Q6) For the given recurrence equation, derive its time complexity, by using the Substitution Method. Make sure you show **<u>at least 3</u>** exact equations before you define the generalized statement.

$$T(n) = \begin{cases} 5T(n-4) + 1 & ,n > 0 \\ 1 & ,n = 0 \end{cases}$$

$$\boxed{T(n) = 5T(n-4) + 1} \quad (1)$$

Get $T(n-4)$: $\quad T(n-4) = 5T(n-8) + 1$

Substitute in (1): $\quad \boxed{T(n) = 25\,T(n-8) + 5 \times 1 + 1} \quad (2)$

Get $T(n-8)$: $\quad T(n-8) = 5T(n-12) + 1$

Substitute in (2): $\quad \boxed{T(n) = 5^3\,T(n-12) + 25 \times 1 + 5 \times 1 + 1} \quad (3)$

k^{th} step: $\quad \boxed{T(n) = 5^k\,T(n-4k) + 5^{k-1} + 5^{k-2} + \cdots + 5^1 + 5^0} \quad (4)$

Base Condition: $\quad T(0) = 1 \therefore n - 4k = 1 \Rightarrow k = \frac{n}{4}$

Substitute in (4):

$$T(n) = 5^{\frac{n}{4}}\,T\left(n - \cancel{4}\left(\frac{n}{\cancel{4}}\right)\right) + \underbrace{5^{\frac{n}{4}-1} + 5^{\frac{n}{4}-2} + \cdots + 5^1 + 5^0}_{\text{Geometric Series: } a=1,\ r=5,\ n=\frac{n}{4}-1}$$

$$= 5^{\frac{n}{4}}\,T(n-n)$$

$$\text{Sum} = \frac{1\left(1 - 5^{\frac{n}{4}-1}\right)}{1-5} = -\frac{1}{4} + \frac{5^{\frac{n}{4}-1}}{4}$$

$$T(n) = 5^{\frac{n}{4}}\,T(0) + \frac{5^{\frac{n}{4}-1}}{4} - \frac{1}{4}$$

$$= 5^{\frac{n}{4}}(1) + \frac{5^{\frac{n}{4}}}{4 \times 5} - \frac{1}{4}$$

$$= 5^{\frac{n}{4}} + \frac{5^{\frac{n}{4}}}{20} - \frac{1}{4}$$

$$= \frac{21}{20} \cdot \underbrace{5^{\frac{n}{4}}}_{\text{Dominant term}} - \frac{1}{4}$$

$$\therefore O\left(5^{\frac{n}{4}}\right)$$