SOLUTION KEY

Name:
HU ID:
Section:

HABIB UNIVERSITY

Math 102 Test 1

Spring Semester 2023

INSTRUCTIONS:

Please show all your work wherever possible and attempt all questions. You may use a calculator, unless stated otherwise in the question. Show the work and explain your thinking wherever possible/applicable. You have 60 minutes. Good luck!

1. The parametric equations of a cycloid are given as [2]

$$x = 9t - 9\sin(t),\ y = 9 - 9\cos(t)$$

Find the speed of this cycloid at points where the tangent line is horizontal.

Speed: $|\vec{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$\frac{dx}{dt} = 9 - 9\cos t$

$\frac{dy}{dt} = 9\sin t$

If tangent line is horizontal:

$\frac{dx}{dt} \neq 0 \Rightarrow 1 - \cos t \neq 0$

$\Rightarrow \cos t \neq 1 \Rightarrow t \neq 0, \pm 2\pi,$

and $\frac{dy}{dt} = 0 \Rightarrow \sin t = 0$

$\Rightarrow t = 0, \pm\pi, \pm 2\pi, \dots$

Therefore, t must be odd coeff. of π.

$t = \pm\pi, \pm 3\pi, \pm 5\pi, \dots$

Speed: $|\vec{v}| = \sqrt{(9 - 9\cos t)^2 + (9\sin t)^2}$

At $t = \pi$, $|\vec{v}| = \sqrt{(9 - 9\cos\pi)^2 + (9\sin\pi)^2}$

$= \sqrt{(9 - (-9))^2 + 0}$

$= \sqrt{(18)^2}$

$|\vec{v}| = 18$

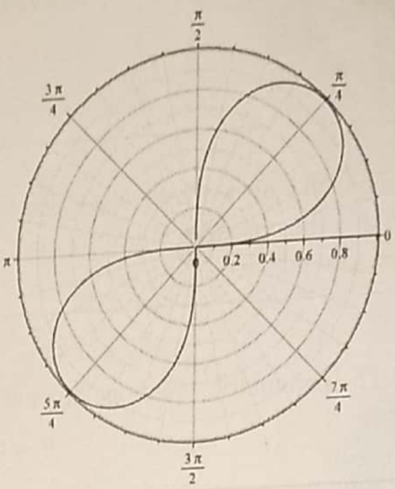
At $t = -\pi$, $|\vec{v}| = \sqrt{(9 - 9\cos(-\pi))^2 + (9\sin(-\pi))^2}$

$|\vec{v}| = \sqrt{(9 - (-9))^2 + 0} = 18$

speed of the cycloid is 18 for $t = \pm\pi, \pm 3\pi, \dots$

Scanned with CamScanner

2. Find the area enclosed by the curve $r = \sqrt{\sin(2\theta)}$.



$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

$$A = \frac{1}{2}\left[2\int_0^{\pi/2} (\sin(2\theta))\, d\theta\right]$$

$$A = \int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$A = -\frac{\cos 2\theta}{2}\Big|_0^{\pi/2}$$

$$A = -\frac{1}{2}\left(\cos \cancel{2}\left(\frac{\pi}{\cancel{2}}\right) - \cos(0)\right) \Rightarrow A = -\frac{1}{2}(-1-1) \Rightarrow A = \frac{+2}{2}$$

$$\Rightarrow \boxed{A = 1}$$

3. Find an equation of the largest sphere contained in the cube determined by the planes [2]

$$x = 2, x = 6; y = 5, y = 9; \text{and } z = -1, z = 3.$$

Finding center of the cube (which is also the center of the sphere):

$$x_0 = \frac{x_1 + x_2}{2} \Rightarrow x_0 = \frac{2+6}{2} \Rightarrow x_0 = 4$$

$$y_0 = \frac{y_1 + y_2}{2} \Rightarrow y_0 = \frac{5+9}{2} \Rightarrow y_0 = 7$$

$$z_0 = \frac{z_1 + z_2}{2} \Rightarrow z_0 = \frac{-1+3}{2} \Rightarrow z_0 = 1$$

center at $(4, 7, 1)$

Radius of the sphere: $r = \frac{|x_2 - x_1|}{2} \Rightarrow r = \frac{|6-2|}{2} \Rightarrow \boxed{r = 2}$

Equation of a sphere: $(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$

$$\Rightarrow \boxed{(x-4)^2 + (y-7)^2 + (z-1)^2 = 2^2}$$

4. In Cartesian coordinates, write an equation for the tangent line to $r = {}^{1}/_{\theta}$ at $\theta = {}^{\pi}/_{2}$. [2]

$x = r\cos\theta \Rightarrow x = \frac{\cos\theta}{\theta}$

$y = r\sin\theta \Rightarrow y = \frac{\sin\theta}{\theta}$

$\frac{dx}{d\theta} = \frac{-\theta\sin\theta - \cos\theta}{\theta^2}$

$\frac{dy}{d\theta} = \frac{\theta\cos\theta - \sin\theta}{\theta^2}$

$\frac{dy}{dx} = \frac{\sin\theta - \theta\cos\theta}{\cos\theta + \theta\sin\theta}$

At $\theta = \pi/2$

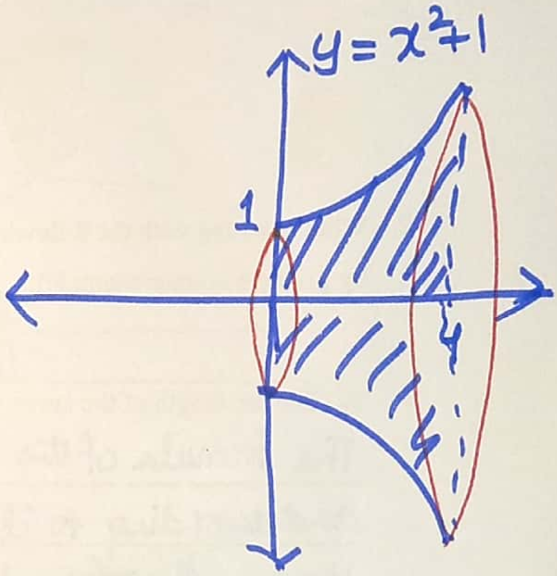
$\frac{dy}{dx} = \frac{1 - (\pi/2)(0)}{0 + (\pi/2)\,1} = \frac{2}{\pi}$

At $\theta = \pi/2$, $x = 0$, $y = 2/\pi$

Eq. of tangent line:

$\boxed{y = \frac{2}{\pi}x + \frac{2}{\pi}}$

5. Find the region bounded by $y = x^2 + 1, x = 0, x = 4$ and the $x-$axis and rotated about $y = 0$. [4]



$\int_0^4 \pi\left[(x^2+1)\right]^2 dx$

$= \pi \int_0^4 (x^4 + 2x^2 + 1)\, dx$

$= \pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_0^4$

$= \pi \left[\frac{4^5}{5} + \frac{2(4)^3}{3} + 4\right] \Rightarrow \pi\left(\frac{3772}{15}\right) = \frac{3772\pi}{15}$

$\Rightarrow \boxed{790.006}$

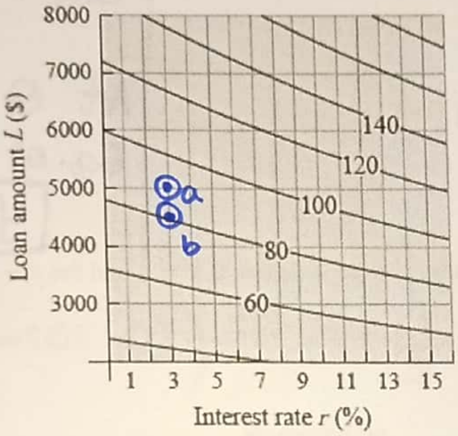
6. Let $f(r, L)$ be the montly payment of a 5-year car loan as a function of the interest rate r and the loan amount L. The figure given below is a contour plot of $f(r, L)$. Use this plot in each part to:

a) Estimate the monthly payment on a loan of \$5000 at an interest rate of 3%. [1]

\$90.

b) Estimate the loan amount if the monthly payment is \$80 and the interest rate is 3%. [1]

\$4500



7. What is wrong with the following statements (Explain your answer by stating the correct answer or by giving a counterexample): [3]

a) The arc length of the curve $y = \sin(x)$ from $x = 0$ to $x = \pi/4$ is $\int_0^{\pi/4} \sqrt{1 + \sin^2(x)}dx$.

The formula of the arc length is $\int_a^b \sqrt{1+[f'(x)]^2}dx$ and according to it, $f'(x) = \cos x$.
Hence, the formula should be $\int_0^{\pi/4} \sqrt{1+\cos^2 x}\, dx$.

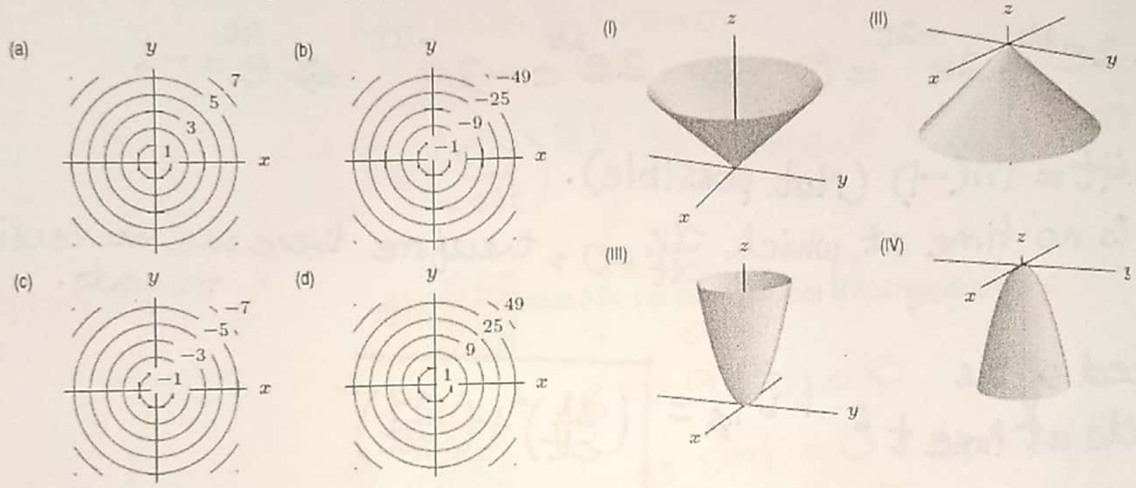
b) The solid obtained by rotating the region bounded by the curves $y = 2x$ and $y = 3x$ betweeen $x = 0$ and $x = 5$ around the $x - axis$ has volume $\int_0^5 \pi (3x - 2x)^2 dx$.

According to the formula, it should be
$\int_0^5 \pi [(3x)^2 - (2x)^2] dx$

c) Any polar curve that is symmetric about both the x and y axes must be a circle, centered at the origin.

Not always true. Counterexamples: curves that have petals symmetric to x and y axes (like $\sin 4\theta$, or in general, $\cos n\theta$ and $\sin n\theta$ when n is even)

8. Match the contour diagrams (a)-(d) with the surfaces (I)-(IV). [2]



a) I

b) IV

c) II

d) III

9. The motion of two particles, A and B, is given by the following parametric equations, where t is the time in seconds.

Particle A:	Particle B:
$x(t) = e^{2t} - e^{-2t}$	$x(t) = 4e^{t}$
$y(t) = 3e^{-2t} + e^{2t}$	$y(t) = 7e^{-t}$

a) Does the particle A's curve ever have a vertical tangent to its curve? Explain. [2]

For a vertical tangent $\frac{dy}{dt} \neq 0$ and $\frac{dx}{dt} = 0$.

$\frac{dx}{dt} = 2e^{2t} + 2e^{-2t} = 0 \Rightarrow 2e^{2t} = -2e^{-2t} \Rightarrow e^{4t} = -1$

$\Rightarrow 4t = \ln(-1)$ (Not possible).

There is no time at which $\frac{dx}{dt} = 0$, therefore there are no vertical tangents.

b) Which particle is moving faster at $t = 4$ seconds? [2]

Speed of the particle at time t: $|\vec{v}|_A = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$|\vec{v}|_A$ at $t=4$ $= \sqrt{(2e^{8} + 2e^{-8})^2 + (-6e^{-8} + 2e^{8})^2} = \sqrt{35544450.08 + 35544418.08}$

≈ 8431.42 units/sec

$|\vec{v}|_B$ at $t=4$ $= \sqrt{(4e^{4})^2 + (-7e^{-4})^2} = \sqrt{47695.34423} \approx 218.39$ units/sec.

$|\vec{v}|_A > |\vec{v}|_B$ at $t=4$ $\rightarrow$ $\boxed{\text{Particle A is faster at } t=4}$

c) Write a definite integral expressing the distance covered by the particle B in the first 4 seconds. You do not need to evaluate the integral. [1]

$\int_0^4 v\,dt = \int_0^4 \sqrt{(4e^{t})^2 + (-7e^{-t})^2}\,dt$