LINEAR ALGEBRA PROOFS

Q) Show that if AB and BA are both defined, then AB and BA are square matrix.

<u>Solution:</u> if AB is defined, a column of A is equal to rows of B; if BA is defined it means a column of B equals rows of A. This means that AB and BA have equal rows & columns and thus are square matrix.

Q) Show that if A is $m \times n$ matrix and $A(BA)$ is defined, then B is an $n \times m$ matrix.

<u>Mathematically</u>

→ let A is $m \times n$ matrix and B is $m_1 \times n_1$ matrix

→ if $A(BA)$ is defined then first (BA) should be defined so, $n_1 = m$ so (BA) becomes $m_1 \times n$ size

→ Now if $A(BA)$ is defined then $n = m_1$ so the size of matrix becomes $m \times n$

→ So this implies matrix B becomes $n \times m$.

Q) Show that if A has row of zeros and B is any matrix for which AB is defined, then AB also has row of zeros.

Assume that the entries of i-th row of A are all zeros. We claim that the i-th row of AB is a row of zeros. To see this pick an entry c_{ij} in the i-th row of AB. By definition of multiplication of AB, we have

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Since i-th row of A is zero, we have $a_{i1} = a_{i2} = a_{in} = 0$

$$c_{ij} = 0b_{1j} + 0b_{2j} + \dots + 0b_{nj} = \sum_{k=1}^{n} 0b_{kj} = 0$$

Hence i-th row of AB is a row of zeros.

★ NO SOLUTION → INCONSISTENT

★ ATLEAST ONE SOLUTION → CONSISTENT

★ NON-SINGULAR/INVERTIBLE → invese of A exists and $\det(A) \neq 0$

★ SINGULAR → no invesse exists and $\det(A) = 0$

★ HOMOGENOUS → in $AX = B$, $B = 0$ → CONSISTENT

★ NON-HOMOGENOUS → $B \neq 0$

Q) If A and B are two square matrices of same size then find the condition such that

$$(A+B)^2 = A^2 + B^2 + 2AB.$$

$$(A+B)^2 = (A+B)(A+B)$$

$$= A^2 + \underbrace{AB + BA} + B^2$$

→ if A and B camute then

$$AB = BA$$

$$\therefore (A+B)^2 = A^2 + 2AB + B^2$$

Q) If B and C are both inverses of matrix A then $B = C$

$\therefore$ B is an inverse of A we have,

$$BA = I \quad — ①$$

$$C(BA) = I(C)$$

but $(BA)C = B(AC)$ by ASSOCIATIVE LAW

$$B(AC) = I(C)$$

$$BI = CI \qquad \because AC = I$$

$$B = C \quad \text{(proved)}$$

Q) If A and B are invertible matrices of same size, then $(AB)^{-1} = B^{-1}A^{-1}$

Consider $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$$= A(I)A^{-1}$$

$$= AA^{-1} = I \quad — ①$$

$$(B^{-1}\underbrace{A^{-1})(A}B) = B^{-1}(A^{-1}A)B$$

$$= B^{-1}B = I \quad — ②$$

from ① and ②

$$(AB)^{-1} = B^{-1}A^{-1} \quad \text{(proved)}$$

★ TO BE CONSISTENT A ROW HAS TO BE COMPLETELY ZERO.

★ For an equation to be linear

① Variables should have no power ② Do not indulge any products or roots of variables ③ Variables do not appear as arguments for trigonometry, logarithmic or exponential function.

Q) If A and B are of same size then $(A+B)^T = A^T + B^T$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$$(A+B)^T = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}^T \quad \text{— ①}$$

also $$A^T + B^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix}$$

$$\therefore (A+B)^T = A^T + B^T$$

→ A square matrix A is called SYMMETRIC if $A^T = A$

→ A square matrix A is called SKEWSYMMETRIC if $A^T = -A$

↳ diagnol always ZERO.

→ If A is invertible then $Ax = 0$ has trivial solution.

Q) If $AX = B$ represents a system of n equations in n variables then prove that solution is unique if A is invertible.

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B \quad \text{is unique} \because (AA^{-1} = I, IX = X)$$

→ ELEMENTARY ROW OPERATIONS

① Multiply a row by a non zero

② Interchange two rows

③ Add a multiple of one row to another

→ Equivalent Matrices is when one or more ERO is applied on matrix

→ Elementary Matrices is obtained by performing one single ERO on identity matrix.

~~Q) Since $Ax=0$ has only $x=0$ solution. Theorem 1.6.4 guarantees that A is invertible~~.

Q) Let $Bx=0$ be a homogenous system of linear equation such that coefficient matrix is a square matrix and the system has only trivial solution. Show that if m is positive integer, then the system $B^m x=0$ also has only trivial solution.

Given $Bx=0$ then $B^m x=0$

using theorem $(B^m)^{-1} = (B^{-1})^m$

$$B^m x = 0$$

$$(B^m)^{-1} B^m x = (B^m)^{-1} 0$$

$$\underbrace{B^{-1} \cdots B^{-1}}_{m \text{ times}} \cdot \underbrace{B \cdots B}_{m \text{ time}} x = 0$$

$$IX = 0$$

$$X = 0$$

Q) Let $Ax=0$ be a homogenous solution of n linear equation in n unknown and let Q be an invertible $n \times n$ matrix. Show that $Ax=0$ has just a trivial solution if & only if $(QA)x=0$ has just trivial sol.

Let $Ax=0$ holds.

$$Q(Ax) = Q0$$

$$(QA)x = 0 \quad \because \text{Associative property}$$

Now we let $(QA)x=0$ and we apply Q^{-1}

$$\underline{Q^{-1}Q}Ax = Q^{-1}0$$

$$IAx = 0$$

$$Ax = 0.$$

Q) Suppose that x_1 is a fixed matrix ~~number~~ which satisfies the equation $Ax_1 = b$. Further let x be any matrix whatsoever which satisfies $Ax = b$. We must then show that there is a matrix x_0 which satisfies both eq $x = x_1 + x_0$ and $Ax_0 = 0$. Clearly the ① eq implies that $x_0 = x - x_1$

Given $X = X_1 + X_0$

$$X_0 = X_1 - X$$

$$A(X_0) = A(X_1 - X)$$

$$A(X_0) = A(X_1) - A(X)$$

$$= b - b = 0$$

$$A(X_0) = 0$$

Now we show that $A(X_1 + X_0)$ has solution.

$$A(X_1 + X_0) = A(X_1) + A(X_0)$$

$$= b + 0$$

$$A(X_1 + X_0) = b$$

→ When $\det(A) = 0$ then there will be NO SOLUTION (parallel) / INFINITE (coincident) SOLUTION

→ When $\det(A) \neq 0$ then there is UNIQUE SOLUTION (intersection of lines)

→ If A is NOT INVERTIBLE, then $\det(A) = 0$

→ If two columns/rows are IDENTICAL then $\det(A) = 0$

→ If a square matrix with two propotional rows/column, then $\det(A) = 0$

→ If a row/column of zeros, then $\det(A) = 0$

→ If a scalar k is multiplied $\det(B) = k\det(A)$

→ Adding row/column then determinat remais SAME.

→ If row/column interchanging then $\det(B) = -\det(A)$

→ Determinat of identity matrix, $\det(I) = 0$

Q) Let A and B be $n \times n$ matrices. Show that if A is invertible then $\det(B) = \det(A^{-1}BA)$

$$\det(A^{-1}BA) = \det(A^{-1})\det(B)\det(A)$$
$$= \frac{1}{\cancel{\det(A)}} \cdot \det(B)\cancel{\det(A)}$$
$$= \det(B) \text{ (proved)}$$

Q) Let A and B be $n \times n$ matrices. You know from earlier work that AB is invertible if A and B are invertible. What can you say about the invertibility of AB if one or both of the factors are singular?

If either A or B is singular, then either $\det(A)$ or $\det(B)$ is zero. Hence $\det(AB) = \det(A)\det(B) = 0$. Thus AB is also singular.

Q) Prove that if $\det(A) = 1$ and ~~the~~ all entries in A are integers, then all the entries in A^{-1} ae integers

This follows from Theorem 2.1.2 and the fact that the cofactors of A are integer if A has only integer entries since & integer are under multiplication addition & subtraction $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \operatorname{adj}(A)$

Q) If B is $n \times n$ matrix and E is an elementary matrices of order $n \times n$ then $\det(EB) = \det(E)\det(B)$.

If E is performed by multiplying row of I_n by scalar of k then this means that

$$EB = C \text{ (say)}$$
$$\det(EB) = \det(C)$$
$$\det(EB) = k\det(B)$$

Since $\det(E) = k\det(I) = k$

$$\det(EB) = \det(E)\det(B).$$

Q) If A and B are square matrices of same size then

$\det(AB) = \det(A)\det(B)$

CASE 1 If A is NOT INVERTIBLE

We have $\det(AB) = 0$ by Theorem 1.6.5 and $\det(A) = 0$

This implies

$$\det(AB) = 0 \cdot \det(B)$$

$$\det(AB) = \det(A)\det(B).$$

CASE 2 If A is INVERTIBLE

We take A as product of elementary matrices

$$A = E_r E_{r-1} \cdots E_2 E_1$$

$$AB = E_r E_{r-1} \cdots E_2 E_1 B$$

$$\det(AB) = \det(E_r E_{r-1} \cdots E_2 E_1)\det(B)$$

$$\det(AB) = \det(E_r)\det(E_{r-1}) \cdots \det(E_2)\det(E_1)\det(B)$$

$$\det(AB) = \det(A)\det(B)$$

Q) Show that a matrix with a row of zeros cannot have an inverse OR Show that a matrix with column of zeros cannot have an inverse.

Let A denote a matrix which has an entire row or an entire column of zeros. Then B is any matrix, either AB has an entire row of zeros or BA has entire column of zeros. Hence neither AB nor BA can be identity matrix, therefore A cannot have inverse.

Q) Show that the diagnol entries of 3×3 skew symmetric matrix are all zero.

let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $A^T = -A$ (Skew Symmetric)

$$-\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2a_{11} & a_{21}+a_{12} & a_{31}+a_{13} \\ a_{12}+a_{21} & 2a_{22} & a_{32}+a_{23} \\ a_{13}+a_{31} & a_{23}+a_{32} & 2a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Equating the components of the matrices, so we have diagnol entries a_{11}, a_{22}, a_{33} equals to 0.

Q) consider $Ax=0$ has only trivial solution & want to prove that A can be tranformed into I_n or raw reduced echelon form.

$$\left[\begin{array}{cccc|c} a_{12} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ a_{31} & a_{32} & \cdots & a_{3n} & 0 \end{array}\right]$$

This we can achive with the idea of ERO which will tranform the augmented matrix into

$$\left[\begin{array}{ccccc} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right]$$

If we disregard the last column, we can conclude that the matrix is in raw reduced echelon form.

Q) If A and B are $n \times n$ matrices, then $tr(A+B) = tr(A) + tr(B)$

Note that the (i,i)-th entry of $A+B$ is $a_{ii} + b_{ii}$ the (i,i)-th entry of A^T is a_{ii} and the (i,i)-th entry of B^T is b_{ii}. Then consider the following

$$
\begin{aligned}
Tr(A+B) &= (a_{11} + b_{11}) + \dots + (a_{nn} + b_{nn}) \\
&= (a_{11} + \dots + a_{nn}) + (b_{11} + \dots + b_{nn}) \\
&= Tr(A) + Tr(B)
\end{aligned}
$$

Q) Let A be any $m \times n$ matrix and let 0 be the $m \times n$ matrix each of whose entries is zero. Show that if $kA = 0$ then $k = 0$ or $A = 0$

Given $kA = 0$, and suppose $k \neq 0$ so

$$kA = 0$$

$$\frac{1}{k} kA = \frac{1}{k} 0$$

$$A = 0$$

Now suppose $A \neq 0$, if $kA = 0$ and A is nonzero then the only possibility this outcome $(kA = 0)$ is $k = 0$.

Q) Let $Ax = b$ be a system of n linear equations in n unknowns with integer coefficients & integer ~~and~~ constants. Prove that if $\det(A) = 1$, the solution x has integer entries.

Solution to $Ax = b$ is $x = A^{-1}b$, so A^{-1} is from the previous solution (the fact that cofactors of A are integers if A has only integer entries $A^{-1} = \frac{1}{\det(A)} adj(A) = adj(A)$ and multiplication, addition & subtraction of integer are closed. Hence x has also integer entries) is an integer, so A^{-1} multiplied by b is also integer.

Q) Use relationship b/w elementary matrices & elementary row operation (ERO) to show that (EROS) that transform A to I will also transform I to A^{-1}

Let Q be any ERO and E be the elementary matrix corresponding to Q then,

$$Q(A) = EA = B$$

Let $Q_1, Q_2, \ldots, Q_n$ be n number of EROS and $E_1, E_2, \ldots, E_n$ are elementary matrices such that when applied on an invertible matrix 'A' give 'I'

$$Q_n \ldots Q_2 Q_1 (A) = E_n \cdots E_2 E_1 A = I$$

where $P = E_n \cdots E_2 E_1 =$

$$PA = I \quad \Rightarrow A \sim I$$

$$PAA^{-1} = IA^{-1}$$

$$PI = A^{-1} \quad \Rightarrow I \sim A^{-1}$$

Q) Show that if A is invertible then prove that $(A^{-1})^T = (A^T)^{-1}$

We know that

$$(AA^{-1}) = I$$

Apply transpose on both sides

$$((A^{-1})(A))^T = I^T$$

$$A^T (A^{-1})^T = I$$

$$\underbrace{(A^T)^{-1} A^T}(A^{-1})^T = I (A^T)^{-1}$$

$$I (A^{-1})^T = (A^T)^{-1}$$

$$(A^{-1})^T = (A^T)^{-1} \quad \text{(proved)}$$

Q) Let A be a non-singular $n \times n$ matrix. Show that $A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$

[You may assume that if $i \neq j$, then $a_{i1}C_{j1} + a_{i2}C_{j2} + \dots + a_{in}C_{jn} = 0$]

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

$\operatorname{Adj} A = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ C_{1n} & & & C_{nn} \end{bmatrix}$

$A \operatorname{Adj} A = \begin{bmatrix} a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n} & \cdots \\ \vdots & \\ & \end{bmatrix}$

using if $i \neq j$ then $a_{i1}C_{j1} + a_{i2}C_{j2} + \dots a_{in}C_{jn} = 0$. All off dignol gets zero. Only diagnol entries remaing

$$A \operatorname{Adj} A = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & & \\ \vdots & & \ddots & \\ 0 & & & \det(A) \end{bmatrix} = \det(A)\, I_n$$

Hence $A \operatorname{Adj} A = \det(A)\, I_n \Rightarrow A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$ $\therefore$ A is non-singula

Q) Let A be $n \times n$ matrix. Then prove that if reduced row-echelon form of A is I_n then A is expressible as product of elmetary matrices.

If $A \xrightarrow{EROS} I_n$

then $A = E_i \dots E_k$

To show,

let $E_1 \dots E_k$ is applied on A and it becomes I_n

$$E_n \dots E_1 A = I_n$$

$$E_n^{-1}(E_n E_{n-1} \cdots E_1) A = E_n^{-1} I_n$$

$$(E_n^{-1} E_n)(E_{n-1} \cdots E_1) A = E_n^{-1}$$

$$(I_n)(E_{n-1} \cdots E_1) A = E_n^{-1}$$

Similarly for all E_i

$$A = E_1^{-1} E_2^{-1} \cdots E_{n-1}^{-1} E_n^{-1} \quad \text{(proved)}$$

→ $\det(kA) = k^n \det(A)$ where n is number of rows.

→ Minor is $ad-bc$

→ Cofactor is $(-1)^{i+j} M_{ij}$

→ Determinant of 3x3 Matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$|A| = a(ei-fh) - b(di-gf) + c(dh-eg)$

here minor is $\begin{vmatrix} e & f \\ h & i \end{vmatrix}$ of a.

→ Adjoint (A)

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

→ taking transpose of cofactors is adjoint of A.

→ To find $A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$.

$\rightarrow \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$

$\rightarrow$ length of vector u, $\|u\| = \sqrt{u_1^2 + u_2^2}$

$\rightarrow$ unit vector $= \dfrac{v}{\|v\|}$

$\rightarrow u \cdot v = \|u\| \, \|v\| \cos\theta$

$\rightarrow$ orthogonal vector $\vec{u} \cdot \vec{v} = 0$

$\rightarrow e_i \cdot e_j = 0$ if $i \neq j$

$\rightarrow e_i \cdot e_j = 1$ if $i = j$

Q) Prove that any vector in 3-dimensional e.g $u = (u_1, u_2, u_3)$ can be written as linear combination of e_1, e_2 and e_3.

$$u = u_1 e_1 + u_2 e_2 + u_3 e_3$$

$$e_1 = (1,0,0) \quad e_2 = (0,1,0) \quad e_3 = (0,0,1)$$

$$= u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$$

$$= (u_1,0,0) + (0,u_2,0) + (0,0,u_3) \Rightarrow (u_1, u_2, u_3) = \overrightarrow{OP} = u.$$

Q) If $u = (u_1, u_2, u_3) = u_1 e_1 + u_2 e_2 + u_3 e_3$ and $v = (v_1, v_2, v_3) = v_1 e_1 + v_2 e_2 + v_3 e_3$ then $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$.

$$u \cdot v = (u_1 e_1 + u_2 e_2 + u_3 e_3) \cdot (u_1 v_1 + u_2 v_2 + u_3 v_3)$$

$$= v_1 u_1 e_1 \cdot e_1 + u_1 v_2 \cancel{e_1 \cdot e_2}^{\,0} + u_1 v_3 \cancel{e_1 \cdot e_3}^{\,0} + u_2 v_2 e_2 \cdot e_2$$

$$+ u_3 v_3 e_3 e_3$$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3.$$

$\rightarrow$ Dot product of vector with itself is square of it's norm.

$\rightarrow$ V is called vector in space if following properties are satisfied.

1. For all $u, v \in V$, $u + v \in V$ which means V is closed under VECTOR ADDITION
2. $u + v = v + u$ (commutative property)
3. $u + (v + w) = (u + v) + w$ (associative property)
4. $0 + u = u + 0 = u$
5. Negative of u / Additive inverse of u $\quad u + (-u) = (-u) + u = 0$
6. ku, if k is any scalar & u is an object then V is closed under SCALAR MULTIPLICATION.
7. $k(u + v) = ku + kv$
8. $(k + l)u = ku + lu$
9. $k(lu) = (kl)u$
10. $1u = u$.

Q) $V = R^n$

$V_1 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad V_2 = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$V_1 + V_2 = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix} \in R^n$

Q) $(u+v)\cdot w = u\cdot w + v\cdot w$

$(u+v)\cdot w = (u_1+v_1, u_2+v_2, \dots u_n+v_n)\cdot(w_1, w_2, \dots, w_n)$

$= (u_1+v_1)w_1 + (u_2+v_2)w_2 + \dots + (u_n+v_n)w_n$

$= (u_1w_1 + u_2w_2 + \dots + u_nw_n) + (v_1w_1 + v_2w_2 + \dots + v_nw_n)$

$= u\cdot w + v\cdot w$

Q) $u\cdot u \geq 0$. Further if $u\cdot u = 0$ if and only if $u = 0$.

we have $u\cdot u = u_1^2 + u_2^2 + \dots + u_n^2 \geq 0$
further equality holds if and if $u_1 = u_2 = \dots = u_n = 0$ - that is if and only if $u = 0$.

Q) If $u = (u_1, u_2, \dots, u_n)$ then $ku = (ku_1, ku_2, \dots ku_n)$, prove that $\|ku\| = \|k\|\,\|u\|$

$\|ku\| = \sqrt{(ku_1)^2 + (ku_2)^2 + \dots + (ku_n)^2}$

$= |k|\sqrt{(u_1)^2 + (u_2)^2 + \dots (u_n)^2}$

$= |k|\,\|u\|$

Q) Prove $\|u+v\| \leq \|u\| + \|v\|$

$\|u+v\|^2 = (u+v)(u+v)$

$= (u\cdot u) + 2(u\cdot v) + (v\cdot v)$

$= \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$

$= (\|u\| + \|v\|)^2$

$\|u+v\| = \|u\| + \|v\|$

Q) $d(\vec{u},\vec{v}) \leq d(\vec{u},\vec{w}) + d(\vec{w},\vec{v})$

$d(\vec{u},\vec{v}) = d(\vec{u},\vec{v})$

$= \|\vec{u} - \vec{v}\|$

$= \|(\vec{u} - \vec{w}) + (\vec{w} - \vec{v})\|$

$= \|\vec{u} - \vec{w}\| + \|\vec{w} - \vec{v}\|$

$= d(\vec{u},\vec{w}) + d(\vec{w},\vec{v})$

Q) Prove $\vec{u}\cdot\vec{v} = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$

$\frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2 = \frac{1}{4}(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) - \frac{1}{4}(\vec{u}-\vec{v})(\vec{u}-\vec{v})$

$= \frac{1}{4}(\vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v})$

$- \frac{1}{4}(\vec{u}\cdot\vec{u} - \vec{v}\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v})$

$= \frac{1}{\cancel{4}}(\cancel{4}\,\vec{u}\cdot\vec{v}) = \vec{u}\cdot\vec{v}$

Q) $(A\vec{u})\cdot\vec{v} = \vec{u}\cdot(A^T\vec{v})$

$(A\vec{u})\cdot\vec{v} = (A\vec{u})^T\cdot\vec{v}$

$= u^TA^T\vec{v} \quad \because u^T = u.$

$(A\vec{u})\cdot\vec{v} = u\cdot(A^T\vec{v})$

Q) Prove $0\vec{u} = 0$.

Ax 8 $\Rightarrow$ $(k+l)\vec{u} = k\vec{u} + l\vec{u}$

$0\vec{u} + 0\vec{u} = (0+0)\vec{u}$

Ax 4 $\Rightarrow$ $u \in V \iff -u \in V$

$0\vec{u} + 0\vec{u} + (-0\vec{u}) = 0\vec{u} + (-0\vec{u})$

Ax 3 $\Rightarrow$ $0\vec{u} + (0\vec{u} + (-0\vec{u})) = 0\vec{u} + (-0\vec{u})$

Ax 5 $\Rightarrow$ $0\vec{u} + 0 = 0$

Ax 4 $\Rightarrow$ $0\vec{u} = 0$.

Q) $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$

$\|u+v\|^2 = (u+v)(u+v)$

$= \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$

$\|u-v\|^2 = (u-v)(u-v)$

$= (u\cdot u) - 2(u\cdot v) + (v\cdot v)$

$= \|u\|^2 - 2\|u\|\|v\| + \|v\|^2$

Adding both eqs.

$\|u\|^2 + \cancel{2\|u\|\|v\|} + \|v\|^2 + \|u\|^2 - \cancel{2\|u\|\|v\|} + \|v\|^2$

$= 2(\|u\|^2 + \|v\|^2)$

(b) The result states a theorem about parallelogram in R^2, what is the theorem?

It stats that sum of squares of legth of four sides of a parallelogram equals sum of squares of legth of two diagnols.

Q) Prove $(-1)\vec{u} = -\vec{u}$

$\vec{u} + (-1)\vec{u} = 0$

$1\vec{u} + (-1)\vec{u} = 0$

$(1+(-1))\vec{u} = 0$

$0\vec{u} \Rightarrow 0$

Q) Prove: If u and v are vectors in R^n and k is any scalar, then

$u\cdot(kv) = k(u\cdot v)$

$\vec{u}\cdot(k(\vec{v})) = (u_1, u_2, \dots, u_n)\cdot(k(v_1, v_2, \dots v_n))$

$= (u_1, u_2, \dots, u_n)\cdot(kv_1, kv_2, \dots, kv_n)$

$= u_1(kv_1) + u_2(kv_2) + \dots + u_n(kv_n)$

$= ku_1v_1 + ku_2v_2 + \dots + ku_nv_n$

$= k(u_1v_1 + u_2v_2 + \dots + u_nv_n)$

$= k(\vec{u}\cdot\vec{v})$

Q) Show that u and v are orthogonal vectors in R^n if $\|u+v\| = \|u-v\|$.

$u\cdot v = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$. Now substitute $\|u+v\| = \|u-v\|$, we have $u\cdot v = \frac{1}{4}\|u-v\|^2 - \frac{1}{4}\|u-v\|^2$, so $u\cdot v = 0$. Hence it is orthogonal

Q) Prove $d(u,v) \geq 0$.

$d(u,v) = \|u-v\| = \sqrt{(u_1-v_1)^2 + \dots + (u_n-v_n)^2}$

hence by property of squaring each component it must be greater and equal to zero.

Q) Prove the following generalization of Theorem 4.1.7. If $v_1, v_2, \dots, v_n$ are pairwise orthogonal vectors in R^2, then

$$\|v_1+v_2+\dots+v_r\|^2 = \|v_1\|^2 + \|v_2\|^2 + \dots + \|v_r\|^2$$

$$\|v_1+v_2+\dots+v_r\|^2 = \|v_1 + (v_2+\dots+v_r)\|^2$$

$$= \|v_1\|^2 + \|v_2+v_3+\dots+v_r\|^2$$

We know $\|v_1+v_2\|^2 = \|v_1\|^2 + \|v_2\|^2$ $\because v_i v_j$ are orthogonal

Apply this procedure $r-1$ times

⋮

⋮

$$= \|v_1\|^2 + \|v_2\|^2 + \dots + \|v_r\|^2$$

Q) Prove: If u and v are $n\times 1$ matrices and A is $n\times n$ an matrix then $(v^TA^TAu)^2 \le (u^TA^TAu)(v^TA^TAv)$

Let $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ be vectors and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in R^n and let

A be an $n\times n$ matrix. So, letting $u = Au$ and $v = Av$ and this multiplication possible. Hence the inner product is $(a,b) = a\cdot b = b^Ta$

$$(u,v) = Au \cdot Av = (Av)^T Au$$

So we have $(u,v) = v^TA^TAu$

In special case we have $(u,u) = u^TA^TAu$ and $(u,v) = v^TA^TAv$

The Cauchy-Schwarz inequality is

$$(u,v)^2 \le (u,u)(u,v)$$

By substituting the previous we get

$$(u^TA^TAu)^2 \le (u^TA^TAu)(v^TA^TAv).$$

→ $Proj_a u = \frac{(u \cdot a)}{\|a\|^2} a$

→ Vector camparent of u arthogonal to a $u - proj_a u$

Q) If u and v are vectors in 2-space or 3-space and 4 $a \neq 0$ then $proj_a u = \frac{(u \cdot a)}{\|a\|^2} a$ and $u - proj_a u = u - \frac{(u \cdot a) a}{\|a\|^2}$

$u = w_1 + w_2$

$\because w_1 = ka$

$u = ka + w_2$

$w_2 = u - ka$

taking dot product with a

$w_2 \cdot a = (u - ka) \cdot a = 0$

$u \cdot a - ka \cdot a = 0$

$k = \frac{u \cdot a}{a \cdot a} = \frac{(u \cdot a)}{\|a\|^2} = k$

$\because w_1 = \frac{(u \cdot a)}{\|a\|^2} a$ and $w_2 = u - \frac{(u \cdot a)}{\|a\|^2} a.$

Q) Let V be vector space and the zero vector of V is 0. If k is some scalar, then prove that $k0 = 0$.

$k0 + ku = k(0 + u)$ ∴ Axiom 7

$k0 + ku = k(u)$ ∴ Axiom 4

$(k0 + ku) + (-ku) = k(u) + (-ku)$

$k0 + (ku + (-ku)) = (ku + (-ku))$

$k0 + 0 = 0$

$k0 = 0$ (proved)

Q) Show that it is not possible for vector u in vector space V to have two different additive inverse both of which satisfy axiom 5 for vector space.

let there exists two inverses of u.

$u + u_1 = 0$ and $u + u_2 = 0$

$u + u_1 = 0$

$u_2 + (u + u_1) = u_2 + 0$

$(u_2 + u) + u_1 = u_2$

$0 + u_1 = u_2$

$u_1 = u_2$