Quiz 3

CS/CE 412/471 Algorithms: Design and Analysis, Spring 2025

12 Mar, 2025. 4 questions, 20 points, 3 printed sides

Reference

DFS(G)

```
for each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
for each vertex u ∈ G.V
    if u.color == WHITE
        DFS-VISIT(G, u)
```

DFS-VISIT(G, u)

```
time = time + 1
u.d = time
u.color = GRAY
for each vertex v ∈ G.adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
time = time + 1
u.f = time
u.color = BLACK
```

The input to a *shortest-paths problem* is a weighted, directed graph $G = (V, E)$, with a weight function $w : E \to \mathbb{R}$ mapping edges to real-valued weights. The *weight* $w(p)$ of path $p = \langle v_0, v_1, \ldots, v_k \rangle$ is the sum of the weights of its constituent edges: $w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$.
We define the *shortest-path weight* $\delta(u, v)$ from u to v by

$$\delta(u,v) = \begin{cases} \min\{w(p) : u \rightsquigarrow v\} & \text{if there is a path from } u \text{ to } v, \\ \infty & \text{otherwise.} \end{cases}$$

A *shortest path* from vertex u to vertex v is then defined as any path p with weight $w(p) = \delta(u, v)$.
The *predecessor subgraph* $G_\pi = (V_\pi, E_\pi)$ induced by the π values is defined as:

$$V_\pi = \{v \in V : v.\pi \neq \text{NIL}\} \cup \{s\}$$
$$E_\pi = \{(v.\pi, v) \in E : v \in V_\pi - \{s\}\}$$

INITIALIZE-SINGLE-SOURCE(G, s)

```
for each vertex v ∈ G.V
    v.d = ∞
    v.π = NIL
s.d = 0
```

RELAX(u, v, w)

```
if v.d > u.d + w(u, v)
    v.d = u.d + w(u, v)
    v.π = u
```

Computation of shortest path begins with a call to INITIALIZE-SINGLE-SOURCE(G, s) and the d attribute is *only* changed through calls to RELAX(u, v, w).

Triangle inequality
For any edge $(u, v) \in E$, we have $\delta(s, v) \leq \delta(s, u) + w(u, v)$.

Upper-bound property
We always have $v.d \geq \delta(s, v)$ for all vertices $v \in V$, and once $v.d$ achieves the value $\delta(s, v)$, it never changes.

No-path property
If there is no path from s to v, then we always have $v.d = \delta(s, v) = \infty$.

Convergence property
If $s \rightsquigarrow u \to v$ is a shortest path in G for some $u, v \in V$, and if $u.d = \delta(s, u)$ at any time prior to relaxing edge (u, v), then $v.d = \delta(s, v)$ at all times afterward.

Path-relaxation property
If $p = \langle v_0, v_1, \ldots, v_k \rangle$ is a shortest path from $s = v_0$ to v_k, and the edges of p are relaxed in the order $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$.
This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations of the edges of p.

Predecessor-subgraph property
Once $v.d = \delta(s, v)$ for all $v \in V$, the predecessor subgraph is a shortest-paths tree rooted at s.

```
BELLMAN-FORD(G, w, s)
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  for i = 1 to |G.V| - 1
3      for each edge (u, v) ∈ G.E
4          RELAX(u, v, w)
5  for each edge (u, v) ∈ G.E
6      if v.d > u.d + w(u, v)
7          return FALSE
8  return TRUE
```

Lemma 22.2
Let $G = (V, E)$ be a weighted, directed graph with source vertex s and weight function $w : E \to \mathbb{R}$, and assume that G contains no negative-weight cycles that are reachable from s. Then, after the $|V| - 1$ iterations of the **for** loop of lines 2–4 of BELLMAN-FORD, $v.d = \delta(s, v)$ for all vertices v that are reachable from s.

Corollary 22.3
Let $G = (V, E)$ be a weighted, directed graph with source vertex s and weight function $w : E \to \mathbb{R}$. Then, for each vertex $v \in V$, there is a path from s to v if and only if BELLMAN-FORD terminates with $v.d < \infty$ when it is run on G.

Problems

Do any $n = 4$ problems. In case you do more, I will only check the first n. In proving any inequality, property, or corollary from above, you may assume all other inequalities, properties, or lemmas that precede it.

1. (5 points) Prove an inequality or property above.

> **Solution:** Reference proofs of the Triangle Inequality and of the Upper-bound, No-path, Convergence, Path-relaxation, and Predecessor-subgraph properties are provided in Section 22.5 of the book.

5. (5 points) Prove Corollary 22.3.

> **Solution:**
>
> *Proof.* $s \rightsquigarrow v \iff$ BELLMAN-FORD(G) terminates with $v.d < \infty$.
>
> Case: $\Longrightarrow$
> At initialization, $v.d = \infty$ and $s.d = 0$. If $v = s$, then the claim holds.
> Otherwise, there exists a path from s to v containing k edges where $1 \leq k \leq |G.V| - 1$.
> Let the path be $\langle v_0, v_1, v_2, \ldots, v_k \rangle$ where $v_0 = s$ and $v_k = v$.

Consider iterations of the For loop on lines 2 to 4.
At the end of the 1st iteration, $v_1.d = \delta(s, v_1) < \infty$.
At the end of the 2nd iteration, $\textsc{Relax}(v_1, v_2)$ has executed and $v_2.d < \infty$.
Inductively, at the end of the k-th iteration, $\textsc{Relax}(v_{k-1}, v_k)$ has executed and $v_k.d < \infty$.
Any subsequent iterations either do not change $v_k.d$ or reduce it.

Case: $\Longleftarrow$
At initialization, $v.d = \infty$ and $s.d = 0$. If $v = s$, then the claim holds.
Otherwise, consider iterations of the For loop on lines 2 to 4.
At the end of the 1st iteration, $v.d < \infty$ only if there is a path from s to v of length at most 1.
At the end of the 2nd iteration, $v.d < \infty$ only if there is a path from s to v of length at most 2.
Inductively, after iteration $i \leq |G.V| - 1$, it holds that $v.d < infty \implies s \leadsto^i v$. □

6. (5 points) Show how DFS(G) can be used to detect a cycle in G.

Solution: A cycle exists in G if, in executing DFS(G), a vertex is encountered at line 5 of DFS-VISIT(G, u) whose color is GRAY. This corresponds to a `back edge` as described in the book.

7. (5 points) Give a counterexample to the conjecture that if a directed graph G contains a path from u to v, and if $u.d < v.d$ in a depth-first search of G, then v is a descendant of u in the depth-first forest produced.

Solution: DFS on the directed graph, G, on the left yields the depth-first forest on the right. G contains a path from u to v, and $u.d < v.d$ in a DFS of G, but v is not a descendant of u in the depth-first forest produced.

8. (5 points) Give a counterexample to the conjecture that if a directed graph G contains a path from u to v, then any depth-first search must result in $v.d \leq u.f$.

Solution: DFS on the above graph is a counterexample. G contains a path from u to v, but $v.d > u.f$.