for $CYCLE$. The certificate for a given G is a cycle, c, in G.

Then, V: on input $\langle G, c \rangle$:

- If c is a cycle containing only vertices from G, *accept*.

Let n be the number of vertices in G. Verifying that c only contains vertices from G can be done in polynomial time in n. Checking whether c is a cycle can also be done in polynomial time in n, e.g. using DFS. □

Sample Solution