

## Quiz 4

CS/CE 412/471 Algorithms: Design and Analysis, Spring 2025

26 Mar, 2025. 4 questions, 25 points, 3 printed sides

### Reference

**Definition 1** (Flow Network). A *flow network*  $G = (V, E)$  is a directed graph in which each edge  $(u, v) \in E$  has a nonnegative *capacity*  $c(u, v) \geq 0$ .

Each flow network contains two distinguished vertices: a *source*  $s$  and a *sink*  $t$ . Each vertex lies on some path from the source to the sink. That is, for each vertex  $v \in V$ , the flow network contains a path  $s \rightsquigarrow v \rightsquigarrow t$ .

**Definition 2** (Flow). Let  $G = (V, E)$  be a flow network with a capacity function  $c$ . Let  $s$  be the source of the network, and let  $t$  be the sink. A *flow* in  $G$  is a real-valued function  $f : V \times V \rightarrow \mathbb{R}$  that satisfies the following two properties:

Capacity constraint: For all  $u, v \in V$ ,  $0 \leq f(u, v) \leq c(u, v)$ .

Flow conservation: For all  $u \in V - \{s, t\}$ ,  $\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$ .

The *value*  $|f|$  of a flow  $f$  is defined as  $|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$ .

**Definition 3** (Residual Capacity). For a flow network  $G = (V, E)$  with source  $s$ , sink  $t$ , and a flow  $f$ , consider a pair of vertices  $u, v \in V$ . We define the *residual capacity*  $c_f(u, v)$  by

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E, \\ f(v, u) & \text{if } (v, u) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 4** (Residual Network). Given a flow network  $G = (V, E)$  and a flow  $f$ , the *residual network* of  $G$  induced by  $f$  is  $G_f = (V, E_f)$ , where  $E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$ .

**Definition 5** (Flow Augmentation). If  $f$  is a flow in  $G$  and  $f'$  is a flow in the corresponding residual network  $G_f$ , we define  $f \uparrow f'$ , the *augmentation* of flow  $f$  by  $f'$ , to be a function from  $V \times V$  to  $\mathbb{R}$ , defined by

$$(f \uparrow f')(u, v) = \begin{cases} f(u, v) + f'(u, v) - f'(v, u) & \text{if } (u, v) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 6** (Cut of a Flow Network). A *cut*  $(S, T)$  of flow network  $G = (V, E)$  is a partition of  $V$  into  $S$  and  $T = V - S$  such that  $s \in S$  and  $t \in T$ . If  $f$  is a flow, then the *net flow*  $f(S, T)$  across the cut  $(S, T)$  is defined to be

$$f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u).$$

The *capacity* of the cut  $(S, T)$  is

$$c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v).$$

A *minimum cut* of a network is a cut whose capacity is minimum over all cuts of the network.

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FORD-FULKERSON-METHOD( $G, s, t$ )

- 1 initialize flow  $f$  to 0
- 2 **while** there exists an augmenting path  $p$  in the residual network  $G_f$
- 3     augment flow  $f$  along  $p$
- 4 **return**  $f$

**Lemma 24.1.** Let  $G = (V, E)$  be a flow network with source  $s$  and sink  $t$ , and let  $f$  be a flow in  $G$ . Let  $G_f$  be the residual network of  $G$  induced by  $f$ , and let  $f'$  be a flow in  $G_f$ . Then the function  $f \uparrow f'$  defined in equation (24.4) is a flow in  $G$  with value  $|f \uparrow f'| = |f| + |f'|$ .

**Lemma 24.2.** Let  $G = (V, E)$  be a flow network, let  $f$  be a flow in  $G$ , and let  $p$  be an augmenting path in  $G_f$ . Define a function  $f_p : V \times V \rightarrow \mathbb{R}$  by

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u, v) \text{ is on } p, \\ 0 & \text{otherwise.} \end{cases}$$

Then,  $f_p$  is a flow in  $G_f$  with value  $|f_p| = c_f(p) > 0$ .

**Corollary 24.3.** Let  $G = (V, E)$  be a flow network, let  $f$  be a flow in  $G$ , and let  $p$  be an augmenting path in  $G_f$ . Let  $f_p$  be defined as in equation (24.7), and suppose that  $f$  is augmented by  $f_p$ . Then the function  $f \uparrow f_p$  is a flow in  $G$  with value  $|f \uparrow f_p| = |f| + |f_p| > |f|$ .

**Lemma 24.4.** Let  $f$  be a flow in a flow network  $G$  with source  $s$  and sink  $t$ , and let  $(S, T)$  be any cut of  $G$ . Then the net flow across  $(S, T)$  is  $f(S, T) = |f|$ .

**Corollary 24.5.** The value of any flow  $f$  in a flow network  $G$  is bounded from above by the capacity of any cut of  $G$ .

**Theorem 24.6** (Max-flow min-cut theorem). If  $f$  is a flow in a flow network  $G = (V, E)$  with source  $s$  and sink  $t$ , then the following conditions are equivalent:

1.  $f$  is a maximum flow in  $G$ .
2. The residual network  $G_f$  contains no augmenting paths.
3.  $|f| = c(S, T)$  for some cut  $(S, T)$  of  $G$ .

## Problems

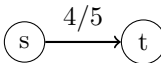
1. (5 points) Suppose that, in addition to edge capacities, a flow network has *vertex capacities*. That is, each vertex  $v$  has a limit  $l(v)$  on how much flow can pass through  $v$ . Show how to transform a flow network  $G = (V, E)$  with vertex capacities into an equivalent flow network  $G' = (V', E')$  without vertex capacities, such that a maximum flow in  $G'$  has the same value as a maximum flow in  $G$ .

**Solution:**  $G' = (V', E')$  can be obtained from  $G = (V, E)$  as follows. For every vertex  $v$  in  $V$ , add a vertex  $v'$  and an edge  $(v, v')$  such that  $c(v, v') = l(v)$ . Furthermore, change the starting vertex of all outgoing edges from  $v$  to  $v'$ . When  $v = t$ , then the newly added vertex becomes the new sink.

2. (5 points) Prove or disprove: Augmenting a flow  $f$  in a flow network  $G$  by another flow  $f'$ , also in  $G$ , yields a valid flow.

**Solution:** We use a counterexample to show that the claim is false.

*Proof.* The claim is false.

Consider the following flow,  $f$ . 

Augmenting  $f$  by itself will result in a flow of 8 along the edge  $(s, t)$  which violates the capacity constraint.  $\square$

3. (5 points) Prove or disprove: Every iteration of the Ford-Fulkerson method increases  $f$ .

**Solution:** We refer to a corollary above to prove that the claim is true.

*Proof.* The claim is true.

This is exactly the claim in Corollary 24.3.  $\square$

4. (10 points) We are given 2 flows  $f$  and  $f^*$  in a flow network  $G$  where  $|f^*|$  is maximal and  $|f| < |f^*|$ . Prove whether  $(S, T)$  cuts of  $G$  exist such that: (do any 5)

- (a)  $c(S, T) < |f|$    (b)  $c(S, T) = |f|$    (c)  $c(S, T) > |f|$    (d)  $c(S, T) < |f^*|$    (e)  $c(S, T) = |f^*|$   
 (f)  $c(S, T) > |f^*|$

**Solution:** We use results from above to prove or disprove the existence of cuts for each of the parts. In one instance, we use a counterexample.

*Proof.* The claims in (a) and (d) are False. That is, no cut  $(S, T)$  exists for which they are true. The claims contradict Corollary 24.5  $\square$

*Proof.* The claim in (b) is False. That is, no cut  $(S, T)$  exists for which it is true.

As  $f$  is not the maximum flow, the claim contradicts Theorem 24.6.  $\square$

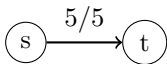
*Proof.* The claim in (c) is True. That is, some cut  $(S, T)$  exists for which it is true.

From Corollary 24.5, this claim is true for every cut of  $G$ .  $\square$

*Proof.* The claim in (e) is True. That is, some cut  $(S, T)$  exists for which it is true.

The claim follows from Theorem 24.6.  $\square$

*Proof.* The claim in (f) is False. That is, there are networks in which no cut  $(S, T)$  exists for which the claim is true.

Consider the following maximum flow,  $f^*$ . 

There is no cut  $(S, T)$  in this network such that  $c(S, T) > |f^*|$ .  $\square$

Note: As the question only asks for existence, and not universality, an existence proof for (f) will also be considered correct.