1)

LINEAR ALGEBRA

LECTURE 15

MATH 205

REVISION:

① $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

IS CALLED STANDARD BASIS FOR $\boxed{M_{22}}$.

② $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ IS STANDARD BASIS FOR $\boxed{R^3}$.

③ $\{1, x, x^2, \dots, x^n\}$ IS STANDARD BASIS FOR $\boxed{P_n}$. (DONE LAST TIME)

④ $\left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ IS A BASIS BUT NOT A STANDARD BASIS FOR M_{22}, SINCE (FIRST TWO) ELEMENTS ARE DIFFERENT FROM $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ AND $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

⑤ STANDARD BASIS FOR R^2 IS $\{\underline{e}_1, \underline{e}_2\}$, WHERE $\underline{e}_1 = (1,0)$, $\underline{e}_2 = (0,1)$

R] **RESULT:** IF $S = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ IS A SET OF $\boxed{n}$ VECTORS IN AN n-DIMENSIONAL SPACE $\boxed{V}$, THEN $\boxed{S}$ IS A BASIS FOR $\boxed{V}$ IF EITHER $\boxed{S}$ SPANS $\boxed{V}$ $\boxed{\text{OR}}$ $\boxed{S}$ IS LINEARLY INDEPENDENT. (P. 241 8TH ED.)/(P. 253 7TH ED.)

EXAMPLE: SHOW THAT $\{(-3,7),(5,5)\}$ IS A BASIS FOR R^2 (EASY ONE)

SOLUTION: TWO METHODS

METHOD ①:

SINCE R^2 IS $\boxed{\text{TWO}}$ DIMENSIONAL SPACE WHY?

BECAUSE $\{(1,0),(0,1)\}$ IS THE STANDARD BASIS FOR R^2 WHICH CONTAINS $\boxed{\text{TWO}}$ ELEMENTS, THEREFORE WE

ONLY PROVE THE GIVEN SET

3 TO BE LINEARLY INDEPENDENT

CONSIDER $\underline{K_1(-3,7) + K_2(5,5) = (0,0)}$

COMPARING BOTH SIDES

$$\Rightarrow \boxed{\begin{aligned} -3K_1 + 5K_2 &= 0 \\ 7K_1 + 5K_2 &= 0 \end{aligned}}$$

$$\Rightarrow \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{—} (1)$$

$$\det \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix} = -15 - 35 \neq 0$$

$\Rightarrow \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix}$ IS INVERTIBLE

$$(1) \Rightarrow \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \boxed{K_1 = K_2 = 0}$ $\therefore$ LIN. INDEPENDENT AND FINALLY $\{(-3,7), (5,5)\}$ IS A BASIS FOR R^2.

METHOD (2):

SINCE R^2 IS TWO DIMENSIONAL, THEREFORE WE ONLY PROVE THAT THE

(4) GIVEN SET $\{(-3,7), (5,5)\}$ SPANS R^2

CONSIDER

UNKNOWNS

$$(x,y) = k_1(-3,7) + k_2(5,5)$$

$$\Rightarrow x = -3k_1 + 5k_2$$

KNOWNS

$$y = 7k_1 + 5k_2$$

$$\Rightarrow \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix}^{-1}\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{—(1)}$$

$\therefore \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix}$ IS INVERTIBLE

AS $\det \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix} = -15 - 35$
$= -50 \neq 0$

$$\therefore \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix}^{-1} = -\frac{1}{50}\begin{bmatrix} 5 & -5 \\ -7 & -3 \end{bmatrix}$$

$$\therefore (1) \Rightarrow \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = -\frac{1}{50}\begin{bmatrix} 5x - 5y \\ -7x - 3y \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} k_1 &= -\frac{1}{10}(x - y) \\ k_2 &= \frac{1}{50}(7x + 3y) \end{aligned}}$$

UNIQUE SOLUTION

$\therefore$ ANY ELEMENT $(x, y) \in R^2$

CAN BE WRITTEN AS A LINEAR COMBINATION OF $(-3, 7)$ AND $(5, 5)$ $\therefore$ $\{(-3, 7), (5, 5)\}$ SPANS R^2 $\therefore$ A BASIS FOR R^2.

NOTE: $\{(-3, 7), (5, 5)\}$ IS A BASIS BUT NOT A STANDARD BASIS SINCE DIFFERENT FROM $\{(1, 0), (0, 1)\}$.

TRY THE FOLLOWING:

IF $S = \{\underline{v}_1, \underline{v}_2, \ldots\ldots, \underline{v}_n\}$ IS A BASIS FOR A VECTOR SPACE V, THEN EVERY VECTOR $\underline{v}$ IN V CAN BE EXPRESSED IN THE FORM

$$\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_n\underline{v}_n$$

IN EXACTLY ONE WAY.

DEFINITION: (P.244 7th ED) (P.233 8th ED.) P.233

HERE $(c_1, c_2, \ldots, c_n)$ IS CALLED THE COORDINATE VECTOR OF $\underline{v}$ RELATIVE TO S.