Instructor comments do not write in this section

EM MIDTERM

SOLUTIONS RUBRIC

FALL 2022

PROBLEM-1 $\vec{F} = x^2y\,\hat{i} + xyz\,\hat{j} - x^2y^2\,\hat{k}$

a) $\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-x^2y^2)$ — 1

$= 2xy + xz$ — 1

b) i) $(\vec{\nabla}\cdot\vec{F})_{P_1(0,0,0)} = 0$ neither expanding nor contracting — 0.5

ii) $(\vec{\nabla}\cdot\vec{F})_{P_2(1,1,1)} = 3$ expanding — 0.5

iii) $(\vec{\nabla}\cdot\vec{F})_{P_3(-1,-1,-1)} = 3$ expanding — 0.5

c) $\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xyz & -x^2y^2 \end{vmatrix}$

$= \left(\frac{\partial}{\partial y}(-x^2y^2) - \frac{\partial}{\partial z}(xyz)\right)\hat{i} - \left(\frac{\partial(-x^2y^2)}{\partial x} - \frac{\partial}{\partial z}(x^2y)\right)\hat{j}$

$+ \left(\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(x^2y)\right)\hat{k}$ — 1

$= (-2x^2y - xy)\,\hat{i} + 2xy^2\,\hat{j} + (yz - x^2)\,\hat{k}$ — 1

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d) i) $(\vec{\nabla} \times \vec{F})_{P_1(0,0,0)} = 0\hat{i} + 0\hat{j} + 0\hat{k}$ 0.5

not rotating at all

ii) $(\vec{\nabla} \times \vec{F})_{P_2(1,1,1)} = -3\hat{i} + 2\hat{j} + 0\hat{k}$ 0.5

iii) $(\vec{\nabla} \times \vec{F})_{P_3(-1,-1,-1)} = \hat{i} - 2\hat{j} + 0\hat{k}$ 0.5

The paddle is rotating but clockwise/anticlockwise has no meaning in 3D.

e) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$

For any vector field $\vec{F}$, the divergence of curl is always zero. This was proven in HW 2. -1

OR

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = \vec{\nabla} \cdot (-2x^2y - xy,\ 2xy^2,\ yz - x^2)$$

$$= -4xy - y + 4xy + y$$

$$= 0.$$

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PROBLEM 2

i) $y = 2 - x^2 \;,\; y = 1$

The curves intersect at $2 - x^2 = 1$ i.e $y = 1$

$$x^2 = 1$$

$$x = \pm 1 \quad — 1$$

Points of intersection

$P_1\ (-1, 1) \rightarrow 1$

$P_2\ (1, 1)$

ii) $\vec{F} = (x^2 + y^2)\,\hat{i} + (x^2 - y^2)\,\hat{j}$

$\cancel{\vec{\nabla} \cdot \vec{F} \Rightarrow 2x - 2y}$ $\quad \text{Curl}\,\vec{F} = \dfrac{\partial}{\partial x}(x^2 - y^2) - \dfrac{\partial}{\partial y}(x^2 + y^2)$

By Green's Theorem $\quad = 2x - 2y \quad — 1$

$$\oint_C \vec{F}(r) \cdot d\vec{r} = \iint_R (\text{Curl}\,F)\, dA \quad — 1$$

$$= \int_{-1}^{1} \int_{1}^{2-x^2} 2x - 2y \; dy\, dx \quad — 0.5 \text{ correct limit} \quad — 0.5$$

$$= \int_{-1}^{1} \left(2xy - y^2 \Big|_{1}^{2-x^2} \right) dx \quad — 0.5$$

$$= \int_{-1}^{1} (4x - 2x^3 - 4 + 4x^2 - x^4 - 2x + 1)\, dx$$

$$= \int_{-1}^{1} (2x - 2x^3 - 3 + 4x^2 - x^4)\, dx \quad — 0.5$$

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$$= x^2 - \frac{1}{2}x^4 - 3x + \frac{4}{3}x^3 - \frac{x^5}{5}\Big|_{-1}^{1}$$ −0.5

$$= \left(\cancel{1} - \cancel{\tfrac{1}{2}} - 3 + \frac{4}{3} - \frac{1}{5}\right) - \left(\cancel{1} - \cancel{\tfrac{1}{2}} + 3 - \frac{4}{3} + \frac{1}{5}\right)$$

$$= -6 + \frac{8}{3} - \frac{2}{5} = \frac{-90 + 40 - 6}{15}$$

$$= -\frac{56}{15}$$ −0.5

Note: If any one has attempted to find $\int_C \vec{F} \cdot d\vec{r}$ without green's theorem, they donot get any points.

PROBLEM 3: $F = z\hat{j}$

i) $x = 1$, a) $\hat{n} = \hat{i}$, b) $\vec{F} \cdot \hat{n} = 0$

$\Rightarrow$ flux through this face is zero.

ii) $x = 0$, a) $\hat{n} = -\hat{i}$ b) $\vec{F} \cdot \hat{n} = 0$

$\Rightarrow$ flux $= 0$

iii) $y = 1$ a) $\hat{n} = \hat{j}$ b) $\vec{F} \cdot \hat{n} = z \geq 0$

$z \geq 0$ in first octant

$\Rightarrow$ Flux out is positive

iv) $y = 0$ a) $\hat{n} = -\hat{j}$ b) $\vec{F} \cdot \hat{n} = -z \leq 0$

$-z \leq 0$ in first octant

$\Rightarrow$ Flux out negative

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v) $z = 1$ a) $\hat{n} = k$ b) $\vec{F} \cdot \hat{n} = 0$

$\Rightarrow$ flux $= 0$

vi) $z = -1$ a) $\hat{n} = -k$ b) $\vec{F} \cdot \hat{n} = 0$

$\Rightarrow$ flux $= 0$

0·25 points for each correct answer to (a) & (b)

PROBLEM 4:

Show that for any curve on which the Green's Theorem applies the following is true.

$$\underbrace{\frac{1}{2}\oint_C x^2\,dy}_{①} = \underbrace{-\oint_C xy\,dx}_{②} = \underbrace{\frac{1}{3}\oint_C x^2dy - xy\,dx}_{③}$$

SOLUTION: In ①, $\vec{F}(x,y) = \frac{1}{2}x^2\hat{j}$

$2D\text{-curl}(\vec{F}) = x$ — 0·5

In ② $\vec{F}(xy) = -xy\,\hat{i}$

$2D\text{-curl}(\vec{F}) = x$ — 0·5

In ③ $\vec{F}(x,y) = -\frac{1}{3}xy\,\hat{i} + \frac{1}{3}x^2\hat{j}$

$2D\text{-curl}(\vec{F}) = \frac{2}{3}x + \frac{1}{3}x = x$ — 0·5

Since All three vector fields have same curl and the curve meets the conditions of Green's Theore 0·5
The line integral of these fields over $C = \iint_R x\,dA$ where R is region bounded by C.

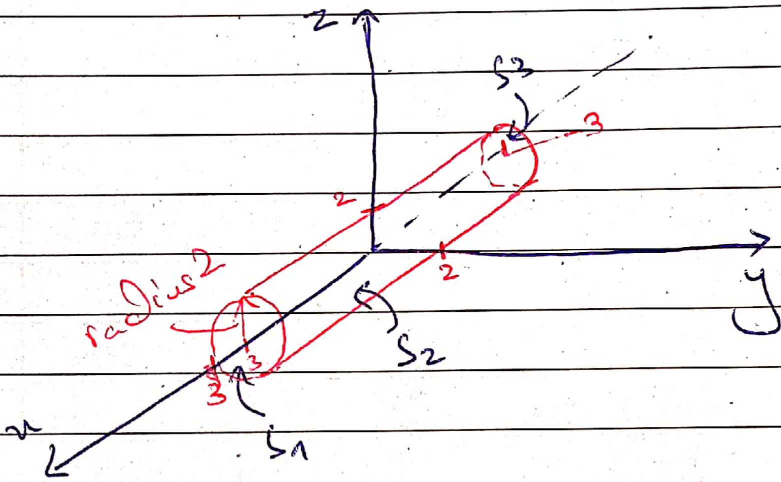
PROBLEM 5:

S boundary of $R: y^2+z^2 \leq 4, \; -3 \leq x \leq 3$

$\vec{F} = (z-y)\,\hat{i} + y^3\,\hat{j} + z^3\,\hat{k}$

a) Closed Cylinder — 1

b)



1 point for correct axis alignment
1 point for mentioning range of x and radius

c) 1 piecewise smooth, it is the union of three smooth surfaces

$S_1: x = 3, \; y^2+z^2 \leq 4$

$S_2: y^2+z^2 = 4, \; -3 \leq x \leq 3$

$S_3: x = -3, \; y^2+z^2 \leq 4$

d) Outward flux through S = Sum of outward flux from S_1, S_2 & S_3

Outward flux through S_1 : $x = 3$, $y^2+z^2 \leq 4$

Unit normal vector from S_1 in outward direction $= \hat{i}$ $\quad$ (grad $(S_1) = \frac{\partial}{\partial x}(x)\hat{i} = \hat{i}$) $\quad$ 0.5

$$\vec{F}\cdot\hat{n} = \left((z-y)\hat{i} + y^3\hat{j} + z^3\hat{k}\right)\cdot\hat{i}$$

$$= z - y \quad 0.5$$

$\vec{F}\cdot\hat{n}$ in cylindrical coordinates $= r\sin\theta - r\cos\theta$

Since we have $y^2+z^2 \leq 4$

$\Rightarrow r^2 \leq 4$

$\Rightarrow r \leq 2$, $0 \leq \theta \leq 2\pi$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\, dS = \int_0^{2\pi}\int_0^2 (r\sin\theta - r\cos\theta)\, r\, dr\, d\theta \quad 1$$

$$= \int_0^{2\pi} \left.\frac{r^3}{3}\right|_0^2 (\sin\theta - \cos\theta)\, d\theta$$

$$= \frac{8}{3}\int_0^{2\pi} (\sin\theta - \cos\theta)\, d\theta$$

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Similarly for S_3 : $x = -3$, $y^2 + z^2 \leq 4$

Unit normal vector 'outward' from S_3 $= -\hat{i}$ — 0.5

$\vec{F} \cdot \hat{n} = y - z$ — 0.5

$\vec{F} \cdot \hat{n}$ in cylindrical coordinates $= r(\cos\theta - \sin\theta)$

$$\iint_{S_3} \vec{F} \cdot \hat{n} \, dS = \int_0^{2\pi}\int_0^{2} r(\cos\theta - \sin\theta) r dr d\theta$$ — 1

$$= \frac{8}{3}\int_0^{2\pi} (\cos\theta - \sin\theta) . d\theta$$

For S_2 : $y^2 + z^2 = 4$, $-3 \leq x \leq 3$

Normal $= \text{grad}(S_2) = 2y\hat{j} + 2z\hat{k}$

$$\hat{n} = \frac{y\hat{j} + z\hat{k}}{\sqrt{y^2+z^2}} = \frac{y\hat{j} + z\hat{k}}{2}$$ — 0.5

$\vec{F} \cdot \hat{n} = \frac{1}{2}(y^4 + z^4)$, — 0.5

In Cylindrical Coordinates $\vec{F} \cdot \hat{n} = \frac{2^4}{2}(\cos^4\theta + \sin^4\theta)$

$dS = 2\, d\theta\, dx.$

$$\iint_{S_2} \vec{F} \cdot \hat{n} \, dS = 16\int_{-3}^{3}\int_0^{2\pi} (\cos^4\theta + \sin^4\theta) \, d\theta \, dx$$ — 1

$$= \int_{-3}^{3} dx \int_0^{2\pi} (\cos^4\theta + \sin^4\theta)\, d\theta$$

$$= x\Big|_{-3}^{3} \times 16 \int_0^{2\pi} (\cos^4\theta + \sin^4\theta)\, d\theta$$

$$= 96 \int_0^{2\pi} (\cos^4\theta + \sin^4\theta)\, d\theta$$

$$\text{Flux through } S = \iint_S \vec{F}\cdot n\, dS = \iint_{S_1} \vec{F}\cdot n\, dS + \iint_{S_2} \vec{F}\cdot n\, dS + \iint_{S_3} \vec{F}\cdot \hat{n}\, ds$$

$$= \iint_{S_2} \vec{F}\cdot \hat{n}\, dS$$ (flux from S_1 & S_3 cancel out)

$$= 96 \int_0^{2\pi} (\cos^4\theta + \sin^4\theta)\, d\theta$$

$$= \frac{96 \times 3}{2}\pi = 144\pi$$

OR

Since S is simple, closed, piecewise smooth we can apply the Divergence Theorem -1

$$\mathrm{Div}(\vec{F}) = 3y^2 + 3z^2 = 3(y^2 + z^2)$$ -1

In the region $y^2+z^2 \le 4$, $-3 \le x \le 3$

In cylindrical coordinates

$$r^2 \le 4 \Rightarrow 0 \le r \le 2$$

$$0 \le \theta \le 2\pi \quad -2$$

$$-3 \le x \le 3$$

$$y = r\cos\theta, \quad z = r\sin\theta$$

Thus $\displaystyle \iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_R \mathrm{Div}(F)\,dV = \int_{-3}^{3}\int_{0}^{2\pi}\int_{0}^{2} 3r^2\, r\,dr\,d\theta\,dx \quad -2$

$$= \int_{-3}^{3} dx \int_{0}^{2\pi} d\theta \int_{0}^{2} 3r^3\,dr$$

$$= \cancel{9\times}\ 6 \times 2\pi \times 3\left.\frac{r^4}{4}\right|_0^2$$

$$= 18\pi \times 2\left(\frac{16}{4}\right)$$

$$= 144\pi$$