1 LINEAR ALGEBRA

LECTURE 22

MATH 205

RESULT:

P. 338, 7TH ED.

IF $\boxed{A}$ IS AN ORTHOGONAL MATRIX THEN

$$\boxed{\det(A) = \pm 1}$$

P. 320 8TH ED.

PROOF:

$$\because A^{-1} = A^{T}$$

$$\Rightarrow \det(A^{-1}) = \det(A^{T})$$

$$\Rightarrow \frac{1}{\det(A)} = \det(A^{T}) = \det(A)$$

$$\Rightarrow \frac{1}{\det(A)} = \det(A)$$

$$\Rightarrow 1 = [\det(A)]^{2}$$

$$\Rightarrow \boxed{\det(A) = \pm 1}$$

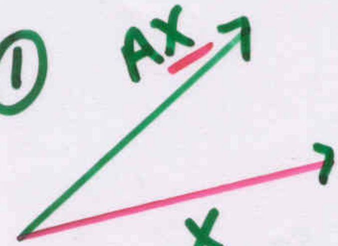
NOTE: IF $\boxed{A^{-1} = A^{T}}$ AND $\det(A) = 1$ THEN $\boxed{A}$ IS CALLED PROPER ORTHOGONAL MATRIX.

EIGENVALUES / EIGENVECTORS: (P. 338 8TH ED.) (P. 355 7TH ED.)

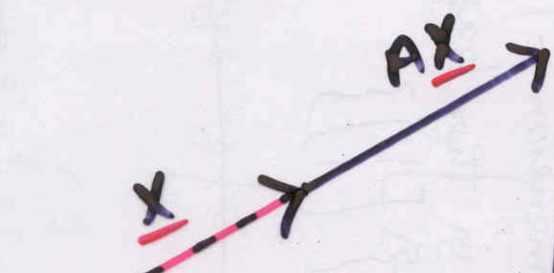
(P. 338 8TH ED.) (P. 355 7TH ED.)

NOTE: IF $\boxed{A}$ IS AN $\boxed{n \times n}$ MATRIX AND $\boxed{X \neq 0}$ IS A COLUMN VECTOR OF ORDER $\boxed{n \times 1}$, THEN $\boxed{AX = b}$ IS ALSO A COLUMN VECTOR OF ORDER $\boxed{n \times 1}$. BUT IS THERE ANY GEOMETRIC RELATIONSHIP BETWEEN X AND AX?

SEE THE FOLLOWING FIGURES:

① 

$\boxed{\text{NO}}$ GENERAL GEOMETRIC RELATIONSHIP BETWEEN X AND AX.

② 

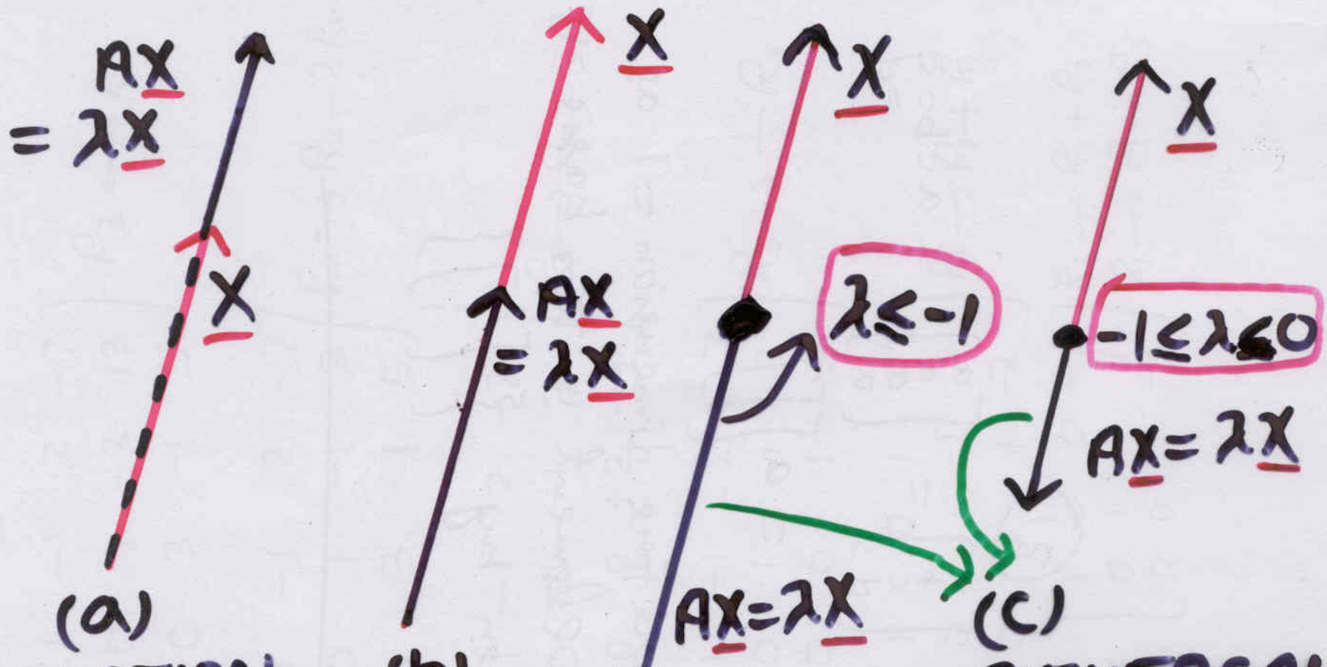
X AND AX ARE SCALAR MULTIPLES OF EACHOTHER.

DEFINITION: IF $\boxed{A}$ IS AN $n \times n$ MATRIX, THEN A $\boxed{\text{NONZERO}}$ VECTOR $\boxed{X}$ IS CALLED AN EIGENVECTOR OF $\boxed{A}$ IF $\boxed{AX}$ IS A SCALAR MULTIPLE OF X.

3) THAT IS, $\boxed{A\underline{X} = \lambda \underline{X}}$ FOR SOME SCALAR $\boxed{\lambda}$. THE SCALAR $\boxed{\lambda}$ IS CALLED AN EIGENVALUE OF $\boxed{A}$, AND $\boxed{\underline{X}}$ IS SAID TO BE AN EIGENVECTOR OF $\boxed{A}$ CORRESPONDING TO $\boxed{\lambda}$.

GEOMETRIC INTERPRETATION

IF $A\underline{X} = \lambda \underline{X}$, MULTIPLICATION BY $\boxed{A}$ DILATES $\underline{X}$, CONTRACTS $\underline{X}$, OR REVERSES THE DIRECTION OF $\underline{X}$, DEPENDING ON THE VALUE OF $\boxed{\lambda}$ (AS SHOWN BELOW)



(a) DILATION $(\lambda > 1)$

(b) CONTRACTION $(0 < \lambda < 1)$

(c) REVERSAL OF DIRECTION $(\lambda < 0)$

TWO PROBLEMS

HOW TO FIND EIGENVALUES?

HOW TO FIND EIGENVECTORS?

METHOD TO FIND EIGENVALUES :-

$\because A\underline{X} = \lambda \underline{X} \Rightarrow A\underline{X} = \lambda I \underline{X}$ — (1),

WHERE $\boxed{I}$ IS THE IDENTITY MATRIX

(1) $\Rightarrow (A - \lambda I)\underline{X} = \underline{0}$ — (2), $\underline{X} \neq \underline{0}$

SINCE EIGENVECTOR IS $\boxed{A}$ NONZERO VECTOR THEREFORE FOR $\boxed{\lambda}$ TO BE AN EIGENVALUE OF $\boxed{A}$, NONZERO (NONTRIVIAL) SOLUTIONS OF (2) EXIST IF

$$\boxed{\det(A - \lambda I) = 0} \longrightarrow (*)$$

(*) IS CALLED THE CHARACTERISTIC EQUATION OF $\boxed{A}$. AFTER SOLVING (*), WE GET THE EIGENVALUES (λ's) OF $\boxed{A}$.

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EXAMPLE: FIND THE $\boxed{\text{EIGENVALUES}}$ OF

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

SOLUTION:

CONSIDER $\boxed{A - \lambda I}$

$$= \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix}$$

NOTE THIS STEP

$\boxed{A - \lambda I}$ IS OBTAINED BY SUBTRACTING $\boxed{\lambda}$ FROM THE DIAGONAL ENTRIES OF $\boxed{A}$.

THE $\boxed{\text{CHARACTERISTIC}}$ EQUATION OF $\boxed{A}$ IS

$$\boxed{\det(A - \lambda I) = 0}, \quad A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0}$$

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$$\Rightarrow \boxed{(4-\lambda)(1-\lambda)^2 + 2(1-\lambda) = 0} \rightarrow ①$$

NOTE: THIS IS AN EQUATION OF $\boxed{\text{DEGREE 3}}$ = ORDER OF $\boxed{A}$.
THEREFORE $\boxed{\text{NO.}}$ OF EIGENVALUES OF $\boxed{A}$ ARE = ORDER OF $\boxed{A}$.

$$① \Rightarrow \boxed{(4-\lambda)(1-\lambda)^2 + 2(1-\lambda) = 0}$$

$$\Rightarrow (1-\lambda)\left[(4-\lambda)(1-\lambda) + 2\right] = 0$$

NOTE THIS STEP

$$\Rightarrow 1-\lambda = 0 \Rightarrow \boxed{\lambda = 1}$$

$$(4-\lambda)(1-\lambda) + 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda-2)(\lambda-3) = 0$$

$$\Rightarrow \boxed{\lambda = 2, \lambda = 3}$$

OR $\boxed{\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3}$

ARE THE EIGENVALUES OF $\boxed{A}$.

NOTE: FOR AN $\boxed{n \times n}$ MATRIX $\boxed{A}$, THE CHARACTERISTIC EQUATION $\boxed{\det(A - \lambda I) = 0}$ IS OF DEGREE $\boxed{n}$ AND NUMBER OF

EIGENVALUES OF $A = n$ (A FEW MAY BE REPEATED).

HOW TO FIND EIGENVECTORS OF A?

EXAMPLE: FIND THE EIGENVECTORS OF $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

SOLUTION: IF $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ IS AN EIGENVECTOR OF A CORRESPONDING TO λ THEN WE HAVE

$$A\underline{X} = \lambda \underline{X} = \lambda I \underline{X}$$

$$\Rightarrow \boxed{(A - \lambda I)\underline{X} = \underline{0}} \longrightarrow (*)$$

THEREFORE IN ORDER TO OBTAIN THE EIGENVECTORS WE HAVE TO SOLVE $(*)$

$$\therefore (*) \Rightarrow \begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

NOW WE HAVE TO OBTAIN THE EIGENVECTORS CORRESPONDING TO $\lambda = 1, 2,$ AND 3. (ALREADY OBTAINED)

$$\Rightarrow \begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

STEP (1) :- EIGENVECTOR CORRESPONDING TO $\boxed{\lambda = 1}$

PUTTING $\boxed{\lambda = 1}$ IN (1)

$$\Rightarrow \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_1 + x_3 = 0, \ -2x_1 = 0$$

$$\Rightarrow x_1 = x_3 = 0, \ \boxed{x_2 = t}, \ \because \underline{X} \neq \underline{0}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ t \neq 0$$

$\therefore$ FOR $\lambda = 1$, EIGENVECTOR IS $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ OR ANY OF ITS NONZERO MULTIPLE.

$\boxed{\text{CHECK:}}$

$$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A \underline{X} = \lambda \underline{X}$$

STEP (2) FOR $\boxed{\lambda = 2}$

$$\begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 + x_3 = 0 \Rightarrow \boxed{x_3 = -2x_1}$$
$$-2x_1 - x_2 = 0 \Rightarrow \boxed{x_2 = -2x_1}$$
$$-2x_1 - x_3 = 0$$

SAME

FOR $x_1 = t$, $x_2 = -2t$, $x_3 = -2t$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ -2t \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, \quad t \neq 0$$

$\therefore \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$ OR ANY OF ITS NONZERO MULTIPLES ARE EIGENVECTORS OF $\boxed{A}$ CORRESPONDING TO $\boxed{\lambda = 2}$

STEP (3) FINALLY FOR $\lambda = 3$, EIGENVECTOR $= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ OR ANY OF ITS NONZERO MULTIPLES (CHECK)

NOTE:

(1) $$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

(2) $$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$A\underline{X} = \lambda \underline{X}$$