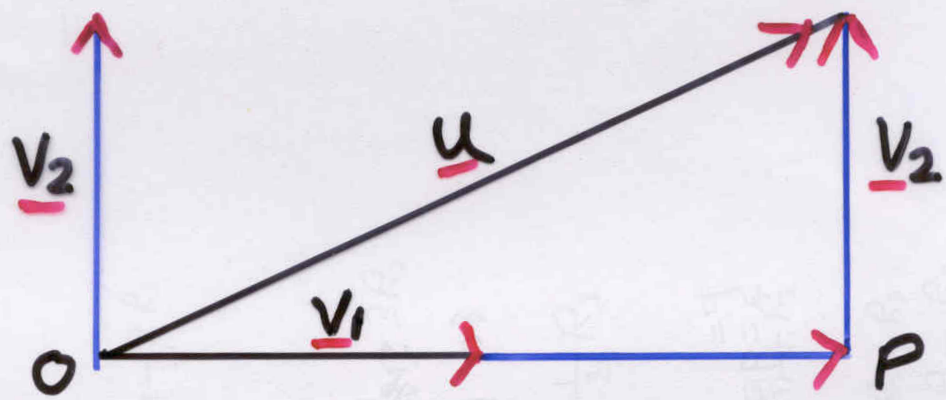
1 LINEAR ALGEBRA | LECTURE 21 | MATH 205

PREVIOUS RESULTS:

(1) IF $\boxed{\underline{v}_2}$ IS THE VECTOR PROJECTION OF $\boxed{\underline{u}}$ ORTHOGONAL TO $\boxed{\underline{v}_1}$ THEN $\boxed{\underline{v}_2}$ IS GIVEN BY

$$\underline{v}_2 = \underline{u} - \underbrace{\frac{\langle \underline{u}, \underline{v}_1 \rangle \underline{v}_1}{\|\underline{v}_1\|^2}}_{\overrightarrow{OP}}$$

, AS SHOWN BELOW:



(2) SIMILARLY THE VECTOR PROJECTION $\boxed{\underline{v}_3}$ OF $\boxed{\underline{v}}$, WHICH IS ORTHOGONAL TO BOTH $\boxed{\underline{v}_1}$ AND $\boxed{\underline{v}_2}$ IS GIVEN BY

$$\boxed{\underline{v}_3 = \underline{v} - \frac{\langle \underline{v}, \underline{v}_1 \rangle \underline{v}_1}{\|\underline{v}_1\|^2} - \frac{\langle \underline{v}, \underline{v}_2 \rangle \underline{v}_2}{\|\underline{v}_2\|^2}}$$

$\rightarrow$ (*)

SUMMARY:

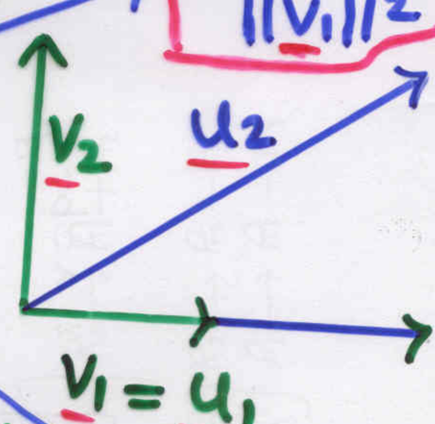
$\boxed{2}$ IF $\boxed{V}$ IS AN INNER PRODUCT SPACE AND $\{u_1, u_2, \ldots, u_n\}$ IS A $\boxed{\text{BASIS}}$ FOR $\boxed{V}$ THEN WE CAN FIND THE ORTHOGONAL BASIS BY FOLLOWING THESE STEPS:

(1) LET $\boxed{v_1 = u_1} \rightarrow$ 1

(2) TO FIND v_2 ORTHOGONAL TO v_1 BY COMPUTING THE COMPONENT OF u_2 THAT IS ORTHOGONAL TO v_1.

$$\boxed{v_2 = u_2 - \text{Proj}_{v_1} u_2} \rightarrow 2$$

$$\text{Proj}_{v_1} u_2 = \boxed{\frac{\langle u_2, v_1 \rangle v_1}{\|v_1\|^2}}$$



(3) TO FIND v_3 ORTHOGONAL TO BOTH v_1 AND v_2 BY COMPUTING THE COMPONENT OF u_3 ORTHOGONAL TO THE $\boxed{\text{PLANE}}$ SPANNED BY v_1 AND v_2 AND IS GIVEN BY

$$\boxed{v_3 = u_3 - \frac{\langle u_3, v_1 \rangle v_1}{\|v_1\|^2} - \frac{\langle u_3, v_2 \rangle v_2}{\|v_2\|^2}} \rightarrow 3$$

$\boxed{3}$ REPLACE $\underline{v}$ BY $\underline{u}_3$ IN (*) ON SLIDE ONE IN ORDER TO GET (3). SO WE OBTAINED $\underline{v}_1, \underline{v}_2, \underline{v}_3, \ldots\ldots$ SO ON UNTIL WE GET $\underline{v}_n$.

THE PRECEDING STEP-BY-STEP CONSTRUCTION FOR CONVERTING AN ARBITRARY BASIS INTO AN ORTHOGONAL BASIS IS CALLED THE GRAM-SCHMIDT PROCESS.

→ P. 304 (8TH ED.) OR P. 318 (7TH ED.)

EXAMPLE:

LET THE VECTOR SPACE $\boxed{P_2}$ HAVE THE INNER PRODUCT

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

APPLY THE GRAM-SCHMIDT PROCESS TO TRANSFORM THE STANDARD BASIS $S = \{1, x, x^2\}$ INTO AN ORTHONORMAL BASIS.

4)

SOLUTION: (HINTS)

$$\langle p, q \rangle = \int_{-1}^{1} p(x)\,q(x)\,dx$$

$S = \{1, x, x^2\}$, HERE

$\underline{u}_1 = 1$, $\underline{u}_2 = x$, $\underline{u}_3 = x^2$

① $\underline{u}_1 = \underline{v}_1 = 1$

② $\|\underline{v}_1\| = \|1\| = \langle \underline{v}_1, \underline{v}_1 \rangle^{\frac{1}{2}}$

$$= \langle 1, 1 \rangle^{1/2} = \left(\int_{-1}^{1} 1\,dx \right)^{\frac{1}{2}} = \sqrt{2}$$

③ $$\underline{v}_2 = \underline{u}_2 - \frac{\langle \underline{u}_2, \underline{v}_1 \rangle \underline{v}_1}{\|\underline{v}_1\|^2}$$

$$= x - \frac{\cancelto{0}{\langle x, 1 \rangle}}{\|1\|^2} = x$$

④ $$\|\underline{v}_2\| = \|x\| = \left(\int_{-1}^{1} x^2\,dx \right)^{\frac{1}{2}}$$

$$= \sqrt{\frac{2}{3}}$$

⑤ $$\underline{v}_3 = \underline{u}_3 - \frac{\langle \underline{u}_3, \underline{v}_1 \rangle}{\|\underline{v}_1\|^2}\underline{v}_1 - \frac{\langle \underline{u}_3, \underline{v}_2 \rangle \underline{v}_2}{\|\underline{v}_2\|^2}$$

BUT $\langle \underline{u}_3, \underline{v}_1 \rangle = \int_{-1}^{1} x^2 dx$

$$= \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3}[1-(-1)] = \frac{2}{3}$$

AND $\langle \underline{u}_3, \underline{v}_2 \rangle = \langle x^2, x \rangle$

$$= \int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0$$

$$\therefore \underline{v}_3 = \underline{u}_3 - \frac{\langle \underline{u}_3, \underline{v}_1 \rangle}{\|\underline{v}_1\|^2}\underline{v}_1 - \frac{\langle \underline{u}_3, \underline{v}_2 \rangle}{\|\underline{v}_2\|^2}\underline{v}_2$$

0

$$= x^2 - \frac{2}{3}\cdot\frac{1}{2} = x^2 - \frac{1}{3} = \underline{v}_3$$

(6) $\|\underline{v}_3\| = \langle \underline{v}_3, \underline{v}_3 \rangle^{\frac{1}{2}}$

$$= \left[\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx\right]^{1/2} = \sqrt{\frac{8}{45}} \text{ (CHECK)}$$

(7) REQUIRED ORTHONORMAL BASIS

$$\text{IS} = \left\{ \frac{\underline{v}_1}{\|\underline{v}_1\|}, \frac{\underline{v}_2}{\|\underline{v}_2\|}, \frac{\underline{v}_3}{\|\underline{v}_3\|} \right\}$$

$$= \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x, \left(x^2 - \frac{1}{3}\right)\sqrt{\frac{45}{8}} \right\}$$

$$= \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x, \frac{(3x^2-1)}{\cancel{3}} \sqrt{\frac{\cancel{9}\times 5}{8}} \right\}$$

$$= \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x, (3x^2-1)\frac{\sqrt{5}}{2\sqrt{2}} \right\}$$

TRY THE FOLLOWING:

IF $S = \{ \underline{v}_1, \underline{v}_2, \ldots\ldots, \underline{v}_n \}$ IS AN ORTHOGONAL SET OF NONZERO VECTORS IN AN INNER PRODUCT SPACE, THEN $\boxed{S}$ IS LINEARLY INDEPENDENT.

HINT: ASSUME

$$k_1\underline{v}_1 + k_2\underline{v}_2 + \cdots + k_n\underline{v}_n = \underline{0}$$

AND PROVE

$$k_1 = k_2 = \cdots = k_n = 0$$

ALSO THIS IS TH. 6.3.3. (P. 301 8TH ED.) OR (P. 315 7TH ED.)

7

PROOF: LET $k_1\underline{v_1}+k_2\underline{v_2}+\cdots+k_n\underline{v_n}=\underline{0}$
TAKING THE INNER PRODUCT WITH $\underline{v_i}$ ON BOTH SIDES, $(1\le i\le n)$

$$<k_1\underline{v_1}+k_2\underline{v_2}+\cdots+k_n\underline{v_n},\underline{v_i}>=0$$

$\because <\underline{0},\underline{v_i}>=<\underline{0}+\underline{0},\underline{v_i}>=<\underline{0},\underline{v_i}>+<\underline{0},\underline{v_i}> \Rightarrow <\underline{0},\underline{v_i}>=0$

$$k_1<\underline{v_1},\underline{v_i}>+k_2<\underline{v_2},\underline{v_i}>+\cdots+k_i<\underline{v_i},\underline{v_i}>+\cdots+k_n<\underline{v_n},\underline{v_i}>=0 \quad (1)$$

BUT $S=\{\underline{v_1},\underline{v_2},\ldots,\underline{v_n}\}$ IS AN ORTHOGONAL SET THEREFORE

$<\underline{v_i},\underline{v_j}>=0$ WHEN $i\neq j$

SO THAT THE LAST EQUATION (1) REDUCES TO $k_i<\underline{v_i},\underline{v_i}>=0$

BUT $\underline{v_i}\neq\underline{0}$ (GIVEN)

THEREFORE $<\underline{v_i},\underline{v_i}>=\|\underline{v_i}\|^2>0$

SO THAT $\boxed{k_i=0}$. SINCE THE SUBSCRIPT $\boxed{i}$ IS ARBITRARY, WE HAVE $\boxed{k_1=k_2=\cdots=k_n=0}$;

THUS, $\boxed{S}$ IS LINEARLY INDEPENDENT.

ASSIGNMENT 5(b)

(1) Q. 17 (P. 244 8TH ED) / P. 256 7TH ED.

(2) PROVE CAUCHY-SCHWARZ INEQUALITY IN CASE OF EUCLIDEAN INNER PRODUCT.

<u>HINT</u>: SEE P. 300 7TH ED. OR P. 287 8TH ED.

(3) ALSO PROVE (2) IN GENERAL.

(4) IF $\underline{u}$ AND $\underline{v}$ ARE TWO VECTORS IN AN INNER PRODUCT SPACE THEN

(i) $\|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$ } TRIANGLE INEQUALITY

(ii) USING THE FACT THAT $\underline{u}$ AND $\underline{v}$ ARE ORTHOGONAL PROVE THAT $\|\underline{u}+\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$ WHICH IS ALSO CALLED A GENERALIZED THEOREM OF PYTHAGORAS.

(5) Q. NO. 6(a), 11, P. 295 (8TH ED.)
OR P. 310 (7TH ED.)

(6) LET $f(x)$ AND $g(x)$ BE CONTINUOUS FUNCTIONS ON $[0,1]$
PROVE

$$\left[\int_0^1 f(x)g(x)\,dx\right]^2 \leq \left[\int_0^1 f^2(x)\,dx\right]\left[\int_0^1 g^2(x)\,dx\right]$$

(7) GRAM-SCHMIDT PROCESS AND ORTHONORMALITY
17(a), 18, 25, 26, 27, 29
P. 310 (8th ED.) / P. 326-327 (7TH ED.)

(8) Q. NO. 32, P. 297 (8th ED.) OR
Q. NO. 30, P. 311 (7th. ED.)

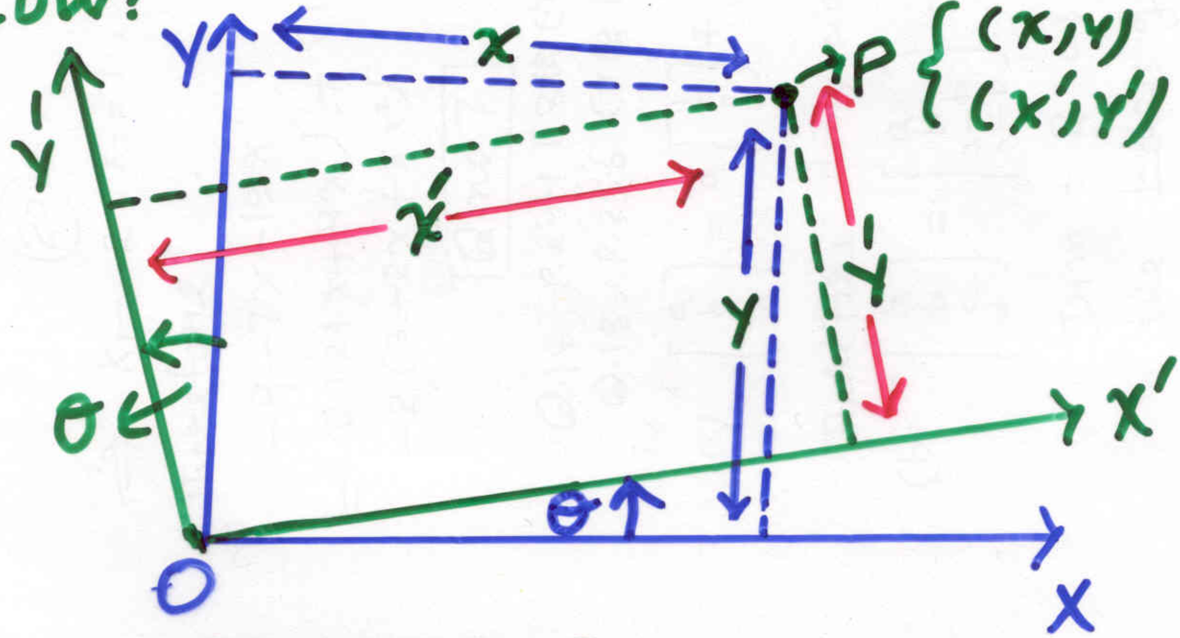
<u>HINT:</u> $\cos(\alpha+B) = \cos\alpha\cos B - \sin\alpha\sin B$
$\cos(\alpha-B) = \cos\alpha\cos B + \sin\alpha\sin B$

ADDING $\cos(\alpha+B) + \cos(\alpha-B)$
$= 2\cos\alpha\cos B$

ROTATION OF AXES:

10

CONSIDER A ROTATION OF THE AXES ABOUT THE ORIGIN AS SHOWN BELOW:



X COORDINATE GIVES DISTANCE FROM Y AXIS, Y COORDINATE GIVES DISTANCE FROM X AXIS. IF THE AXES ARE ROTATED THROUGH AN ANGLE θ, THEN EVERY POINT OF THE PLANE HAS $\boxed{\text{TWO}}$ REPRESENTATIONS: (x, y) IN THE ORIGINAL COORDINATE SYSTEM AND (x', y') IN THE NEW COORDINATE SYSTEM.

PROBLEM: WHAT IS THE RELATIONSHIP BETWEEN THE $\boxed{x}$ AND $\boxed{y}$ OF ONE COORDINATE SYSTEM

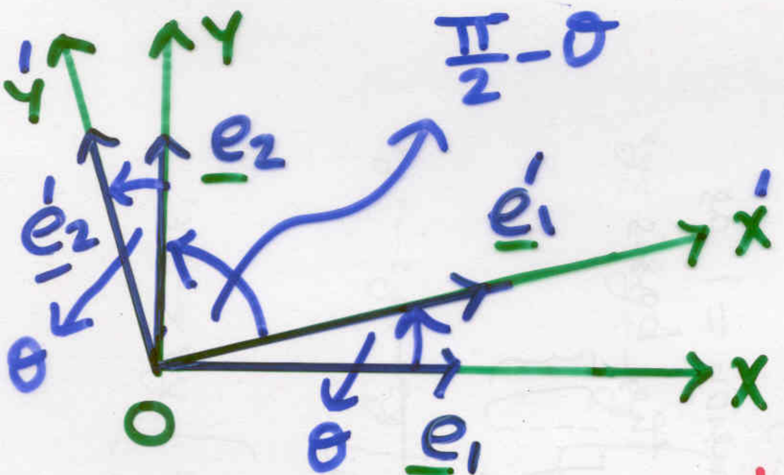
AND THE $\boxed{x'}$ AND $\boxed{y'}$ OF THE OTHER?

11 CONSIDER THE VECTOR $\overrightarrow{OP}$ WHICH IS GIVEN BY $\boxed{\overrightarrow{OP} = (x, y) = x\underline{e}_1 + y\underline{e}_2}$ IN THE ORIGINAL COORDINATE SYSTEM AND ALSO

$$\boxed{\overrightarrow{OP} = (x', y') = x'\underline{e}_1' + y'\underline{e}_2'}$$

IN THE NEW COORDINATE SYSTEM ($\underline{e}_1'$ AND $\underline{e}_2'$ ARE ALSO UNIT VECTORS).

CONSIDER THE FOLLOWING FIGURE:



$\therefore \{\underline{e}_1, \underline{e}_2\}$ IS A BASIS

$\therefore$

$$\boxed{\underline{e}_1' = k_1\underline{e}_1 + k_2\underline{e}_2}$$

$$\Rightarrow \underline{e}_1' \cdot \underline{e}_1 = k_1 \underbrace{\underline{e}_1 \cdot \underline{e}_1}_{1} + k_2 \underbrace{\underline{e}_2 \cdot \underline{e}_1}_{0}$$

$$\Rightarrow \boxed{k_1 = \underline{e}_1' \cdot \underline{e}_1 = \cos\theta}$$

SIMILARLY $\boxed{k_2 = \sin\theta}$ (CHECK).

$$\therefore \boxed{\underline{e}_1' = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2}$$

$$= (\cos\theta, \sin\theta)$$

$\boxed{12}$

SIMILARLY

$$\boxed{\underline{e}_2' = -\sin\theta\,\underline{e}_1 + \cos\theta\,\underline{e}_2} \quad \text{(CHECK)}$$

$$\to = (-\sin\theta, \cos\theta)$$

THEREFORE

$$\vec{OP} = x'\underline{e}_1' + y'\underline{e}_2'$$

$$= x'(\cos\theta\,\underline{e}_1 + \sin\theta\,\underline{e}_2)$$

$$+ y'(-\sin\theta\,\underline{e}_1 + \cos\theta\,\underline{e}_2)$$

$$= (x'\cos\theta - y'\sin\theta)\underline{e}_1 +$$

$$(x'\sin\theta + y'\cos\theta)\underline{e}_2$$

$$= x\underline{e}_1 + y\underline{e}_2 = \vec{OP} \quad —①$$

① SHOWS THAT

$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$

$$\Rightarrow \boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}}$$

THE MATRIX $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = R$ (SAY)

WHICH GIVES ROTATION THROUGH AN ANGLE $\boxed{\theta}$ (COUNTER-CLOCKWISE) IS CALLED A ROTATION MATRIX.

ITS COLUMN VECTORS ARE NEW BASIS VECTORS i.e. $\left[[\underline{e}_1'] \; [\underline{e}_2'] \right]$ AND ARE ORTHONORMAL WITH THE EUCLIDEAN INNER PRODUCT. ALSO ITS $\boxed{\text{ROW}}$ VECTORS ARE ORTHONORMAL WITH THE EUCLIDEAN INNER PRODUCT.

CHECK: $(\cos\theta, -\sin\theta) \cdot (\sin\theta, \cos\theta)$
$= 0$

NOTES: ① $RR^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ INTO

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= R^T R \quad \Rightarrow \quad \boxed{R^T = R^{-1}}$$

② $\det(R) = \cos^2\theta + \sin^2\theta = 1$

③ WHEN THERE IS NO ROTATION THEN $R = I = \left[[\underline{e}_1] [\underline{e}_2] \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ FOR $\theta = 0$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

DEFINITION:

A SQUARE MATRIX $\boxed{A}$ WITH THE PROPERTY $\boxed{A^{-1} = A^T}$ IS SAID TO BE AN ORTHOGONAL MATRIX.

TRY THE FOLLOWING:

IF $A^T = A^{-1}$ THEN WHAT ARE THE POSSIBLE VALUES OF $\det(A)$?

ANSWER: ± 1