LINEAR ALGEBRA

day / date:

the branch of mathematics concerned with linear equations, matrices, determinants, vector spaces, etc.

LINEAR → RELATING TO THE FIRST DEGREE; HAVING NO VARIABLE RAISED TO ANY POWER

LINEAR EQUATIONS: A line in the xy-plane is an equation of the form

$$a_1 x + a_2 y = b \quad \text{—— (1)}$$

Equation (1) is called linear equation in the variables x and y. We define a linear equation in the n variables $x_1, x_2, \ldots, x_n$ to be one that can be expressed in the form

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b \quad \text{— (2)}$$

where $a_1, a_2, \ldots, a_n$ and b are REAL CONSTANTS. The variables in a linear equation are called the unknowns and in (2) unknowns are

$$x_1, x_2, x_3, \ldots, x_n$$

the real constants $a_1, a_2, \ldots, a_n$ are also called coefficients

* variables should have no power
* variables do not appear as arguments for trigonometry, logarithmic, or exponential functions.
* do not indulge any products or roots of variables

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A general system of m linear equations in n unknowns will be written

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad (3)$$

the first subscript i $(1 \le i \le m)$ on the coefficient a_{ij} indicates the equation in which the coefficients occurs and the second subscript j $(1 \le j \le n)$ indicates which unknown it multiplies linear system (3) can be abbreviated by writing only the rectangular array of numbers.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

This is called AUGMENTED MATRIX due to $\boxed{b_i\text{'s}}$ $(1 \le i \le m)$

AUGMENT → increase / enlarge due to presence of the entries on RHS of the linear system (3).

* if a system of equations that has NO SOLUTION is said to be inconsistent. If there is atleast one solution then it is CONSISTENT

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MATRICES

MATRIX: A matrix is a rectangular array of numbers enclosed in brackets. The numbers in ~~brackets~~ array are called entries in the matrix. (2) the size of a matrix is described by specifying the number of rows (horizontal lines) and columns (vertical lines)

Size : row x column

↳ if a matrix has row = column then it is a SQUARE MATRIX

↳ if a matrix has only one column is also called a COLUMN VECTOR. and similarly a matrix which has only one row is called a ROW VECTOR.

→ the entries $a_{11}, a_{22}, \ldots, a_{nn}$ in a square matrix A of order n, are said to be on the main diagnol of A as shown here in following matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & a_{22} & \cdots & a_{2n} \\ & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

DIAGNOL OF A.

→ a square matrix in which all entries off the main dignol are zero is called DIAGNOL MATRIX

$$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

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→ if all entries are zero it is called a zero or null matrix, denoted by 0.

→ if A and B are any two matrices of same size then the sum A+B is the matrix obtained by ADDING together the corresponding entries in two matrices

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 4 & -2 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 5 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Find A+B and B+C.

SOLUTION A+B is NOT defined since A & B are of different size.

$$B + C = \begin{bmatrix} 3+1 & 2+0 \\ 5+0 & 6+1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 2 \\ 5 & 7 \end{bmatrix}$$

By the definition of EQUAL MATRICES the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

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can be written as

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{m\times 1}$$

further the $m \times 1$ matrix on the left hand side can be written as the product of two matrices given by

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m \times n \text{ matrix}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{m\times 1}$$

if we designate these matrices by A, X and B, respectively, the original system of m equations in n unknowns has been replaced by the single matrix equation.

$AX = B$, where A is called the coefficient matrix. But note the following

$$\underset{m\times n}{A} \quad \underset{n\times 1}{X} = \underset{m\times 1}{B}$$

INSIDE

OUTSIDE

No of columns of A = No of rows of X = n

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→ Two matrices A and B can be multiplied if the number of columns of A ~~and~~ are equal to the number of rows of B.

$$\underset{m\times r}{A}\ \underset{r\times n}{B} = \underset{m\times n}{AB}$$

INSIDE

OUTSIDE

Find AB, where

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \rightarrow 2\times 3, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} \rightarrow 3\times 4$$

The product AB is 2×4 matrix

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix} = \underset{2\times 4}{C} \text{ (SAY)}$$

where

$$c_{11} = 1(4) + 2(0) + 4(2) = 12$$

$$c_{12} = 1(1) + 2(-1) + 4(7) = 27$$

$$c_{13} = 1(4) + 2(3) + 4(5) = 30$$

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then

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -9 & 26 & 12 \end{bmatrix}$$

Two matrices A and B comute if $AB = BA$ but in general $AB \neq BA$

→ if A and B ae two square matrices of same size, then find the condition such that

$$(A+B)^2 = A^2 + B^2 + 2AB$$

<u>NOTE</u>: in such type of questions don't use matrix entries. Just use matrix symbols.

$$(A+B)^2 = (A+B)(A+B)$$

$$= A^2 + AB + BA + B^2$$

↳ if A and B commute then

$$AB = BA$$

$$\therefore (A+B)^2 = A^2 + 2AB + B^2$$

→ if A, B and C are matrices such that AB and BC are defined then $A(BC) = (AB)C$ which is also called ASSOCIATE LAW for multiplication.

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→ If A is $m \times n$ matrix and D is diagnol matrix of order m then find the multiplication rule for DA.

Solution: DA

$$\underbrace{\begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \dots & \dots & d_m \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_{A}$$

$$= \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & \dots & d_1 a_{1n} \\ d_2 a_{21} & d_2 a_{22} & \dots & d_2 a_{2n} \\ \vdots & \vdots & & \vdots \\ d_m a_{m1} & d_m a_{m2} & \dots & d_m a_{mn} \end{bmatrix}$$

PROOF

(1) Show that if AB and BA are both defined, then AB and BA are square matrices.

$\underbrace{A_{m_1 \times n_1} \quad \& \quad B_{m_2 \times n_2}}$

$n = n_1 = m_2 \longrightarrow \underbrace{A_{m_1 \times n} \text{ is } B_{n \times n_2}}$

$m_1 = n_2 = m$

$A_{m \times n} \; B_{n \times m} = AB_{m \times m}$ * this is square matrix

LINEAR COMBINATIONS

we showed in Example 5 that

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

The column matrices of AB can be expressed as liner carbinations of column matrices of A as follows.

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

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Express each row matrix of AB as a linear combination of the row matrices of B.

$$AB = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} \begin{bmatrix} 6 & -2 & 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} + \dots$$

PROOF (b) Show that if A is an $m \times n$ matrix & A(BA) is defined, then B is an $n \times m$ matrix.

Mathematically:

→ Let A is $m \times n$ matrix & B is $m_1 \times n_1$ matrix

→ If A(BA) is defined then first (BA) should be defined so $n_1 = m$ so (BA) becomes $m_1 \times n$ size matrix

→ Now if A(BA) is defined then $n = m_1$ so the size becomes $m \times n$.

→ So this implies matrix B becomes $n \times m$.

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PROOF (a) Show that if A has a raw of zeros and B is any matrix for which AB is defined, then AB also has raw of zeros.

Solution

Assume that the entries of i-th raw of A are all zeros. We claim that the ith-raw of AB is a raw of zeros.

To see this, pick an entry c_{ij} in i-th raw of AB. By definition of multiplication of AB, we have.

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Since the ith raw of A is zero we have $a_{i1} = a_{i2} = a_{in}$

$$c_{ij} = 0b_{1j} + 0b_{2j} + \dots + 0b_{nj} = \sum_{k=1}^{n} 0b_{kj} = 0$$

Hence i-th row of AB is a raw of zeros.

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→ If A is matrix and C is any scalar, then the product CA is the matrix obtained by multiplying each entry of A by c.

If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}$ then $2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}$

NOTE: Two matrices A and B commute if AB = BA but in general AB ≠ BA ~~but in general~~ ~~AB ≠ BA~~

NOTE: If A, B and C are matrices such that AB and BC are defined then $A(BC) = (AB)C$ which is also called ASSOCIATIVE LAW for multiplication

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BASIC RESULTS:

① If A is $m \times n$ matrix and d is dignol matrix of order m the find the multiplication rule for DA.

SOLUTION DA

$$= \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & \cdots & \cdots & d_m \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & \cdots & d_1 a_{1n} \\ d_2 a_{21} & d_2 a_{22} & \cdots & d_2 a_{2n} \\ \vdots & \vdots & & \vdots \\ a_m a_{m1} & d_m a_{m2} & \cdots & d_m a_{mn} \end{bmatrix}$$

$\therefore$ DA is obtained by multiplying d_1 with the first raw of A d_2 with secnd raw of A, ... and finally d_m with mth raw of A.

② If A is $m \times n$ matrix and $E = diag(e_1, e_2, \ldots, e_n)$ then AE is obtained by multiplying the first column of A by e_1, ..., nth column of A by e_n.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} e_1 & \cdots & \cdots & 0 \\ 0 & e_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & e_n \end{bmatrix}$$

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$$\begin{bmatrix} e_1 a_{11} & e_2 a_{12} & \cdots & e_n a_{1n} \\ & & & \vdots \\ e_1 a_{m1} & e_2 a_{m2} & \cdots & e_n a_{mn} \end{bmatrix}$$

(3) $AE = A$ and $DA = A$, what should be the ~~results~~ values of $d_1, d_2, \ldots d_m$ and $e_1, e_2, \ldots, e_n$

<u>ANSWER</u> only possible values are $d_1 = d_2 = \ldots = d_m = 1$ and $e_1 = e_2 = \ldots = e_n = 1$ in both cases the diagnol matrices E and D became special matrix which is called IDENTITY MATRIX.

Notation: $E = I_n$ OR Just I
$D = I_m$ OR Just I

diagnol matrix = idetity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & \cdots & & 1 \end{bmatrix}$$

(4) If A is a matrix of order n then $AI_n = I_nA = A$, this means that I_n COMMUTES with every matrix with which it multiplies.

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INVERSE MATRICES:

→ If Q is square matrix and A is also a square matrix (Both of order n) then Q is said to be an inverse of A if and only if → converse holds

$$QA = AQ = I_n = I$$

is denoted by A^{-1} i.e $Q = A^{-1}$

** Note: ① The matrix A is described as NONSINGULAR OR INVERTIBLE if an inverse of A exists and singular if $\boxed{\text{no}}$ inverse of A exists.

② Both Q and A are square matrices of order n because they are commuting to give I_n.

**

→ If B and C are both inverses of matrix A then $B = C$

PROOF ~~proof~~ ∵ B is an inverse of A, we have $BA = I$ — ①

multiplying both sides (of ①) on the right by C gives

$$(BA)C = IC = C \quad — ②$$

but $(BA)C = B(AC)$ ∵ $AC = I$

$$= B(I) = B \quad — ③$$

so that $C = B$ from ② and ③

*RESULT: Inverse of A NONSINGULAR Matrix is unique.

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**

→ If A and B are invertible matrices of same size, then $(AB)^{-1} = B^{-1}A^{-1}$

PROOF: ~~ARRAY~~ :

Consider $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= A(I)A^{-1}$

$= AA^{-1} = I$ — ①

also

$(B^{-1}A^{-1})(AB) = B^{-1}\underbrace{(A^{-1}A)}_{I}B = B^{-1}IB$

$= B^{-1}B = I$ — ②

from ① and ②

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

RESULT: hence if $A_1, A_2, \ldots, A_n$ are invertible matrices of same size then

$(A_1 A_2 \ldots A_n)^{-1}$

$= A_n^{-1} A_{n-1}^{-1} \ldots A_2^{-1} A_1^{-1}$

SUMMARY

① A product of any number of invertible matrices is invertible

② The inverse of the product is the product of inverses in the reverse order.

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(A) If A is an invertible matrix then:

$$(A^{-1})^{-1} = A$$

PROOF:

$$\because AA^{-1} = A^{-1}A = I$$

$$(A^{-1})^{-1} = A$$

(*) $(A^n)^{-1} = (A^{-1})^n$ for $n = 0, 1, 2, \cdots$

TRANSPOSE OF A MATRIX

→ If A is $m \times n$ matrix then a transpose of A (Denoted by A^T) is obtained by interchanging rows and columns of A.

order (size) of $A^T = n \times m$

Example: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \rightarrow 2 \times 3$

* first row of A becomes first column of A^T

$A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix} \rightarrow 3 \times 2$

$(A^T)^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

$(A^T)^T = A$

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** → If A and B are of <u>same size</u> then

$(A+B)^T = A^T + B^T$

<u>Example</u> : let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$(A+B)^T = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}^T$ — (1)

also $A^T + B^T$

$= \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}$

$= \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix}$ — (2)

$\boxed{\therefore (A+B)^T = A^T + B^T}$

* in transpose left to right diagnol remains same while ~~to~~ right to left gets swapped.

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→ A square matrix A is called SYMMETRIC if $\boxed{A^T = A}$

(1) Identity matrix is symmetric

$\because I^T = I$ (ALWAYS)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) every dignol matrix is symmetric

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}^T = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

when taking the transpose, dignol entries of a square matrix don't change their position.

→ A square matrix A is called SKEW-SYMMETRIC if $A^T = -A$

example $B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$ is skew symmetric

$\because B^T = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = -B$

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dignol entries in skew symmetric matrix are always zero.

* Theorem says that if A is invertible then $Ax = 0$ has trivial solution.

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→ If $AX = B$ represents a system of n equations in n variables then prove that solution is unique if A is invertible.

→ ability to take inverse of matrix

PROOF

~~proof~~ let say we have two solutions X_1 and X_2

$$AX_1 = B \quad , \quad AX_2 = B$$

$$AX_1 = AX_2$$

Because A is invertible.

$$A^{-1}AX_1 = A^{-1}AX_2 \qquad \because A^{-1}A = I$$

$$IX_1 = IX_2$$

$$\boxed{X_1 = X_2} \qquad \text{(one solution = unique solution)}$$

$\therefore X = A^{-1}B$ is a solution of $AX = B$ (provided A is invertible)

OR

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B \text{ is UNIQUE}$$

HOW TO FIND A^{-1} ?

For this we start elementary row operations.

Elementary row operations are the following:

✓ ① Multiply a row by a ~~e~~ nonzero constant.

e.g
$$B = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

R_n : nth row

consider $R_2 \rightarrow 2R_2$ gives.

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 4 & -2 & 6 & 12 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

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(2) Interchange two rows

e.g consider ($R_1 \leftrightarrow R_2$) gives

$$C = \begin{bmatrix} 2 & -1 & 3 & 6 \\ 1 & 0 & 2 & 3 \\ 1 & 4 & 4 & 0 \end{bmatrix} \text{ from } B$$

(3) Add a multiple of one row to another row

e.g. for $B = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$

consider $R_1 \rightarrow R_1 + 2R_2$ gives

$$D = \begin{bmatrix} 5 & -2 & 8 & 15 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

EQUIVALENT MATRICES

→ If B is a matrix and A is a matrix obtained from B by one or more elementary row operations then A is called ROW EQUIVALENT to B and vice versa and is denoted by $A \sim B$ or $B \sim A$

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ELEMENTARY MATRICES

An $n \times n$ matrix is called an elementary matrix if it can be obtained from the $n \times n$ $\boxed{\text{identity matrix}}$ I_n by performing a single elementary row operations.

e.g $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ is an elementary matrix

$$\because \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \underset{R_3+3R_1}{R_3 \rightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

i.e E is obtained from I_3 by performing the ERO

$$R_3 \rightarrow R_3 + 3R_1$$

* Apply $R_3 \rightarrow R_3 + 2R_1$ on identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = E$$

when we perform EA we get B.

* if you apply certain change in A you get B.

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let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 1 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 4 & 10 \end{bmatrix} = B$

$$R_3 \to R_3 + 3R_1$$

CONSIDER

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 1 & 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 4 & 10 \end{bmatrix} = B$$

the ERO has the same effect on a matrix as premultiplication of an elementary matrix

or in mathematical language if θ be any ** ERO and E be the elementary matrix corresponding to θ then

$$\boxed{\theta(A) = EA = B} \quad , \quad A \sim B$$

let $\theta_1, \theta_2, \dots, \theta_n$ be n number of EROS and

$E_1, E_2, \cdots, E_n$ are the corresponding elementary matrices such that when applied on an invertible matrix 'A' give 'I' (identity matrix)

ie $\theta_n \ldots \theta_2 \theta_1 (A) = \underbrace{E_n \cdots E_2 E_1}_{} A = I$ or $PA = I$,

$P = E_n \ldots E_2 E_1$ $\therefore$ A is invertible

$\Rightarrow PAA^{-1} = IA^{-1} \Rightarrow PI = A^{-1}$

now $PA = I \Rightarrow A \sim I$ $\qquad I = A^{-1}$

and $PI = A^{-1} \Rightarrow I \sim A^{-1}$

$\therefore$ EROS which transformed A into I also transformed I into A^{-1} $\therefore$ A^{-1} can be found by

$[A \,\vdots\, I] \xrightarrow{E.R.O.S} [I \,\vdots\, A^{-1}]$

METHOD TO FIND A^{-1}

→ $AX = B$ is called a consistent system of linear equations if there is atleast one solution, otherwise its called inconsistent

→ If $B \neq 0$ in $AX = B$, then $AX = B$ is called a NON-HOMOGENOUS SYSTEM

→ if $B = 0$ then $AX = B = 0$ is called a HOMOGENOUS SYSTEM

NOTE:- Homogenous system is ALWAYS consistent

$\because X = 0$ is always a solution of $AX = 0$ called a zero solution or trivial solution

→ to be consistent a row has to be completely zeros

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**

Q: Solve the following system (NON-HOMOGENOUS) of linear equations by finding the inverse of the coefficient matrix.

$$3x_1 + 4x_2 + 5x_3 = 12$$

$$x_1 - x_2 + 2x_3 = 2$$

$$2x_1 + x_2 + 3x_3 = 6$$

$AX = B$

$$\Rightarrow \underbrace{\begin{bmatrix} 3 & 4 & 5 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}}_{\text{COEFFICIENT MATRIX}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 2 \\ 6 \end{bmatrix}$$

consider

$$[A \,\vdots\, I] = \left[\begin{array}{ccc|ccc} 3 & 4 & 5 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

1st Row 1st COL

STEP 1: Make the entry $(1,1) = 1$ by any ERO for this perform $R_1 \leftrightarrow R_2$ to get

$$[A \,\vdots\, I] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 3 & 4 & 5 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

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STEP 2: Make entries $(2,1)$ and $(3,1) = 0$
for this perform the following two ERO operations

$R_2 \to R_2 - 3R_1$ $\qquad$ $R_3 \to R_3 - 2R_1$

to get

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 7 & -1 & 1 & -3 & 0 \\ 0 & 3 & -1 & 0 & -2 & 1 \end{array}\right]$$

STEP 3: Next to make the entry $(2,2) = 1$
for this perform $R_2 \to R_2 - 2R_3$ to

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & -2 \\ 0 & 3 & -1 & 0 & -2 & 1 \end{array}\right]$$

STEP 4: Make entries $(1,2)$ and $(3,2) = 0$, for this perform

$R_1 \to R_1 + R_2$ $\qquad$ $R_3 \to R_3 - 3R_2$ to get

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 2 & -2 \\ 0 & 1 & 1 & 1 & 1 & -2 \\ 0 & 0 & -4 & -3 & -5 & 7 \end{array}\right]$$

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STEP 5: Make entry (3,3) = 1 for this perform $R_3 \to \frac{-1}{4} R_3$ to get

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 2 & -2 \\ 0 & 1 & 1 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3/4 & 5/4 & -7/4 \end{array}\right]$$

STEP 6: In the final step perform $R_1 \to R_1 - 3R_3$ and $R_2 \to R_2 - R_3$ to get

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/4 & -7/4 & 13/4 \\ 0 & 1 & 0 & 1/4 & -1/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & 5/4 & -7/4 \end{array}\right]$$

$[I \,|\, A^{-1}]$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} -5 & -7 & 13 \\ 1 & -1 & -1 \\ 3 & 5 & -7 \end{bmatrix}$$

$$X = A^{-1}B$$

$$X = \frac{1}{4}\begin{bmatrix} -5 & -7 & 13 \\ 1 & -1 & -1 \\ 3 & 5 & -7 \end{bmatrix}\begin{bmatrix} 12 \\ 2 \\ 6 \end{bmatrix} \qquad \underline{X = A^{-1}B}$$

$$= \frac{1}{4}\begin{bmatrix} -60 - 14 + 78 \\ 12 - 2 - 6 \\ 36 + 10 - 42 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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PROOF

(d) ** Show that if A is invertible then prove that $(A^{-1})^T = (A^T)^{-1}$

$$(A^{-1})^T = (A^T)^{-1}$$

We know that $I = I$

$$(AA^{-1}) = I$$

Apply transpose on both sides

$$((A^{-1})(A))^T = I^T$$

$$A^T (A^{-1})^T = I$$

$$\underbrace{(A^T)^{-1} A^T}(A^{-1})^T = (A^T)^{-1} I$$

$$I (A^{-1})^T = (A^T)^{-1}$$

$$\boxed{(A^{-1})^T = (A^T)^{-1}}$$ ∴ hence proven

Q10) consider the matrix $A = \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix} = I_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5/2 & 1/2 \end{bmatrix}$$

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GAUSSIAN ELIMINATION: (Echelon form) A matrix having the following properties is said to be in row-echelon form

(1) If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. (we call this leading 1)

(2) If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

(3) In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

EXAMPLE: Solve the following system by gaussian elimination method

$$3x_1 + 4x_2 + 5x_3 = 12$$

$$x_1 - x_2 + 2x_3 = 1$$

$$2x_1 + x_2 + 3x_3 = 6$$

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SOLUTION:

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & 12 \\ 1 & -1 & 2 & 2 \\ 2 & 1 & 3 & 6 \end{array}\right]$$

STEP 1 Make $(1,1) = 1$, $R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 3 & 4 & 5 & 12 \\ 2 & 1 & 3 & 6 \end{array}\right]$$

STEP 2 Make $(2,1) = 0$ and $(3,1) = 0$

$R_2 \rightarrow R_2 - 3R_1$, $R_3 \rightarrow R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 7 & -1 & 6 \\ 0 & 3 & -1 & 2 \end{array}\right]$$

STEP 3 Make $(2,2) = 1$, $R_2 \rightarrow R_2 - 2R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & -1 & 2 \end{array}\right]$$

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STEP 4 Make $(3,2)=0$, $R_3 \to R_3 - 3R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -4 \end{array}\right]$$

STEP 5 Make $(3,3)=1$, $R_3 \to \frac{-1}{4} R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

which is the required echelon form, so the given linear system is reduced to

$$x_1 - x_2 + 2x_3 = 2 \quad \text{— (1)}$$
$$x_2 + x_3 = 2 \quad \text{— (2)}$$
$$x_3 = 1 \quad \text{— (3)}$$

$x_1 = 1$, $x_2 = 1$, $x_3 = 1$

NOTES (1) In row reduction process don't perform any steps by which you lose zeros OR 1's
(2) If possible then avoid the formation of fractions.

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A matrix is in reduced raw echelan form if

(1) it is already in echelan form
(2) each column that contains a leading 1 has zeros everywhere else. idetity matrix

EXAMPLES:- The following matrices are in the reduced raw-echelan form

① $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

② $$\begin{bmatrix} 0 & 1 & -2 & \boxed{0} & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

EXAMPLE Solve the following linear system by reducing the augmented matrix to reduced row-echelan form

$$3x_1 + 4x_2 + 5x_3 = 12$$
$$x_1 - x_2 + 2x_3 = 2$$
$$2x_1 + x_2 + 3x_3 = 6$$

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SOLUTION Here the augmented matrix is

$$\begin{bmatrix} 3 & 4 & 5 & 12 \\ 1 & -1 & 2 & 2 \\ 2 & 1 & 3 & 6 \end{bmatrix}$$

and its echelon form is given by

$$\begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(derived last time)

now to reduce it to reduced row-echelon form we proceed as follows:

$$\begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_1 \to R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

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now rewriting the linear ~~algebra~~ system again we get

$$x_1 = 1, \; x_2 = 1, \; x_3 = 1$$

required reduced row echelon form. In this case entries in the last (4th) column form the solution.

F

FUNDAMENTAL OF LINEAR ALGEBRA

If A is a matrix of order $m \times m$ then, there statement are equivalent

(a) A is invertible

(b) $Ax = 0$ has only trivial solution

(c) A can be transformed into row-reduced echelon form which is equal to I_m.

(d) A can be expressed as the product of any elementary matrices.

$a \Rightarrow b$

PROOF A is $m \times m$ matrix ~~where (m×m)~~ consider A is invertible and want to prove that $Ax = 0$ has only trivial solution

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Solution

$$AA^{-1}x = A^{-1}0 \qquad \because AA^{-1} = I$$

$$Ix = 0 \qquad \because Ix = x$$

$$x = 0$$

$b \Rightarrow c$

PROOF

consider $Ax = 0$ has only trivial solution and want to prove that A can be transformed into I_n or row-reduced echelon form.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 \end{array}\right]$$

By performing EROS A can be transformed

$$\left[\begin{array}{cccc|c} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \dots & 1 & 0 \end{array}\right] I_m$$

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$c \Rightarrow d$

PROOF consider A can be rewritten in the row-reduced echelon form and we want to show that A can be expressed as the product of elementary matrix.

$$[A \,\vdots\, I] \xRightarrow{EROs} [I \,\vdots\, A^{-1}]$$

$$\begin{array}{ccc} A & I & \\ \downarrow \text{ERO} & \downarrow \text{ERO} & EA = B \\ B & E & \end{array}$$

$$\underbrace{E_k E_{k-1} \cdots E_2 E_1}_{} A = I$$

$$PA = I$$

$$AA^{-1}P = A^{-1}I$$

$$IP = A^{-1}$$

$$\underline{I} \sim A^{-1}$$

$$P = E_k^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k$$

$d \Rightarrow a$

PROOF By using $(AB)^{-1} = A^{-1}B^{-1}$ theorem

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QUESTION:

For what values of 'K' does the following system have

(a) NO SOLUTION (b) ONLY ONE SOLUTION

(c) INFINITELY MANY SOLUTIONS.

$$kx + y = 1$$
$$x + ky = 1$$

SOLUTION The augmented matrix of the given system of equation is given by

$$A = \left[\begin{array}{cc|c} k & 1 & 1 \\ 1 & k & 1 \end{array}\right]$$, let us try to

find the echelon form of this matrix.

$$\Rightarrow A \sim \left[\begin{array}{cc|c} 1 & k & 1 \\ k & 1 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

→ after making diagnol 1 make zero ~~too~~

$$\sim \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & \underline{1-k^2} & 1-k \end{array}\right] \quad R_2 \rightarrow R_2 - kR_1$$

Let us discuss different cases;

(1) for exactly one solution $1 - k^2 \neq 0$

① must be transformed into the echelon

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form by making the entry (2,2) one by performing $R_2 \to \frac{R_2}{1-k^2}$ to get

$$\sim \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1 & 1/1+k \end{array}\right]$$

provided $1-k^2 \neq 0$
$k^2 \neq 1$

$\Rightarrow k \neq \pm 1 \to$ for $\boxed{\text{one solution}}$

using $k=-1$ in ① gives

$$\left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1-k^2 & 1-k \end{array}\right] = \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 0 & 2 \end{array}\right] \quad k=-1$$

$0 = 2$ (no solution)

so we have $\boxed{\text{no solution}}$ for $k=-1$
second row gives $0=2$ which is not possible

using $k=+1$ in ① gives.

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$, rewriting the linear

$0=0$ (infinitely many solutions)

system gives $x_1 + x_2 = 1$

here no of unknowns $= 2$, no of equations $= 1$,

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no of equations < no of unknowns
→ infinite solutions.

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which is less than no of unknowns. This gives $\boxed{\text{infinite solutions}}$ for $k = +1$.

NOTE: Here (infinite solutions case) x_1 is called a leading variable which corresponds to the leading 1 in the echelon form matrix of augmented

ie $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$ and x_2 is called a free variable which does not correspond to the leading 1!

We can write down solution like

$$x_1 = 1 - x_2 = 1 - t \quad \text{where} \quad x_2 = t$$

we have infinite values for x_1 and x_2

$\therefore$ we have infinite solutions for $k=1$.

Question 16

Find the condition that the b's must satisfy for the system to be consistent

$$6x_1 - 4x_2 = b_1$$
$$3x_1 - 2x_2 = b_2$$

solution:

OR $R_1 \to R_1 - 2R_2$

$$\left[\begin{array}{cc|c} 6 & -4 & b_1 \\ 3 & -2 & b_2 \end{array}\right] \xrightarrow{R_2 \rightleftarrows 2R_1} \left[\begin{array}{cc|c} 3 & -2 & b_2 \\ 6 & -4 & b_1 \end{array}\right]$$

$$b_1 - 2b_2 = 0$$
$$2b_2 = b_1$$

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$$\xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 3 & -2 & b_2 \\ 0 & 0 & b_1 - 2b_2 \\ & & \end{array}\right]$$

one can see that, to be consistent second raw has to get completely zeros.

$$\boxed{b_1 = 2b_2}$$

Question 23

since $Ax =$ has only $x = 0$ as a solution Theorem 1.6.4 guarantees that A is invertible. By Theorem 1.4.8 (b), A^k is also invertible.

$$(A^k)^{-1} = (A^{-1})^k$$

$$\underbrace{A^{-1}A^{-1}\ldots A^{-1}}_{k \text{ factors}} \underbrace{AA\ldots A}_{k \text{ factors}} = I$$

Because A^k is invertible, Theorem 1.6.4 allows us to conclude that $A^k x = 0$ has only trivial solution.

$$A^k x = 0$$

$$(A^k)^{-1} A^k x = (A^k)^{-1} 0$$

$$\underbrace{A^{-1}A^{-1}\ldots A^{-1}}_{k \text{ factors}} \underbrace{AA\ldots A}_{k \text{ factors.}} X = 0$$

$$IX = 0 \qquad \because IA = A$$

$X = 0$ has only trivial sol.

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Question 24

Let $Ax=0$ be a homogeneous solution of n linear eqs in n unknown and let Q be an invertible $n \times n$ matrix. Show that $Ax=0$ has just the trivial solution if and only if $(QA)x=0$ has just trivial solution.

PROOF

Let $Ax=0$ holds. Now we apply Q matrix from LHS we will have

$$Q(Ax) = Q0$$

$$(QA)x = 0 \quad \because \text{Associative property.}$$

Now we let $(QA)x=0$ and we apply Q^{-1}

$$\underbrace{Q^{-1}Q}Ax = Q^{-1}0$$

$$IAx = 0$$

$$Ax = 0$$

Question 25

Suppose that x_1 is a fixed matrix which satisfies the equation $Ax_1=b$. Further let x be any matrix whatsoever which satisfies $Ax=b$. We must then show that there is a matrix x_0 which satisfies both eq $x = x_1 + x_0$

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and $Ax_0 = 0$. clearly the ① eq implies that

$$x_0 = x - x_1$$

Given

$$x = x_1 + x_0$$

$$x_0 = x_1 - x$$

$$A(x_0) = A(x_1 - x)$$

$$A(x_0) = A(x_1) - A(x)$$

$$= b - b = 0$$

$$A(x_0) = 0$$

Now we show that $A(x_1 + x_0)$ has solution

$$A(x_1 + x_0) = A(x_1) + A(x_0)$$

$$A(x_1 + x_0) = b + 0$$

$$A(x_1 + x_0) = b$$

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REVISION :-

If A is an invertible square matrix then A^{-1} could be evaluated by the following method:

$$[A \,\vdots\, I] \xrightarrow{EROS} [I \,\vdots\, A^{-1}]$$

EROS → elementary row operations.

↳ identity matrix

** PROOF

Example

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then using the technique

$[A \,\vdots\, I] \xrightarrow{EROS} [I \,\vdots\, A^{-1}]$ prove that

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad ad-bc \neq 0$$

Solution

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right] \quad R_1 \to \frac{R_1}{a}$$

$$\sim \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d - cb/a & -c/a & 1 \end{array}\right] \quad R_2 \to R_2 - cR_1$$

$$\left(\frac{ad-bc}{a}\right)$$

$$= \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & \frac{ad-bc}{a} & -c/a & 1 \end{array}\right]$$

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$$\sim \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$R_2 \rightarrow \frac{a}{(ad-bc)} R_2$, $ad-bc \neq 0$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$R_1 \rightarrow R_1 - \frac{b}{a} R_2$

which completes the proof.

RESULT

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided $ad-bc \neq 0$

this number $ad-bc$ is ca

matrix A and is written

$$ad-bc = \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = |A|$$

DETERMINANT OF 3x3 MATRIX

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$|A| = a(ei-fh) - b(di-gf) + c(dh-eg)$$

$$\left[a \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} \right] - \left[b \times \begin{vmatrix} d & f \\ g & i \end{vmatrix} \right] + \left[c \times \begin{vmatrix} d & e \\ g & h \end{vmatrix} \right]$$

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here A is invertible

$\therefore A$ is a square matrix so we have the following result:

determinant of a nonsingular nonsquare matrix is not defined also if A^{-1} exists then $\det(A) \neq 0$

RESULT: If $\boxed{AX = B}$ is the system of n linear equations in n unknowns then:

① $\boxed{\text{Unique solution}}$ if A is invertible $(\det(A) \neq 0)$ and $X = A^{-1}B$

② If A^{-1} doesn't exist

(a) NO SOLUTION

(b) INFINITE SOLUTIONS

Let us consider examples when $AX = B$ is the system of two linear equations in two unknowns.

① For $\boxed{\text{unique solution}}$ consider the following system $\boxed{(X = A^{-1}B)}$

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consider

$$3x_1 + 4x_2 = 7 \quad \text{— (1)}$$

$$x_1 - x_2 = 0 \quad \text{— (2)}$$

* taking determinant of coefficient matrix

$$\Rightarrow \underbrace{\begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 7 \\ 0 \end{bmatrix}}_{B}$$

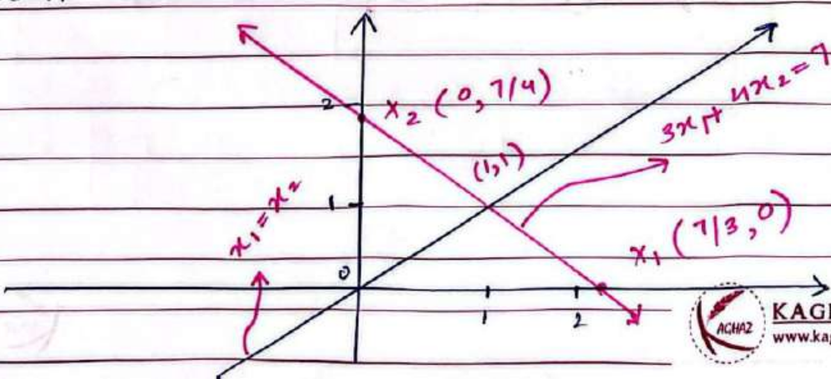
now $\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} = -3-4 = -7 \neq 0 \Rightarrow |A|$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 0 \end{bmatrix} \quad (X = A^{-1}B)$$

$$= \frac{-1}{7} \begin{bmatrix} -1 & -4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \end{bmatrix} = - \begin{bmatrix} -1 & -4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ ie point of (unique solution)

intersection of lines (1) and (2) is $(1,1)$ as shown below.



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(2a) consider $x_1 + x_2 = 4$
$x_1 + x_2 = 1 \longrightarrow 4 \neq 1$

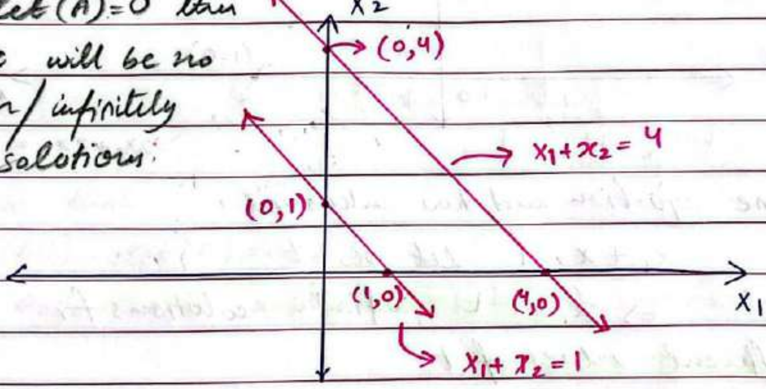
$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{B}$$

(NO SOLUTION)

$$|A| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0$$

ie $\boxed{\text{parallel line}}$ don't intersect.

when det(A)=0 then there will be no solution / infinitely many solutions.



(2b) $x_1 + x_2 = 1$, $2x_1 + 2x_2 = 2$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(INFINITE MANY SOLUTIONS)

$$\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2 - 2 = 0$$

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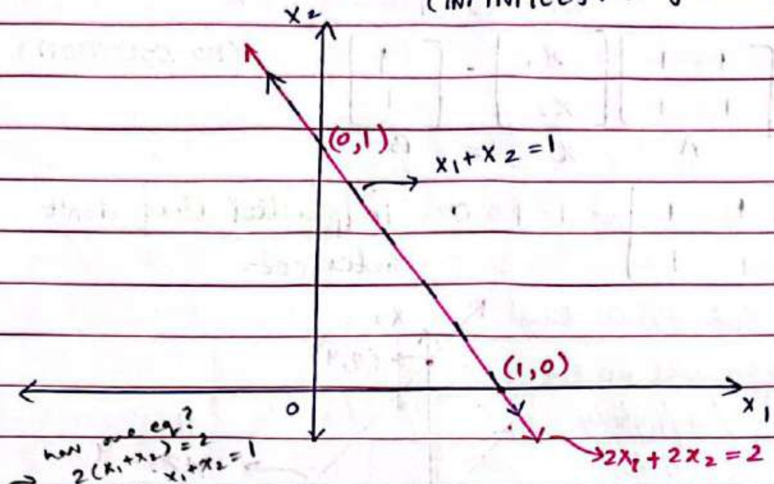
both lines are $\boxed{\text{COINCIDENT}}$

(INFINITELY MANY SOLUTIONS)



one equation and two unknowns

$x_1 + x_2 = 1$, let $x_2 = t$

$\Rightarrow x_1 = 1 - t$, infinite solutions for different values of t.

NOTE :-

Here $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$ $R_2 \to R_2 - 2R_1$

$\nsim \left[\begin{array}{cc|c} 1 & 0 & A^{-1} \\ 0 & 1 & \end{array}\right]$, A^{-1} doesn't exist

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PROPERTIES OF DETERMINAN[T]

① If A and B are squar[e] ... same size then det (AB ...

$\det(E_1 E_2 \dots E_r B) = d$...

→ if A is invertible the matrix A is expressible as product of elementary matrices.

$A = E_1 E_2 \dots E_r$ so

$AB = E_1 E_2 \dots E_r B$

$\det(AB) = \det(E_1)\det(E_2)\dots\det(E_r)\det(B)$

$\det(AB) = \underline{\det(E_1 E_2 \dots E_r)}\det(B)$

$\det(AB) = \det(A)\det(B)$

PROOF ①

CHECK FOR 2×2 MATRICES

for $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, $\det(B) = b_{11}b_{22} - b_{12}b_{21}$

PROOF ②

prove that

$\det(AB) = \det(A)\det(B)$

$= a_{11}a_{22}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{2} - a_{12}a_{21}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21}$

② If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\det(I) = 1$ which is true for identity matrix of any order.

PROVE THE FOLLOWING.

If A is square matrix A is $\boxed{\text{invertible}}$ then

$\boxed{\det(A) \neq 0}$

→ if A is NOT invertible then neither is the product AB.

↳ we have $\det(AB) = 0$ and $\det(A) = 0$ so it follows that

$\det(AB) = \det(A)\det(B)$

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$\det(I)$

RESULT If A^{-1} exists then $\det(A) \neq 0$.

$AA^{-1} = I \Rightarrow \det(A^{-1}A) = I$

$\det(A^{-1})\det(A) = 1 = \det(A^{-1}) = \frac{1}{\det(A)}$

$\det(AB) = \det(A)\det(B)$

$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

③ $\det(A) = \det(A^T)$

for 2×2, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

$\det(A) = \det(A^T) = a_{11}a_{22} - a_{21}a_{12}$

④ If two columns or two rows of a matrix are identical then $\boxed{\det(A) = 0}$

Check for 2×2 matrix

$$\begin{vmatrix} a & a \\ b & b \end{vmatrix} = ab - ba = 0$$

OR

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0$$

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(5) Adding rows (or columns) together makes $\boxed{\text{no difference}}$ to the determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

consider $(C_1 \rightarrow C_1 + C_2)$ $\begin{vmatrix} a_{11}+a_{12} & a_{12} \\ a_{21}+a_{22} & a_{22} \end{vmatrix}$

$\Rightarrow a_{11}a_{22} - a_{12}a_{21}$ (still SAME)

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

also

$R_1 \rightarrow R_1 + R_2$ $\begin{vmatrix} a_{11}+a_{21} & a_{12}+a_{22} \\ a_{21} & a_{22} \end{vmatrix}$

$\Rightarrow a_{11}a_{22} - a_{12}a_{21}$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$\det(kA) = k^n \det(A)$
$\therefore$ where n is the number of rows.

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(6) If B is the matrix that results when a single raw or single column of A is multiplied by a scalar k, then $\det(B) = k\det(A)$

Let $\det(B) = \begin{vmatrix} ka_{11} & a_{12} \\ ka_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & ka_{12} \\ a_{21} & ka_{22} \end{vmatrix}$

$= \begin{vmatrix} ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ ka_{21} & ka_{22} \end{vmatrix}$

$= k(a_{11}a_{22} - a_{21}a_{12}) = k \begin{vmatrix} a_{11} & a_{12} \end{vmatrix}$

$\boxed{\det(B) = k\det(A)}$

→ A is NOT INVERTIBLE, then $\det(A) = 0$
→ if two columns/raws are IDENTICAL, then $\det(A) = 0$
→ if a square matrix with two proportional raws/columns, then $\det(A) = 0$
→ if A has raw/column of zeros, then $\det(A) = 0$
→ if a scalar k is multiplied $\det(B) = k\det(A)$
→ Adding raws/column, the determinat remain SAME.

TRY THE FOLLOWING

Find $\det(A)$ where

$A = \begin{bmatrix} b+c & c+a \\ a & b \\ 1 & 1 \end{bmatrix}$

$R_1 \longrightarrow R_1 + R_2$

now taking $a+b+c$ common. we get

$\begin{bmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix} a+b+c.$

* since two raws are identical we get $\det(A) = 0$

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(7) If A is a square matrix with two proportial rows or two proportional columns, then det (A) = 0

$\longrightarrow C_2 = 2C_3$

EXAMPLE :- $\det\left(\begin{bmatrix} -2 & 8 & 4 \\ 3 & 2 & 1 \\ 1 & 10 & 5 \end{bmatrix}\right) = 0$

$$\because \begin{vmatrix} -2 & 8 & 4 \\ 3 & 2 & 1 \\ 1 & 10 & 5 \end{vmatrix} = 2 \begin{vmatrix} -2 & 4 & 4 \\ 3 & 1 & 1 \\ 1 & 5 & 5 \end{vmatrix} = 0$$

* In determinant the constant 'k' not necessarily needs to be multiplied by each row/col.

$\rightarrow C_2$ is identical to C_3.

(8) If A is a square matrix such that A has a row of zeros or a column of zeros then det (A) = 0

$$\begin{vmatrix} a & 0 \\ b & 0 \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix}$$

$$= a(0) - b(0) = 0$$

NOTE :- If A is any square matrix that contains a row of zeros or a column of zeros then A is singular.

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MORE PROPERTIES

① if A and B are square matrices of same size then $\det(A) + \det(B) \neq \det(A+B)$

PROOF

EXAMPLE For $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $A+B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$

$\det(A) = 1$, $\det(B) = 8$

$\det(A+B) = 23$

$9 \neq 23$ hence proved

TRY THE FOLLOWING

let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}$

find $\det(A) + \det(B) = ?$

$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

$\det(B) = \det \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix} \Rightarrow a_{11}b_{22} - a_{12}b_{21}$

ANS $a_{11}(a_{22} + b_{22}) - a_{12}(a_{21} + b_{21}) \Rightarrow \det(A)\det(B)$

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which can be written as.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{vmatrix}, \quad \therefore \text{ we have}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{vmatrix} \quad \det(A+B)$$

(2) (ADDITION RULE) Let A, B and C be $n \times n$ matrices that differ only in single row (say r^{th}) and assume that r^{th} row of C can be obtained by adding corresponding entries in r^{th} rows of A and B. Then

$$\det(C) = \det(A) + \det(B)$$

(3) If B is the matrix that results when two rows or two columns of A are $\boxed{\text{interchanged}}$, then

$$\det(B) = -\det(A)$$

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PROOF

CHECK for 2×2 matrix

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$
$$= -\underbrace{(a_{12}a_{21} - a_{11}a_{22})}$$
$$= -\det(B)$$

$$\boxed{\det(A) = -\det(B)}$$

TRY THE FOLLOWING

If $A = \begin{bmatrix} \boxed{a \quad b \quad c} \\ d \quad e \quad f \\ g \quad h \quad i \end{bmatrix}$, let $\det(A) = -7$

find $\det\begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix} = \begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = ?$

SOLUTION $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$ taking transpose.

$= -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\det(A) = -(-7) = 7$

$R_1 \leftrightarrow R_3$

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(4) If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then

$$\det(A) = \det(B) \rightarrow (*)$$

CHECK for 2×2 case.

Take $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$; $B = \begin{bmatrix} a_{11}+ka_{12} & a_{12} \\ a_{21}+ka_{22} & a_{22} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}+ka_{12} & a_{12} \\ a_{21}+ka_{22} & a_{22} \end{bmatrix}, \quad k \text{ is any scalar } k \neq 0$$

$C_1 \rightarrow C_1 + kC_2$

DETAIL

$$\det(B) = \begin{vmatrix} a_{11}+ka_{12} & a_{12} \\ a_{21}+ka_{22} & a_{22} \end{vmatrix} = (a_{11}+\cancel{ka_{12}})a_{22} - a_{12}(a_{21}+\cancel{ka_{22}})$$
$$= a_{11}a_{22} - a_{12}a_{21}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \det(A)$$

(5) If any matrix A is singular then $\det(A) = 0$

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CRAMERS RULE

We shall ~~as~~ discuss another method to find the unique solution of n equations in n unknowns, provided determinant of the coefficient matrix $\neq 0$ in $AX = B$, $B \neq 0$ i.e. linear system is NONHOMOGENOUS.

SIMPLE CASE

system of two equations in two unknowns:

here $AX = B \Rightarrow$ $\quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad — (1)$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \quad — (2)$$

from (1)

$$x_2 = \frac{b_1 - a_{11}x_1}{a_{12}} \quad — (*)$$

using (*) in (2) gives the following

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$, similarly we can obtain

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$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$ or just

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}$$, where $|A|$ is

the ~~different~~ determinant of coefficient matrix

$|A_1| = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$ is obtained by replacing the entries in the 1st column of $|A|$ by the entries in B, etc. similarly we can extend this method to three equations in three unknowns.

But first we must know the expansion or method to expand a determinant of a matrix of order 3.

EXPANSION BY FIRST ROW:

consider

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

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$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$
$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

If A is a square matrix then the minor of entry a_{ij} is denoted by M_{ij} and is defined to be the determinant that remains after i TH row and J TH column are deleted from A the number $(-1)^{i+j} M_{ij}$ is denoted C_{ij} and is called the cofactor of entry a_{ij}.

EXAMPE $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

M_{11} minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

C_{11} = cofactor of a_{11}

$= (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

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M_{12} = minor of a_{12} $= \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = ad - bc$

$\Rightarrow C_{12} = (-1)^{1+2} M_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

similarly

$C_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

NOTE: $det(A) = det(A)$

$= a_{11} (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$+ a_{13} (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$

TRY THE FOLLOWING

Find $C_{11}, C_{12}, C_{21}, C_{22}$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Solution: $C_{11} = (-1)^{1+1} d = d$

$C_{12} = (-1)^{1+2} c = -c$

$C_{21} = (-1)^{1+2} b = -b$

$C_{22} = (-1)^{2+2} a = a.$

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consider $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T$

$= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, recall that $\longrightarrow$ (1)

$\Rightarrow A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \longrightarrow$ (2)

$\Rightarrow A^{-1} = \frac{1}{\det(A)} \underbrace{\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T}$ from (1) and (2)

$\longrightarrow$ this matrix is called Adj(A) or Adjoint of A.

INVERSE OF MATRIX USING ITS ADJOINT

If A is an invertible matrix then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Example: Find the $\boxed{\text{inverse of } A =}$ $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$ by adjoint method

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STEP1 Find det(A) which is given by

$3(0+12) - 2(-6) - 1(-4-12) = 36+12+16 = 64$

$\det(A) \neq 0$

STEP 2 Find adj(A) which is

$$= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \mathrm{adj}(A)$$

cofactor

*after taking transpose.

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

C_{11} = cofactor of $(a_{11} = 3)$ is given by

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} = 12$$

Similarly $C_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 6,$

check the following

$$C_{13} = \begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix} = -4-12 = -16$$

$\vdots$

$$\therefore A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A) = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

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PROOF

Question 16 Let A and B be $n \times n$ matrices. Show that if A is invertible, then $\det(B) = \det(A^{-1}BA)$

solution:

$$\det(A^{-1}BA) = \det(A^{-1})\det(B)\det(A)$$

$$= \frac{1}{\cancel{\det(A)}}\det(B)\cancel{\det(A)}$$

$$\det(A^{-1}BA) = \det(B) \quad \text{(proven)}$$

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PROOF

Question 21 Let A and B be $n \times n$ matrices. You know from earlier work that AB is invertible if A and B are invertible. What can you say about the invertibility of AB if one or both of the factors are singular? Explain your reasoning.

Solution If either A or B is singular, then either $\det(A)$ or $\det(B)$ is zero. Hence $\det(AB) = \det(A)\det(B) = 0$. Thus AB is also singular.

PROOF

Question 25

Prove that if $\det(A)=1$ and all entries in A are integers, then all the entries in A^{-1} are integers.

Solution This follows from Theorem 2.1.2 and the fact that the cofactors of A are integers if A has only integer entries since integers are closed under multiplication addition & subtraction. $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \operatorname{adj}(A)$

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PROOF

If B is $n \times n$ matrix and E is an elemetary matrix of order $n \times n$ then $\det(EB) = \det(E)\det(B)$

If E is performed by multiplying a raw of I_n by scalar of k then, this means that

$$EB = C \ (say)$$
$$\det(EB) = \det(C)$$
$$\det(EB) = k\det(B)$$

since $\det(E) = k\det(I) = k$

$$\det(EB) = \det(E)\det(B)$$

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TRIANGULAR MATRICES

(1) A square matrix in which all the entries above the main diagnol are zero is called lower triangle.

Example:

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is lower triangle here $\det(A) = a_{11}a_{22}a_{33}$ ie the product of diagnol entries which is true for any lower triangular matrix.

(2) Similarly a square matrix in which all the entries below the main diagnol are zero is called UPPER TRIANGULAR ~~METHOD~~. MATRIX.

Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$ is upper triangular matrix.

here $\det(A) = a_{11}a_{22}a_{33}$ ie product of diagnol entries.

RESULT: A matrix that is either upper or lower triangular is called triangular and

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determinant of any triangular matrix is equal to product of it's diagnol entries.

NOTE: A square matrix in raw echelon form is upper triangular since it has zeros below the main diagnol.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

are in raw-echelon form and also upper triangular matrices.

NOTE: Diagnol matrices are both upper triangular and lower triangular e.g $I \rightarrow$ Identity matrix etc.

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VECTORS

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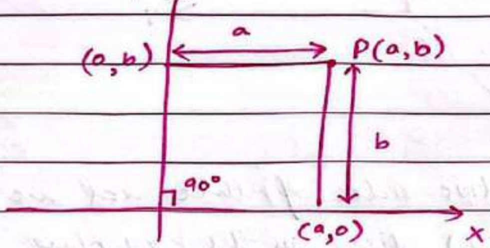
2-DIMENSIONAL SPACE

let us consider xy-plane in which any point P is denoted by two numbers, uniquely associated with P, called coordinates e.g if P is (a,b) then a is the x-coordinate which gives distance from y-axis and b is the y-coordinate which gives distance from x-axis as shown below.



along x-axis, $y=0$ along y-axis, $x=0$.

Now let us consider 3 dimensional space we begin with a set of three lines, called axes concurrent at a point (origin)

The three lines (axes) are : (3-dimensional)

① NOT COPLANAR ie they not all lie in same plane.

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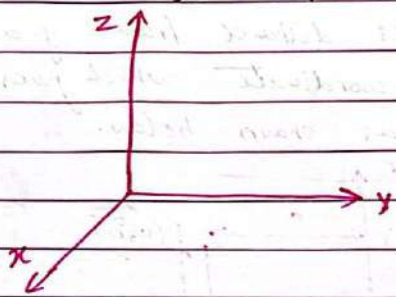
(2) MUTUALLY PERPENDICULAR ie angle b/w them is 90°

(3) labeled x, y and z determine a set of three number called coordinates, consider a point $P(x, y, z)$ is space below



(4) Only positive sides of three axes are shown.

(5) $Q(x, y, 0)$ lies in the xy-plane which is formed due to the intersection of x and y axes.

(6) In xy-plane z coordinate = 0 because z coord gives distance from xyplane

(7) yz plane is determined from intersection of y and z axes and x coord is 0.

(8) xz plane is determined due to intersection of x and z axes.

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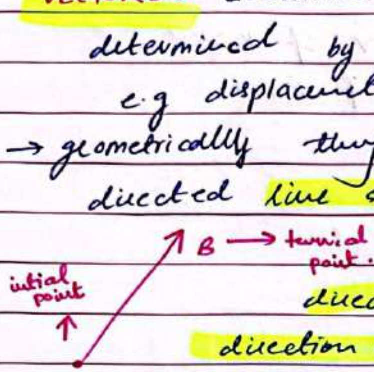
VECTORS: Quantities which ~~[illegible]~~ are completely determined by magnitude and direction e.g displacement, velocity ~~are~~ etc.

→ geometrically they can be represented as directed line segments or arrows.

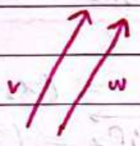


direction of arrow ~~re~~specifies the direction of vector and length of arrow describes the magnitude.

Equal (Equivalent Vectors)

Vectors with same length and magnitude.

$v = v = w$



Absolute value of scalar.

$|k| \geq 0$, $|k| = k$, $k \geq 0$

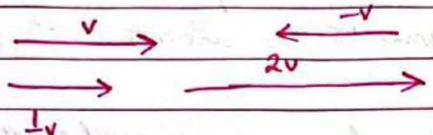
$|k| = -k$, $k < 0$

$|2| = 2$, $|-2| = -(-2) = 2$.

$k \rightarrow$ scalar (real number)

If k is a NONZERO scalar and v is the nonzero vector then kv is the vector whose length is $|k|$

times the length of v and whose direction is the same as that of v if $k>0$ and opposite to that of v if $k<0$.



VECTOR ADDITION:-

$$\overrightarrow{OP_2} = \overrightarrow{OP_1} + \overrightarrow{P_1P_2}$$

$$\boxed{\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}}$$

$$= (x_2-0,\ y_2-0,\ z_2-0) - (x_1, y_1, z_1)$$

$$\overrightarrow{P_1P_2} = (x_2-x_1,\ y_2-y_1,\ z_2-z_1)$$

The components of $\overrightarrow{P_1P_2}$ are obtained by subtracting the coordinates of the initial point from the coordinates of the terminal point.

The length of a vector u is ~~affected~~ often called the norm of u and is denoted by $\|u\|$ in 2 dimensional space for $v = (u_1, u_2)$

$$\boxed{\|v\| = \sqrt{u_1^2 + u_2^2} \qquad \text{(PYTHAGORAS THEOREM)}}$$

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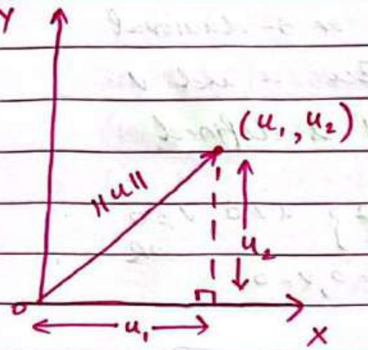
UNIT VECTOR $\rightarrow \dfrac{V}{\|V\|}$

day / date:

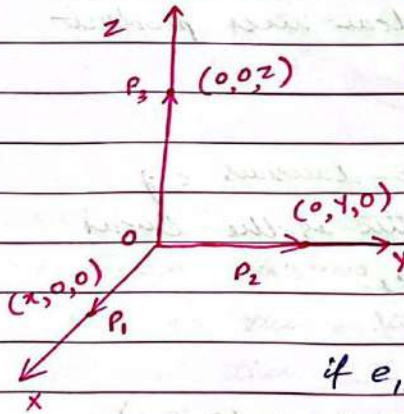


if $\|u\| = 1$ ie u is a vector of norm 1 then u is called a unit vector.

RESULT: In three dimensional space for $u = (u_1, u_2, u_3)$

$$\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Unit Vectors along x, y and z axes in 3-dimensional space.



$\overrightarrow{OP_1} = (x, 0, 0)$

$\|\overrightarrow{OP_1}\| = \sqrt{x^2} = x \neq 0$

$\overrightarrow{OP_2} = (0, y, 0)$

$\|\overrightarrow{OP_2}\| = y \neq 0$

$\|\overrightarrow{OP_3}\| = z \neq 0$

$\overrightarrow{OP_3} = (0, 0, z)$

if e_1 is the unit vector along x-axis then

$$e_1 = \frac{\overrightarrow{OP_1}}{\|\overrightarrow{OP_1}\|} = \frac{(x,0,0)}{x} = (1,0,0)$$

$\|e_1\| = 1$, similarly unit vectors along y and z axes are given by $(0,1,0) = e_2$ and $(0,0,1) = e_3$

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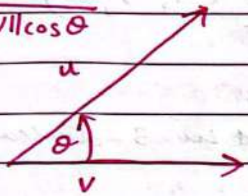
If u and v are vectors in 2 or 3-dimensional space and θ is angle between u and v then the dot product $u \cdot v$ is defined by

$$u \cdot v = \begin{cases} \|u\| \, \|v\| \cos\theta & u \neq 0, v \neq 0 \\ & \text{OR} \\ 0 & u = 0, v = 0 \end{cases}$$

DOT PRODUCT

$\|u\| \|v\| \cos\theta$



$\boxed{u \cdot v \text{ is scalar quantity.}}$

$u \cdot v$ is also called euclidean inner product

PROOF ✱ TRY THE FOLLOWING

Prove that any vector in 3-dimensions e.g $u = (u_1, u_2, u_3)$ can be written as the linear combination of e_1, e_2 and e_3.

CHECK $u = u_1 e_1 + u_2 e_2 + u_3 e_3$

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.

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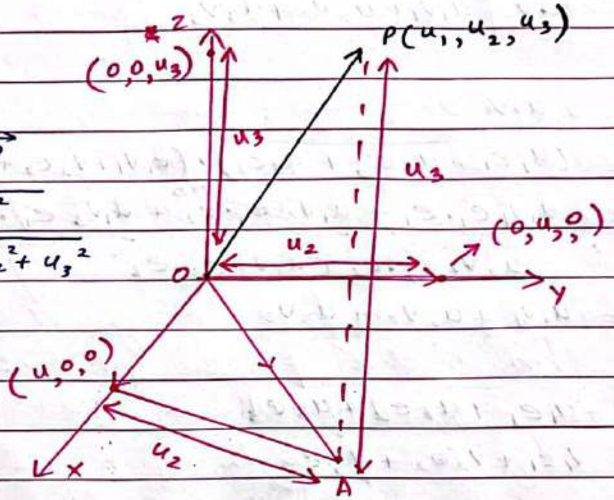
DOT PRODUCT $\rightarrow$ $a \cdot b = a_x b_x + a_y b_y + a_z b_z$ day / date:

$$\because u_1 e_1 + u_2 e_2 + u_3 e_3$$
$$= u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$$
$$= (u_1,0,0) + (0,u_2,0) + (0,0,u_3)$$
$$= (u_1,u_2,u_3) = \overrightarrow{OP} = u.$$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$\|\overrightarrow{OA}\| = \sqrt{u_1^2 + u_2^2}$

$\|\overrightarrow{OP}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$



ORTHOGONAL VECTORS

Dot product b/w two vectors is zero when the angle b/w them = 90°. Such vectors are perpendicular to each other and they are also called orthogonal vectors $\vec{u} \cdot \vec{v} = 0$

$\therefore e_1, e_2, e_3$ are orthogonal vectors

$$\boxed{\begin{aligned} e_i \cdot e_j &= 1 \quad \text{if } i = j \qquad i,j \in \{1,2,3\} \\ &= 0 \quad \text{if } i \neq j \end{aligned}}$$

e.g. 1 $e_1 \cdot e_2 = \|e_1\|\|e_2\| \cos 90° = 0$

$e_1 \cdot e_1 = \|e_1\|\|e_2\| \cos 0° = 1$

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PROVE THE FOLLOWING

If $u = (u_1, u_2, u_3) = u_1 e_1 + u_2 e_2 + u_3 e_3$

$v = (v_1, v_2, v_3) = v_1 e_1 + v_2 e_2 + v_3 e_3$

then $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$

PROOF $u \cdot v$

$$= (u_1 e_1 + u_2 e_2 + u_3 e_3) \cdot (e_1 v_1 + v_2 e_2 + v_3 e_3)$$

$$= u_1 v_1 \underbrace{e_1 \cdot e_1} + \cancelto{0}{u_1 v_2 e_1 \cdot e_2} + \cancelto{0}{u_1 v_3 \underbrace{e_1 \cdot e_3}} +$$

$e_i \cdot e_j = 1$ if $i = j$ $\qquad u_2 v_2 e_2 \cdot e_2 + u_3 v_3 e_3 \cdot e_3 \qquad e_i \cdot e_j = 0$ if $i \neq j$

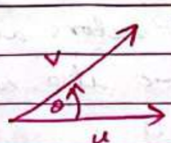
$$= u_1 v_1 + u_2 v_2 + u_3 v_3$$

for $u = u_1 e_1 + u_2 e_2 + u_3 e_3$

$v = v_1 e_1 + v_2 e_2 + v_3 e_3$

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$= \|u\| \, \|v\| \cos\theta, \quad u \neq 0, \; v \neq 0$$



$$\boxed{u \cdot u = u_1^2 + u_2^2 + u_3^3 = \|u\|^2}$$

The dot product of vector with itself is the square of it's norm.

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TRY THE FOLLOWING

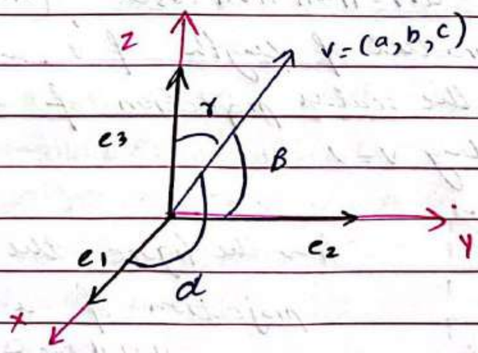
① If $u = (2, -1, 1)$, $v = (1, 1, 2)$ then the angle b/w them $= 60°$ $u \cdot v = \|u\| \|v\| \cos\theta$

$2 + (-1) + 2 = 6\cos\theta$ $\cos\theta = 3/6 \Rightarrow \theta = 60°$

② Let e_1, e_2, e_3 be the unit vectors along x, y and z axes respectively in 3-dimensional space. If $v = (a, b, c)$ making angles α, β and γ with e_1, e_2 and e_3 respectively then

(i) find $\cos\alpha$, $\cos\beta$, $\cos\gamma$ prove that

(ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$



Note: α, β, γ are called the direction angles of v and $\cos\alpha$, $\cos\gamma$ and $\cos\beta$ are called the direction cosines of v.

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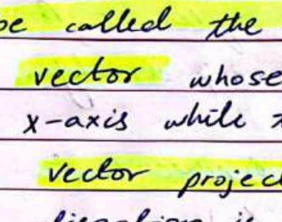
SCALAR & VECTOR PROJECTIONS.

let $u = u_1e_1 + u_2e_2 + u_3e_3$ we shall call u_1e_1, u_2e_2 and u_3e_3 the vector components of u and the scalars u_1, u_2 and u_3 the scalar components of x, y and z components of u.

In $u = u_1e_1 + u_2e_2 + u_3e_3$ the scalar u_1 may be called the scalar projection of u on any vector whose direction is that of positive x-axis while the vector u_1e_1 may be called vector projection of u on any vector whose direction is that of the positive x-axis

Therefore $u \cdot v = ||u|| \, ||v|| \cos\theta$, $(u \neq 0, v \neq 0)$

= product of length of v and the scalar projection of u on v or of u along $v = v$



from the figure the scalar projection of u along v $= ||u|| \cos\theta = OP$

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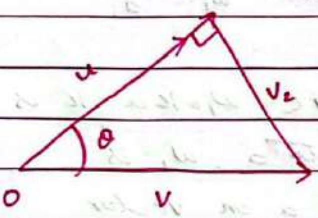
Similarly

$u \cdot v = ||u|| \, ||v|| \cos\theta$

= product of length of u and the scalar projection of v on u.



$\overrightarrow{OP} = v_1$

$v = v_1 + v_2$

$\cos\theta = \frac{OP}{||v||} \Rightarrow OP ||v|| \cos\theta$

which is the scalar projection of v along u.

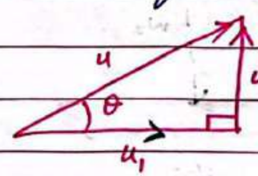
Notice that $u \cdot v = v \cdot u$

SUMMARY

$u \cdot v = u_1v_1 + u_2v_2 + u_3v_3 = ||u|| \, ||v|| \cos\theta$ $\quad u \neq 0, \; v \neq 0$

if vector components of a vector are at 90° to each other they they are also called $\boxed{\text{rectangular vector}}$ components

e.g $u = u_1 + u_2$, $\boxed{u_1 \text{ and } u_2 \text{ are rectangular vector components of } u}$ and their norms are



$||u_1|| = ||u|| \cos\theta$

$||u_2|| = ||u|| \sin\theta$

$0 < \theta < \pi/2$, $u_1 = u_2 \neq 0$, $\cos\theta > 0$ and $\sin\theta > 0$

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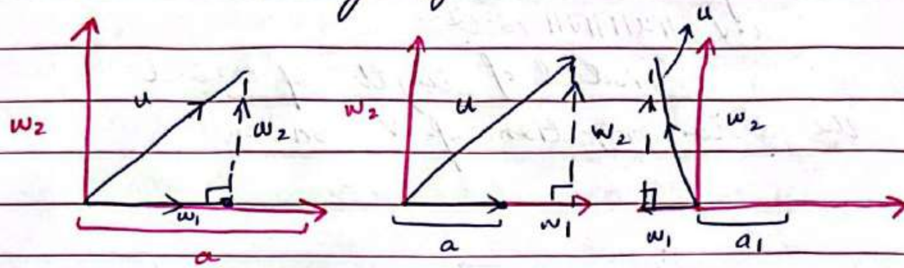
Consider the following figures:



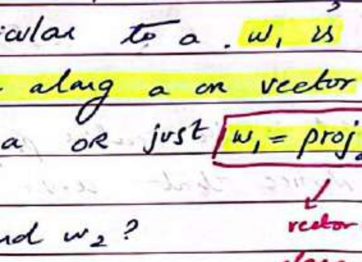
$u = w_1 + w_2$, w_1 is along a therefore $w_1 = ka$, k is a scalar. w_2 is perpendicular to a. w_1 is orthogonal projection of u along a or vector component of u along a or just $\boxed{w_1 = proj_a\, u}$

vector com of u along a.

PROBLEM: How to find w_1 and w_2?

If $a \neq 0$ then $w_1 = proj_a\, u = \frac{(u \cdot a) \cdot a}{\|a\|^2}$, $a \neq 0$

PROOF



$u = w_1 + w_2$

$\because w_1 = ka$

$\therefore u = ka + w_2$

$\Rightarrow w_2 = u - ka$

15.a

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$\text{Proj}_a u \rightarrow \frac{(u \cdot a)}{\|a\|^2} \cdot a$

perpendicular $w_2 \cdot a = 0$

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taking dot product with a $\cancel{w_2 \cdot a}^{\,0} = (u - ka) \cdot a$

$\Rightarrow u \cdot a - k\, a \cdot a = 0$

$\Rightarrow k = \frac{u \cdot a}{a \cdot a} = \frac{u \cdot a}{\|a\|^2} = k$ $\qquad a \cdot a = \|a\|^2$

$\therefore w_1 = ka = \frac{(u \cdot a)}{\|a\|^2} a$

$\because w_1 = \frac{(u \cdot a)}{\|a\|^2} a$ AND

$\text{proj}_a u.$

$w_2 = u - \frac{(u \cdot a)}{\|a\|^2} a \rightarrow$ vector component of u orthogonal to a.

$w_2 = u - w_1$

also note that $\boxed{w_2 \cdot a = 0}$

PROBLEM:

Let $u = (2, -1, 3)$ $a = (4, -1, 2)$ find:

(1) Vector component of u along a ($\text{Proj}_a u$) and

(2) The vector component of u orthogonal to a

Ans $(\text{Proj}_a u) = \left(\frac{20}{7}, \frac{-5}{7}, 10/7\right)$ and the one orthogonal to $a = \left(-6/7, -2/7, 11/7\right)$

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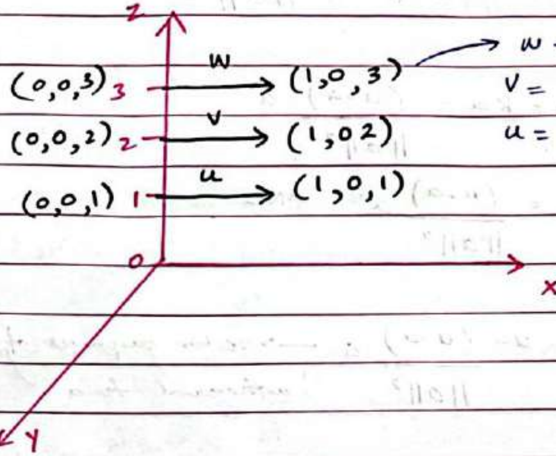
Recall that two vectors are equal (equivalently) if they have same length and direction. Consider the following vectors in xz-plane.



$w = (1,0,0) = e_1$

$v = (1,0,0) = e_1$

$u = (1,0,0) = e_1$

VECTOR SPACES

We shall state some properties, if satisfied by a set of objects will entitle those objects to be called vectors.

These new type of vectors will include various kinds of matrices and functions etc.

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DEFINITION: let V be non-empty set of objects on which two operations are defined addition & multiplication by scalars (real numbers)

By addition we mean for u, u $\in V$ an object $u+V$ OR $u \oplus V$

By scalar multiplication we mean for each scalar k and each object $u \in V$ an object ku called the scalar multiple of u by k. OR $k \odot u$

$u+v \rightarrow$ vector addition

$ku \rightarrow$ scalar multiplication.

V is called a VECTOR SPACE for u, v, w $\in V$ and for all scalars k and L if the following 10 properties are satisfied:

① for all u, v $\in V$, $u+v \in V$ which means V is closed under VECTOR ADDITION

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(2) $u+v = v+u$ {COMMUTATIVE PROPERTY OF VECTOR ADDITION}

(3) $(u)+(v+w) = (u+v)+w$ {ASSOCIATIVE PROPERTY OF VECTOR ADDITION}

(4) There is an object 0 in V called ZERO VECTOR such that $0+u = u+0 = u$ for all $u \in V$

$0 \rightarrow$ additive identity OR identity element with respect to vector addition.

(5) for each $u \in V$ there is an object $-u \in V$ called the negative of u or additive inverse of u such that

$$u+(-u) = (-u)+u = 0$$

(6) if k is any scalar and u is any object in V, then ku is in V which shows that V is closed under scalar multiplication.

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(7) $k(u+v) = ku + kv$

(8) $(k+l)u = ku + lu$

(9) $k(lu) = (kl)u$

(10) $1u = u$

due to property no (4) V
ie vector space is a nonempty set.

α —————— α

<u>Example</u>: Prove that set V of all 2×2 matrices (vectors) with real entries is a vector space under the matrix addition (as vector addition) and matrix scalar multiplication

NOTE: IN ORDER TO AVOID THE CONFUSION WITH ORDINARY VECTORS USE α, β, γ AS ELEMENTS OF V.

<u>SOLUTION</u>:

let $\alpha = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $\beta = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

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① $\alpha + \beta = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix} \in V$

$\because$ $\alpha + \beta$ is also a 2 × 2 MATRIX

② $\alpha + \beta = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix}$

$= \begin{bmatrix} b_1+a_1 & b_2+a_2 \\ b_3+a_3 & b_4+a_4 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$

$= \beta + \alpha$

③ $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ obvious check this by taking $\gamma = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$

④ $0 + \alpha = \alpha + 0 = \alpha$ obvious since

$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $0 + \alpha = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$

$= \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \alpha + 0 = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$

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(5) $\alpha + (-\alpha) = (-\alpha) + \alpha = 0 \rightarrow (*)$

for $-\alpha = \begin{bmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{bmatrix}$ $(*)$ is satisfied

$\hookrightarrow$ 2×2 matrix

(6) $k\alpha = k\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix} \in V$

(7) $k(\alpha + \beta) = k\alpha + k\beta$

(8) $(k+L)\alpha = k\alpha + L\alpha$

(9) $k(L\alpha) = k\begin{bmatrix} La_1 & La_2 \\ La_3 & La_4 \end{bmatrix}$

$= k(L)\begin{bmatrix} a_1 & a_4 \\ a_3 & a_5 \end{bmatrix} = (kL)\alpha$

(10) $1\alpha = \alpha.$

TRY THE FOLLOWING

PROOF

Determine whether the set of all 2×2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with matrix addition and scalar multiplication is vector space.

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$\mathbb{R} \ni a$

$(a,b) \in R^2$

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$(a_1, a_2, \dots, a_n) \in R^n \Rightarrow$ n tuples

$$A = \begin{bmatrix} a_{11} & 1 \\ 1 & b_{12} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & 1 \\ 1 & b_{22} \end{bmatrix}$$

$$A+B = \begin{bmatrix} a_{11}+b_{11} & 2 \\ 2 & b_{12}+b_{22} \end{bmatrix} \notin V$$

Not a vector space, axiom (1) fails

Example V = set of all real-valued functional defined on the real line $(-\infty, \infty)$

PROOF

Proving 10^{th} property.

$f \in V$

$f = x^2 + 3x + 1$

$1f = f \in V$

Example $V = R^n$

PROOF

$$V_1 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad V_2 = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$V_1 + V_2 = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix} \in R^n$$

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$(u+v)\cdot w = u\cdot w + v\cdot w$

PROOF let $u = (u_1, u_2, \dots, u_n)$, $v = (v_1, v_2, \dots, v_n)$ and $w = (w_1, w_2, \dots, w_n)$ thus

$$(u+v)\cdot w = (u_1+v_1, u_2+v_2, \dots, u_n+v_n)\cdot(w_1, w_2, \dots, w_n)$$
$$= (u_1+v_1)(w_1) + (u_2+v_2)w_2 + \dots + (u_n+v_n)w_n$$
$$= (u_1w_1 + u_2w_2 + \dots + u_nw_n) + (v_1w_1 + v_2w_2 + \dots + v_nw_n)$$
$$= u\cdot w + v\cdot w$$

$v\cdot v \geq 0$. Further $v\cdot v = 0$ if and only if $v = 0$

PROOF

We have $v\cdot v = v_1^2 + v_2^2 + \dots + v_n^2 \geq 0$. Further equality holds if and if $v_1 = v_2 = \dots = v_n = 0$ – that is, if and only if $v = 0$.

To solve the vector equation $x + u = v$ for x we can add $-u$ to both sides

$$(x+u) + (-u) = v + (-u)$$
$$x + (u-u) = v - u$$
$$x + 0 = v - u$$
$$x = v - u$$

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PROPERTIES OF LENGTH IN R^n

(a) $\|u\| \geq 0$

(b) $\|u\| = 0$ if and only if $u = 0$

(c) $\|ku\| = |k| \, \|u\|$

(d) $\|u+v\| \leq \|u\| + \|v\|$

PROOF (c) If $u = (u_1, u_2, \dots, u_n)$ then $ku = (ku_1, ku_2 \dots ku_n)$

so

$$\|ku\| = \sqrt{(ku_1)^2 + (ku_2)^2 + \dots + (ku_n)^2}$$

$$= |k| \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

$$= |k| \, \|u\|$$

PROOF (d)

$$\|u+v\|^2 = (u+v) \cdot (u+v)$$

$$\|u+v\|^2 = (u \cdot u) + 2(u \cdot v) + v \cdot v$$

$$\|u+v\|^2 = \|u\|^2 + 2(u \cdot v) + \|v\|^2$$

$$\|u+v\|^2 = \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$

$$\|u+v\|^2 = (\|u\| + \|v\|)^2$$

$$\|u+v\| = \|u\| + \|v\|$$

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PROOF

$$d(\vec{u},\vec{v}) \leq d(\vec{u},\vec{w}) + d(\vec{w},\vec{v})$$

$$d(\vec{u},\vec{v}) = d(\vec{u},\vec{v})$$
$$= \|\vec{u}-\vec{v}\|$$
$$= \|(\vec{u}-\vec{w}) + (\vec{w}-\vec{v})\|$$
$$= \|\vec{u}-\vec{w}\| + \|\vec{w}-\vec{v}\|$$
$$d(\vec{u},\vec{v}) = d(\vec{u},\vec{w}) + d(\vec{w},\vec{v})$$

PROOF

$$\vec{u}\cdot\vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2$$

$$\frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2 = \frac{1}{4}(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) - \frac{1}{4}(\vec{u}-\vec{v})(u-v$$

$$= \frac{1}{4}((\vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v})$$

$$= \frac{-1}{4}(\vec{u}\cdot\vec{u} - \vec{v}\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v})$$

$$= \frac{1}{\cancel{4}}(\cancel{4}\,\vec{u}\cdot\vec{v}) \Rightarrow \vec{u}\cdot\vec{v}$$

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PROOF :-

$$(A\vec{u})\cdot\vec{v} = \vec{u}\cdot(\vec{A}^T\vec{v})$$

$$(A\vec{u})\cdot\vec{v} = (A\vec{u})^T\vec{v} \quad \because \vec{u}\cdot\vec{v} = \vec{u}^T v$$

$$= u^T A^T \vec{v}$$

$$= u^T (A^T \vec{v})$$

$$(A\vec{u})\cdot\vec{v} = u\cdot(A^T\vec{v})$$

PROVE :- $0\vec{u} = 0$

PROOF

$$Ax\ 8 \Rightarrow (k+l)\vec{u} = k\vec{u} + l\vec{u}$$

$$0\vec{u} + 0\vec{u} = (0+0)\vec{u}$$

$$0\vec{u} + 0\vec{u} = 0\vec{u}$$

$$Ax\ 4 \Rightarrow u \in V \Leftrightarrow -u \in V$$

$$0\vec{u} + 0\vec{u} + (-0\vec{u}) = 0\vec{u} + (-0\vec{u})$$

$$Ax\ 3 \Rightarrow 0\vec{u} + (0\vec{u} + (-0\vec{u})) = 0\vec{u} + (-0\vec{u})$$

$$Ax\ 5 \Rightarrow 0\vec{u} + 0 = 0$$

$$Ax\ 4 \Rightarrow 0\vec{u} = 0$$

PROVE :- $(-1)\vec{u} = -\vec{u} \quad \forall u \in V$

PROOF

$$(-1)\vec{u} = -\vec{u}$$

$$\Rightarrow \vec{u} + (-1)\vec{u} = 0$$

$$\vec{u} + (-1)\vec{u}$$

$$1\vec{u} + (-1)\vec{u}$$

$$(1 + (-1))\vec{u}$$

$$0\vec{u} \Rightarrow 0$$

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TRANSPOSE DOT PRODUCT

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$v^T u = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 + u_2 v_2 + \dots + u_n v_n \end{bmatrix}$$

$$= [u \cdot v] = u \cdot v$$

$$\boxed{u \cdot v = v^T u} \text{ — eq } (7)$$

$$Au \cdot v = u \cdot A^T v \text{ — eq } (8)$$

PROOF

$$\underbrace{Au}_{u} \cdot \underbrace{v}_{v} = (\overbrace{v^T A}^{v^T})\, \overbrace{u}^{u} \quad \text{using eq } (7)$$

$$= \underbrace{(A^T v)^T u}_{v^T \cdot u} = \underbrace{u \cdot (A^T v)}_{u \cdot v}$$

$$u \cdot Av = A^T u \cdot v \text{ — eq } (9)$$

PROOF

$$u \cdot Av = (Av)^T u = v^T (A^T u) = A^T u \cdot v$$

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If u and v are vectors in R^n with the Euclidean product, then

$$u \cdot v = \frac{1}{4} \|u+v\|^2 - \frac{1}{4} \|u-v\|^2$$

(orthogonal vectors)

EUCLIDEAN DISTANCE

Euclidean Norm

$$\|u\| = (u \cdot u)^{1/2} = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

Euclidean Distance

$$d(u, v) = \|u - v\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots (u_n - v_n)^2}$$

If u and v are orthogonal vectors in R^n with Euclidean inner product, then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$ Pythagoras Theorem.

CAUCHY SWHARZ

$$(u, v)^2 \leq (u, u)(v, v)$$

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Determine wether a given (V) set is a vector space under the given operations

The set of all pairs of real numbers (x, y) with operations

$$\underbrace{(x, y) + (x', y') = (x + x', y + y')}_{\text{VECTOR ADDITION}} \text{ AND}$$

$$\underbrace{k(x, y) = (2kx, 2ky)}_{\text{SCALAR MULTIPLICATION}}$$

Solution:

(1) $\overbrace{(x, y)}^{u} + \overbrace{(x', y')}^{v} = (x + x', y + y') \in V$

VECTOR ADD since this is also an order pair of REAL NUMBER

(2) $\overbrace{(x, y) + (x', y')}^{u+v} = (x + x', y + y')$ $v + u$

$= (x' + x, y' + y) = (x', y') + (x, y)$

$\Rightarrow u + v = v + u$ for $u = (x, y)$ AND $v = (x', y')$

(3) $\overset{u}{(x, y)} + [\overbrace{(x', y')}^{v} + \overbrace{(x'', y'')}^{w}] \Rightarrow u + (v + w)$

$= (x, y) + (x' + x'', y' + y'')$

$= \{x + (x' + x''), y + (y' + y'')\}$

$= \{(\overset{u+v}{x + x'}) + x'', (y + y') + y''\}$

$= (x + x', y + y') + (x'', y'')$

$(u + v) + w = [(x, y) + (x', y')] + (x'', y'')$

$\Rightarrow u + (v + w) = (u + v) + w$

for $w = (x'', y'')$ . . . .

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$u + 0 = 0 + u$

(4) $(x,y) + (0,0) = (0,0) + (x,y)$

$= (x+0, y+0) = (x,y)$

$\Rightarrow 0 = (0,0)$

✓

(5) If $u = (x,y)$, $-u = (x',y')$

then $u + (-u) = (-u) + u = 0$

$(x,y) + (x',y') = (0,0)$ — ①

but $(x,y) + (x',y') = (\underbrace{x+x'}_{=0}, \underbrace{y+y'}_{=0})$

from ① and ②

$x + x' = 0 \Rightarrow x' = -x$

$y + y' = 0 \Rightarrow y' = -y$

$\therefore -u = (-x, -y)$

$\because (x,y) + (-x,-y) = (-x,-y) + (x,y)$

$\Rightarrow (0,0)$

✓

(6) $k(x,y) = (2kx, 2ky) \in V$ SCALAR MULTIPLICATON

obvious $\Rightarrow ku \in V$

Axiom ⑦

(7) $k(u+v) = k[\overset{u}{(x,y)} + \overset{v}{(x',y')}]$

$= k[(\underline{x+x'}, \underline{y+y'})]$

$= 2k(x+x'), 2k(y+y')$

$= (2kx + 2kx', 2ky + 2ky')$

$= (2kx, 2ky) + (2kx', 2ky')$

$= k(x,y) + k(x',y') = ku + kv$

$k(u+v) = ku + kv$

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(8) $(k+l)u = (k+l)(x,y)$

$$= [2(k+l)x, 2(k+l)y] \quad -①$$

also $[2x(k+l), 2y(k+l)]$

$ku + lu = k(x,y) + l(x,y)$

$$= (2kx, 2ky) + (2lx, 2ly)$$

$$= [2(k+l)x, 2(k+l)y] \quad — ②$$

from ① and ②

$$(k+l)u = ku + lu$$

$k[lu]$

(9) $k(lu) = k[l(x,y)]$

$$= k(2lx, 2ly) = (4lkx, 4kly) — ①$$

$(kl)u = (kl)(x,y) = (2klx, 2kly) — ②$

from ① and ② $k(lu) \neq (kl)u$

So $\boxed{\text{NOT}}$ a vector space.

(10) $1u = 1(x,y) = (2x, 2y) \neq (x,y)$

$$1u \neq u$$

$\because k(x,y) = (2kx, 2ky)$

Axiom 9 and 10 fails.

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SUBSPACES

A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Theorem 5.2.1

If W is a set of one or more vectors from vector space V, then W is a subspace of the W is a subspace of V if and only if the following condition holds.

(a) If u and v are vectors in W then ~~with~~ $u+v$ is in W.

(b) If k is any scalar and u is any vector in W then $\boxed{ku \in W}$

PROOF

If W is a subspace of V then all the vector space axioms or properties are satisfied including (1) and (6) which are same as (a) and (b) above.

Conversely, assume conditions (a) and (b) hold. Since they are vector space axioms 1 and 6 we need to show that other 8 are satisfied.

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Axioms 2, 3, 7, 8, 9 and 10 are satisfied ~~including~~ ~~(1)~~ automatically satisfied by vectors in w since they are satisfied by all vectors in V.

Therefore to complete the proof, we need only to verify that axioms 4 and 5 are satisfied by vectors in w.

let u be any vector in w. By condition (b), $ku \in W$ for every scalar k.

setting $k=0$, $ku = 0u = 0$

but $ku \in W \Rightarrow 0 \in W$,

and setting $k=-1$, it follows that

$(-1)u = -u \in W$

RESULT w is subspace of v if and only if w is closed under addition and closed under multiplication.

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PROOF Prove that if $S = \{v_1, v_2, \cdots, v_n\}$ is a basis for same vector space V, then any vector $u \in V$ can be written as linear combination.

let u vector has two different representation

$u = c_1 v_1 + \cdots + c_n v_n$ — ①

$u = k_1 v_1 + \cdots + k_n v_n$ — ②

eq ① - eq ②

$u - u = (c_1 v_1 + \cdots + c_n v_n) - (k_1 v_1 + \cdots + k_n v_n)$

$u = (c_1 - k_1) v_1 + \cdots + (c_n - k_n) v_n$

By using linear independent property

$c_1 - k_1 = 0, \; c_2 - k_2 = 0, \; \ldots, \; c_n - k_n = 0$

$c_i = k_i$

hence

u vector has exactly one representation.

PROOF Show that the set W of $\boxed{\text{all polynomials of degree} \le n}$ (including zero polynomial) is a subspace of real-valued functions under addition and scalar multiplication.

→ W is not an empty set since it contains $\boxed{\text{zero polynomial}}$

$0 = x^0 + 0x^1 + \ldots + 0x^n$

let $u_1, v_1 \in W$

so $u + v \in W$

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$u = a_0 + a_1 x + \ldots + a_n x^n$ and $v = b_0 + b_1 x + \ldots + b_n x^n$

so $\boxed{u+v}$ is

$(a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n$

$c_0 + c_1 x + \cdots + c_n x^n \in W$

Next

$ku \in W$, k is any scalar

$ku = k(a_0 + a_1 x + \cdots + a_n x^n)$

$= ka_0 + ka_1 x + \ldots + ka_n x^n$

$= d_0 + d_1 x + \ldots + d_n x^n \in W$

hence it is a subspace.

PROOF Prove that for a homogenous ~~solution~~ system of equations, with an coefficient matrix, the solution space is subspace of R^n

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show that set W of all 2×2 matrices having zeros on the main diagnol is a subspace of vector space M_{22}.

Solution

$$\text{Let } u = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}, \quad v = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}$$

$$u+v = \begin{bmatrix} 0 & a+c \\ b+d & 0 \end{bmatrix} \in W \text{ and } ku = \begin{bmatrix} 0 & ka \\ kb & 0 \end{bmatrix} \in W$$

since both $u+v$ and ku contain zeros on the main diagnol $\therefore$ $\boxed{W \text{ is a subspace of } M_{22}}$

LINEAR COMBINATIONS

A vector w is called a LINEAR COMBINATION of the vectors $v_1, v_2, \ldots, v_n$ if it can be expressed in the form $w = k_1v_1 + k_2v_2 + \ldots + k_nv_n$ where $k_1, k_2, \ldots, k_n$ are $\boxed{\text{scalars}}$

EXAMPLES

① Any vector in 3-dimensional space can be expressed as linear combination of the vectors e_1, e_2 and e_3

$$u = u_1e_1 + u_2e_2 + u_3e_3$$
$$= u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1) \Rightarrow (u_1, u_2, u_3)$$

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ATLEAST ONE SOLUTION $\Longrightarrow$ LINEARLY INDEPENDENT day / date:

(2) Any polynomial of degree n can be written as a linear combination of the following n+1 elements

$\{1, x, x^2, \ldots, x^n\}$ as

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

(3) Any two by two 2x2 matrix can be written as linear combination of

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

→ If $S = \{V_1, V_2, \ldots, V_n\}$ is a nonempty set of vectors then the vector equation

$k_1V_1 + k_2V_2 + \ldots + k_nV_n = 0$ has atleast one solution

namely $k_1 = k_2 = k_n = 0$

if this is the only solution then S is called a linearly independent set and vectors in set S are called LINEARLY INDEPENDENT VECTORS.

Examples

(1) The set S given by $\{e_1, e_2, e_3\}$ is LINEARLY INDEPENDENT and the vectors e_1, e_2 and e_3 are linearly independent vectors.

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CHECK:

Consider $k e_1 + k_2 e_2 + k_3 e_3 = 0 = (0,0,0)$

$\Rightarrow k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) = (0,0,0)$

$\Rightarrow (k_1, k_2, k_3) = (0,0,0)$

$\Rightarrow k_1 = 0, k_2 = 0, k_3 = 0$

(2) $S = \{1, x, x^2, \ldots, x^n\}$ is LINEARLY INDEPENDENT

since for

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = 0 \quad — (1)$$

$\forall x$

$\Rightarrow a_0 = a_1 = \ldots = a_n = 0$

since (1) is satisfied by infinite values of x

otherwise (1) has at most n distinct roots

if all or some of the coefficients $\neq 0$

(3) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is LINEARLY INDEPENDENT

SOLUTION:-

$$k_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow k_1 = k_2 = k_3 = k_4 = 0$ $\quad \therefore S$ is LINEARLY INDEPENDENT

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If $V_1, V_2, \ldots, V_n$ are vectas in a vecta space V and if every vector in V is expressible as a LINEAR COMBINATION of these vectors, then we say that $V_1, V_2, \ldots, V_n$ SPAN V.

Exaples

① $\{e_1, e_2, e_3\}$ spans R^3 (3-dimensional space) since $(u_1, u_2, u_3) = u = u_1 e_1 + u_2 e_2 + u_3 e_3$ $\forall u \in R^3$ for all.

② $\{1, x, x^2, \ldots, x^n\}$ spans the vecta space P_n since each polynomial $p(x)$ in P_n can be written as
$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ (LINEAR COMBINATION)

③ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ span M_{22}

(all matrices of order 2) since

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}$$

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If V is any vector space and $S = \{v_1, v_2, \ldots, v_n\}$ is a $\boxed{\text{set of vectors}}$ in V, then S is called a basis for V if the following two condition hold:

(a) S is linealy independent

(b) S spans V.

Let us consider some examples of sets which are bases ie they are LINEARLY INDEPENDENT as well as SPAN DIFFERENT VECTOR SPACES

Examples (OF BASES)

BASES → Plural of Basis.

① $\{e_1, e_2, e_3\}$ is a basis for R^3 because it's linearly independent as well as spans R^3 similarly

② $\{1, x, x^2, \ldots, x^n\}$ is basis for P_n and

③ $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is basis for M_{22}

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DIMENSION: The dimension of vector space V is defined to be the number of vectors in a basis for V.

REMARKS ① Dimension of $R^3 = 3$ since there are three vectors in $\{e_1, e_2, e_3\}$

② Dimension of $P_n = n+1$

③ Dimension of $M_{22} = 4$.

TRY THE FOLLOWING

Check weather

$$\left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for M_{22}?

HINT (i) First check that given matrices are linearly independent put

$$a_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and see if $a_1 = a_2 = a_3 = a_4 = 0$

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(ii) Take an arbitary ~~case~~ element of M_{22} as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and check if it can be written as a linear combination of given matrices.

For this put $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$= k_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and try to find k_1, k_2, k_3, k_4 in terms of a, b, c and d.

$$\left.\begin{matrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{matrix}\right\} \text{unknowns} \qquad \left.\begin{matrix} a \\ b \\ c \\ d \end{matrix}\right\} \text{knowns.}$$

Answer : YES it is a basis

$\because a_1 = a_2 = a_3 = a_4 = 0$ and

$k_1 = \frac{b-a}{2}$, $k_2 = \frac{a+b}{2}$

$k_3 = c$, $k_4 = d$.

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REMARK: A vector space may have more than one basis. But in all the bases (plural) the number of elements (vectors) are same. As we saw that

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and $\left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

are bases for M_{22} and both contain 4 vectors = dimension of M_{22}

Note: ① $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

is called STANDARD BASES for M_{22} and similarly $\{e_1, e_2, e_3\}$ and $\{1, x, x^2, \dots, x^n\}$ are standard bases for R^3 and P_n respectively.

② $\left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis but not STANDARD BASIS for M_{22}.

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REVISION

① $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$

is called STANDARD BASES for M_{22}.

② $\{e_1, e_2, e_3\}$ is STANDARD BASIS for R^3

③ $\{1, x, x^2, \dots, x^n\}$ is STANDARD BASIS for P_n

④ $\left\{\begin{bmatrix}-1&1\\0&0\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ is basis

but not a standard basis for M_{22} since

(first two elements are different from $\begin{bmatrix}1&0\\0&0\end{bmatrix}$ and

$\begin{bmatrix}0&1\\0&0\end{bmatrix}$

⑤ Standard basis for R^2 is $\{e_1, e_2\}$ where $e_1 = (1,0)$,
$e_2 = (0,1)$

RESULT: If $S = \{v_1, v_2, \dots, v_n\}$ is a set of n vectors in an n-dimensional space V, then S is a basis for V if either S spans V or S if either S spans V or S is linearly independent.

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∴ Example: Show that $\{(-3,7), (5,5)\}$ is basis for R^2.

solution Two methods

① Since R^2 is two dimensional space why?
Because $\{(1,0),(0,1)\}$ is the STANDARD BASIS for R^2 which contains two elements, therefore we only prove the given set to be linearly independent consider $k_1(-3,7) + k_2(5,5) = (0,0)$

$$\Rightarrow -3k_1 + 5k_2 = 0$$
$$7k_1 + 5k_2 = 0$$

$$\Rightarrow \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad - ①$$

$$\det \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix} = -15 - 35 \neq 0$$

$$\Rightarrow \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix} \text{ is INVERTIBLE}$$

$$① \Rightarrow \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow k_1 = k_2 = 0$ ∴ Lin Indp and finally $\{(-3,7), (5,5)\}$ is basis for R^2

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(2) Since R^2 is two dimensional, the
we only prove that the given set
spans R^2

consider

$$(x,y) = k_1(-3,7) + k_2(5,5)$$

$$\Rightarrow x = -3k_1 + 5k_2$$

$$y = 7k_1 + 5k_2$$

$$\Rightarrow \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \quad -(1)$$

$\because \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix}$ is INVERTIBLE as

$$\det \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix} = -15 - 35 \neq 0$$

$$\therefore \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix}^{-1} = \frac{-1}{50} \begin{bmatrix} 5 & -5 \\ -7 & -3 \end{bmatrix}$$

$$(1) \Rightarrow \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \frac{-1}{50} \begin{bmatrix} 5x - 5y \\ -7x - 3y \end{bmatrix}$$

$$k_1 = \frac{-1}{10}(x-y) \qquad k_2 = \frac{1}{50}(7x+3y)$$

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$\therefore$ Any element $(x,y) \in R^2$ can be written as LINEAR COMBINATION of $(-3,7)$ and $(5,5)$

$\therefore$ $\{(-3,7),(5,5)\}$ spans R^2

NOTE $\{(-3,7),(5,5)\}$ is a basis but not a standard basis since different from $\{(1,0),(0,1)\}$

Q11) show that if $S:\{v_1, v_2, \dots, v_r\}$ is a linearly independent set of vectors, then so is every non-empty subset of S.

PROOF

Suppose that S has a linearly dependent subset T. Denote it's vectors by $w_1, \dots, w_m$. Then

$$k_1 w_1 + \dots + k_m w_m = 0$$

But if we let $u_1, \dots, u_{n-m}$ denote the vectors which are in S but not in T, then

$$k_1 w_1 + k_m w_m + 0u_1 + \dots + 0u_{n-m} = 0$$

Thus we have linear combination of vectors $v_1, \dots, v_n$ which equals 0. Since not all constants are zero, it follows that S is not linearly independent set of vectors, contrary to the hypothesis. That is, if S is linearly independent set, then so is every non-empty subset. T.

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Q13) Show that if $\{v_1, v_2, \ldots, v_n\}$ is linearly dependent set of vectors in a vector space V and if $v_{r+1}, \ldots, v_n$ are any vectors in V, then $\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ is also linearly dependent.

PROOF

Since $\{v_1, v_2, \ldots, v_n)$ is linearly dependent set of vectors, there exist constants $c_1, c_2, \ldots, c_r$ not all zero such that.

$$c_1v_1 + c_2v_2 + \ldots + c_rv_r = 0$$

But then

$$c_1v_1 + c_2v_2 + \ldots + c_rv_r + 0v_{r+1} + \ldots + 0v_n = 0$$

The above equation implies that vector $v_1, \ldots, v_n$ are linearly dependent.

Q15) Show that if is $\{v_1, v_2\}$ linearly independent and v_3 does not lie in span $\{v_1, v_2\}$ then $\{v_1, v_2, v_3)$ is linearly independent.

PROOF:

Suppose that $\{v_1, v_2, v_3\}$ is linearly independent. Then there exists constants a, b and c not all zero such that

$$(*) \quad av_1 + bv_2 + cv_3 = 0$$

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CASE 1 $c=0$. Then (*) becomes

$$av_1 + bv_2 = 0$$

where not both a and b are zero.

CASE 2 $c \neq 0$. Then solving (*) for v_3 yields.

$$v_3 = -\frac{a}{c} v_1 - \frac{b}{c} v_2$$

This equation implies that v_3 is in span $\{v_1, v_2\}$ contrary to hypothesis. Thus $\{v_1, v_2, v_3\}$ is linearly independent.

RESULT: If $S = \{v_1, v_2, \dots, v_n\}$ is a basis for a vector space V then every vector v in V can be expressed in form

$$v = c_1v_1 + c_2v_2 + \dots + c_nv_n$$

in exactly one way.

PROOF Let $v = c_1v_1 + c_2v_2 + \dots c_nv_n$ and $v = k_1v_1 + k_2v_2 + \dots + k_nv_n$ subtracting the second equation from the first gives

$$0 = (c_1 - k_1)v_1 + (c_2 - k_2)v_2 + \dots + (c_n - k_n)v_n$$

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the linear independence of vectors in $\{v_1, v_2, \dots, v_n\}$ implies that

$$c_1 - k_1 = 0,\ c_2 - k_2 = 0,\ \dots,\ c_n - k_n = 0$$

$$\Rightarrow c_1 = k_1,\ c_2 = k_2,\ \dots,\ c_n = k_n$$

PREVIOUS RESULT ① $\{(1,0), (0,1)\}$ is standard basis for $\mathbb{R}^2$

② $\{(-3,7), (5,5)\}$ is basis but not standard basis for $\mathbb{R}^2$

RECALL $(x,y) = x(1,0) + y(0,1)$ and

$$(x,y) = \frac{(y-x)}{10}(-3,7) + \frac{(7x+3y)}{50}(5,5)$$

HOMOGENOUS LINEAR SYSTEM

for $Ax = 0$, exactly one of the following is true:

① system has only trivial solution (zero sol) if A is invertible $\det(A) \neq 0$ (linear indp)

② system has infinitely many solutions in addition to trivial solution.

→ If $S = \{v_1, v_2, \dots, v_r\}$ is a non-empty set of vectors then in vector equation.

$$k_1v_1 + k_2v_2 + \dots + k_rv_r = 0$$

if any one of scalars $k_i \neq 0$, $1 \leq i \leq r$ then S is linearly dependent and all vectors v_i are linearly dependent.

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EXAMPLE check wether $\{(2,2),(1,1)\}$ is linear dependent or independent?

$$k_1(2,2) + k_2(1,1) = (0,0)$$

$$2k_1 + k_2 = 0$$

$$2k_1 + k_2 = 0$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \therefore \det\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = 0$$

$\therefore$ nontrivial solution exist hence given vectors are dependent.

NOTE Recall that if $Ax=0$ represents a homogeneous system of equation then infinitely many nontrivial solution exists if A is singular $\therefore A^{-1}$ doesnt exist

RECALL ① $\{(1,1),(2,2)\}$ is linearly dependent
② $\{(5,5),(-3,7)\}$ is " " independent.

GEOMETRIC INTERPRETATION. in $\mathbb{R}^2$

→ linearly dependent if two vectors lie on SAME LINE.

→ linearly Independent if they lie on different line.

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NOTE: Also for any vector space V, the set $\{v_1, v_2\}$ is dependent if v_1, v_2 are scalar multiples of each other and $\{v_1, v_2\}$ is independent if and only if neither vector is scalar multiple.

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RESULT A set S with two or more vectors is
(a) linearly dependent if and only if atleast one of the vectors in S is expressible as linear combination of other vectors in S.
e.g. in last eg $\{v_1, v_2, v_3\}$ is dependent and $v_3 = v_2 + 3v_1$. ie v_3 is linear combination of v_1 and v_2. scalar multiple.
(b) " " independent if and only if no vector in S is expressible as linear combination of other vector in S.
e.g $\{e_1, e_2, e_3\}$ is linearly independent $\because$ none of e_1, e_2, e_3 is linear combination of other two.
Let $e_3 = k_1e_1 + k_2e_2$
$(0,0,1) = k_1(1,0,0) + k_2(0,1,0)$
$(0,0,1) = (k_1, k_2, 0)$ hence NOT possible.

LECTURE 17

RESULTS: ① If v_1 is a nonzero vector then $\{v_1\}$ is always independent
$\because kv_1 = 0 \Rightarrow k = 0 \quad \because v_1 \neq 0$
here $\{v_1\}$ is a set containing only one nonzero vector.

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(2) If 0 is a zero vector then $\{0\}$ is always dependent $\because k0=0$ is satisfied by infinite nonzero values of k.

→ The solution space of homogenous system of equations $Ax=0$, which is a subspace of $\mathbb{R}^n$ is called NULLSPACE of A

EXAMPLE: Null Space of $A=\begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \text{ t is any real number.}$$

→ The $\boxed{\text{dimension of nullspace of A}}$ is called the NULLITY of A and is denoted by NULLITY(A).

Example: in above example NULLITY(A) = 1 this is number of elements in the basis of nullspace of A i.e number of elements in $\left\{\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}\right\}$ which is ONE.

DEFINITION for an $m \times n$ matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$ the vectors (in $\mathbb{R}^n$)

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$r_1 = (a_{11}, a_{12}, \ldots, a_{1n})$

$r_2 = (a_{21}, a_{22}, \ldots, a_{2n})$

$\vdots \qquad \vdots$

$r_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$

formed from rows of A are called ROW VECTORS of A

and vectors

$$c_1 = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad c_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \quad c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \text{ in } R^m$$

formed from columns of A are called COLUMN VECTOR of A

→ If A is an $m \times n$ matrix, then subspace of R^n spanned by row vectors of A is called row space of A and subspace of R^m spanned by column vectors is called column space of A.

→ The dimensions of row space or column space of matrix A is called the RANK of A and denoted by RANK(A)

EXAMPLE

Consider the following $A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 5 & -1 \\ 7 & -1 & 5 & 8 \end{bmatrix}$

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* largest possible value for Rank(A) is min(m, n)
* min value of nullity of any matrix of order m×n is No of Col − Rank of Ma

(ii) find ROW(RANK) of A?

ROW(RANK) ⟶ dimension of A row space or find number of linearly independent row vector so

let $k_1 r_1 + k_2 r_2 + k_3 r_3 = 0$ ⟶

$k_1(2, -1, 0, 3) + k_2(1, 2, 5, -1) + k_3(7, -1, 5, 8) = (0, 0, 0, 0)$

but last time it was proved that

$k_1 = 3, \; k_2 = 1$ and $k_3 = -1$

∴ 3 ROW VECTORS are linearly dependent

∴ ROW(RANK) ≠ 3

→ if scalar multiple of other vector then linear dependent

now ignore $(7, -1, 5, 8)$ which is linear combination of first two row vectors s.t $3r_1 + r_2 = r_3$

$a_1 r_1 + a_2 r_2 = 0$

$\Rightarrow a_1(2, -1, 0, 3) + a_2(1, 2, 5, -1) = (0, 0, 0, 0)$

$\Rightarrow 2a_1 + a_2 = 0$

$-a_1 + 2a_2 = 0$

$5a_2 = 0$

$3a_1 - a_2 = 0 \Rightarrow a_1 = a_2 = 0$ $\boxed{\text{first two}}$ are linearly independent.

ROW(RANK) = 2 of given matrix

$\text{rank}(A^T) + \text{nullity}(A^T) = m \rightarrow$ num of rows. day / date:

$\text{rank}(A^T) = \text{rank}(A)$

NOTE ① Raw space of A is spanned by $\{(2,-1,0,3),(1,2,5,-1)\}$ which is a linearly independent set and hence basis for raw space

$\therefore$ ROW(RANK) = RANK(A) = 2 = dimension of raw space = no of linearly independent raw vectors of A

② Similarly we can prove that column space of A is spanned by $\left\{ \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}$ which is linear independent set of vector and hence basis of column space

$\therefore$ RANK(A) = COLUMN(RANK) = 2

(b) find NULLITY(A)

for this we solve $Ax = 0$

$$\Rightarrow \underbrace{\begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 5 & -1 \\ 7 & -1 & 5 & 8 \end{bmatrix}}_{3\times 4} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{4\times 1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{3\times 1}$$

augmented matrix is given by

$$\begin{bmatrix} 2 & -1 & 0 & 3 & 0 \\ 1 & 2 & 5 & -1 & 0 \\ 7 & -1 & 5 & 8 & 0 \end{bmatrix}$$

now reduce this to reduced echelon form

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* system consistent if rank(A) = rank(A | b) ~~< 3~~

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$$\sim \begin{bmatrix} 1 & 2 & 5 & -1 & 0 \\ 2 & -1 & 0 & 3 & 0 \\ 7 & -1 & 5 & 8 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 5 & -1 & 0 \\ 0 & -5 & -10 & 5 & 0 \\ 0 & -15 & -30 & 15 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 7R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to -1/5 R_2 \\ R_3 \to -1/15 R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad R_1 \to R_1 - 2R_2$$

$\Rightarrow x_1 + x_3 + x_4 = 0$ — (2)

$x_2 + 2x_3 - x_4 = 0$ — (3)

→ 2 equations and four unknowns.

let $x_3 = t$, $x_4 = r$ $\boxed{x_1 = -t - r}$ from (2)

$\Rightarrow x_2 = -2x_3 + x_4$

$x_4 - 2x_3 = \boxed{s - 2t = x_2}$

FREE VARIABLE

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t-s \\ -2t+s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ basis for null space of $A = \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\therefore$ NULLITY of $A = 2$

= number of free variables = dimension of null space of A

$x_3 = t$, $x_4 = s$ $\longrightarrow$ free variables.

leading variables $\leftarrow \begin{cases} x_1 = -t-s \\ x_2 = -2t+s \end{cases}$ $\Longrightarrow$ RANK$(A) = 2$

DIMENSION THEOREM $\qquad 2+2=4$

If A has n columns then RANK(A) + NULLITY$(A) = n$

$n \rightarrow$ no of columns = total variables

Result

In $Ax = 0$ $\left\} \begin{matrix} \overset{2}{m} \rightarrow \text{equations in} \\ \underset{4}{n} \rightarrow \text{unknowns} \end{matrix} \right.$ with $m < n$ ($2 < 4$) and if there

are $\overset{3}{r}$ nonzero rows in the reduced row-echelon form of augmented matrix then number of free variables are $n - r$ $\quad 4-3=1$

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Example $\quad m=2 \quad n=4$

That for the following matrix $\begin{bmatrix} 2 & -1 & 0 & 3 & 0 \\ 1 & 2 & 5 & -1 & 0 \\ 7 & -1 & 5 & 8 & 0 \end{bmatrix}$

reduced row-echelon form is $\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $r=2$

which contains two non-zero rows $\Rightarrow r=2$

here $m = \overset{2<4}{3} < 4 = n$

$\therefore n-r = 4-2 = 2$ free variables

As we say that $x_3 = t$ and $x_4 = s$ were free variables and their no $= 2$ also x_1 and x_2 were leading variables given by

$$\left.\begin{aligned} x_1 &= -t-s \\ x_2 &= s-2t \end{aligned}\right]$$

and are $r = 2 =$ RANK where r indicates non-zero rows in reduced row echelon of augmented matrix

ANOTHER METHOD TO FIND THE BASIS FOR ROW SPACE OF MATRIX

In order to do this we refer to following

THEOREM The nonzero row vectors in any row echelon form of matrix form a basis for row space of that matrix.

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EXAMPLE: The reduced row echelon form of

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 5 & -1 \\ 7 & -1 & 5 & 8 \end{bmatrix}$$ is given by

$$R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ the two nonzero row vectors in R form basis for row ~~shape~~ space of A and R, therefore the basis for row space of R and A is given by

$\{(1,0,1,1), (0,1,2,-1)\}$

elementary row operations do not change the row space of a matrix because

→ If A is an $m \times n$ matrix and B is row equivalent to A then

① Row space is a subspace of R^n

② If the row operation is a row interchange ★, then B and A have same row vectors

③ If the row operation is multiplication of row by a nonzero scalar or addition of multiple of one row to another then the row vectors ... $, r_m'$ of B are linear combinations of ... thus they lie in the row space of ...

... (subspace of R^n) is closed under

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addition and scalar multiplication.

METHOD TO FIND BASIS FOR COLUMN SPACE OF AN $m \times n$ MATRIX A.

If A is an $m \times n$ matrix and R is it's row-~~reduced~~ echelon form then the column vectors with leading 1's of row vectors form a basis for column space of R. and the corresponding column vector in A form BASIS.

EXAMPLE:

for $A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 5 & -1 \\ 7 & -1 & 5 & 8 \end{bmatrix}$ the reduced row echelon

form is given by

$R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ RANK(A) = 2

the first two column vectors in R which contain the leading 1's form the basis for the column space of R (but not A) and the corresponding column vectors in A form the basis for the column space of A (but not R) because elementary row operations usually change the column space.

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REMARS

① Basis for the column space for $R = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

② Basis for the column space for $A = \left\{ \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\}$

③ Elementary Row operations can ~~make~~ change the column space:

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ $\therefore$ the column space for A

consists of all scalar multiples of the first column vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Now $A \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

$\therefore$ The column space for $B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ consists of all scalar multiples of first column vector i.e $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. ~~space of A~~.

this not same as column space of A.

REMARK

If A is a matrix and R is it's echelon form then the number of nonzero rows or number of column vector that contain leading 1's in R is rank of matrix A.

LECTURE 19

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REVISION

If u and v are vectors in 2-space or 3space and θ is the angle b/w u and v then the dot product or euclidean inner product $u \cdot v$ is defined by

$$u \cdot v = \begin{cases} \|u\| \, \|v\| \cos\theta \\ 0 \quad \text{if } u=0 \text{ or } v=0 \end{cases}$$

(i) $u \cdot v = v \cdot u$

(ii) $u \cdot (v+w) = u \cdot v + u \cdot w$

(iii) $k(u \cdot v) = (ku) \cdot v = u \cdot (kv)$

(iv) $v \cdot v > 0$ if $v \neq 0$ and $v \cdot v = 0$ if $v = 0$

$v \cdot v = \|v^2\| > 0$

The distance b/w two points is defined by

$$\|u - v\| = \|v - u\|$$
$$= [(v-u) \cdot (v-u)]^{1/2}$$
$$= \sqrt{(v_1-u_1)^2 + (v_2-u_2)^2 + (v_3-u_3)^2}$$

INNER PRODUCT

An inner product on a real vector space V is a real number $\langle u, v \rangle$ which satisfies following axioms for ~~all~~ ~~al~~ all vectors u, v and $w \in V$ and all scalar k.

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① $\langle u, v \rangle = \langle v, u \rangle \longrightarrow u \cdot v = v \cdot u$

② $\langle u+v, w \rangle = \langle w, u+v \rangle \underset{=}{\longrightarrow} u \cdot (v+w) = u \cdot v + u \cdot w$

$\langle u, w \rangle + \langle v, w \rangle = \langle w, u \rangle + \langle w, v \rangle.$

③ $\langle ku, v \rangle = k \langle u, v \rangle \longrightarrow k(u \cdot v) = (ku) \cdot v$

④ $\langle v, v \rangle \geq 0 \longrightarrow v \cdot v > 0 \ , \ v \neq 0$

→ A real vector space with an inner product is called a REAL INNER PRODUCT SPACE.

<u>Example</u>: If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are vectors in R^3, then the formula

$\langle u, v \rangle = u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$

→ If V is an inner product space then norm of vector u in V is ~~dif~~ denoted by $\|u\|$ and is defined by

$\|u\| = \langle u, u \rangle^{1/2}$

another notation of euclidean inner product:

for $u, v \in R^3$

$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $\quad \langle u, v \rangle = u \cdot v = v^t u$ WHY?

$\because v^t u = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

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$= u_1v_1 + u_2v_2 + u_3v_3 = u \cdot v$

$v^t \to$ TRANSPOSE OF V

TRY THE FOLLOWING

Let $f(x)$, $g = g(x)$ be two functions (continuos) then decide wether

$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx$ defines an inner product on $C[a,b]$ INNER PRODUCT

HINT Check ① $\langle f, g \rangle = \langle g, f \rangle$

② $\langle f + g, s \rangle = \langle f, r \rangle + \langle g, r \rangle$

③ $\langle kf, g \rangle = k \langle f, g \rangle$

④ $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0$ if and only if $f = 0$.

SOLUTION :- let $f(x)$ and $g(x)$ be two functions. such that $f, g \in C[a,b]$, $\{f = f(x), g = g(x)\}$

$C[a,b] \to$ all continuos functions defined on interval $[a,b]$ consider

① $\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx = \int_a^b g(x) f(x) dx = \langle g, f$

② $\langle f + g, s \rangle = \int_a^b [f(x) + g(x)]\, s(x)\, dx$

$= \int_a^b f(x)\, s(x)\, dx + \int_a^b g(x)\, s(x)\, dx$

$= \langle f, s \rangle + \langle g, s \rangle$

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$, g> = \int_a^b kf(x)\,g(x)\,dx$

$= k\int_a^b f(x)g(x)\,dx = k<f,g>$

④ $<f,f> = \int_a^b f(x)f(x)\,dx$

$= \int_a^b f^2(x)\,dx, \quad \because f \in C[a,b]$

$\therefore f^2$ is banded by, $\because f^2(x) \geq 0$

$\Rightarrow \min . f^2(x) = 0$ AND

$\Rightarrow \max . f^2(x) = M$

$\Rightarrow 0 \leq f^2(x) \leq M$

integrating

$\int_a^b 0\,dx \leq \int_a^b f^2(x) \leq \int_a^b M\,dx$

$\Rightarrow 0 \leq \int_a^b f^2(x)\,dx \leq M(b-a)$

$\int_a^b 0\,dx = c \Big|_a^b = c - c = 0$

NOTICE THAT

① $\int_a^b 0\,dx$ is the area b/w x-axis and x-axis and $= 0$

② $\int_a^b f^2(x)\,dx$ is the area b/w $y = f^2(x)$ and x-axis.

③ $\int_a^b M\,dx = M(b-a)$ area b/w $f^2(x) = M$ and x-axis on $[a,b]$

$\therefore <f,g> = \int_a^b f(x)\,g(x)\,dx$

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DEFINITION : If V is an inner product sp and u, v ∈ V then u and v are called $\boxed{\text{ORTHOGONAL VECTORS}}$ if

$\langle u, v \rangle = 0$, further if

$\|u\| = \|v\| = 1$ then u and v are

called ORTHOGONAL VECTORS.

TRY THE FOLLOWING

If $f(x) = \frac{1}{\sqrt{2}}$ and $g(x) = \sqrt{\frac{3}{2}}\, x$

(a) then according to the inner product defined in last example.

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

f(x) and g(x) are orthogonal and orthonormal on [-1, 1] ??

IMPORTANT NOTES

① for $\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$

$$\langle f, f \rangle^{1/2} = \|f\| = \sqrt{\int_a^b f^2(x)\, dx}$$

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u and v are vectors from R^2
$_3$ and

$\langle u,v\rangle = u\cdot v = 0$ then u, v are orthogonal as well as perpendicular.

(3) If V is any inner product space and $\langle u,v\rangle \neq u\cdot v$ then $\langle u,v\rangle = 0$ means that u, v are orthogonal but not perpendicular.

LECTURE 20

PROBLEM: GRAM - SCHMIDT PROCESS

$V \longrightarrow$ inner product space

Given: $\{u_1, u_2, \dots, u_n\}$ be any basis for V

Then have to produce an orthogonal basis $\{v_1, v_2, \dots, v_n\}$ for V?

ie $\langle v_i, v_j\rangle = 0$, $i \neq j$

$1 \le j \le n$, $1 \le i \le n$

which can be normalised to produce an ~~orthogonal~~ orthonormal basis ie

$$\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \dots, \frac{v_n}{\|v_n\|} \right\}$$

ie norm of each vector $= 1$

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Recall that for euclidean inner product (DOT PRODUCT)

$proj_a u = \frac{(u \cdot a) a}{\|a\|^2}$ and vector projection

(component) of u perpendicular to a is given by

$$u - proj_a u = u - \frac{(u \cdot a) a}{\|a\|^2}$$

similarly if v_1 and v_2 are orthogonal vectors in an inner product space V and $u \in V$ such that horizontal ($\overrightarrow{OP}$) projection of u lies along v_1 then

$$v_2 = u - \frac{\langle u, v_1 \rangle v_1}{\|v_1\|^2}$$

PROOF

$$u = v_2 + \overrightarrow{OP}$$

$$\because u = kv_1 + v_2 = \overrightarrow{OP} + v_2$$

$$\because \overrightarrow{OP} = kv_1$$

$$\Rightarrow v_2 = u - \overrightarrow{OP} = u - kv_1 \quad \text{— (1)}$$

taking inner product on both sides by v_1, we get

$$\overset{0}{\cancel{\langle v_2, v_1 \rangle}} = \langle u, v_1 \rangle - \langle kv_1, v_1 \rangle$$

$$\Rightarrow \langle u, v_1 \rangle = \langle kv_1, v_1 \rangle$$

$$\langle u, v_1 \rangle = k \langle v_1, v_1 \rangle$$

$$k = \frac{\langle u, v_1 \rangle}{\|v_1\|^2}$$

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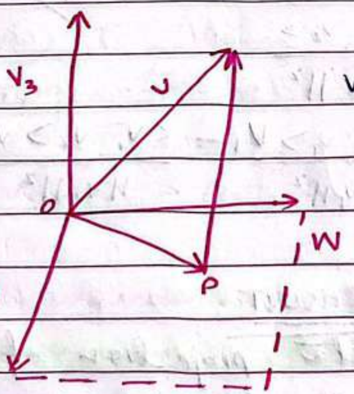
$$V_2 = u - \frac{\langle u, v_1 \rangle}{\|v_1\|^2} v_1 \quad \text{proj}_{v_1} u$$

<u>NOTE</u>: Vectors $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$ span the xy plane because any vector in the xy-plane can be written as their linear combination.

$$(x, y, 0) = x(1,0,0) + y(0,1,0)$$
$$= xe_1 + ye_2$$

similarly if v_1 and v_2 are orthogonal vectors spanning a plane W as shown below,



$V_3 = ?$

$v = \overrightarrow{OP} + V_3$

V_3 is orthogonal to both v_1 and v_2. $\overrightarrow{OP}$ is the projection or component of v in W. $\because \overrightarrow{OP}$ lies in W (spanned by v_1 and v_2)

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$\overrightarrow{OP}$ is linear combination of v_1 and v_2

$\therefore \overrightarrow{OP} = k_1v_1 + k_2v_2$ but $v = \overrightarrow{OP} + v_3 \Rightarrow v_3 = v - \overrightarrow{OP}$

$\Rightarrow v_3 = v - k_1v_1 - k_2v_2$ — (1)

to find k_1 take inner product with v_1

$$\Rightarrow \cancel{\langle v_3, v_1\rangle}^{0} = \langle v, v_1\rangle - k_1\underbrace{\langle v_1, v_1\rangle}_{\|v_1\|^2} - k_2 \cancel{\langle v_2, v_1\rangle}^{0}$$

v_i, v_j

$i \neq j = 0$

$$\Rightarrow k_1 = \frac{\langle v, v_1\rangle}{\|v_1\|^2}$$

similarly to find k_2 take inner product with v_2

$$\Rightarrow \cancel{\langle v_3, v_2\rangle}^{0} = \langle v, v_2\rangle - k_1 \cancel{\langle v_1, v_2\rangle}^{0} - k_2\langle v_2, v_2\rangle$$

$$\Rightarrow k_2 = \frac{\langle v, v_2\rangle}{\|v_2\|^2}$$

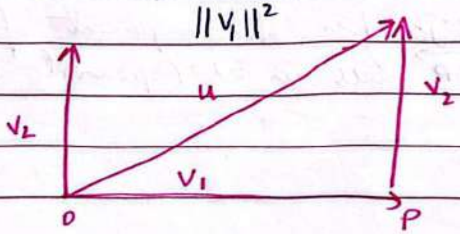
$$v_3 = v - \frac{\langle v, v_1\rangle}{\|v_1\|^2}v_1 - \frac{\langle v_1, v_2\rangle}{\|v_2\|^2}v_2$$

LECTURE 21

PREVIOUS RESULT

(1) If v_2 is the vector projection of u orthogonal to v_1 then v_2 is given by

$$v_2 = u - \frac{\langle u, v_1\rangle}{\|v_1\|^2}v_1 \quad \text{as shown}$$



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$u - proj_a u = u - \frac{(u \cdot a)a}{\|a\|^2}$

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(2) similarly, the vector projection v_3 of v which is orthogonal to both v_1 and v_2 is given by

$$v_3 = v - \frac{\langle v, v_1 \rangle v_1}{\|v_1\|^2} - \frac{\langle v, v_2 \rangle v_2}{\|v_2\|^2} \longrightarrow (*)$$

SUMMARY

If V is an inner product space and $\{u_1, u_2, \dots, u_n\}$ is a basis for V then we can find the ~~operation~~ orthogonal basis by following these steps.

(1) let $u_1 = v_1 \rightarrow$ (1)

(2) to find v_2 orthogonal to v_1 by computing the component of u_2 that is orthogonal to v_1

$$v_2 = u_2 - proj_{v_1} u_2 \longrightarrow \frac{\langle u_2, v_1 \rangle v_1}{\|v_1\|^2}$$

(3) To find v_3 orthogonal to both v_1 and v_2 by computing component of u_3 orthogonal to plane spanned by v_1 and v_2 and is given by

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle v_1}{\|v_1\|^2} - \frac{\langle u_3, v_2 \rangle v_2}{\|v_2\|^2}$$

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replace v by u_3 in (*) in order to get ③ so we obtained $v_1, v_2, v_3, \dots, v_n$ so on until we get v_n.

The preceding step by step construction for converting an arbitrary basis into an orthogonal basis is called. GRAM – SCHMIDT PROCESS

Example :
Let vector space P_2 have inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x) q(x)\, dx$$

apply the gram-schmidt process to transform the standard basis $S = \{1, x, x^2\}$ into an arbitrary orthonormal basis.

SOLUTION :

$$\langle p, q \rangle = \int_{-1}^{1} p(x) q(x)\, dx$$

$$S = \{1, x, x^2\}$$

$$u_1 = 1, \quad u_2 = x, \quad u_3 = x^2$$

① $u_1 = v_1 = 1$

② $\|v_1\| = \|1\| = \langle v_1, v_1 \rangle^{1/2}$

$$= \langle 1, 1 \rangle^{1/2} = \left(\int_{-1}^{1} 1\, dx \right)^{1/2} \Rightarrow \sqrt{2}$$

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(3) $v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

$v_2 = x - \frac{\cancel{\langle x, 1 \rangle}^{\,0}}{\|1\|^2} = x$

(4) $\|v_2\| = \|x\| = \left(\int_{-1}^{1} x^2 dx\right)^{1/2}$

$\langle x, x \rangle^{1/2} = \sqrt{2/3}$

(5) $v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$

but $\langle u_3, v_1 \rangle = \int_{-1}^{1} x^2 dx$

$= \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$

and $\langle u_3, v_2 \rangle = \langle x^2, x \rangle$

$= \int_{-1}^{1} x^3 dx = 0$

$\therefore v_3 = u_3 - \frac{\langle u_3, v_1 \rangle v_1}{\|v_1\|^2} - \cancel{\frac{\langle u_3, v_2 \rangle v_2}{\|v_2\|^2}}^{\,0}$

$= x^2 - \frac{2}{3} \cdot \frac{1}{2} = x^2 - \frac{1}{3} = v_3$

(6) $\|v_3\| = \langle v_3, v_3 \rangle^{1/2}$

$\left(\int_{-1}^{1} (x^2 - 1/3) dx\right) \Rightarrow \sqrt{8/45}$

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(7) Required orthonormal bases is = $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$

$$= \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x, \left(x^2 - 1/3\right)\sqrt{\frac{45}{8}} \right\}$$

$$= \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x, (3x^2-1)\frac{\sqrt{5}}{2\sqrt{2}} \right\}$$

TRY THE FOLLOWING

If $S = \{v_1, v_2, \dots, v_n\}$ is an orthogonal set of nonzero vector in an inner product space, then S is linearly independent.

HINT Assume $k_1 v_1 + k_2 v_2 + \dots + k_n v_n = 0$ and prove

$$k_1 = k_2 = \dots = k_n = 0$$

PROOF Let $k_1 v_1 + k_2 v_2 + \dots + k_n v_n = 0$ taking inner product with v_i on both sides,

$$\langle k_1 v_1 + k_2 v_2 + \dots + k_n v_n, v_i \rangle = 0$$

$$\because \langle 0, v_i \rangle = \langle 0+0, v_i \rangle = \langle 0, v_i \rangle + \langle 0, v_i \rangle$$

$$\Longrightarrow \langle 0, v_i \rangle = 0$$

$$k_1 \langle v_1, v_i \rangle + k_2 \langle v_2, v_i \rangle + \dots + k_i \langle v_i, v_i \rangle + \dots + k_n \langle v_n, v_i \rangle = 0 \quad \text{--- (1)}$$

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But $S = \{v_1, v_2, \dots v_n\}$ is an orthogonal set therefore $\langle v_i, v_j \rangle = 0$ when $i \neq j$ so that ① equation reduces to
$k_i \langle v_i, v_i \rangle = 0$ but $v_i \neq 0$
therefore $\langle v_i, v_i \rangle = \|v_i\|^2 > 0$ so that $k_i = 0$. Since the subscript i is arbitrary we have $k_1 = k_2 = \dots = k_n = 0$; thus, S is linearly independent.

LECTURE 22

RESULT:

If A is an orthogonal matrix then
$$\det(A) = \pm 1$$

PROOF:

$$\because A^{-1} = A^T$$
$$\Rightarrow \det(A^{-1}) = \det(A^T)$$
$$\Rightarrow \frac{1}{\det(A)} = \det(A^T) = \det(A)$$
$$\Rightarrow \frac{1}{\det(A)} = \det(A)$$
$$\Rightarrow 1 = [\det(A)]^2$$
$$\Rightarrow \det(A) = \pm 1$$

NOTE: if $A^{-1} = A^T$ and $\det(A) = 1$ then A is called proper orthogonal matrix

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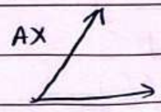
EIGENVALUES / EIGENVECTORS.

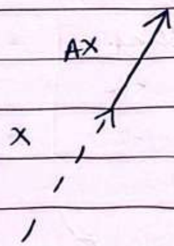
NOTE: If A is an $n \times n$
is a column vector of
$Ax = b$ is also a column
$n \times 1$. But is there a
relationship b/w X an

- → To FIND EIGENVALUE
  - ↳ $A - \lambda I$
  - ↳ $\det(A - \lambda I) = 0$
  - ↳ find λ values.
- → TO FIND EIGENVECTOR
  - ↳ $(A - \lambda I)x = 0$
  - ↳ insert λ values
  - ↳ form equations
  - ↳ then find eigenvector.

SEE THE FOLLOWING FIGUR

(1) 

no general relationship between X and AX

(2) 

X and AX are scalar multiples of each other

DEFINITION If A is an $n \times n$ matrix then a non-zero vector X is called EIGENVECTOR of A if $\boxed{AX \text{ is scalar multiple of } X}$. That is $AX = \lambda X$ for some scalar λ. The scalar λ is called EIGENVALUE of A and X is said to be EIGENVECTOR of A corresponding to λ.

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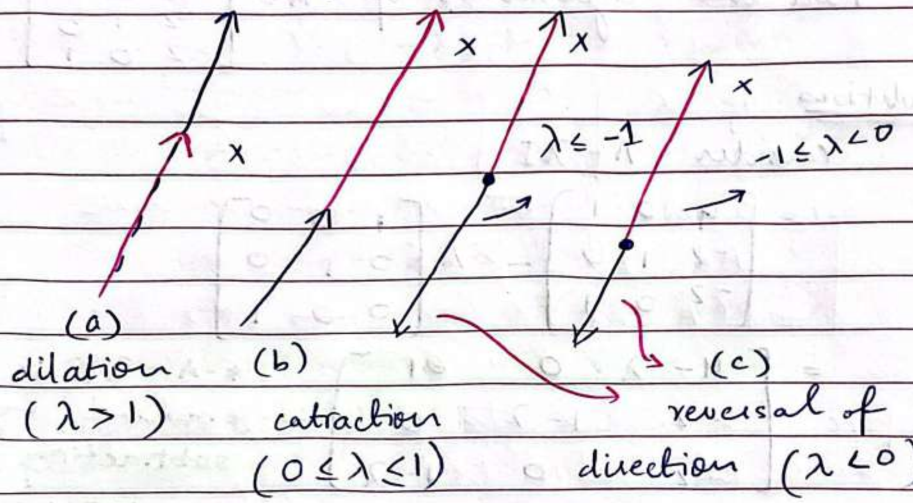
GEOMETRIC INTERPRETATION

If $AX = \lambda X$, multiplication by A dilates X, contracts X, or reverses the direction of X, depending on the value of λ



METHOD TO FIND EIGENVALUES

$\because AX = \lambda X \Rightarrow AX = \lambda I X$ —①

where I is identity matrix

① $\Longrightarrow (A - \lambda I)X = 0$ — ②, $X \neq 0$

since eigenvector is A nonzero vector therefore for λ to be an eigenvalue of A, nonzero (non-trivial) solutions of ② exist if

$\det(A - \lambda I) = 0 \longrightarrow (*)$

(*) is called the characteristic equation of A

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After solving (*), we get the eigenvalues (λ's) of A.

EXAMPLE

Find the eigenvalues of $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

Solution

Consider $A - \lambda I$

$$= \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix}$$

* $A - \lambda I$ is obtained by subtracting λ from DIAGNOL ENTRIES of A.

The characteristic equation of A is

$\det(A - \lambda I) = 0$, $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$+$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 0 & \boxed{1} \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)(1-\lambda)^2 + 2(1-\lambda) = 0$$

$(+)\,1\,(-2(1-\lambda))$

$-2(1-\lambda)$

$(-2 \times 0) +$

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Order of Matrix = No of Eigen Values. day / date:

NOTE: This is an equation of degree 3 = order of A.

Therefore no. of eigenvalues of A are = order of A / 3

$$① \Rightarrow (4-\lambda)(1-\lambda)^2 + 2(1-\lambda) = 0$$

$$\Rightarrow \underbrace{(1-\lambda)}[(4-\lambda)(1-\lambda)+2] = 0$$

$$\longrightarrow 1-\lambda = 0, \quad \lambda = 1 \quad ①$$

$$(4-\lambda)(1-\lambda)+2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda-2)(\lambda-3) = 0$$

$$\Rightarrow \lambda = 2, \lambda = 3 \quad ② \text{ and } ③$$

OR $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ are the eigenvalues of A.

NOTE: for an $n \times n$ matrix A, the characteristic equation $\det(A - \lambda I) = 0$ is of degree n and number of eigenvalues of $A = n$ (a few may be repeated)

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METHOD TO FIND EIGENVECTORS

EXAMPLE

Find the eigenvectors of $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

Solution If $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector of A corresponding to λ then we have

$$AX = \lambda X = \lambda I X$$

$$\Rightarrow (A - \lambda I) X = 0 \longrightarrow (*)$$

Therefore in order to obtain the eignvectors we have to solve $(*)$

$$(*) \Rightarrow \begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we have to obtain the eigenvectors corresponding to $\lambda = 1$, $\lambda = 2$ and $\lambda = 3$

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STEP1 eigenvector corresponding to $\lambda = 1$
putting $\lambda = 1$ in (1)

$$\Rightarrow \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 3x_1 + x_3 = 0 \quad -2x_1 = 0$

$\Rightarrow x_1 = x_3 = 0, \; x_2 = t \qquad \because x \neq 0$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda = 1$, eigenvector is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ or any of it's nonzero multiple.

STEP 2 for $\lambda = 2$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x_1 + x_3 = 0 \Rightarrow x_3 = -2x_1$

$-2x_1 - x_2 = 0 \Rightarrow x_2 = -2x_1$

$-2x_1 - x_3 = 0 \qquad x_1 = x_1$

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for $x_1 = t$, $x_2 = -2t$, $x_3 - 2t$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ -2t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

↑ eigenvector.

STEP 3 finally for $\lambda = 3$, eigenvector is

$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ or any of multiples.

LECTURE 23

EIGENSPACE: The eigenvectors corresponding to λ are the nonzero vectors in the solution space of $AX = \lambda X$ or $(A - \lambda I)X = 0$. We call this solution space the eigenspace of A corresponding to λ.

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QUESTION Find the bases for eigenspaces of

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Solution: Eigenvalues of A are 1, 2, 3 therefore there are THREE eigenspaces of A corresponding to $\lambda = 1, 2,$ and 3 so we proceed as follows.

① eigenspace corresponding to $\lambda = 1$ is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{BASIS.}}$$ eigenvector = eigenspace.

② and ③ Therefore bases for eigenspaces corresponding to $\lambda = 2$ and $\lambda = 3$ are given by $\left\{ \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$ respectively.

NOTE: Dimension of each $\boxed{\text{eigenspace of A}}$ described above = 1 ∵ each has only one basis vector.

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*Eigenvalues of Diagnol and triangular matrices are diagnol entries.

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LECTURE 24

TRACE: If A is a square matrix then the trace of A is denoted by $tr(A)$ and is defined to be the sum of entries on the main diagonal of A.

EXAMPLE: for $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$$tr(A) = 4 + 1 + 1 = 6$$

last time we saw that eigenvalues of $A = 1, 2, 3$

consider

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 2 + 3 = 6 = tr(A)$$

Result

If A is square matrix then $tr(A) =$ sum of it's eigenvalues also note that from ①

$$det(A) = 4 + 1(2) = 6 = \lambda_1 \lambda_2 \lambda_3$$

$$\lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 \lambda_2 \lambda_3$$

RESULT

If A is square matrix then $det(A) =$ product of it's eigenvalues.

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TRY THE FOLLOWING

(a) Show that the characteristic equation of 2×2 matrix A can be expressed as.

$$\lambda^2 - tr(A)\lambda + det(A) = 0 \longrightarrow (1)$$

(b) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the solutions of characteristic equation of A are.

$$\lambda = \frac{1}{2}\left[(a+d) \pm \sqrt{(a-d)^2 + 4bc}\right]$$

NOTE: If λ_1, λ_2 are roots of (1) then

$$\lambda_1 + \lambda_2 = tr(A), \quad \lambda_1\lambda_2 = det(A)$$

RECALL

for α, B as roots of $ax^2 + bx + c = 0$, $a \neq 0$

$$\alpha + B = \frac{-b}{a} \quad \text{and} \quad \alpha B = \frac{c}{a}$$

DIAGONALIZATION

A square matrix A is called diagonalizable if there is an invertible matrix P such that $P^{-1}AP$ is diagonal matrix; the matrix P is said to diagonalize A.

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$AP = DP$ * if not to find P^{-1} day / date:

EXAMPLE

$P^{-1} A P$ → bases.

$$= \begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & -2 & 1 \\ -1 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ -3 & -4 & 1 \\ -3 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

P is the matrix having eigenvectors of A as it's column vectors and $P^{-1}AP$ is a diagnol matrix having eigen values as main diagnol.

APPLICATION OF DIAGONALISATION.

The Eigenvector Problem.

Given an $n \times n$ matrix A, does there exist a basis for R^n consisting of eigenvectors of A?

If A is an $n \times n$ matrix then the following are equivalent

(A) A is diagonalizable.

(B) A has n linearly independent EIGENVECTORS.

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PROOF (b) $\Rightarrow$ (a)

Assume that A has n linearly independent eigenvector $P_1, P_2, \ldots, P_n$, ~~which~~ with corresponding eigenvalues. $\lambda_1, \lambda_2, \ldots, \lambda_n$

Consider the matrix P with $P_1, P_2, \ldots, P_n$ as it's column vectors.

i.e

$$P = \begin{array}{c} \begin{array}{cccc} \downarrow P_1 & \downarrow P_2 & & \downarrow P_n \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \end{array}$$

now the columns of the product AP are $AP_1, AP_2, \ldots, AP_n$

EXTRA: Consider $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$,

$$AP = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

$$\Rightarrow AP = \begin{bmatrix} a_{11}P_{11} + a_{12}P_{21} & a_{11}P_{12} + a_{12}P_{22} \\ a_{21}P_{11} + a_{22}P_{21} & a_{21}P_{12} + a_{22}P_{22} \end{bmatrix}$$

$\hookrightarrow P_{11}$ $\quad \hookrightarrow P_{21}$

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First column of AP is $\begin{bmatrix} a_{11}P_{11} + a_{12}P_{21} \\ a_{21}P_{11} + a_{22}P_{21} \end{bmatrix}$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix} = AP_1$$

x ——————— x

But $AP_1 = \lambda_1 P_1$, $AP_2 = \lambda_2 P_2$, ..., $AP_n = \lambda_n P_n$ so that

$$AP = \begin{bmatrix} \lambda_1 P_{11} & \lambda_2 P_{12} & \cdots & \lambda_n P_{1n} \\ \lambda_1 P_{21} & \lambda_2 P_{22} & \cdots & \lambda_n P_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda_1 P_{n1} & \lambda_2 P_{n2} & \cdots & \lambda_n P_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$= PD$

$\Rightarrow AP = PD$ where D is the diagnol matrix having the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the main diagnol. Since the column vectors of P are LINEARLY INDEPENDENT Therefore

$\text{RANK}(P) = n$ so that

$\det(P) \neq 0$

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Recall that RANK is also defined as the highest order of nonzero determinant

$\therefore P$ is invertible

thus $AP = PD$ can be written as $P^{-1}AP = D$; that is A is diagnolizable.

Converse is also true.

(A) $\Longrightarrow$ (B)

If A is diagnolizable then A has n ~~of~~ linearly independent eigenvectors and they form a basis for R^n.

* unique values of eigen values in diagnol matrix then DIAGONALIZABLE.
* if eigenvectors is not equal to order of matrix then it is NOT DIAGONALIZABLE.

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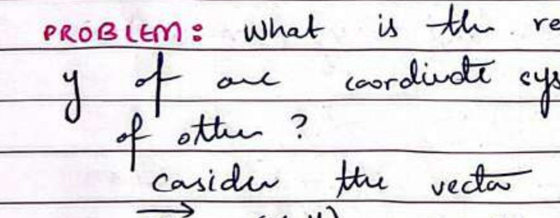
ROTATION OF AXES.

x-coordinate gives distance from y-axis, y-coord gives distance from x-axis. If the axes are rotated through an angle θ, then every point of plane has two representation:
(x, y) in original coordinate system and
(x', y') in new system.

PROBLEM: What is the relationship b/w x and y of one coordinate system and x' and y' of other?

Consider the vector $\overrightarrow{OP}$ which is given by $\overrightarrow{OP} = (x, y) = xe_1 + ye_2$ in original coordinate system and also
$\overrightarrow{OP} = (x', y') = x'e_1' + y'e_2'$ in new coordinate system (e_1' and e_2' are also unit vector)



$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

$$\Rightarrow e_1' \cdot e_1 = k_1 e_1 \cdot e_1 + k_2 \cancel{e_2 \cdot e_1}$$

$$k_1 = e_1' \cdot e_1 = \cos\theta$$

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similarly $k_2 = \sin\theta$

$\therefore e_1' = \cos\theta e_1 + \sin\theta e_2$

Similarly

$e_2' = -\sin\theta e_1 + \cos\theta e_2$

$\rightarrow (-\sin\theta, \cos\theta)$

$$\overrightarrow{OP} = x'e_1' + y'e_2'$$
$$= x'(\cos\theta e_1 + \sin\theta e_2) + y'(-\sin\theta e_1 + \cos\theta e_2)$$
$$= (x'\cos\theta - y'\sin\theta)e_1 + (x'\sin\theta + y'\cos\theta)e_2$$
$$= xe_1 + ye_2 = \overrightarrow{OP} \quad \text{—} (1).$$

Shows that

$x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

The matrix $\begin{array}{cc} \begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}0\\1\end{bmatrix} \\ \end{array}$ $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = R$ (say) $\begin{array}{cc} e_1 & e_2 \end{array}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

which gives rotation through an angle θ (counterclockwise) is called ROTATION MATRIX.

It's column vectors are new basic vector ie $[[e_1'], [e_2']]$ and are orthonormal with euclidean inner product.

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Also it's raw vector are orthonormal with the euclidean inner product.

NOTES :

(1) $RR^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ into

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= R^T R \Rightarrow R^T = R^{-1}$$

(2) $\det(R) = \cos^2\theta + \sin^2\theta = 1$

(3) when there is no rotation then

$R = I = \begin{bmatrix} [e_1] [e_2] \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for $\theta = 0$

$$\because \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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→ A square matrix A with the property $A^{-1} = A^T$ is said to be an ~~orthonormal~~ orthogonal matrix.

OR

$$AA^T = A^TA = 1$$

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LECTURE 25

REVISION

Last time we saw that the eigenvalues of $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ were distinct and $= 1, 2, 3$

corresponding eigenvectors are given by $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ respectively which are linearly independent $\therefore$ They form a basis for R^3 because $\boxed{A \text{ is diagonalizable}}$ as shown below:

$$P^{-1} A P,$$

$$\begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & -2 & 1 \\ -1 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ If $v_1, v_2, \ldots, v_k$ are eigenvectors of A corresponding to $\boxed{\text{distinct eigenvalues } \lambda_1, \lambda_2}, \ldots, \lambda_k$ thus $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent set.

* distinct eigenvalues → so eigen vectors are linearly independent.

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What will happen if eigenvalues are not distinct?

CONSIDER THE FOLLOWING EXAMPLE

Let $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ eigenvalues of A are

obtained by solving $\det(A - \lambda I)$

$$= \begin{vmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)[(4-\lambda)^2 - 4] - 2[2(4-\lambda) - 4] + 2[4 - 2(4-\lambda)] = 0$$

$$\Rightarrow (4-\lambda)[\lambda^2 - 8\lambda + 12] - 4[2(4-\lambda) - 4] = 0$$

$$\Rightarrow (4-\lambda)[(\lambda-6)(\lambda-2)] - 8[4-\lambda-2] = 0$$

$$\Rightarrow (4-\lambda)(\lambda-6)(\lambda-2) + 8(\lambda-2) = 0$$

$$\lambda = 2 \quad \text{OR} \quad (4-\lambda)(\lambda-6) + 8 = 0$$

$$-\lambda^2 + 10\lambda - 24 + 8 = 0$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda-8)(\lambda-2) = 0$$

$$\lambda = 2, 2, 8$$

repeated twice.

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EIGENVECTOR CORRESPONDING TO $\lambda = 8$

$$\begin{bmatrix} 4-8 & 2 & 2 \\ 2 & 4-8 & 2 \\ 2 & 2 & 4-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + x_2 + x_3 = 0 \quad \text{--- } (1)$$

$$x_1 - 2x_2 + x_3 = 0 \quad \text{--- } (2)$$

$$x_1 + x_2 - 2x_3 = 0 \quad \text{--- } (3)$$

$$(1) + 2(2) \Rightarrow x_2 - 4x_2 + x_3 + 2x_3 = 0$$

$$\Rightarrow -3x_2 = -3x_3 \Rightarrow \boxed{x_2 = x_3}$$

$$(2) + 2(3) \Rightarrow 3x_1 + x_3 - 4x_3 = 0$$

$$\Rightarrow x_1 = x_3$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

CONSIDER FOR $\lambda = 2$

$$\begin{bmatrix} 4-2 & 2 & 2 \\ 2 & 4-2 & 2 \\ 2 & 2 & 4-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 + x_3 = 0$$

$$\Rightarrow x_1 = -t - s \quad \text{for} \quad x_2 = t, \; x_3 = s$$

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$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t-s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ eigenvectors caresponding to $\lambda = 2$ are of form given by (4) ie $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the eigenspace corresponding to $\lambda = 2$ therefore they are linealy independent. Therefore eigenvectors caresponding to $\lambda = 2$ are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ (for $s=0, t=1$), $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (for $s=1, t=0$)

or linear cabination of $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Notice that $\lambda = 2$ is repeated twice and the caresponding eigenspace is also two dimesional.

$\therefore$ BASIS $= \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ in this case

A is diagonalizable and we could easily check that $P^{-1}AP = D$

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ie $P^{-1}AP$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

NOTE:- If an eigenvalue λ of A is repeated k times and the eigenspace corresponding to λ is k-dimensional then the set consisting of basis vectors $\{v_1, \ldots, v_k\}$ is linearly independent and they are also eigenvectors corresponding to λ as we saw in last example.

EXAMPLE:- Eigenvalues of A are

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} = A \quad \text{are } \lambda_1 = 3, \lambda_2 = \lambda_3 = 2$$

→ triangular matrix. It's eigenvalues are main diagonal entries for $\lambda = 2$: eigenvectors

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = x_2 = 0$$

$x_3 = t$ (SAY)

The eigenspace corresponding to $\lambda = 2$ is one dimensional but $\lambda = 2$ is repeated twice, so A is not diagonalizable as only two out of three eigenvector of A are linearly independent

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad 1 \neq 2$$

LECTURE 26

ORTHOGONAL DIAGONALIZATION.

The ORTHONORMAL eigenvector problem:

Given an $n \times n$ matrix A, does there exist an orthonormal basis for R^n with the euclidean inner product consisting of eigenvectors of A?

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DEFINITION: A square matrix A is called ORTHOGONALLY DIAGONALIZABLE if there is an ORTHOGONAL MATRIX P such that

$$P^{-1}AP = P^t AP \quad \text{is a diagnol matrix,}$$

the matrix P is said to ORTHOGONALLY DIAGONALIZE A.

NOTE: Recall that for an ORTHOGONAL MATRIX P we have

$$P^t = P^{-1} \quad \text{OR} \quad P^tP = PP^t = I$$

TRY THE FOLLOWING

If A is orthogonally diagonalizable then prove that A is a symmetric matrix.

PROOF $\because P^tAP = D$ where D is diagnol matrix and $P^tP = I$

$$\therefore \underbrace{PP^t}_{I} A \underbrace{PP^t}_{I} = PDP^t$$

$$\Rightarrow A = PDP^t \quad \text{—— (1)}$$

$\rightarrow D^t = D$

$$\Rightarrow A^t = (PDP^t)^t = (P^t)^t D^t P^t$$

$$\Rightarrow PDP^t = A \text{ from (1)}$$

$$\Rightarrow A^t = A$$

NOTE: SYMMETRIC is always diagonalizable.

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PROBLEM: find an orthogonal matrix P that diagonalizes

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}, \quad A^t = A$$

STEPS ① find the eigenvalues of A they are given by $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = 8$

② find the basis for the eigenspace corresponding to $\lambda = 2$ and is given by

$$\{u_1, u_2\} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

NOTE: $u_1 \cdot u_2 = 1 \neq 0$ so u_1 is not orthogonal to u_2. * if they are orthogonal then only make them unit vector.

③ Apply Gram-Schmidt process to $\{u_1, u_2\}$ to get an orthonormal basis ie $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|} \right\}$

$$v_1 = u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = w_1 \text{ (SAY)}$$

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$$v_2 = u_2 - \frac{(u_2 \cdot v_1)\, v_1}{\|v_1\|^2} = \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \quad \text{(CHECK)}$$

$$\|v_2\| = \frac{\sqrt{6}}{2}, \quad \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}}(-1, -1, 2) = w_2$$

(4) find the basis for eigenspace corresponding to $\lambda = 8$. In this case

$$\text{BASIS} = \{u_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(5) Apply gram-schmidt process to u_3 to get

$$w_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_3 = u_3$$

NOTE: No need to find v_3 by using u_1 and u_2 in step (3) $\because$ eigenspaces are different.

(6) finally using w_1, w_2 and w_3 as column vectors we obtain

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ +1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

for $\lambda = 2$

$\lambda = 8$

which orthogonally diagonalizes A.

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CHECK

PP^t

$$= \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ $\therefore$ P is an athogonal matrix

further P^tAP

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} = D$$

for the symmetric matrix $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$, basis

for the eigenspace which corresponds to $\lambda = 2$ is given by $\{u_1, u_2\} = \{(-1,1,0), (-1,0,1)\}$ and

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the basis for the eigenspace corresponding to $\lambda = 8 = \{u_3\} = \{(1,1,1)\}$ notice that

$$u_1 \cdot u_3 = (-1,1,0) \cdot (1,1,1) = 0$$

and $$u_2 \cdot u_3 = (-1,0,1) \cdot (1,1,1) = 0$$

→ If A is symmetric matrix then eigenvectors from different eigenspaces are orthogonal.

→ If A and B are square matrix we say B is similar to A if there is an invertible matrix P such that $B = P^{-1}AP$

LINEAR TRANSFORMATIONS.

$y = f(x)$

dependent variable ← y; x → independent variable

BOTH ARE SCALAR

We shall begin the study of functions of the form $w = f(v)$ where the independent variable v and dependent variable w are both vectors.

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We shall study functions which are called LINEAR TRANSFORMATIONS. Consider the following

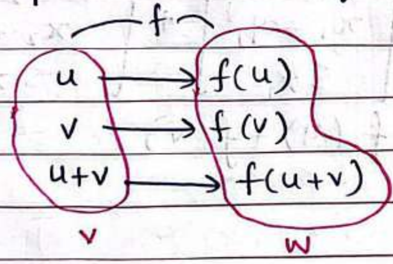
→ If $f : V \longrightarrow W$ is a function from the vector space V into vector space W, then f is called a LINEAR TRANSFORMATION if

(a) $f(u+v) = f(u) + f(v)$ for all $u, v \in V$

(b) $f(ku) = kf(u)$ for all $u \in V$ and all scalars k

$f : V \longrightarrow W$

domain of f ↙ ↘ image space of f.



EXAMPLE

let $f : R^2 \longrightarrow R^3$ be given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \\ 5x \end{bmatrix} \longrightarrow (*)$$

is f linear? or is f a linear transformation / mapping from $R^2 \rightarrow R^3$?

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$\boxed{\text{SOLUTION}}$

Let $u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$$f(u+v) = f\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

$$= f\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} (x_1+x_2) - (y_1+y_2) \\ (x_1+x_2) + (y_1+y_2) \\ 5(x_1+x_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - y_1 \\ x_1 + y_1 \\ 5x_1 \end{bmatrix} + \begin{bmatrix} x_2 - y_2 \\ x_2 + y_2 \\ 5x_2 \end{bmatrix}$$

$$= f(u) + f(v)$$

Now consider

$$f(ku) = f\left(k\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$= f\left(\begin{bmatrix} kx \\ ky \end{bmatrix}\right)$$

$$= \begin{bmatrix} kx - ky \\ kx + ky \\ 5x \end{bmatrix} \quad \because f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \\ 5x \end{bmatrix}$$

$$= k\begin{bmatrix} x-y \\ x+y \\ 5x \end{bmatrix} = kf(u)$$

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from (1) and (2) f is a linear tranformation from $R^2 \rightarrow R^3$

TRY THE FOLLOWING

Let $D: W \rightarrow V$ be the tranformation that maps $f = f(x)$ into it's derivative that is

$$D(f) = f'(x)$$

is D linear?

SOLUTION

$$D(f+g) = (f(x)+g(x))'$$

$$= \frac{d}{dx}\left(f(x)+g(x)\right)$$

$$= \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

$$= f'(x) + g'(x)$$

$$D(f+g) = D(f) + D(g)$$

$$D(kf) = (kf(x))' = \frac{d}{dx}(kf(x))$$

$$= k\frac{d}{dx}(f(x))$$

$$= kD(f)$$

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TRY THE FOLLOWING

let $V = C[0,1]$ (continuos function from 0 to 1)

let $J: V \to R$ be defined by

real number space.

$$J(f) = \int_0^1 f(x)\,dx$$

prove that J is linear transformation from V to R.

SOLUTION:

let $f, g \in V$

$$J(f+g) = \int_0^1 (f+g)(x)\,dx$$

$$= \int_0^1 \big(f(x)+g(x)\big)\,dx$$

$$= \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx$$

$$J(f+g) = J(f) + J(g)$$

also $J(kf) = \int_0^1 k f(x)\,dx = k\int_0^1 f(x)\,dx$

$$J(kf) = k\,J(f)$$

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No... RY THE FOLLOWING

$R^n \to R^m$ given by

$T(x) = Ax = b$

$A \to m \times n$ matrix

$x \Rightarrow x \to n \times 1$ matrix (column vector)

$b \to m \times 1$ matrix (column vector)

Check wether T is linear?

NOTE $$Ax = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m \times n} \overbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}^{n \times 1} = b \to m \times 1$$

$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \quad x \in R^n, \quad b \in R^m$$

$T: R^n \to R^m$

Let $x_1, x_2 \in R^n$

$T(x_1 + x_2) = A(x_1 + x_2)$

$= Ax_1 + Ax_2 = T(x_1) + T(x_2)$

also $T(kx_1) = A(kx_1) = kAx_1$

$= kT(x_1)$

$\therefore T(x) = Ax$ is linear

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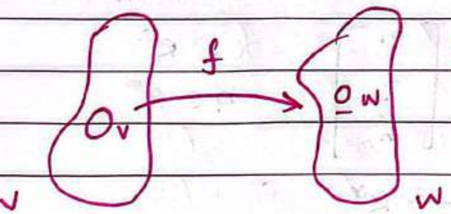
→ $T(x) = Ax$ is linear and is also called MATRIX TRANSFORMATION or a LINEAR transformation called multiplication by A.
Here A in $T(x) = Ax$ is called MATRIX OF LINEAR TRANSFORMATION.

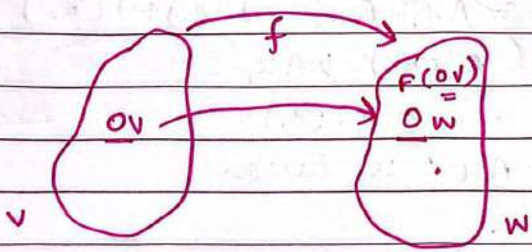
LECTURE 27

If f is a linear transformation from $V \to W$, then the zero vector of the vector space V is always going to map on zero vector of the vector space W as shown below:



$f(0v) = 0w$

$0v \to$ zero vector of V
$0w \to$ zero vector of W



$f(0v) = 0w$.

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Note: In future we shall use 0 instead of 0_V or 0_W

EXAMPLES :

① $T : R^n \longrightarrow R^m$

$T(x) = Ax$ is linear

$T(0) = A0 = 0$

$\Rightarrow T(0) = 0$

we already proved the following transformations as linear.

② $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \\ 5x \end{bmatrix}$

it is seen that

$F\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$ zero of R^3

↓ zero of R^2

③ $J : V \rightarrow R$ $V = C[0,1]$

$J(f) = \int_0^1 f(x)\,dx$

$J(0) = \int_0^1 0\,dx = 0$

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- mapping vector into 0 (kernel / nullspace)

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(4) $D : W \to V$

$$D(f) = f'(x) = \frac{d}{dx}(f(x))$$

$$D(0) = \frac{d}{dx}(0) = 0$$

but how to prove in general?

PROOF

If $T: V \to W$ is a linear tranformation, then $T(0) = 0$.

PROOF

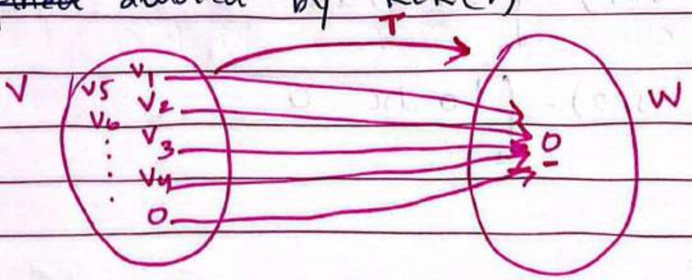
$$T(\overset{\text{vector}}{0}) = T(\overset{\text{scalar}}{0}v) = 0T(v) = 0$$

$v \in V$ $\quad \because T(kv) = kT(v)$

OR for any $v \in V$

$$T(0) = T(v - v) = T(v + (-v))$$
$$= T(v) + T(-v) = T(v) - T(v) = 0$$

→ If $T: V \to W$ is a linear traformation, then the set of vectors in V that maps into 0 is called the $\boxed{\text{kernel}}$ (or nullspace) of T; it is ~~defined~~ denoted by $\text{KER}(T)$.



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$KERT(T) = \{v_1, v_2, v_3, v_4, 0\}$

$T(v_1) = T(v_2) = T(v_3) = T(v_4) = T(0) = 0$

EXAMPLES

(1) $D : V \longrightarrow W$

$\Rightarrow D(f) = f'(x)$

$KER(D)$ = set of all functions s.t. $D(f) = 0$

$\Rightarrow f'(x) = 0 \Rightarrow f(x) = k$

$k \rightarrow$ constant

$\therefore$ $KER(D)$ = set of all constant functions in V. *differentiating a constant gives you 0.

(2) $J : P_1 \longrightarrow R$ $\quad$ $P_1(x)$ } polynomials of degree = 1

Find $Ker(J)$, where $J(P) = \int_0^1 P(x)\, dx$

ANSWER $KER(J)$ constants of all polynomials of form $P(x) = kx$

$k \rightarrow$ constant

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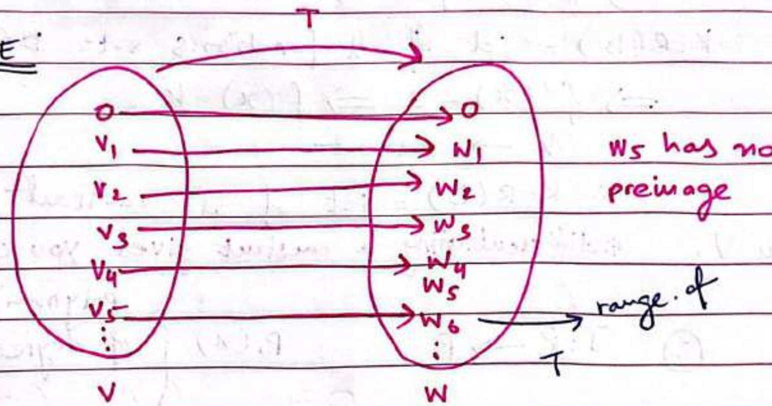
RANGE OF LINEAR TRANSFORMATION

If $T: V \rightarrow W$ is linear then set of all vectors in W that are images under T of atleast one vector in V is called range of T. It is denoted by $R(T)$

EXAMPLE



$R(T) = \{0, w_1, w_2, w_3, w_4, w_6, w_7, \ldots\}$

IDENTITY TRANSFORMATION

Let V be any vector space the mapping $I: V \rightarrow V$ ~~determined~~ defined by $I(V) = V$ is called IDENTITY OPERATOR as shown in the figure.

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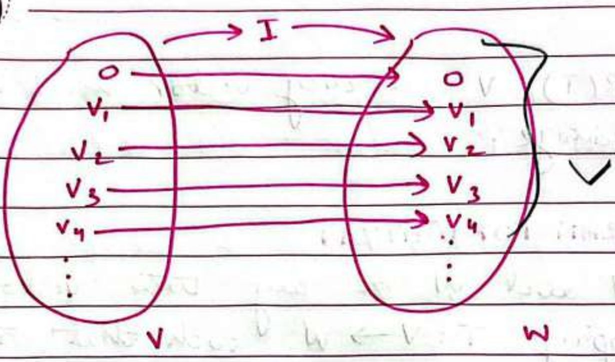

$\Rightarrow I(v_1) = v_1, I(v_2) = v_2, \ldots, I(0) = 0$ etc.

TRY THE FOLLOWING

If $I: V \to V$ is an identity tranformation ie $I(v) = v \quad \forall v \in V$ then
① I is linear
② find $R(I)$, ③ $KER(I)$

SOLUTION

1) let $u, v \in V \Rightarrow ku \in V = u+v \in V$
$\Rightarrow I(u+v) = u+v = I(u) + I(v)$
also $I(ku) = ku \Rightarrow kI(u)$

3) $KER(I) = 0 \because 0$ is the only vector which maps into 0.

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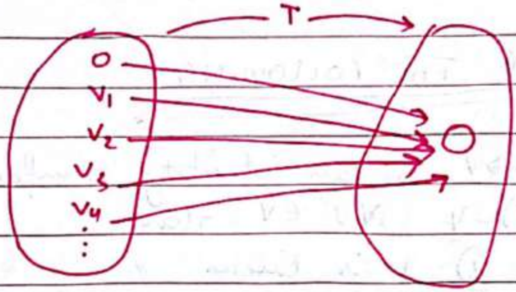
2) $R(I) = V$ ∵ every vector in V has a preimage.

ZERO TRANSFORMATION

Let V and W be any two vector spaces. The mapping $T: V \to W$ such that $T(v) = 0$ for every v in V is called zero transformation.



→ $T(v_1) = T(v_2) = \dots = 0$

→ A linear transformation $T: V \to W$ is said to be ONE-TO-ONE if T maps distinct vectors in V into distinct vectors in W.

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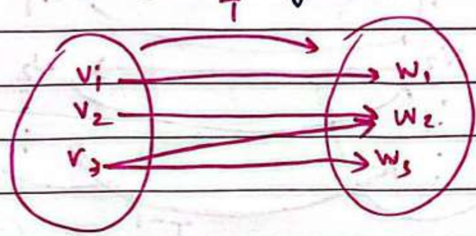
EXAMPLES

IDENTITY TRANSFORMATION is one-to-one, $\forall v \in V$

$I(v) = v$, $I : V \to V$

NOTE: ZERO TRANSFORMATION is not one to one

NOTE: In a mapping every element has only one image

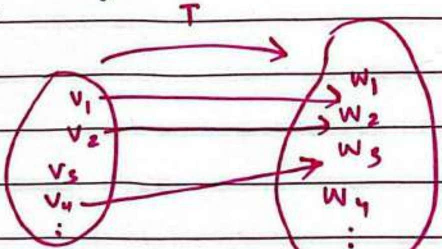


T is $\boxed{\text{NOT}}$ mapping

$\because T(v_3) = w_2$ and $T(v_3) = w_3$

v_3 has two images

In addition every element in domain must have an image.



T is $\boxed{\text{NOT}}$ mapping

$\because v_5$ has no image.

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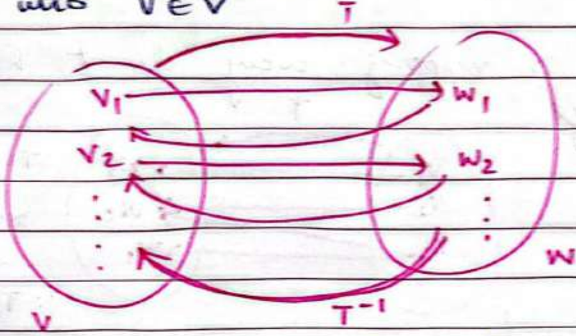
INVERSE LINEAR TRANSFORMATION.

If $T: V \to W$ is linear and one-to-one then the INVERSE LINEAR TRANSFORMATION is given by

$T^{-1}: R(T) \to V$ which maps $w \in R(T)$ back into $v \in V$



$\because$ T is one to one $\therefore$ each vector in $R(T)$ is image of A unique vector in V. $R(T)$ may or may not be all of W.

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