MATH 205

LECTURE 8

LINEAR ALGEBRA

PROPERTIES OF DETERMINANTS (OF MATRICES OF ORDER n) REVISITED.

① $\boxed{\det(AB) = \det(A)\det(B)}$

$\boxed{A, B}$ ARE SQUARE MATRICES OF THE SAME SIZE.

② $\boxed{\det(I) = 1}$, $\boxed{I} \to$ IDENTITY MATRIX

③ IF $\boxed{A^{-1}}$ EXISTS THEN $\boxed{\det(A) \neq 0}$

$$\Rightarrow \boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

④ IF TWO ROWS/COLUMNS OF A MATRIX A ARE IDENTICAL THEN $\boxed{\det(A) = 0}$.

⑤ ADDING ROWS (OR COLUMNS) TOGETHER MAKES NO DIFFERENCE TO THE DETERMINANT.

⑥
$$\begin{vmatrix} ka_{11} & a_{12} & \cdots & a_{1n} \\ ka_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ ka_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

ETC. WHERE $\boxed{k}$ IS ANY SCALAR.

(7) IF $\boxed{A}$ IS A SQUARE MATRIX WITH TWO PROPORTIONAL ROWS OR TWO PROPORTIONAL COLUMNS, THEN $\boxed{\det(A)=0}$.

(8) IF $\boxed{A}$ IS A SQUARE MATRIX SUCH THAT $\boxed{A}$ HAS A ROW OF $\boxed{\text{ZEROS}}$ OR A COLUMN OF ZEROS THEN $\boxed{\det(A)=0}$.

NOTATION: FOR $\boxed{2\times 2}$ CASE

IF $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, THEN

$$\det(A)=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = |A|$$

MORE PROPERTIES

(1) IF $\boxed{A}$ AND $\boxed{B}$ ARE SQUARE MATRICES OF SAME SIZE THEN $\boxed{\det(A)+\det(B)\neq\det(A+B)}$ (IN GENERAL).

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EXAMPLE: FOR $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$,

$B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $A+B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$

VERIFY $\boxed{\det(A) + \det(B) \neq \det(A+B)}$ (P.93 7th ED.) P.96 8th ED.

HINT: $\det(A) = 1$, $\det(B) = 8$,
$\det(A+B) = 23$, $9 \neq 23$

TRY THE FOLLOWING:

LET $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}$

FIND $\det(A) + \det(B) = ?$

HINT: $\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

$\det(B) = \det \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$= \begin{vmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{vmatrix} = a_{11}b_{22} - a_{12}b_{21}$

ANS: $\boxed{a_{11}(a_{22}+b_{22}) - a_{12}(a_{21}+b_{21})}$

WHICH CAN BE WRITTEN AS

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{vmatrix}, \therefore \text{ WE HAVE}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{vmatrix}$$

(2) RESULT: (ADDITION RULE) P.93 (7th ED.) OR P.96 (8th ED.)

LET $\boxed{A}$, $\boxed{B}$, AND $\boxed{C}$ BE $\boxed{n \times n}$ MATRICES THAT DIFFER ONLY IN A SINGLE ROW (SAY rth), AND ASSUME THAT THE rth ROW OF C CAN BE OBTAINED BY ADDING CORRESPONDING ENTRIES IN THE rth ROWS OF $\boxed{A}$ AND $\boxed{B}$. THEN

$$\boxed{\det(C) = \det(A) + \det(B)}$$

IN THE LAST EXAMPLE

$$C = \begin{bmatrix} a_{11} & a_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

THE SAME RESULT HOLDS FOR COLUMNS.

③ P.90 (8th ED.), P.88 (7th ED.)

IF B IS THE MATRIX THAT RESULTS WHEN TWO ROWS OR TWO COLUMNS OF A ARE INTERCHANGED, THEN

$$\boxed{\det(B) = -\det(A)}$$

CHECK: FOR 2X2 CASE

IF $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$

$$\begin{aligned} \det(A) &= a_{11}a_{22} - a_{12}a_{21} \\ &= -(\underbrace{a_{12}a_{21} - a_{11}a_{22}}) \\ &= -\boxed{\det(B)} \end{aligned}$$

$$\Rightarrow \boxed{\det(A) = -\det(B)}$$

6)

TRY THE FOLLOWING:

IF $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, LET $\det(A) = 7$
$= -7$

FIND $\det \begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix} = \begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = ?$

SOLUTION:

$$\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} \quad \boxed{\text{TAKING TRANSPOSE}}$$

$$\underset{R_2 \leftrightarrow R_3}{=} - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\det(A)$$
$$= -(-7) = 7$$

(4) [redacted] (P.90 8th ED.) OR (P.88 7th ED.)

IF $\boxed{B}$ IS THE MATRIX THAT RESULTS WHEN $\boxed{A}$ MULTIPLE OF $\boxed{\text{ONE}}$ ROW OF $\boxed{A}$ IS ADDED TO ANOTHER ROW OR WHEN A MULTIPLE OF ONE COLUMN IS ADDED TO ANOTHER COL-

7) UMN, THEN $\boxed{\det(A) = \det(B)} \rightarrow (*)$

CHECK: FOR 2X2 CASE

TAKE $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} a_{11} + k a_{12} & a_{12} \\ a_{21} + k a_{22} & a_{22} \end{bmatrix}$

$= \begin{bmatrix} a_{11} + k a_{12} & a_{12} \\ a_{21} + k a_{22} & a_{22} \end{bmatrix}$, k IS ANY SCALAR, $k \neq 0$

$C_1 \rightarrow C_1 + kC_2$

DETAIL:

$$\det(B) = \begin{vmatrix} a_{11} + k a_{12} & a_{12} \\ a_{21} + k a_{22} & a_{22} \end{vmatrix}$$

$$= (a_{11} + k\cancel{a_{12}}) a_{22} - a_{12}(a_{21} + k\cancel{a_{22}})$$

$$= a_{11} a_{22} - a_{12} a_{21}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \det(A)$$

ASSIGNMENT 3(a)

Q.no.1 ✓

{Q.no. 24, 25, 28(a)} P.76 8th ED.

OR

P.76-77 (7th ED.)

Q.no.2

Q.no.19 (P.88, 8th ED.) OR P.86, 7th ED.

Q.no.3

Q.no.13 (P.94, 8th ED.) OR (P.92, 7th ED.)

Q.no.4 ✓

[Q.no. 8, 13, 17(a)] P.102-103 8th ED.
OR P. 100-101 (7th ED.)

Q.no.5 ✓

Q.9, 14, 17 P.(116-117) 8th ED.

OR P.(114-116) 7th ED.

CRAMER'S RULE:

P. 111 (8th ED.)
P. 109 (7th ED.)

WE SHALL DISCUSS ANOTHER METHOD TO FIND THE $\boxed{\text{UNIQUE}}$ SOLUTION OF $\boxed{n}$ EQUATIONS IN $\boxed{n}$ UNKNOWNS, PROVIDED DETERMINANT OF THE COEFFICIENT MATRIX $\neq 0$ IN $A\underline{X}=\underline{B}$, $\underline{B}\neq\underline{0}$, i.e. LINEAR SYSTEM IS NONHOMOGENEOUS.

$\boxed{\text{SIMPLE CASE}}$

SYSTEM OF $\boxed{\text{TWO}}$ EQUATIONS IN $\boxed{\text{TWO}}$ UNKNOWNS:

HERE $A\underline{X}=\underline{B} \implies \underline{B}=\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \text{— } ①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \quad \text{— } ②$$

FROM ①

$$x_2 = \frac{b_1 - a_{11}x_1}{a_{12}} \quad \text{— } (*)$$

USING (*) IN ② GIVES THE FOLLOWING:

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11} a_{22} - a_{21} a_{12}}, \Rightarrow$$

$$\boxed{x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}}$$

, SIMILARLY WE CAN OBTAIN

$$\boxed{x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}}$$

OR JUST

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|},$$ WHERE

$|A|$ IS THE DETERMINANT OF COEFFICIENT MATRIX.

$$|A_1| = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$$ IS OBTAINED BY REPLACING THE ENTRIES IN THE 1ST COLUMN OF $|A|$ BY ENTRIES IN B, ETC. SIMILARLY WE CAN EXTEND THIS METHOD TO THREE EQUATIONS IN THREE UNKNOWNS.

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BUT FIRST WE MUST KNOW THE EXPANSION OR METHOD TO EXPAND A DETERMINANT OF A MATRIX OF ORDER $\boxed{3}$.

EXPANSION BY FIRST ROW:

CONSIDER

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$