LINEAR ALGEBRA

LECTURE 9

MATH 205

CRAMER'S RULE

P. 111 (8th ED.)
P. 109 (7th ED.)

REVISION (CONTINUED)

CONSIDER $a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

$A\underline{X} = \underline{B}$

$\underline{B} \neq \underline{0}$

HERE

$$\boxed{x_1} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \text{ AND}$$

$$\boxed{x_2} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \text{ PROVIDED}$$

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$. LET US EXTEND THIS METHOD TO THREE EQUATIONS IN THREE UNKNOWNS

2] CONSIDER

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad -(1)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad -(2)$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad -(3)$$

LET US WRITE (1) AND (2) AS

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 &= c_1 \\ a_{21}x_1 + a_{22}x_2 &= c_2 \end{aligned}\right\} \longrightarrow (*)$$

WHERE $\boxed{\begin{aligned} c_1 &= b_1 - a_{13}x_3 \\ c_2 &= b_2 - a_{23}x_3 \end{aligned}}$ AND

SOLUTION OF (*) IS

$$x_1 = \frac{\begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

$$\Rightarrow \boxed{x_1 = \frac{c_1 a_{22} - a_{12} c_2}{a_{11}a_{22} - a_{12}a_{21}}}, \text{ AND}$$

$$\boxed{x_2 = \frac{c_2 a_{11} - c_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}}$$

SUBSTITUTING x_1, x_2 IN (3) AFTER

3) USING THE VALUES OF $\boxed{C_1}$ AND $\boxed{C_2}$ WE GET THE FOLLOWING VALUE OF $\boxed{x_3}$ AS

$$\boxed{\boxed{x_3} = \frac{\left[\begin{array}{l} a_{11}(b_3 a_{22} - a_{32} b_2) \\ -a_{12}(b_3 a_{21} - a_{31} b_2) \\ +\, b_1(a_{21} a_{32} - a_{31} a_{22}) \end{array}\right]}{\left[\begin{array}{l} a_{11}(a_{22} a_{33} - a_{23} a_{32}) \\ -a_{12}(a_{21} a_{33} - a_{31} a_{23}) \\ +\, a_{13}(a_{32} a_{21} - a_{31} a_{22}) \end{array}\right]}}$$

CONSIDER THE DENOMINATOR

$$a_{11}(a_{22} a_{33} - a_{23} a_{32})$$
$$- a_{12}(a_{21} a_{33} - a_{31} a_{23})$$
$$+ a_{13}(a_{32} a_{21} - a_{31} a_{22})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

DETERMINANT OF THE COEFFICIENT MATRIX $\Leftarrow$

$= |A| \neq 0$

4) SIMILARLY THE NUMERATOR IS EQUAL TO

$$|A_3| = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix},$$

$$\therefore\ x_3 = \frac{|A_3|}{|A|} = \frac{|A_3|}{|A|}, \quad |A| \neq 0$$

COMPLETE SOLUTION:

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad x_3 = \frac{|A_3|}{|A|}$$

WHERE

$$|A_1| = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$|A_2| = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

5)

DEFINITION: P. 104 8th ED.
P. 101 7th ED.

IF $\boxed{A}$ IS A SQUARE MATRIX THEN THE MINOR OF ENTRY $\boxed{a_{ij}}$ IS DENOTED BY $\boxed{M_{ij}}$ AND IS DEFINED TO BE THE DETERMINANT THAT REMAINS AFTER THE $\boxed{i\text{TH}}$ ROW AND $\boxed{j\text{TH}}$ COLUMN ARE $\boxed{\text{DELETED}}$ FROM $\boxed{A}$. THE NUMBER $\boxed{(-1)^{i+j} M_{ij}}$ IS DENOTED BY $\boxed{C_{ij}}$ AND IS CALLED THE COFACTOR OF ENTRY $\boxed{a_{ij}}$.

EXAMPLES: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\boxed{M_{11}} =$ MINOR OF $\boxed{a_{11}} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$\boxed{C_{11}} =$ COFACTOR OF $\boxed{a_{11}}$

$= (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

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M_{12} = MINOR OF a_{12}

$= \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ WHICH IS THE DETERMINANT OBTAINED AFTER IGNORING ROW 1 AND COLUMN 2.

$\Rightarrow \boxed{C_{12}} = (-1)^{1+2} M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

SIMILARLY

$\boxed{C_{13}} = (-1)^{1+3} M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

ETC.

NOTE: DET(A) = DET(A)

$$= a_{11}\left\{(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\right\} + a_{12}\left((-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}\right)$$

$$+ a_{13}\left\{(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right\}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

THIS IS EXPANSION BY FIRST ROW.

7 TRY THE FOLLOWING:

FIND $C_{11}, C_{12}, C_{21}, C_{22}$ FOR

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

SOLUTION: $\boxed{C_{11}} = (-1)^{1+1} d = \boxed{d}$

$\boxed{C_{12}} = (-1)^{1+2} c = \boxed{-c}$,

$\boxed{C_{21}} = (-1)^{1+2} b = \boxed{-b}$, $\boxed{C_{22}} = (-1)^{2+2} a$

$\Rightarrow \boxed{C_{22}} = \boxed{a}$

CONSIDER $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T$

$= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, ① RECALL THAT

$A^{-1} = \dfrac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, ② WHERE $ad - bc = \det(A) \neq 0$

$\Rightarrow A^{-1} = \dfrac{1}{\det(A)} \boxed{\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T}$ FROM ① AND ②.

THIS MATRIX IS CALLED $\boxed{\text{Adj}(A)}$ OR $\boxed{\text{Adjoint}}$ of $\boxed{A}$.

NOTATION

INVERSE OF A MATRIX USING ITS ADJOINT:

P.109 8th ED.
P.106 7th ED.

IF $\boxed{A}$ IS AN INVERTIBLE MATRIX, THEN

$$\boxed{A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)}$$

EXAMPLE: FIND THE INVERSE OF $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$ BY ADJOINT METHOD

SOLUTION: STEP (1): FIND $\det(A)$ WHICH IS GIVEN BY

$$3(0+12)-2(-6)-1(-4-12) = 36+12+16 = 64 = \det(A) \neq 0.$$

STEP(2): FIND $\boxed{\operatorname{Adj}(A)}$ WHICH IS

$$= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \operatorname{adj}(A)$$

C_{11} = COFACTOR OF $(a_{11} = 3)$ IS GIVEN BY

$$\boxed{C_{11}} = (-1)^{1+1} M_{11} = (-1)^{1+1} \begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} = \boxed{12}$$

SIMILARLY

$$\boxed{C_{12}} = (-1)^{1+2} M_{12} = -\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = \boxed{6},$$

CHECK THE FOLLOWING:

$$\boxed{C_{13}} = \begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix} = -4 - 12 = \boxed{-16},$$

$$\boxed{C_{21}} = -\begin{vmatrix} 2 & -1 \\ -4 & 0 \end{vmatrix} = -(-4) = \boxed{4},$$

$$\boxed{C_{22}} = \boxed{2}, \quad \boxed{C_{23}} = -(-16) = \boxed{16}, \quad \boxed{C_{31} = 12},$$

$$\boxed{C_{32}} = -(10) = \boxed{-10}, \quad \boxed{C_{33} = 16},$$

$$\therefore A^{-1} = \frac{1}{\det(A)} \operatorname{Adj}(A) = \frac{1}{64} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$\Rightarrow A^{-1} = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$