Habib University - City Campus Course Title: Math202: Engineering Mathematics Instructor's name: Class ID:	
Examination: Final Term	Exam Date: 12/14/2019, Saturday
Total Marks: 49	Time/ Duration: 9:00AM - 12:00PM/ 3 hours

Instructions for students: (example)

- ATTEMPT ALL QUESTIONS.
- Use of a calculator is NOT allowed. You can leave the answers in surd form. For example, $\ln 2$ or $\sin \pi/3$ can be left just like that.
- All answers must be given on answer script.
- The solutions must contain all necessary steps and explanations.
- If you make assumptions, make them clear in your solution.
- If you introduce additional symbols, define them properly before using them.
- If no method is specified, you can use any method to solve the problem.
- Numerical solution of integrals is not required.

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1. **(7 Points)** Consider a tank, which is being filled with water at a constant rate 30 l/min and leaking water at a rate twice the amount of water inside the tank. The rate of change of volume of water, $V = V(t)$, in the tank can be given by the differential equation,
$$\frac{dV}{dt} = 30 - 2V$$
Solve the differential equation and determine the amount of water $V(t)$ inside the tank at any time t, given that $V(t = 0) = 10$.
2. **(09 Points)** Our goal is to solve the initial value problem:
$$y'' + y = 4x + 10 \sin x \tag{1}$$
with the initial conditions: $y(\pi) = 0, y'(\pi) = 2$.
    a. (3 Points) Find the general solution, y_c, of the associated homogeneous equation $y'' + y = 0$.
    b. (1 Point) Suggest a guess function y_p for the non-homogeneous part $r(x) = 4x + 10 \sin x$. Justify your answer.
    c. (3 Points) Solve for y_p using the Method of Undetermined Coefficients.
    d. (0.5 Points) Write down the general solution of the second-order non-homogeneous equation (1).
    e. (1.5 Point) Solve the initial value problem.
3. **(11 Points)** Use the power series method to find the solution, $y = y(x)$, to the differential equation,
$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y = 0$$
Expand the solution to x^5.
4. **(8 Points)** Find the transforms,
    a. $\mathcal{L}\{e^{2-t}u(t-2)\}$
    b. $\mathcal{L}\left\{\frac{1}{s^2+s-20}\right\}$
5. **(08 Points)** Solve the ODE for $y = y(x)$,

PLEASE TURN OVER!!

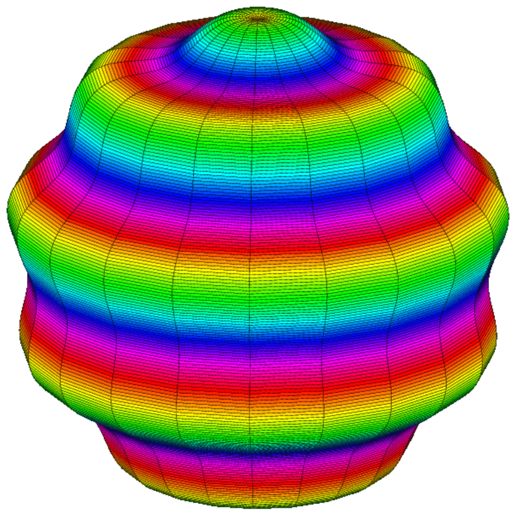
$$y'' - 6y' + 9y = t^2 e^{3t}, y(0) = 2, y'(0) = 6$$

Using the Laplace Transform method. Hint: You can use the shifting theorems to make your life a little easier.

6. **(6 Points)** Let k be a constant. Calculate the flux of the vector field,

$$\vec{F} = \vec{F}(x,y,z) = \left(2x + \tan^2 \frac{\pi}{k}(y+z)^2\right)\hat{a}_x + (\ln|\sin x + \cos z| + 5y)\hat{a}_y + (z)\hat{a}_z$$

through the following closed surface, $r = r(\theta, \phi)$.



The volume enclosed in the closed surface is V.

<u>List of Important Formulae</u>

- Gradient of a scalar function $\phi = \phi(x,y,z)$,

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{a}_x + \frac{\partial \phi}{\partial y}\hat{a}_y + \frac{\partial \phi}{\partial z}\hat{a}_z$$

- Divergence of a vector field $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$,

$$\vec{\nabla}.\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

- Curl of a vector field $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$,

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

- Line integral of a vector field $\vec{F}$,

$$\int_C \vec{F}.d\vec{r}$$

- Closed line integral,

$$\oint_C \vec{F}.d\vec{r}$$

- Surface integral of vector field $\vec{F}$ over an open surface S, (u,v) are the parameters on the surface.

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$$\iint_{S} \vec{F}.\vec{N}dudv \iint_{S} \vec{F}.\hat{N}dA$$

- Surface integral of vector field $\vec{F}$ over a closed surface S, (u, v) are the parameters on the closed surface.

$$\oiint_{S} \vec{F}.\vec{N}dudv = \oiint_{S} \vec{F}.\hat{N}dA$$

- Volume integral of a scalar function $f = f(x, y, z)$,

$$\iiint_{V} f dV$$

- For conservative field $\vec{F}$,
  - For two different curves C and C′ between points (x_a, y_a) and (x_b, y_b),

$$\int_{(x_a,y_a,z_a)_C}^{(x_b,y_b,z_b)} \vec{F}.d\vec{r} = \int_{(x_a,y_a,z_a)_{C'}}^{(x_b,y_b,z_b)} \vec{F}.d\vec{r}$$

  - Over a closed curve C,

$$\oint_{C} \vec{F}.d\vec{r} = 0$$

  - For a scalar potential function ϕ,

$$\vec{F} = \vec{\nabla}\phi$$

  - In 2D plane,

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

  - Fundamental theorem of line integral,

$$\int_{(x_a,y_a,z_a)_C}^{(x_b,y_b,z_b)} \vec{F}.d\vec{r} = \int_{(x_a,y_a,z_a)_C}^{(x_b,y_b,z_b)} \vec{\nabla}\phi.d\vec{r} = \phi(x_a, y_a, z_a) - \phi(x_b, y_b, z_b)$$

- Divergence theorem for a vector field $\vec{F}$,

$$\iiint_{V} \vec{\nabla}.\vec{F}\, dV = \oiint_{S} \vec{F}.d\vec{A}$$

  Here, V is the volume enclosed inside the closed surface S, (u, v) are the parameters on the closed surface.

- Green's theorem for a vector field $\vec{F}$,

$$\iint_{S} \left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x}\right) dxdy = \oint_{C} \vec{F}.d\vec{r}$$

  Here the closed curve C bounds the surface S in a two dimensional xy-plane.

- Stoke's Theorem for a vector field $\vec{F}$,

$$\iint_{S} (\vec{\nabla} \times \vec{F})\,.d\vec{A} = \oint_{C} \vec{F}.d\vec{r}$$

  Here the closed curve C bounds the surface S in 3 dimensional space.

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- Function guesses for Method of Undetermined Coefficient,

$S(x)$	Guess for $y_p(x)$
k = **Constant**	K (Some other constant)
$ke^{\gamma x}$	$Ke^{\gamma x}$
kx^n, (n = positive integer)	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$k \cos \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

Table 1: Short list of guesses of particular solutions for some forms of non-homogeneous part.

- The power series of $y(x)$ about $x_0 = 0$,

$$y(x) = a_0 + a_1(x) + a_2(x)^2 + a_3(x)^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$