MATH 205

LECTURE 25 LINEAR ALGEBRA

REVISION:

LAST TIME WE SAW THAT THE EIGENVALUES OF

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

WERE $\boxed{\text{DISTINCT}}$ AND $= 1, 2, 3$

CORRESPONDING EIGENVECTORS ARE GIVEN BY

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, \text{ AND } \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

RESPECTIVELY

WHICH ARE LINEARLY INDEPENDENT $\because$ THEY FORM A $\boxed{\text{BASIS}}$ FOR $\boxed{R^3}$ BECAUSE $\boxed{A}$ IS DIAGONALIZABLE AS SHOWN BELOW:

$$\overbrace{\begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}}^{P^{-1}} \overbrace{\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} 1 & 1 & 0 \\ -1 & -2 & 1 \\ -1 & -2 & 0 \end{bmatrix}}^{P}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{2}$ THEOREM 7.2.2: } P. 351 (8TH ED.) P. 369 (7TH ED.)

IF $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ ARE EIGENVECTORS OF $\boxed{A}$ CORRESPONDING TO DISTINCT EIGENVALUES $\lambda_1, \lambda_2, \ldots, \lambda_k$, THEN $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ IS A LINEARLY INDEPENDENT SET.

WHAT WILL HAPPEN IF EIGENVALUES ARE $\boxed{\text{NOT}}$ DISTINCT ?

CONSIDER THE FOLLOWING EXAMPLE:-

LET $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

EIGENVALUES OF $\boxed{A}$ ARE OBTAINED BY SOLVING $\boxed{\det(A-\lambda I)}$

$$= \begin{vmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)\left[(4-\lambda)^2 - 4\right] - 2\left[2(4-\lambda) - 4\right] + 2\left[4 - 2(4-\lambda)\right] = 0$$

SAME

3

$$\Rightarrow (4-\lambda)[\underline{\lambda^2-8\lambda+12}]$$
$$-4[2(4-\lambda)-4]=0$$

$$\Rightarrow (4-\lambda)[\underline{(\lambda-6)(\lambda-2)}$$
$$-8[4-\lambda-2]=0$$

$$\Rightarrow (4-\lambda)(\lambda-6)\boxed{(\lambda-2)}$$
$$+8\boxed{(\lambda-2)}=0$$

$$\Rightarrow \boxed{\lambda=2} \text{ OR } (4-\lambda)(\lambda-6)+8=0$$

$$\Rightarrow -\lambda^2+10\lambda-24+8=0$$

$$\Rightarrow \lambda^2-10\lambda+16=0$$

$$\Rightarrow \boxed{(\lambda-8)(\lambda-2)=0}$$

$$\Rightarrow \lambda = 2, 2, 8$$

REPEATED TWICE

EIGENVECTOR CORRESPONDING TO $\lambda=8$, CONSIDER

$$\begin{bmatrix} 4-8 & 2 & 2 \\ 2 & 4-8 & 2 \\ 2 & 2 & 4-8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -2x_1 + x_2 + x_3 = 0 \quad - ①$$

$$x_1 - 2x_2 + x_3 = 0 \quad - ②$$

$$x_1 + x_2 - 2x_3 = 0 \quad - ③$$

$$① + 2② \Rightarrow x_2 - 4x_2 + x_3 + 2x_3 = 0$$

$$\Rightarrow -3x_2 = -3x_3 \Rightarrow \boxed{x_2 = x_3}$$

$$② + 2③ \Rightarrow 3x_1 + x_3 - 4x_3 = 0$$

$$\Rightarrow \boxed{x_1 = x_3}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\boxed{\text{CONSIDER FOR } \lambda = 2}$ REPEATED TWICE

$$\begin{bmatrix} 4-2 & 2 & 2 \\ 2 & 4-2 & 2 \\ 2 & 2 & 4-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + x_2 + x_3 = 0$$

$$\Rightarrow x_1 = -t - s \quad \text{FOR} \quad \boxed{\begin{matrix} x_2 = t \\ x_3 = s \end{matrix}}$$

ONE EQUATION, THREE UNKNOWNS

$$④ \leftarrow \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t-s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

5

$\therefore$ EIGENVECTORS CORRESPONDING TO $\boxed{\lambda = 2}$ ARE OF THE FORM GIVEN BY (4) i.e. $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ FORM A BASIS FOR THE EIGENSPACE CORRESPONDING TO $\lambda = 2$, THEREFORE THEY ARE LINEARLY INDEPENDENT. THEREFORE EIGENVECTORS CORRESPONDING TO $\boxed{\lambda = 2}$ ARE

$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ (FOR $s=0, t=1$), $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (FOR $s=1, t=0$)

OR LINEAR COMBINATION OF $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ AND $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. NOTICE THAT $\lambda = 2$ IS REPEATED TWICE AND THE CORRESPONDING EIGENSPACE IS ALSO TWO DIMENSIONAL

$$\therefore \text{BASIS} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

IN THIS CASE $\boxed{A}$ IS DIAGONALIZABLE AND ONE COULD EASILY CHECK THAT $\boxed{P^{-1}AP = D}$

6 i.e,

$$P^{-1} A P$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

NOTE: IF AN EIGENVALUE λ OF $\boxed{A}$ IS REPEATED K TIMES AND THE EIGENSPACE CORRESPONDING TO $\boxed{\lambda}$ IS K-DIMENSIONAL THEN THE SET CONSISTING OF THE BASIS VECTORS $\{\underline{v}_1, \ldots\ldots, \underline{v}_k\}$ IS LINEARLY INDEPENDENT AND THEY ARE ALSO EIGENVECTORS CORRESPONDING TO $\boxed{\lambda}$ AS WE SAW IN THE LAST EXAMPLE.

EXAMPLE: EIGENVALUES OF

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} = A \text{ ARE } \boxed{\lambda_1 = 3, \lambda_2 = \lambda_3 = 2}$$

CHECK:

→ TRIANGULAR MATRIX. ITS EIGENVALUES ARE MAIN DIAGONAL ENTRIES.

$\boxed{\text{FOR } \lambda = 2 : \text{ EIGENVECTORS}}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{x_1 = x_2 = 0}$$

$\boxed{x_3 = t \text{ (SAY)}}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

THEREFORE THE EIGENSPACE CORRESPONDING TO $\lambda = 2$ IS ONE DIMENSIONAL, BUT $\lambda = 2$ IS REPEATED TWICE, SO $\boxed{A}$ IS NOT DIAGONALIZABLE AS ONLY TWO OUT OF THREE EIGENVECTORS OF $\boxed{A}$ ARE LINEARLY INDEPENDENT.

ASSIGNMENT 6(b) ⑧

① FIND THE EIGENVALUES OF $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ WITHOUT FORMING THE CUBIC EQUATION

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

② (SIMILAR MATRICES)

(i) IF $\boxed{A}$ IS SIMILAR TO MATRIX $\boxed{B}$ ($\boxed{A}$ & $\boxed{B}$ ARE SQUARE MATRICES) THEN $\boxed{B}$ IS ALSO SIMILAR TO $\boxed{A}$.

(ii) SIMILAR MATRICES HAVE THE SAME DETERMINANT (PROVE IT) i.e.,
IF $\boxed{A}$ IS SIMILAR TO $\boxed{B}$ THEN $\det(A) = \det(B)$

③ FIND A MATRIX $\boxed{P}$ THAT DIAGONALIZES $\boxed{A}$,

AND DETERMINE $P^{-1}AP$, WHERE

9

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(4) (Q. no. 11, 12) P. 345 8TH ED. OR P. 363 (7TH ED.)

(5) THEOREM 7.2.1 P. 347 8TH ED. OR P. 365 7TH ED.

(6) (a) FIND A MATRIX $\boxed{P}$ THAT ORTHOGONALLY DIAGONALIZES $\boxed{A}$, AND DETERMINE $P^{-1}AP$, WHERE $\boxed{A}$ IS

(i) $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) WHAT IS THE SIGNIFICANCE OF THE COLUMN VECTORS OF $\boxed{P}$?

(7) PROVE THAT IF $\boxed{A}$ IS A SYMMETRIC MATRIX THEN THE EIGENVECTORS FROM

DIFFERENT EIGENSPACES
ARE ORTHOGONAL.

8 ARE THE FOLLOWING **TRUE**
OR **FALSE**?

(i) IF $\boxed{A}$ IS DIAGONALIZABLE
THEN $\boxed{A}$ HAS $\boxed{n}$ DISTINCT
EIGENVALUES ($A \rightarrow n \times n$ MATRIX)

(ii) $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ IS DIAGONALIZA-
BLE.

(iii) IF $\boxed{A}$ IS A DIAGONALIZABLE
MATRIX, THEN THE RANK OF
$\boxed{A}$ IS THE NUMBER OF
NONZERO EIGENVALUES OF
$\boxed{A}$.

(iv) IF $\boxed{A}$ IS ANY $m \times n$ MATRIX,
THEN $\boxed{A^tA}$ HAS AN ORTHONOR-
MAL SET OF $\boxed{n}$ EIGENVECTO-
RS.

(v) IF $\underline{v}$ IS ANY $n \times 1$ MATRIX AND
$\boxed{I}$ IS THE $n \times n$ IDENTITY MAT-
RIX, THEN $\boxed{I - \underline{v}\underline{v}^t}$ IS ORTHO-
GONALLY DIAGONALIZABLE.

(9) Q. no. 1 (a, b) P. 360 8TH ED.
OR P. 378 (7TH ED.)

(LINEAR TRANSFORMATIONS)

① CONSIDER THE BASIS $S = \{\underline{V_1}, \underline{V_2}, \underline{V_3}\}$ FOR R^3, WHERE $\underline{V_1} = (1,1,1)$, $\underline{V_2} = (1,1,0)$, AND $\underline{V_3} = (1,0,0)$.

LET $T: R^3 \longrightarrow R^2$ BE THE LINEAR TRANSFORMATION SUCH THAT

$T(\underline{V_1}) = (1,0)$, $T(\underline{V_2}) = (2,-1)$, $T(\underline{V_3}) = (4,3)$

FIND A FORMULA FOR $T(x_1, x_2, x_3)$, THEN USE THIS FORMULA TO COMPUTE $T(2,-3,5)$.

$\boxed{\text{Q.no.2}}$ 12

CHECK WHETHER THE FOLLOWING MAPPINGS ARE LINEAR?

(a) $F(x,y) = (2x,y)$

(b) $F(x,y) = (x^2,y)$

(c) $F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = b+c$

(d) $F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y+z \end{bmatrix}$

(e) $T(A) = AB$, $A, B \to$ MATRICES

$\boxed{\text{Q.no.3}}$

(a) IF $T(\underline{e}_1) = (1,1)$,
$T(\underline{e}_2) = (3,0)$ AND
$T(\underline{e}_3) = (4,-7)$ THEN
FIND $\boxed{T(1,3,8)}$ PROVIDED
T IS LINEAR. $(T: R^3 \to R^2)$

ANSWER: $T(1,3,8) = (42,-55)$

(b) ALSO FIND $T(x, y, z)$ by USING INFORMATION IN PART (a).

13

ANS. $\begin{bmatrix} x + 3y + 4z \\ x - 7z \end{bmatrix}$

(c) USING PART (b) FIND THE MATRIX OF LINEAR TRANSFORMATION.

ANS. $A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 0 & -7 \end{bmatrix}$

$\boxed{\text{Q.no.4}}$ Q.29, Q.30 P.375 (8TH ED.)

OR Q.30, Q.31 P.395 (7th ED.)

$\boxed{\text{Q.no.5}}$ (PREVIOUS CONCEPT REVISITED.)

IF $\underline{u}_1 = \underline{v}_1$ AND $\underline{u}_2$ ARE EIGENVECTORS OF A CORRESPONDING TO λ THEN PROVE THAT $\underline{v}_2 = \underline{u}_2 - \frac{(\underline{u}_2 . \underline{v}_1)\underline{v}_1}{\|\underline{v}_1\|^2}$ IS ALSO AN EIGENVECTOR OF A CORRESPONDING TO λ.

$\boxed{\text{Q. no. 6}}$ (14)

(a) IF $T: R^n \longrightarrow R^m$ BE THE MATRIX TRANSFORMATION $T(\underline{x}) = A\underline{x}$, FIND $\boxed{\text{KER}(T)}$ AND $\boxed{\text{RANGE OF } T}$ $(R(T))$.

(b) SEE THE DEFINITIONS OF $\boxed{\text{NULLITY}(T)}$ AND $\boxed{\text{RANK}(T)}$ (P. 378 8TH ED., P. 397 7TH ED.) WHERE $\boxed{T}$ IS A LINEAR TRANSFORMATION.

(C) USING PARTS (a & b) TRY THE FOLLOWING

IF A IS AN $m \times n$ MATRIX AND $T: R^n \longrightarrow R^m$ IS MULTIPLICATION BY A, THEN

(i) NULLITY(A) = NULLITY(T)

(ii) RANK(A) = RANK(T)

(d) USING PART (a), TRY THE FOLLOWING:

LET $T: R^3 \longrightarrow R^3$ BE THE LINEAR OPERATOR DEFINED BY THE FORMULA

$$T(x_1, x_2, x_3) = (x_1 + x_2 + 2x_3,\ x_1 + x_3,\ 2x_1 + x_2 + 3x_3).$$

FIND **BASES** FOR THE KERNEL AND RANGE OF $\boxed{T}$.

HINT: WRITE DOWN

$$= A\underline{X}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ x_1 + x_3 \\ 2x_1 + x_2 + 3x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

AND PROCEED.

$\boxed{\text{Q.no.7}}$

Q.no.19 (P.389 8th ED.)
(P.410 7th ED.)

Q. no. 8

SEE THE STATEMENT OF THE <u>DIMENSION THEOREM</u> FOR <u>LINEAR TRANSFORMATIONS</u> ON {P. 379 (8th ED.) OR P. 398 (7TH ED.)}

THEN PROVE THAT THIS THEOREM AGREES WITH DIMENSION THEOREM FOR MATRICES

IF $\boxed{A}$ IS AN $\boxed{m \times n}$ MATRIX AND $T : R^n \longrightarrow R^m$ IS <u>MULTIPLICATION</u> BY $\boxed{A}$.

~ GOOD LUCK ~