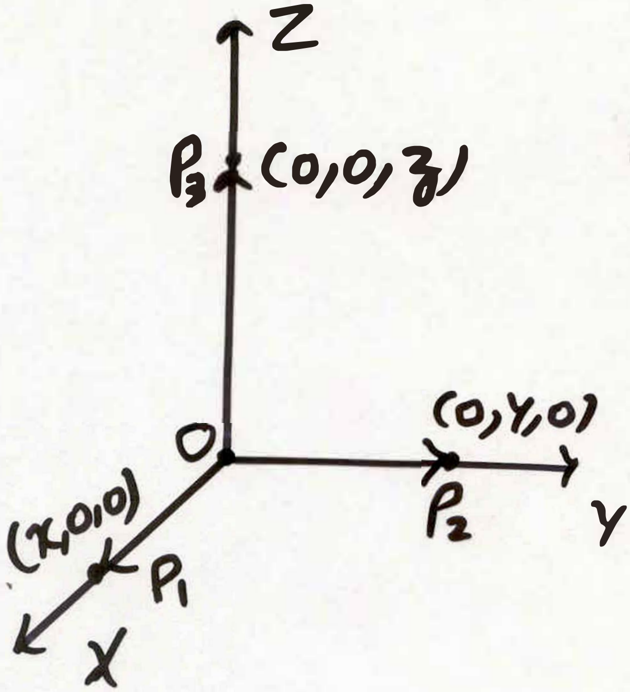
MATH 205

LECTURE 11 LINEAR ALGEBRA

UNIT VECTORS ALONG X, Y AND Z AXES IN 3-DIMENSIONAL SPACE.



$\overrightarrow{OP_1} = (x,0,0)$

$\|\overrightarrow{OP_1}\| = \sqrt{x^2} = x \neq 0$

$\overrightarrow{OP_2} = (0,y,0)$

$\|\overrightarrow{OP_2}\| = y \neq 0$

$\|\overrightarrow{OP_3}\| = z \neq 0$

$\overrightarrow{OP_3} = (0,0,z)$

IF $\underline{e}_1$ IS THE UNIT VECTOR ALONG X-AXIS THEN

$$\underline{e}_1 = \frac{\overrightarrow{OP_1}}{\|\overrightarrow{OP_1}\|} = \frac{(x,0,0)}{x} = (1,0,0)$$

$\|\underline{e}_1\| = 1$, SIMILARLY UNIT VECTORS ALONG Y AND Z AXES ARE GIVEN BY $(0,1,0) = \underline{e}_2$ AND $(0,0,1) = \underline{e}_3$ RESPECTIVELY.

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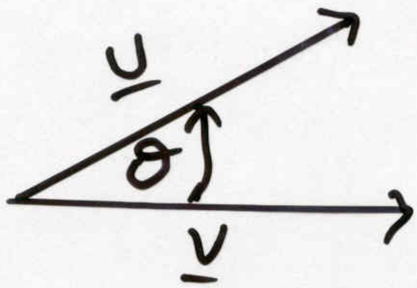
DEFINITION:

IF $\underline{u}$ AND $\underline{v}$ ARE VECTORS IN 2 OR 3-DIMENSIONAL SPACE AND θ IS ANGLE BETWEEN $\underline{u}$ AND $\underline{v}$, THEN THE $\boxed{\text{DOT}}$ PRODUCT $\underline{u}\cdot\underline{v}$ IS DEFINED BY

$$\underline{u}\cdot\underline{v} = \begin{cases} \|\underline{u}\|\,\|\underline{v}\|\cos\theta & \underline{u}\neq\underline{0},\ \underline{v}\neq\underline{0} \\ 0 & \underline{u}=\underline{0} \text{ OR } \underline{v}=\underline{0} \end{cases}$$



$\underline{u}\cdot\underline{v}$ IS A SCALAR QUANTITY.

$\underline{u}\cdot\underline{v}$ IS ALSO CALLED EUCLIDEAN INNER PRODUCT.

TRY THE FOLLOWING:

PROVE THAT ANY VECTOR IN 3-DIMENSIONS E.G. $\underline{u}=(u_1,u_2,u_3)$ CAN BE WRITTEN AS THE LINEAR COMBINATION OF $\underline{e}_1$, $\underline{e}_2$ AND $\underline{e}_3$.

CHECK: $\underline{u} = u_1\underline{e}_1 + u_2\underline{e}_2 + u_3\underline{e}_3$

$\underline{e}_1=(1,0,0)$, $\underline{e}_2=(0,1,0)$, $\underline{e}_3=(0,0,1)$

PREVIOUS RESULT (PROOF):

ANY VECTOR IN 3-DIMENSIONS E.G. $\underline{u} = (u_1, u_2, u_3)$ CAN BE WRITTEN AS THE LINEAR COMBINATION OF $\underline{e}_1, \underline{e}_2, \underline{e}_3$.

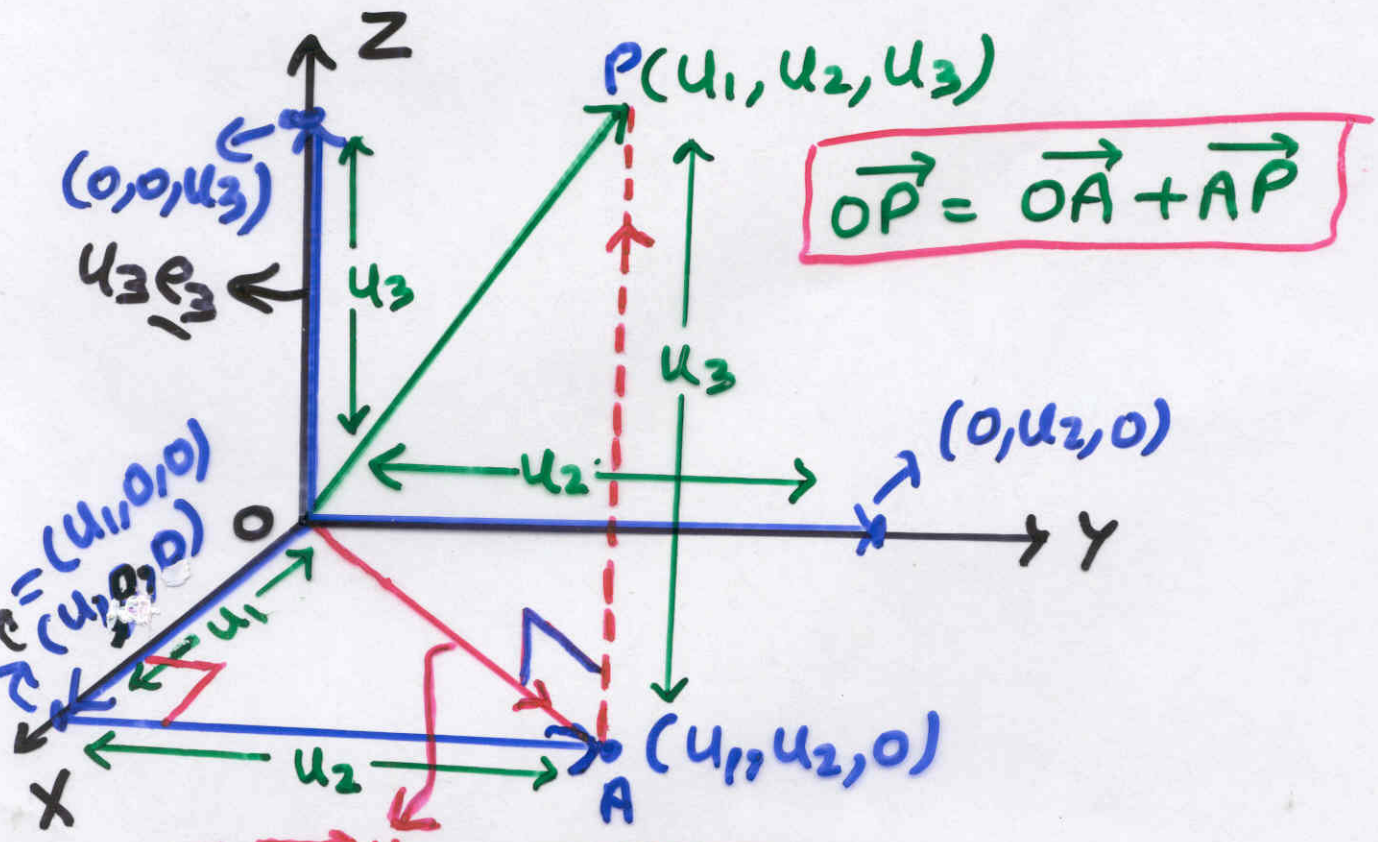
$\because$ $u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3$

$= u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$

$= (u_1,0,0) + (0,u_2,0) + (0,0,u_3)$

$= (u_1, u_2, u_3) = \overrightarrow{OP} = \underline{u}$

ALSO SEE THE FIGURE BELOW:



$\|\overrightarrow{OA}\| = \sqrt{u_1^2 + u_2^2}$

$\|\overrightarrow{OP}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

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DEF: ORTHOGONAL VECTORS

DOT PRODUCT BETWEEN TWO VECTORS IS ZERO WHEN THE ANGLE BETWEEN THEM = $\boxed{90^\circ}$.
SUCH VECTORS ARE PERPENDICULAR TO EACH OTHER AND THEY ARE ALSO CALLED ORTHOGONAL VECTORS.

$\therefore$ $\underline{e}_1, \underline{e}_2, \underline{e}_3$ ARE ORTHOGONAL VECTORS

$$\underline{e}_i \cdot \underline{e}_j = 1 \quad \text{IF } i = j \qquad i, j \in \{1,2,3\}$$
$$= 0 \quad \text{IF } i \neq j$$

E.G. $\underline{e}_1 \cdot \underline{e}_2 = \|\underline{e}_1\| \, \|\underline{e}_2\| \cos 90^\circ = 0$

$\underline{e}_1 \cdot \underline{e}_1 = \|\underline{e}_1\| \, \|\underline{e}_1\| \cos 0^\circ = 1$

PROVE THE FOLLOWING:

IF $\underline{u} = (u_1, u_2, u_3) = u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3$

$\underline{v} = (v_1, v_2, v_3) = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3$

THEN $\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

PROOF: $\underline{u} \cdot \underline{v}$

$$= (u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3) \cdot (v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3)$$

$$= u_1 v_1 \underline{e}_1 \cdot \underline{e}_1 + u_1 v_2 \cancelto{0}{\underline{e}_1 \cdot \underline{e}_2} + u_1 v_3 \cancelto{0}{\underline{e}_1 \cdot \underline{e}_3}$$
$$+ u_2 v_2 \underline{e}_2 \cdot \underline{e}_2 + u_3 v_3 \underline{e}_3 \cdot \underline{e}_3$$
$$= u_1 v_1 + u_2 v_2 + u_3 v_3$$

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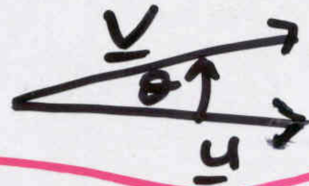
FOR $\underline{u} = u_1\underline{e_1} + u_2\underline{e_2} + u_3\underline{e_3}$

$\underline{v} = v_1\underline{e_1} + v_2\underline{e_2} + v_3\underline{e_3}$

$$\underline{u}\cdot\underline{v} = u_1v_1 + u_2v_2 + u_3v_3$$

$$= \|\underline{u}\|\,\|\underline{v}\| \cos\theta \quad , \underline{u} \neq 0, \underline{v} \neq 0$$



$$\boxed{\underline{u}\cdot\underline{u} = u_1^2 + u_2^2 + u_3^2 = \|\underline{u}\|^2}$$

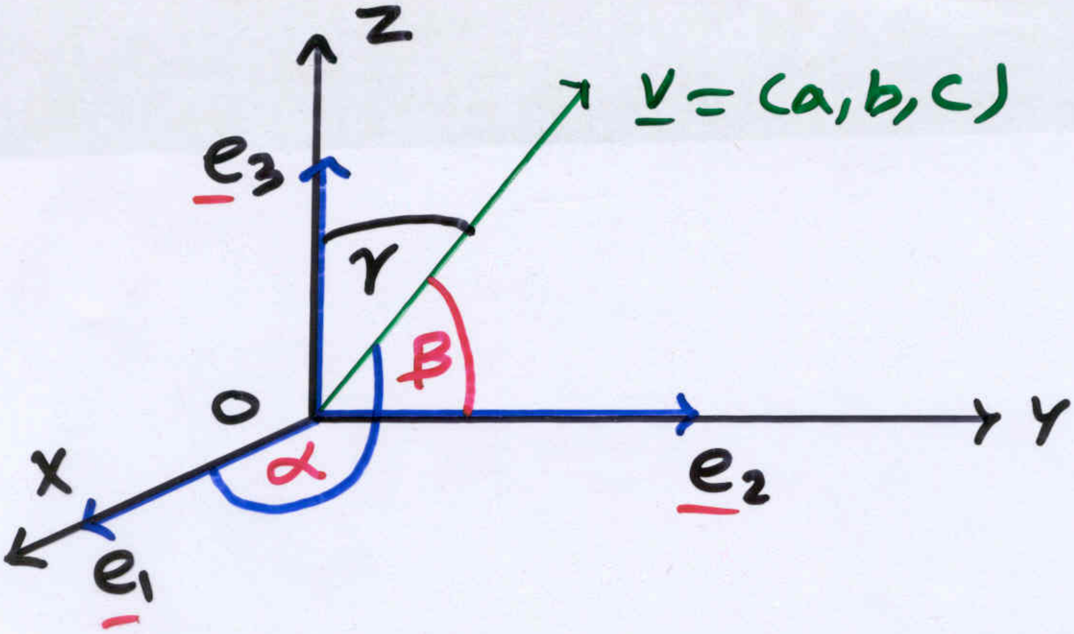
THE DOT PRODUCT OF A VECTOR WITH ITSELF IS THE SQUARE OF ITS NORM.

TRY THE FOLLOWING:

① IF $\underline{u} = (2,-1,1)$, $\underline{v} = (1,1,2)$ THEN THE ANGLE BETWEEN THEM $= 60°$

② LET $\underline{e_1}, \underline{e_2}, \underline{e_3}$ BE UNIT VECTORS ALONG X, Y AND Z AXES RESPECTIVELY IN 3-DIMENSIONAL SPACE. IF $\underline{v} = (a, b, c)$ MAKING ANGLES α, B AND γ WITH $\underline{e_1}, \underline{e_2}$ AND $\underline{e_3}$ RESPECTIVELY, THEN

(i) FIND $\cos\alpha$, $\cos B$, $\cos\gamma$, PROVE THAT (ii) $\cos^2\alpha + \cos^2 B + \cos^2\gamma = 1$



NOTE: α, β, γ ARE CALLED THE DIRECTION ANGLES OF $\underline{V}$ AND $\cos\alpha$, $\cos\gamma$, AND $\cos\beta$ ARE CALLED THE DIRECTION COSINES OF $\underline{V}$.

SCALAR AND VECTOR PROJECTIONS:

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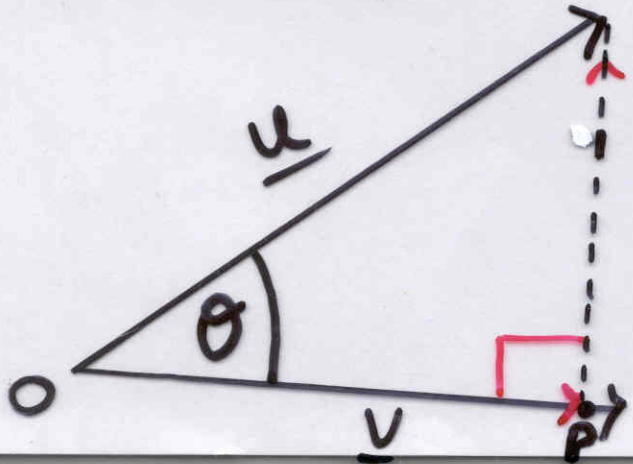
LET $\underline{u} = u_1\underline{e}_1 + u_2\underline{e}_2 + u_3\underline{e}_3$

WE SHALL CALL $u_1\underline{e}_1$, $u_2\underline{e}_2$ AND $u_3\underline{e}_3$ THE VECTOR COMPONENTS OF $\underline{u}$ AND THE SCALARS $\boxed{u_1, u_2 \text{ AND } u_3}$ THE $\boxed{\text{SCALAR}}$ COMPONENTS OR $\boxed{X, Y \text{ AND } Z}$ COMPONENTS OR SIMPLY COMPONENTS OF $\boxed{\underline{u}}$.

(7) IN $\underline{u} = u_1\underline{e_1} + u_2\underline{e_2} + u_3\underline{e_3}$, THE SCALAR u_1 MAY BE CALLED THE $\boxed{\text{SCALAR PROJECTION}}$ OF $\underline{u}$ ON ANY VECTOR WHOSE DIRECTION IS THAT OF THE $\boxed{\text{POSITIVE X-AXIS}}$ WHILE THE VECTOR $\boxed{u_1\underline{e_1}}$ MAY BE CALLED THE $\boxed{\text{VECTOR}}$ PROJECTION OF $\boxed{\underline{u}}$ ON ANY VECTOR WHOSE DIRECTION IS THAT OF THE $\boxed{\text{POSITIVE X-AXIS}}$. THEREFORE

$$\underline{u}\cdot\underline{v} = \|\underline{u}\|\,\|\underline{v}\|\cos\theta, \quad \underline{u} \neq \underline{0},\ \underline{v} \neq \underline{0}$$

= PRODUCT OF $\boxed{\text{LENGTH}}$ OF $\boxed{\underline{v}}$ AND THE SCALAR PROJECTION OF $\boxed{\underline{u}}$ ON $\underline{v}$ OR $\boxed{\underline{u}}$ ALONG $\underline{v} = \underline{v}$. $\underline{u}$ $\boxed{\underline{v}}$ OF



FROM THE FIGURE THE SCALAR PROJECTION OF $\underline{u}$ ALONG $\underline{v}$

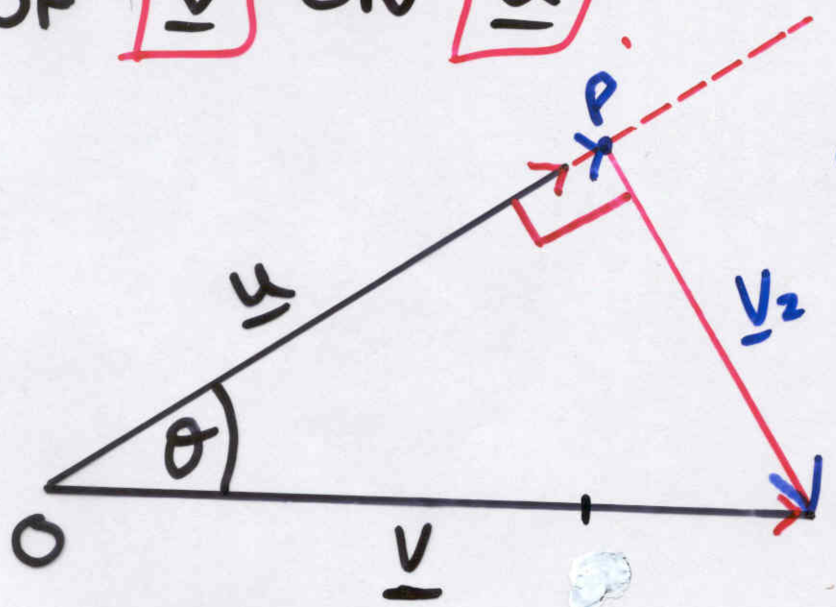
$= \|\underline{u}\|\cos\theta$

$= OP$

SIMILARLY

8 $\underline{u} \cdot \underline{v} = \|\underline{u}\| \, \|\underline{v}\| \cos\theta$

= PRODUCT OF LENGTH OF $\boxed{\underline{u}}$ AND THE SCALAR PROJECTION OF $\boxed{\underline{v}}$ ON $\boxed{\underline{u}}$



$\vec{OP} = \underline{v}_1$

$\underline{v} = \underline{v}_1 + \underline{v}_2$

$\cos\theta = \longrightarrow \dfrac{OP}{\|\underline{v}\|} \Rightarrow OP = \|\underline{v}\|\cos\theta$

WHICH IS THE SCALAR PROJECTION OF $\underline{v}$ ALONG $\underline{u}$.

NOTICE THAT $\boxed{\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}}$

SUMMARY:

$\underline{u} \cdot \underline{v} = u_1v_1 + u_2v_2 + u_3v_3 = \|\underline{u}\| \|\underline{v}\| \cos\theta$

$\underline{u} \neq \underline{0}, \ \underline{v} \neq \underline{0}$

IF VECTOR COMPONENTS OF A VECTOR ARE AT $\boxed{90°}$ TO EACH OTHER THEN THEY

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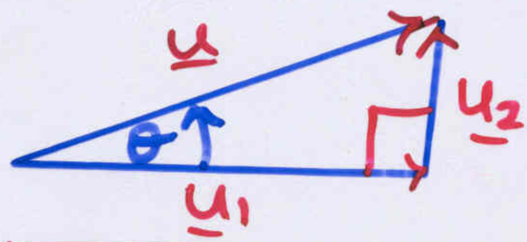
ARE ALSO CALLED THE RECTANGULAR VECTOR COMPONENTS

E.G. $\underline{u} = \underline{u}_1 + \underline{u}_2$, $\underline{u}_1$ AND $\underline{u}_2$ ARE RECTANGULAR VECTOR COMPONENTS OF $\underline{u}$, AND



THEIR $\boxed{\text{NORMS}}$ ARE

$$\|\underline{u}_1\| = \|\underline{u}\| \cos\theta$$
$$\|\underline{u}_2\| = \|\underline{u}\| \sin\theta$$

PROVIDED $\cos\theta > 0$, $\sin\theta > 0$

OR $0 < \theta < \frac{\pi}{2}$, $\underline{u}_1 \neq \underline{0}$, $\underline{u}_2 \neq \underline{0}$

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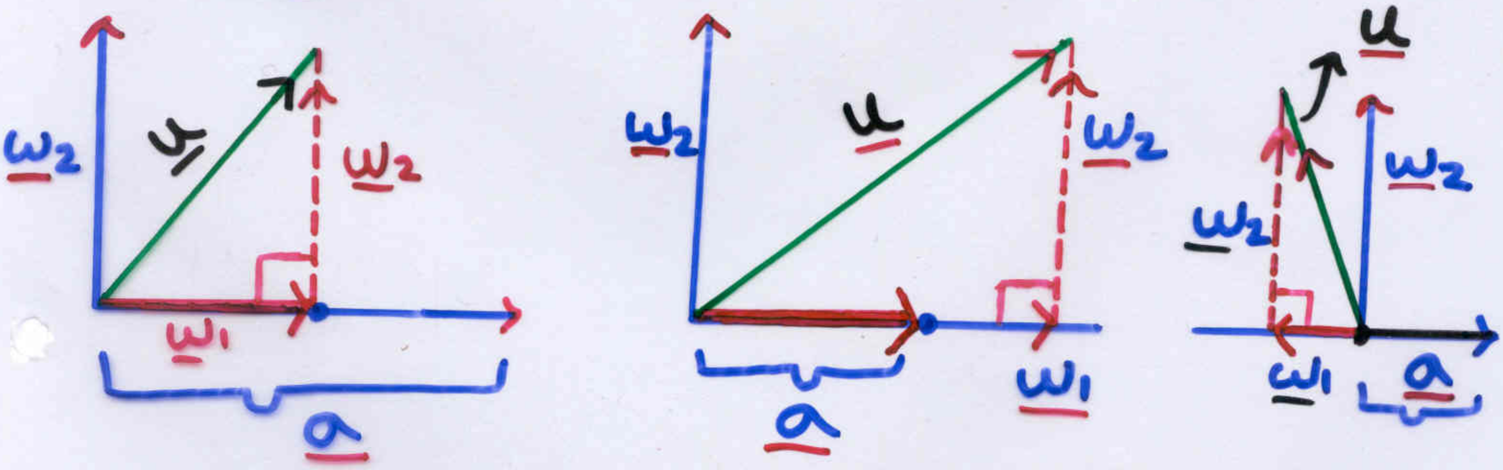
CONSIDER THE FOLLOWING FIGURES:



$\underline{u} = \underline{w}_1 + \underline{w}_2$, $\underline{w}_1$ IS $\boxed{\text{ALONG}}$ $\underline{a}$ THEREFORE $\boxed{\underline{w}_1 = k\underline{a}}$, K IS A SCALAR. $\underline{w}_2$ IS PERPENDICULAR TO $\underline{a}$. $\underline{w}_1$ IS ORTHOGONAL PROJECTION OF $\underline{u}$ ALONG $\underline{a}$ OR VECTOR COMPONENT OF $\underline{u}$ ALONG $\underline{a}$, OR

JUST $\boxed{\underline{w}_1 = \text{Proj}_{\underline{a}} \underline{u}} \rightarrow$ VECTOR COMP. OF $\underline{u}$ ALONG $\underline{a}$

PROBLEM: HOW TO FIND $\underline{w}_1$ AND $\underline{w}_2$?

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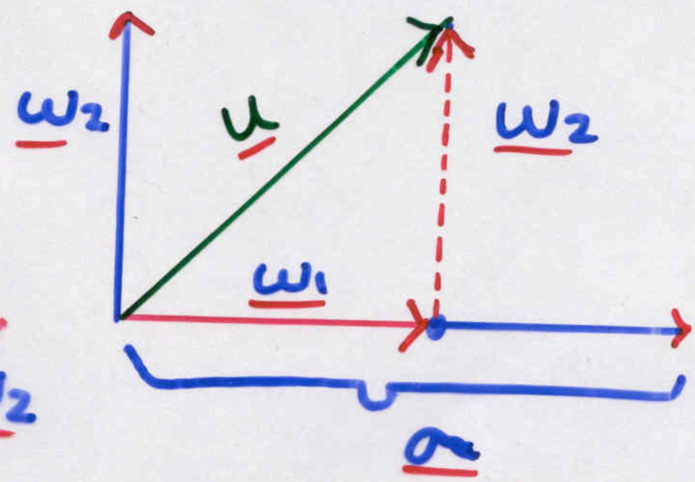
THEOREM (3.3.3)

IF $\underline{a} \neq \underline{0}$ THEN

$$\underline{w}_1 = \text{Proj}_{\underline{a}} \underline{u} = \frac{(\underline{u} \cdot \underline{a})}{\|\underline{a}\|^2} \underline{a}, \quad \underline{a} \neq \underline{0}$$

PROOF:



$\underline{u} = \underline{w}_1 + \underline{w}_2$

$\because \underline{w}_1 = k\underline{a}$

$\therefore \underline{u} = k\underline{a} + \underline{w}_2$

$$\Rightarrow \underline{w}_2 = \underline{u} - k\underline{a}$$

TAKING DOT PRODUCT WITH $\underline{a}$

$$\cancel{\underline{w}_2 \cdot \underline{a}}^{\,0} = (\underline{u} - k\underline{a}) \cdot \underline{a}$$

$$\Rightarrow \underline{u} \cdot \underline{a} - k\underline{a} \cdot \underline{a} = 0$$

$$\Rightarrow k = \frac{\underline{u} \cdot \underline{a}}{\underline{a} \cdot \underline{a}} = \boxed{\frac{\underline{u} \cdot \underline{a}}{\|\underline{a}\|^2} = k}$$

$$\boxed{\therefore \underline{w}_1 = k\underline{a} = \frac{(\underline{u} \cdot \underline{a})\,\underline{a}}{\|\underline{a}\|^2}}$$

(11)

$\because \underline{w}_1 = \dfrac{(\underline{u}\cdot\underline{a})\underline{a}}{\|\underline{a}\|^2}$ AND

Proj$_{\underline{a}}\underline{u}$

$$\boxed{\underline{w}_2 = \underline{u} - \dfrac{(\underline{u}\cdot\underline{a})\underline{a}}{\|\underline{a}\|^2}} \rightarrow$$ VECTOR COMPONENT OF $\underline{u}$ ORTHOGONAL TO $\underline{a}$.

$\because \underline{w}_2 = \underline{u} - \underline{w}_1$

ALSO NOTE THAT $\boxed{\underline{w}_2 \cdot \underline{a} = 0}$

PROBLEM:

LET $\underline{u} = (2,-1,3)$

$\underline{a} = (4,-1,2)$ FIND:

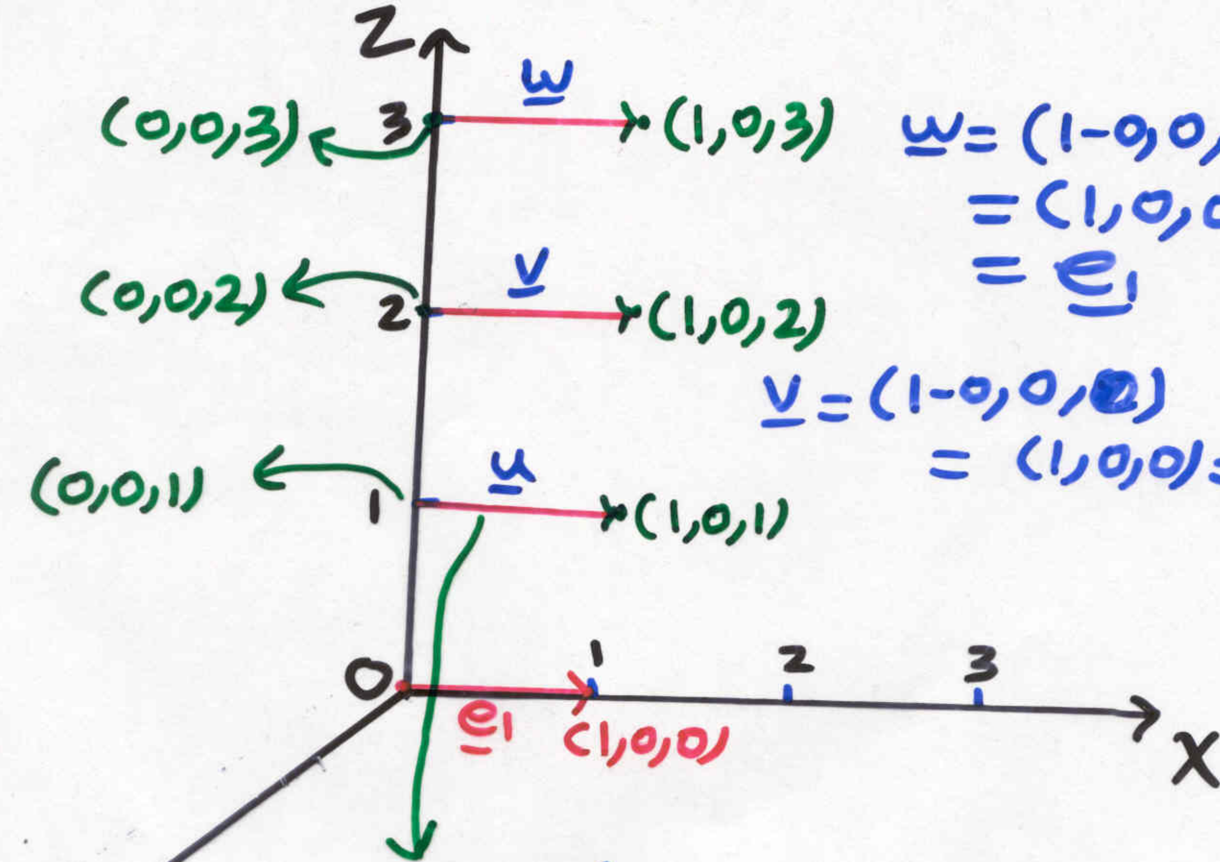
(1) VECTOR COMPONENT OF $\boxed{\underline{u}}$ ALONG $\boxed{\underline{a}}$ (Proj$_{\underline{a}}\underline{u}$) AND

(2) THE VECTOR COMPONENT OF $\boxed{\underline{u}}$ $\boxed{\text{ORTHOGONAL}}$ TO $\boxed{\underline{a}}$.

ANS. PROJ$_{\underline{a}}\underline{u} = \left(\dfrac{20}{7}, -\dfrac{5}{7}, \dfrac{10}{7}\right)$

AND THE ONE ORTHOGONAL TO $\underline{a} = \left(-\dfrac{6}{7}, -\dfrac{2}{7}, \dfrac{11}{7}\right)$

RECALL THAT TWO VECTORS ARE EQUAL (EQUIVALENT) IF THEY HAVE SAME LENGTH AND DIRECTION. CONSIDER THE FOLLOWING VECTORS IN XZ-PLANE.



$\underline{w} = (1-0, 0, 3-3)$
$= (1,0,0)$
$= \underline{e}_1$

$\underline{v} = (1-0, 0,$ [illegible]$)$
$= (1,0,0) = \underline{e}_1$

$\underline{u} = (1-0, 0-0, 1-1)$
$= (1,0,0) = \underline{e}_1$

VECTOR SPACES:

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WE SHALL STATE SOME PROPERTIES, IF SATISFIED BY A SET OF OBJECTS, WILL ENTITLE THOSE OBJECTS TO BE CALLED $\boxed{\text{VECTORS}}$.

THESE NEW TYPE OF VECTORS

(13) WILL INCLUDE VARIOUS KINDS OF MATRICES AND FUNCTIONS ETC.

DEFINITION: LET $\boxed{V}$ BE A NON-EMPTY SET OF OBJECTS ON WHICH $\boxed{\text{TWO}}$ OPERATIONS ARE DEFINED, ADDITION AND MULTIPLICATION BY SCALARS (REAL NUMBERS).

(VECTOR ADDITION)

BY $\boxed{\text{ADDITION}}$ WE MEAN FOR $\underline{u}, \underline{v} \in V$ AN OBJECT $\underline{u}+\underline{v}$ (SOME AUTHORS WRITE $\underline{u} \oplus \underline{v}$ TO DISTINGUISH FROM ADDITION OF REAL NUMBERS)

BY SCALAR MULTIPLICATION WE MEAN FOR EACH SCALAR k AND EACH OBJECT $\underline{u} \in V$ AN OBJECT $k\underline{u}$, CALLED THE SCALAR MULTIPLE OF $\underline{u}$ BY k. (SOME AUTHORS WRITE $k \odot \underline{u}$ TO DISTINGUISH FROM MULTIPLICATION OF REAL NUMBERS).

$\underline{u}+\underline{v} \longrightarrow$ VECTOR ADDITION

$k\underline{u} \longrightarrow$ SCALAR MULTIPLICATION

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$\boxed{V}$ IS CALLED A VECTOR SPACE FOR $\underline{u}, \underline{v}, \underline{w} \in V$ AND FOR ALL SCALARS K AND L IF THE FOLLOWING $\boxed{10}$ PROPERTIES ARE SATISFIED:

(1) FOR ALL $\underline{u}, \underline{v} \in \boxed{V}$, $\underline{u} + \underline{v} \in \boxed{V}$ WHICH MEANS V IS CLOSED UNDER VECTOR ADDITION.

(2) $\underline{u} + \underline{v} = \underline{v} + \underline{u}$ {COMMUTATIVE PROPERTY OF VECTOR ADDITION}.

(3) $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$ {ASSOCIATIVE PROPERTY OF VECTOR ADDITION}

(4) THERE IS AN OBJECT $\underline{0}$ IN $\boxed{V}$, CALLED A ZERO VECTOR SUCH THAT

$\underline{0} + \underline{u} = \underline{u} + \underline{0} = \underline{u}$ FOR ALL $\underline{u} \in \boxed{V}$

$\underline{0} \rightarrow$ ADDITIVE IDENTITY OR IDENTITY ELEMENT WITH RESPECT TO VECTOR ADDITION.

(5) FOR EACH $\underline{u} \in \boxed{V}$ THERE IS AN OBJECT $-\underline{u} \in \boxed{V}$ CALLED A NEGATIVE OF $\underline{u}$ OR ADDITIVE INVERSE OF $\underline{u}$ SUCH THAT

$$\underline{u} + (-\underline{u}) = (-\underline{u}) + \underline{u} = \underline{0}$$

(6) IF K IS ANY SCALAR AND $\underline{u}$ IS ANY OBJECT IN $\boxed{V}$, THEN $K\underline{u}$ IS IN $\boxed{V}$. WHICH SHOWS THAT V IS CLOSED UNDER SCALAR MULTIPLICATION.

(7) $k(\underline{u}+\underline{v}) = k\underline{u} + k\underline{v}$

(8) $(k+l)\underline{u} = k\underline{u} + l\underline{u}$

(9) $k(l\underline{u}) = (kl)\underline{u}$

ONE

(10) $1\underline{u} = \underline{u}$

DUE TO PROPERTY NO. (4)

V i.e. VECTOR SPACE IS A

NONEMPTY SET.