1) LINEAR ALGEBRA

LECTURE NO. 1

MATH 205

LINEAR ALGEBRA:

THE BRANCH OF MATHEMATICS CONCERNED WITH LINEAR EQUATIONS, MATRICES, DETERMINANTS, VECTOR SPACES, ETC.

LINEAR $\rightarrow$ RELATING TO THE FIRST DEGREE; HAVING NO VARIABLE RAISED TO ANY POWER.

(P.2)/8th ED.

INTRODUCTION TO SYSTEMS OF LINEAR EQUATIONS: (P.1)/7th ED.

A LINE IN THE XY-PLANE IS AN EQUATION OF THE FORM

$$\boxed{a_1x + a_2y = b \quad (1)}$$

EQUATION (1) IS CALLED A LINEAR EQUATION IN THE VARIABLES $\boxed{x}$ AND $\boxed{y}$.

WE DEFINE A LINEAR EQUATION IN THE $\boxed{n}$ VARIABLES $x_1, x_2, \ldots,$ x_n TO BE ONE THAT CAN

BE EXPRESSED IN THE FORM

$$\boxed{a_1 x_1 + a_2 x_2 + \cdots\cdots + a_n x_n = b} \quad -(2)$$

WHERE $\boxed{a_1, a_2, \ldots\ldots, a_n,}$ AND $\boxed{b}$ ARE REAL CONSTANTS. THE VARIABLES IN A LINEAR EQUATION ARE CALLED THE UNKNOWNS AND IN (2) UNKNOWNS ARE $\boxed{x_1, x_2, \ldots\ldots, x_n}$.

THE REAL CONSTANTS $a_1, a_2, \ldots\ldots, a_n$ ARE ALSO CALLED COEFFICIENTS.

QUESTION:

ARE THE FOLLOWING LINEAR EQUATIONS?

(1) $x + 3y^2 = 7$

(2) $y - \sin x = 0$

(3) $3x + 2y - z + xz = 4$

(4) $\sqrt{x_1} + 2x_2 + x_3 = 1$

SOLUTION: (NONE OF THEM)

(1) NO. $\because$ IN $x+3y^2=7$, $\boxed{y^2}$ IS PRESENT BUT IN A LINEAR EQUATION ALL VARIABLES OCCUR ONLY TO THE FIRST POWER.

(2) $\boxed{y-\sin x=0}$, NO, $\because$ IN A LINEAR EQUATION VARIABLES DO NOT APPEAR AS ARGUMENTS FOR TRIGNO-METRIC, LOGARITHMIC, OR EXPONENTIAL FUNCTIONS.

(3) AND (4) NO, $\because$ IN $3x+2y-z+xz=4$ & $\sqrt{x_1}+2x_2+x_3=1$, $\boxed{xz}$ AND $\boxed{\sqrt{x_1}}$ ARE PRESENT BUT A LINEAR EQUATION DOES NOT INVOLVE ANY PRODUCTS OR ROOTS OF VARIABLES.

<u>NOTE</u>: A SOLUTION OF A LINEAR EQUATION $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ IS A SET OF NUMBERS $\{s_1, s_2, \ldots, s_n\}$ SUCH THAT THE EQUATION IS <u>SATISFIED</u> WHEN WE SUBSTITUTE

$$x_1 = s_1,\ x_2 = s_2,\ \ldots\ldots,\ x_n = s_n.$$

<u>DEFINITION</u> (P.3)/8th ED. (P.2)/7th ED.

A SYSTEM OF LINEAR EQUATIONS IT IS A FINITE SET OF LINEAR EQUATIONS IN THE VARIABLES $x_1, x_2, \ldots\ldots, x_n$.

A GENERAL SYSTEM OF m LINEAR EQUATIONS IN n UNKNOWNS WILL BE WRITTEN

$$\left.\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2 \\
\quad\vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = b_m
\end{array}\right\} \quad (3)$$

THE FIRST SUBSCRIPT i $(1 \le i \le m)$ ON THE COEFFICIENT a_{ij} INDICATES THE <u>EQUATION</u> IN WHICH THE

COEFFICIENT OCCURS, AND THE SECOND SUBSCRIPT $j (1 \le j \le n)$ INDICATES WHICH UNKNOWN IT MULTIPLIES.

LINEAR SYSTEM (3) CAN BE ABBREVIATED BY WRITING ONLY THE RECTANGULAR ARRAY OF NUMBERS:

ARRAY → ARRANGEMENT

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

THIS IS CALLED THE AUGMENTED MATRIX DUE TO $\boxed{b_i\text{'s}}$ $(1 \le i \le m)$.

AUGMENT → INCREASE / ENLARGE

DUE TO THE PRESENCE OF THE ENTRIES ON R.H.S. OF THE LINEAR SYSTEM (3).

SO IT'S IMPORTANT TO STUDY MATRICES IN DETAIL. LATER ON WE SHALL SEE THAT THEY HELP US IN SOLVING THE SYSTEMS OF LINEAR EQUATIONS

(P.24)/8th ED. (P.25)/7th ED.

MATRIX:

① A MATRIX IS A RECTANGULAR ARRAY OF NUMBERS ENCLOSED IN BRACKETS. THE NUMBERS IN THE ARRAY ARE CALLED THE ENTRIES IN THE MATRIX.

② THE SIZE OF A MATRIX IS DESCRIBED BY SPECIFYING THE NUMBER OF ROWS (HORIZONTAL LINES) AND COLUMNS (VERTICAL LINES).

③ PLURAL OF A MATRIX IS MATRICES.

EXAMPLES: (OF MATRICES)

(1) $A = \begin{bmatrix} 2 & 4 & 8 \\ 5 & -3 & 0 \end{bmatrix}$ 2 ROWS

AND $\begin{bmatrix} 2 & 4 & 8 \\ 5 & -3 & 0 \end{bmatrix}$ 3 COLUMNS

SIZE: 2 × 3 → COLUMNS

↓ ROWS

NOTE: 2 × 3 = 6 GIVES TOTAL NUMBER OF ENTRIES.

NOTE: A MATRIX IN WHICH NO. OF ROWS $\neq$ NO. OF COLUMNS IS CALLED A RECTANGULAR MATRIX.

$$\therefore A = \begin{bmatrix} 2 & 4 & 8 \\ 5 & -3 & 0 \end{bmatrix}$$ IS A RECTANGULAR MATRIX $\because$ SIZE: 2×3 $\boxed{2 \neq 3}$.

(2) LET $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ } $\boxed{\text{2 ROWS, 2 COLUMNS}}$

$\boxed{\text{SIZE: } 2 \times 2}$

HERE $\boxed{B}$ IS A $\boxed{\text{SQUARE}}$ MATRIX OF ORDER $\boxed{2}$ OR 2×2 $\because$ NUMBER OF ROWS ARE = NUMBER OF COLUMNS AND ARE $= \boxed{2}$.

(3) $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$ } SIZE : 2×1 : ORDER

THIS TYPE OF MATRIX WHICH HAS ONLY ONE $\boxed{\text{COLUMN}}$ IS ALSO CALLED A $\boxed{\text{COLUMN VECTOR}}$ AND SIMILARLY A MATRIX WHICH HAS ONLY ONE $\boxed{\text{ROW}}$ IS CALLED A $\boxed{\text{ROW VECTOR}}$ E.G. $[a_1 \;\; a_2 \;\; a_3]$ } 1×3 SIZE OR ORDER 1×3.

IN GENERAL $\boxed{m \times n}$ MATRIX OR A MATRIX OF ORDER $\boxed{m \times n}$ MIGHT BE WRITTEN

$$\begin{array}{ll} & \qquad \text{COLUMN 1} \\ \text{ROW 2} \quad B = & \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \end{array}$$

WITH $\boxed{m}$ ROWS AND $\boxed{n}$ COLUMNS

OR $\boxed{[b_{ij}]_{m \times n}}$

WHERE $\boxed{b_{ij}}$ IS USED TO DENOTE THE ENTRY THAT OCCURS IN ROW i AND COLUMN j OF $\boxed{B}$.

DEFINITIONS: (P.25)/8th ED., (P.26 7th ED)

① THE ENTRIES $a_{11}, a_{22}, \ldots, a_{nn}$ IN A SQUARE MATRIX $\boxed{A}$ OF ORDER $\boxed{n}$, ARE SAID TO BE ON THE MAIN DIAGONAL OF $\boxed{A}$ AS SHOWN HERE IN THE FOLLOWING MATRIX

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

DIAGONAL OF A.

(2) A SQUARE MATRIX IN WHICH ALL ENTRIES OFF THE MAIN DIAGONAL ARE ZERO IS CALLED A DIAGONAL MATRIX.

QUESTION:

ARE THE FOLLOWING MATRICES DIAGONAL?

(1) $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ $\boxed{\text{YES}}$

(2) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\boxed{\text{YES}}$

3. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ → NOT A DIAGONAL

NO ∵ THIS IS NOT A SQUARE MATRIX.

4. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

YES THIS IS A DIAGONAL MATRIX. ITS ALL ENTRIES ARE ZERO AND IS ALSO CALLED A ZERO OR NULL MATRIX, DENOTED BY 0.

DEFINITION: P. 26 / 8th ED., P. 27 / 7th ED.

IF A AND B ARE ANY TWO MATRICES OF THE SAME SIZE, THEN THE SUM $A+B$ IS THE MATRIX OBTAINED BY ADDING TOGETHER THE CORRESPONDING ENTRIES IN THE TWO MATRICES.

CONSIDER THE FOLLOWING EXAMPLE:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 4 & -2 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 5 & 6 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

FIND $\boxed{A+B}$ AND $\boxed{B+C}$

SOLUTION: $\boxed{A+B}$ IS NOT DEFINED $\because$ A & B ARE OF DIFFERENT SIZE.

$$B+C = \begin{bmatrix} 3+1 & 2+0 \\ 5+0 & 6+1 \end{bmatrix}$$

$$\Rightarrow \boxed{B+C = \begin{bmatrix} 4 & 2 \\ 5 & 7 \end{bmatrix}}$$

TRY THE FOLLOWING:

SALES FIGURES FOR THREE PRODUCTS I, II, III IN STORE A ON MONDAY (M), TUESDAY (T), ……… GIVEN BY

12)

M	T	W	Th	F	S	
40	30	81	0	21	47	I
0	12	78	50	50	96	II
10	0	0	27	43	78	III

SIMILARLY THE DATA FOR STORE B IS

M	T	W	Th	F	S	
20	19	18	1	12	74	I
2	21	87	50	49	96	II
10	1	2	72	34	87	III

AND FOR STORE C IS

M	T	W	Th	F	S	
20	3	18	1	12	45	I
1	12	78	5	0	6	II
1	2	3	6	8	99	III

BY USING MATRICES FIND THE TOTAL SALES OF EACH PRODUCT EACH DAY.

13)

SOLUTION:

$$\begin{bmatrix} 40 & 30 & 81 & 0 & 21 & 47 \\ 0 & 12 & 78 & 50 & 50 & 96 \\ 10 & 0 & 0 & 27 & 43 & 78 \end{bmatrix} + \begin{bmatrix} 20 & 19 & 18 & 1 & 12 & 74 \\ 2 & 21 & 87 & 50 & 49 & 96 \\ 10 & 1 & 2 & 72 & 34 & 87 \end{bmatrix} + \begin{bmatrix} 20 & 3 & 18 & 1 & 12 & 45 \\ 1 & 12 & 78 & 5 & 0 & 6 \\ 1 & 2 & 3 & 6 & 8 & 99 \end{bmatrix}$$

$$= \begin{array}{c c} \begin{array}{cccccc} \boxed{M} & \boxed{T} & \boxed{W} & \boxed{Th} & \boxed{F} & \boxed{S} \end{array} & \\ \begin{bmatrix} 80 & 52 & 117 & 2 & 45 & 166 \\ 3 & 45 & 243 & 105 & 99 & 198 \\ 21 & 3 & 5 & 105 & 85 & 264 \end{bmatrix} & \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array} \end{array}$$

THIS MATRIX GIVES THE TOTAL SALES OF EACH PRODUCT EACH DAY.

14] DEFINITION (P. 25) 8th ED. / (P. 27) 7th ED. [1(

TWO MATRICES ARE DEFINED TO BE EQUAL IF THEY HAVE THE SAME SIZE (ORDER) AND THE CORRESPONDING ENTRIES IN THE TWO MATRICES ARE EQUAL.

E.G.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

HERE $A \neq B$, BUT $\boxed{A}$ AND $\boxed{C}$ ARE EQUAL OR $\boxed{A = C.}$