

Quiz 3

CS/CE 412/471 Algorithms: Design and Analysis, Spring 2025

12 Mar, 2025. 4 questions, 20 points, 3 printed sides

Reference

DFS(G)	DFS-VISIT(G, u)
1 for each vertex $u \in G.V$	1 $time = time + 1$
2 $u.color = \text{WHITE}$	2 $u.d = time$
3 $u.\pi = \text{NIL}$	3 $u.color = \text{GRAY}$
4 $time = 0$	4 for each vertex $v \in G.adj[u]$
5 for each vertex $u \in G.V$	5 if $v.color == \text{WHITE}$
6 if $u.color == \text{WHITE}$	6 $v.\pi = u$
7 DFS-VISIT(G, u)	7 DFS-VISIT(G, v)
	8 $time = time + 1$
	9 $u.f = time$
	10 $u.color = \text{BLACK}$

The input to a *shortest-paths problem* is a weighted, directed graph $G = (V, E)$, with a weight function $w : E \rightarrow \mathbb{R}$ mapping edges to real-valued weights. The *weight* $w(p)$ of path $p = \langle v_0, v_1, \dots, v_k \rangle$ is the sum of the weights of its constituent edges: $w(p) = \sum_{i=1}^k w(v_{i-1}, v_i)$. We define the *shortest-path weight* $\delta(u, v)$ from u to v by

$$\delta(u, v) = \begin{cases} \min\{w(p) : u \rightsquigarrow v\} & \text{if there is a path from } u \text{ to } v, \\ \infty & \text{otherwise.} \end{cases}$$

A *shortest path* from vertex u to vertex v is then defined as any path p with weight $w(p) = \delta(u, v)$. The *predecessor subgraph* $G_\pi = (V_\pi, E_\pi)$ induced by the π values is defined as:

$$V_\pi = \{v \in V : v.\pi \neq \text{NIL}\} \cup \{s\}$$
$$E_\pi = \{(v.\pi, v) \in E : v \in V_\pi - \{s\}\}$$

INITIALIZE-SINGLE-SOURCE(G, s)

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1 for each vertex  $v \in G.V$ 
2    $v.d = \infty$ 
3    $v.\pi = \text{NIL}$ 
4  $s.d = 0$ 
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RELAX(u, v, w)

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1 if  $v.d > u.d + w(u, v)$ 
2    $v.d = u.d + w(u, v)$ 
3    $v.\pi = u$ 
```

Computation of shortest path begins with a call to INITIALIZE-SINGLE-SOURCE(G, s) and the d attribute is *only* changed through calls to RELAX(u, v, w).

Triangle inequality

For any edge $(u, v) \in E$, we have $\delta(s, v) \leq \delta(s, u) + w(u, v)$.

Upper-bound property

We always have $v.d \geq \delta(s, v)$ for all vertices $v \in V$, and once $v.d$ achieves the value $\delta(s, v)$, it never changes.

No-path property

If there is no path from s to v , then we always have $v.d = \delta(s, v) = \infty$.

Convergence property

If $s \rightsquigarrow u \rightarrow v$ is a shortest path in G for some $u, v \in V$, and if $u.d = \delta(s, u)$ at any time prior to relaxing edge (u, v) , then $v.d = \delta(s, v)$ at all times afterward.

Path-relaxation property

If $p = \langle v_0, v_1, \dots, v_k \rangle$ is a shortest path from $s = v_0$ to v_k , and the edges of p are relaxed in the order $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$.

This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations of the edges of p .

Predecessor-subgraph property

Once $v.d = \delta(s, v)$ for all $v \in V$, the predecessor subgraph is a shortest-paths tree rooted at s .

BELLMAN-FORD(G, w, s)

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1  INITIALIZE-SINGLE-SOURCE( $G, s$ )
2  for  $i = 1$  to  $|G.V| - 1$ 
3      for each edge  $(u, v) \in G.E$ 
4          RELAX( $u, v, w$ )
5  for each edge  $(u, v) \in G.E$ 
6      if  $v.d > u.d + w(u, v)$ 
7          return FALSE
8  return TRUE
```

Lemma 22.2

Let $G = (V, E)$ be a weighted, directed graph with source vertex s and weight function $w : E \rightarrow \mathbb{R}$, and assume that G contains no negative-weight cycles that are reachable from s . Then, after the $|V| - 1$ iterations of the **for** loop of lines 2–4 of BELLMAN-FORD, $v.d = \delta(s, v)$ for all vertices v that are reachable from s .

Corollary 22.3

Let $G = (V, E)$ be a weighted, directed graph with source vertex s and weight function $w : E \rightarrow \mathbb{R}$. Then, for each vertex $v \in V$, there is a path from s to v if and only if BELLMAN-FORD terminates with $v.d < \infty$ when it is run on G .

Problems

Do any $n = 4$ problems. In case you do more, I will only check the first n . In proving any inequality, property, or corollary from above, you may assume all other inequalities, properties, or lemmas that precede it.

1. (5 points) Prove an inequality or property above.

Solution: Reference proofs of the Triangle Inequality and of the Upper-bound, No-path, Convergence, Path-relaxation, and Predecessor-subgraph properties are provided in Section 22.5 of the book.

5. (5 points) Prove Corollary 22.3.

Solution:

Proof. $s \rightsquigarrow v \iff$ BELLMAN-FORD(G) terminates with $v.d < \infty$.

Case: \implies

At initialization, $v.d = \infty$ and $s.d = 0$. If $v = s$, then the claim holds.

Otherwise, there exists a path from s to v containing k edges where $1 \leq k \leq |G.V| - 1$.

Let the path be $\langle v_0, v_1, v_2, \dots, v_k \rangle$ where $v_0 = s$ and $v_k = v$.

Consider iterations of the For loop on lines 2 to 4.

At the end of the 1st iteration, $v_1.d = \delta(s, v_1) < \infty$.

At the end of the 2nd iteration, $\text{RELAX}(v_1, v_2)$ has executed and $v_2.d < \infty$.

Inductively, at the end of the k -th iteration, $\text{RELAX}(v_{k-1}, v_k)$ has executed and $v_k.d < \infty$.

Any subsequent iterations either do not change $v_k.d$ or reduce it.

Case: \Leftarrow

At initialization, $v.d = \infty$ and $s.d = 0$. If $v = s$, then the claim holds.

Otherwise, consider iterations of the For loop on lines 2 to 4.

At the end of the 1st iteration, $v.d < \infty$ only if there is a path from s to v of length at most 1.

At the end of the 2nd iteration, $v.d < \infty$ only if there is a path from s to v of length at most 2.

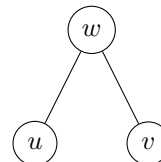
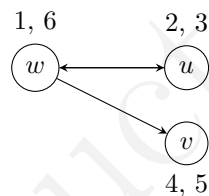
Inductively, after iteration $i \leq |G.V| - 1$, it holds that $v.d < \text{infity} \implies s \rightsquigarrow^i v$. \square

6. (5 points) Show how $\text{DFS}(G)$ can be used to detect a cycle in G .

Solution: A cycle exists in G if, in executing $\text{DFS}(G)$, a vertex is encountered at line 5 of $\text{DFS-VISIT}(G, u)$ whose color is GRAY. This corresponds to a **back edge** as described in the book.

7. (5 points) Give a counterexample to the conjecture that if a directed graph G contains a path from u to v , and if $u.d < v.d$ in a depth-first search of G , then v is a descendant of u in the depth-first forest produced.

Solution: DFS on the directed graph, G , on the left yields the depth-first forest on the right. G contains a path from u to v , and $u.d < v.d$ in a DFS of G , but v is not a descendant of u in the depth-first forest produced.



8. (5 points) Give a counterexample to the conjecture that if a directed graph G contains a path from u to v , then any depth-first search must result in $v.d \leq u.f$.

Solution: DFS on the above graph is a counterexample. G contains a path from u to v , but $v.d > u.f$.