VECTOR FIELDS

→ is a function which assigns … t in

In R-2: $P\hat{i} + Q\hat{j}$

In R-3: $P\hat{i} + Q\hat{j} + R\hat{k}$ where P, Q, R are define…

* plug in (x,y)
- this gives a vector
- plot the vector AT THE POINT

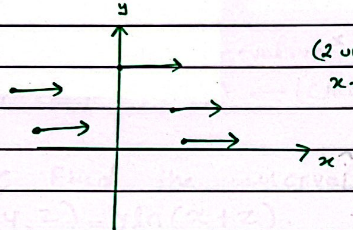
$\hat{i} = x$

$\hat{j} = y$

$\hat{k} = z$

IMPORTANT

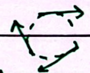
Example 1 $F(x,y) = 2\hat{i}$, $(x,y) \to$ R-2



★ NOTE

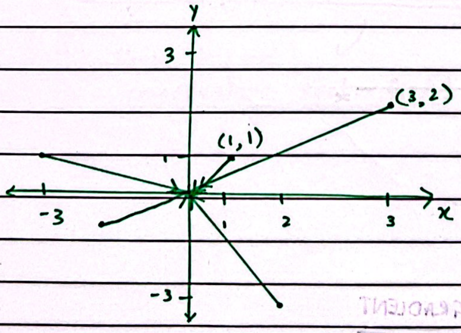
$y\hat{i} - x\hat{j}$

(tangent vector)



Example 2 $F(x,y) = -x\hat{i} - y\hat{j}$ R-2

(1,1) (3,2) (−3,1) (2,−3) (−2,−1)



Example 3 $F(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\hat{i} + \dfrac{y}{\sqrt{x^2+y^2}}\hat{j}$

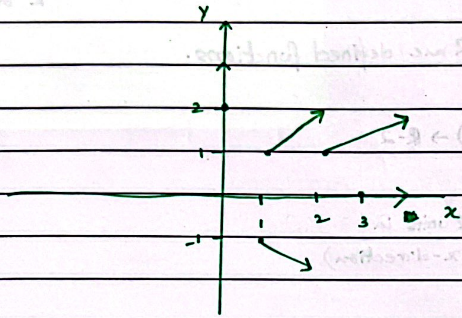
$(1,1) \rightarrow \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

$(1,-1) \rightarrow \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

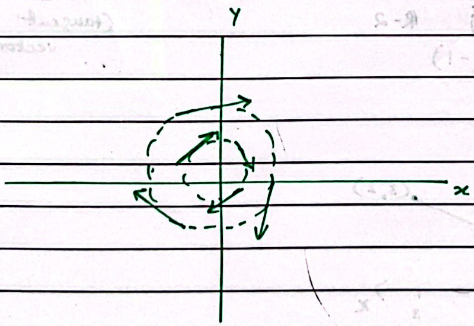
$(2,1) \rightarrow \frac{2}{\sqrt{5}}\hat{i} + \frac{1}{\sqrt{5}}\hat{j}$

$(0,2) \rightarrow +\hat{j}$

$(3,0) \rightarrow \hat{i}$



Example 4 $F(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}\hat{i} - \dfrac{x}{\sqrt{x^2+y^2}}\hat{j}$



$\nabla f(x,y)$ OR $\nabla f(x,y,z)$ GRADIENT

$\longrightarrow$ check next page.

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Example 1 $f(x,y) = x^2y - y^3$

$\nabla f(x,y) = f_x\hat{i} + f_y\hat{j}$ → GRAD F

$\nabla f(x,y) = 2xy\hat{i} + (x^2 - 3y^2)\hat{j}$

* gradient is a type of vectors field!

CONSERVATIVE VECTOR FIELD.

$F(x,y)$ is a ~~cond~~ conservative vector field. if $F(x,y) = \nabla f(x,y)$ for some $f(x,y)$ ← (CALLED A POTENTIAL FUNCTION)

Example Find the conservative vector field, $F(x,y,z)$ for $f(x,y,z) = y\ln(x+z)$.

$$\underbrace{\nabla f(x,y,z) = \left(\frac{y}{x+z}\right)\hat{i} + \ln(x+z)\hat{j} + \left(\frac{y}{x+z}\right)\hat{k}}_{\text{this is a conservative vector field}}$$

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A general vector field can also provide useful information,

→ Local direction of flow / movement: The vector at each point indicates the direction of flow of same quantity.

→ Tendency for convergence / divergence: Divergence tells you if a given region acts as source or sink of flow. Positive divergence indicates spreading out, negative indicates converging.

→ Curl reveals local rotation: Non-zero curl at a point means the vectors are rotating in a clockwise or counterclockwise manner locally.

→ Streamline / Pathline: Streamlines show integral curves that are everywhere tangent to the vectors & indicate likely paths of particles in flow over time.

→ Critical points: Points where vectors is zero act as singularities in field.

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GRADIENT FIELDS

We define the gradient field of a differentiable function $f(x,y,z)$ to be the field of gradient vectors.

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

* grad f gives you F or gradient field.

LINE INTEGRALS

* oriented: (direction of movement is fixed)

$\int_a^b f(x)\,dx$ → gives area under the function.

* integrals gives the area under the curve
* double integrals give the volume under surface

We develop the line integral the way we develop all integrals, in this case by first slicing up the curve C into n small, approximately straight pieces along which $\vec{F}$ is approximately constant.

Two type of 'changes' that can occur at a point in a vector field accelerates/decelerates?

$t_s \leq t \leq t_{end}$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ → position vector.

$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

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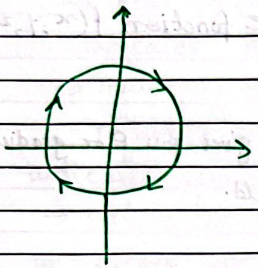
Example :



$\vec{r}(t) = \langle \cos t, \sin t \rangle \Rightarrow \cos t\,\hat{i} + \sin t\,\hat{j}$

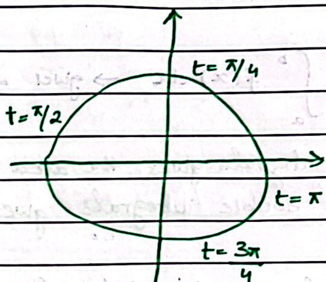
$0 \le t \le 2\pi$

$0 \le t \le 4\pi$ → now the curve is tracing the circle twice.

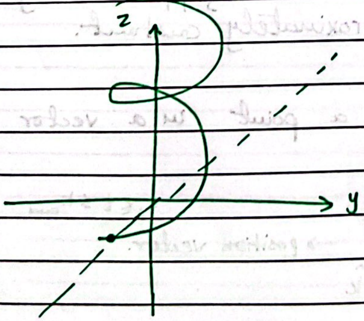
$\vec{r}(t) = \langle \cos 2t, \sin 2t \rangle$

$0 \le t \le \pi$

orginally $\vec{r}(t) = \langle \cos t, \sin t \rangle$



* a curve can have multiple parametrization.



$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$0 \le t \le 4\pi$

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PARAMETRIZATION

$$r(t) = g(t)i + h(t)j + k(t)k \qquad a \le t \le b$$

$$f(x,y) = f(x(t), y(t)) \qquad ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} \qquad \frac{ds}{dt} = |V|$$

$$\vec{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j}$$

$$\|\vec{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$\|\vec{r}'(t)\| = ds$$

(Line Integral over Curve)

$$\boxed{\int_c f(x,y)\, ds = \int_{t=a}^{t=b} f(x(t), y(t))\, \|\vec{r}'(t)\|\, dt}$$

$$L = \int_a^b \|\vec{r}'(t)\|\, dt \quad \text{(Arc length)}$$

$$A = \int_a^b f(x(t), y(t))\, \|\vec{r}'(t)\|\, dt \quad \text{(Area) (Mass)}$$

$$f(r(t)) = f(g(t), h(t), k(t))$$

How to evaluate a line integral?

1. find a smooth parametrization $r(t) = g(t)i + h(t)j + k(t)k$
2. evaluate integral as $\int_a^b f(g(t), h(t), k(t))\, |v(t)|\, dt$

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* **Additivity**: integral over $C_1 \cup C_2$ was obtained by integrating f over each section of path & adding results.

↑ IMPORTANT

Example; $\int_C y\, ds$, $C: \vec{r}(t) = 2t\hat{i} + t^3\hat{j}$ $\quad 0 \le t \le 1$

$\hookrightarrow \int_a^b f(x(t), y(t)) \|\vec{r}\,'(t)\| dt$

$x = 2t \qquad y = t^3$

$x' = 2 \qquad y' = 3t^2$

1) $\vec{r}\,'(t) = 2\hat{i} + 3t^2\hat{j}$

2) $\|\vec{r}\,'(t)\| = \sqrt{2^2 + (3t^2)^2}$

$\|\vec{r}\,'(t)\| = \sqrt{4 + 9t^4}$

$$\int_{t=0}^{t=1} t^3 \sqrt{4+9t^4}\, dt = \frac{1}{54}(13\sqrt{13} - 8)$$

(Mass)

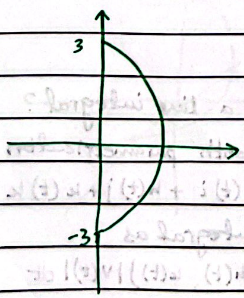
Example; $\int_C (x^2+y^2) ds$ $\qquad C$: the right 1/2 circle $x^2+y^2 = 9$

For CIRCLES

$x = r\cos\theta$, $y = r\sin\theta$

So, $x = 3\cos\theta$ and $y = 3\sin\theta$

$x = 3\cos t$ and $y = 3\sin t$



$\vec{r}(t) = x\hat{i} + y\hat{j}$

$\vec{r}(t) = 3\cos t\,\hat{i} + 3\sin t\,\hat{j}$

$r(t) = xi + yj$

where $x = C_1 + k_1 t$ and $y = C_2 + k_2 t$

1) $\vec{r}\,'(t) = -3\sin t\,\hat{i} + 3\cos t\,\hat{j}$
2) $\|\vec{r}\,'(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t}$

$\|\vec{r}\,'(t)\| = 3$

$$\int_{t=-\pi/2}^{t=\pi/2} (9\cos^2 t + 9\sin^2 t)\cdot(3)\, dt \Rightarrow$$

VECTOR FIELD $\rightarrow$ Spin field.

VECTOR FIELD $\rightarrow$ ~~Gradient~~ Radial Field. (attractive)

LINE INTEGRAL $\rightarrow$

LINE INTEGRAL $\rightarrow$

Eample: $r(t) = ti + t^2 j + t^4 k$

$x = t \quad x'(t) = 1$

$y = t^2 \quad y'(t) = 2t$

$z = t^4 \quad z'(t) = 4t^3$

$r'(t) = i + 2tj + 4t^3 k$

$F = \sqrt{z}\, i - 2xj + \sqrt{y}\, k$

$F(r(t)) = t^2 i - 2tj + tk$

$$\int_0^1 F\cdot\frac{dr}{dt} = \int_0^1 (t^2 - 4t^2 + 4t^4)\, dt$$

$$\int_0^1 (-3t^2 + 4t^4)\, dt$$

$$-t^3 + \frac{4t^5}{4}\Big|_0^1 \Rightarrow -1 + \frac{4}{5} \quad \underline{Ans}$$

Example: $\int_C 2xy\, ds \quad C:$ segment $(-2,-1)$ to $(1,3)$

Note: FOR SEGMENTS WE WANT $0 \le t \le 1$

$X = C_1 + k_1 t$, $Y = C_2 + k_2 t$

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AT $T=0$: $-2 = C_1 + k_1 0 \rightarrow C_1 = -2 \Rightarrow X = -2 + k_1 t$

$1 = -2 + k_1 (1)$

$X = -2 + 3t$

AT $T=0$: $Y = C_2 + k_2 t$

$-1 = C_2 + \cancel{k_2(0)} \rightarrow 3 = -1 + k_2 \cdot 1, \quad k_2 = 4$

$C_2 = -1$

$Y = -1 + 4t$

$Y = -1 + k_2 t$

$\vec{r} = x\hat{i} + y\hat{j}$

$\vec{r}(t) = (-2+3t)\hat{i} + (-1+4t)\hat{j}$

1) $\vec{r}'(t) = 3\hat{i} + 4\hat{j}$

2) $\|\vec{r}\| = \sqrt{9+16} = 5$

$$\int_0^1 2(-2+3t)(-1+4t)\,5dt$$

LINE INTEGRALS WITH RESPECT TO dx, dy, dz

→ no need of $r'(t)$

→ simply substitute x, y, z, dx, dy, dz values

→ & integrate over interval.

Example: $\int_C (x+y)\,dx + (xy)\,dy + y\,dz, \quad C: \vec{r}(t) = e^t\hat{i} + e^{-t}\hat{j} + 2e^{2t}\hat{k}$

$0 \le t \le 1$

$x = e^t, \quad y = e^{-t}, \quad z = 2e^{2t}$

$dx = e^t dt, \quad dy = -e^{-t}dt, \quad dz = 4e^{2t}dt$

$$\int_0^1 (e^t + e^{-t})e^t dt + (e^t \cdot e^{-t})(-e^{-t})dt + e^{-t} \cdot 4e^{2t}dt$$

$$\int_0^1 (e^{2t} + 1 - e^{-t} + 4e^t)\,dt \Rightarrow \frac{1}{2}e^2 + 4e + \frac{1}{e} - \frac{9}{2}$$

Circulation $\rightarrow$ line integration over V.F

Flux $\rightarrow$ dot product b/w $F(r(t))$ and $n = r(t)$ then integrate.

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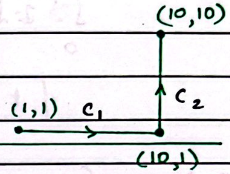
Example: Find the line integral of constant vector field $\vec{F} = \vec{i} + 2\vec{j}$ along the path from $(1,1)$ to $(10,0)$

Let C_1 be horizontal segment of the path going from $(1,1)$ to $(10,1)$.



$$\vec{\Delta r} = \Delta x \vec{i} \quad \text{so} \quad \vec{F} \cdot \vec{\Delta r} = (\vec{i} + 2\vec{j}) \cdot \Delta x \vec{i} = \Delta x$$

$$\int_{C_1} \vec{F} \cdot \vec{dr} = \int_1^{10} dx = 9$$

$$\vec{\Delta r} = \Delta y \vec{j} \quad \text{so} \quad \vec{F} \cdot \vec{\Delta r} = (\vec{i} + 2\vec{j}) \cdot \Delta y \vec{j} = 2\Delta y$$

$$\int_{C_2} \vec{F} \cdot \vec{dr} = \int_1^{10} 2dy = 18$$

Thus, $\int_C \vec{F} \cdot \vec{dr} = \int_{C_1} \vec{F} \cdot \vec{dr} + \int_{C_2} \vec{F} \cdot \vec{dr} = 9 + 18 = 27$

PROPERTIES OF LINE INTEGRALS

1. $\int_C \lambda \vec{F} \cdot \vec{dr} = \lambda \int_C \vec{F} \cdot \vec{dr}$

2. $\int_C (\vec{F} + \vec{G}) \cdot \vec{dr} = \int_C \vec{F} \cdot \vec{dr} + \int_C \vec{G} \cdot \vec{dr}$

3. $\int_{-C} \vec{F} \cdot \vec{dr} = -\int_C \vec{F} \cdot \vec{dr}$

4. $\int_{C_1 + C_2} \vec{F} \cdot \vec{dr} = \int_{C_1} \vec{F} \cdot \vec{dr} + \int_{C_2} \vec{F} \cdot \vec{dr}$

LINE INTEGRAL OVER VECTOR FIELDS :-

1. Find $r'(t)$
2. Substitute $f(r(t))$
3. Take dot product of $r'(t) \cdot f(r(t))$
4. Integrate over interval.

$F_2 = -\cos t\, i + 4\sin t \qquad 0 \le t \le 2\pi$

$r'(t) = -\sin t\, i + 4\cos t\, i$

$F = \nabla f \longrightarrow$ for a function to be "CONTINUOS" it needs to be differentiable.

$\swarrow$ conservative.

continuos

F conservative $\rightleftharpoons F = \nabla f$

differentiable

F

$\Delta f \longrightarrow$ differentiable

conservative vector field

$\longrightarrow$ path independent.

Example:

$u = xy \qquad v = \cos z$

$u' = x \qquad v' = \cancel{\sin z}\ 0$

$$F = \underbrace{yz}_{M}i + \underbrace{xz}_{N}j + \underbrace{xy}_{P}k$$

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

$$x = x \checkmark \qquad y = y \checkmark \qquad z = z \checkmark$$

Example:

$$F = \underbrace{(y\sin z)}_{M} i + \underbrace{(x\sin z)}_{N} j + \underbrace{(xy\cos z)}_{P} k$$

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

$$x\cos z = x \qquad y = y\cos z \qquad \sin z = \sin z$$

$$\int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr = \int_{C_1} F\cdot dr - \int_{C_2} F\cdot dr = 0$$

'F' is conservative if and only if $F = \nabla f$ for some 'f'.

$\int_C F\cdot dr$ The line $\int$ is independent of path!

- so the $\int$ will be the same NO MATTER WHAT CURVE we CHOOSE As long AS END POINTS STAYS THE SAME.
- FOR CONS V.F WE DON'T EVEN DEFINE A CURVE

$F = \nabla f$

→ potential function for F.

→ for function to be CONTINUOS f needs to be differentiable

→ F is conservative.

∗ if F is conservative path independence.

FUNDAMENTAL THEOREM OF CALCULUS.

→ to find conservative vector field

$$\int_C F\cdot dr = f(B) - f(A)$$

Proof of Theorem.

$$\int_C F\cdot dr = \int_A^B \nabla f\cdot dr \quad F = \nabla f$$

$$= \int_{t=a}^{t=b} \nabla f(r(t))\cdot r'(t)\, dt \quad \text{chain rule}$$

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$= \int_a^b \frac{d}{dt} f(r(t))\, dt$ fondamental theorem of calculus.

$= f(r(b)) - f(r(a))$, $r(b) = B$, $r(a) = A$

$= f(B) - f(A)$

CONSERVATIVE FIELDS ARE VECTOR FIELDS

$$F = Mi + Nj + Pk$$

Component Test for Conservative Fields.

$$F = M(x,y,z)i + N(x,y,z)j + P(x,y,z)k$$

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

* field is continuous if the component function M, N, P are continuous, it is differentiable.

Example 2. Find the work done by the conservative field.
$F = yzi + xzj + xyk = \nabla f$, where $f(x,y,z) = xyz$.
along any smooth curve C joining the point $A(-1,3,9)$ to $B(1,6,-4)$

$$\int_C F \cdot dr = \int_A^B \nabla f \cdot dr$$
$$= f(B) - f(A)$$
$$= xyz\Big|_{(1,6,-4)} - xyz\Big|_{(-1,3,9)}$$
$$= (1)(6)(-4) - (-1)(3)(9)$$
$$= -24 + 27 = 3$$

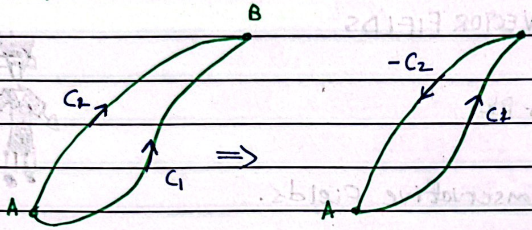
GRADIENT FIELDS

→ We define gradient field of differentiable function $f(x,y,z)$ to be the field
$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

LOOP PROPERTY OF CONSERVATIVE FIELDS

1. $\oint_C F \cdot dr = 0$ around every loop (that is closed curve C) in D.
2. The field F is conservative on D.

Proof that Part 1 ⇒ Part 2. $\oint \rightarrow$ closed curve



$F = \nabla f$ on D

$\Updownarrow$

F conservative on D.

$\Updownarrow$

$\oint_C F \cdot dr = 0$ over any loop in D.

$\oint_C F \cdot dr = 0 \longrightarrow F$ is conservative

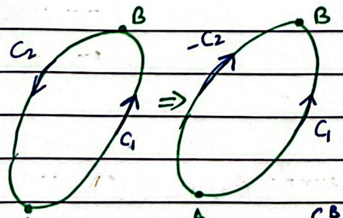
$\downarrow$

$= \int_{C_1} F \cdot dr + \int_{-C_2} F \cdot dr = 0$

$= \int_{C_1} F \cdot dr - \int_{C_2} F \cdot dr = 0$

$= \int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr \longrightarrow$ Conservative Vector field.

Proof that Part 2 ⇒ Part 1



$$\int_{A\,C_1}^{B} F \cdot dr = \int_{C_2\,A}^{B} F \cdot dr$$

$$\int_A^B F \cdot dr - \int_A^B F \cdot dr = 0$$

$$\int_A^B F \cdot dr + \int_B^A F \cdot dr = 0 \Rightarrow \oint_C F \cdot dr = 0$$

conservative VF.

$M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz \rightarrow$ differential form

$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

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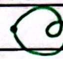
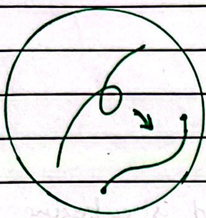
Simply connected: Smooth curve can be drawn in the region

$\oint \Rightarrow$ closed siple curve O

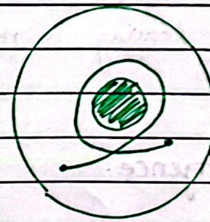




→ simply connected.

Smooth Curve: No overlap.
can be converted into smoth line/point.

not-simply connected ←



divergence → flux density
curl → circulation density

* finding potential function →

$\vec{F} = \nabla f$

$\vec{F} = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

e.g $F = (y\sin z)i + (x\sin z)j + (xy\cos z)k$

$\frac{\partial f}{\partial x} = y\sin z \quad \frac{\partial f}{\partial y} = x\sin z \quad \frac{\partial f}{\partial z} = xy\cos z$

$\int \frac{\partial f}{\partial x} dx \Rightarrow xy\sin z + g(y,z) \rightarrow f(x,y,z)$

$\frac{\partial}{\partial y}(xy\sin z + g(y,z)) = x\sin z + \frac{\partial g}{\partial y} \rightarrow 0$

$f(x,y,z) = xy\sin z + h(z)$

$\frac{\partial}{\partial z}(xy\sin z + h(z)) = xy\cos z + \frac{\partial h}{\partial z} \rightarrow$

$dz)\ \cancel{x^2 z}\ e^y + xz$

$\sin x\cos z \quad \sin y$

$= \sin x \quad v = \cos z$

$\nabla f = \sin x\, i + c$

$= \cos x \quad v = 0$

i	j	k
$\sin x$	$\cos y\sin z$	0
$\sin x\cos z$	$\sin y\sin z$	$\cos x\cos y$

$= \sin y \quad v = \sin z$

$\cos y \quad v' = 0$

$(\cos y\sin z)(\cos x\cos y)\,i - (\sin x)(\cos x\cos y)\,j + [(\sin x)(\sin y\sin z) - (\sin x\cos z)(\cos y\sin z)]$

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Signature

$$\cancel{x^b y} = x^5$$

$$\underbrace{x^b y}_{M}\, i + \underbrace{x^5 y^a}_{N}\, j$$

Magnetic Flux

The flux calculates the rate at which a fluid is entering or leaving a region enclosed by closed curve.

Divergence.

→ if $F(x_0, y_0) > 0$: a gas is EXPANDING. positive divergence

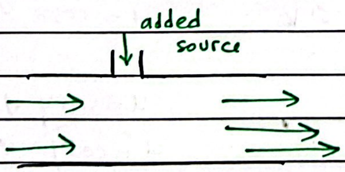
→ if $F(x_0, y_0) < 0$: a gas is COMPRESSING. negative divergence.

→ if $F(x_0, y_0) = 0$: neither expanding nor compressing

1. 

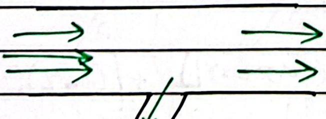
flow in = flow out

⇒ 0 divergence.

2. 

flow in < flow out

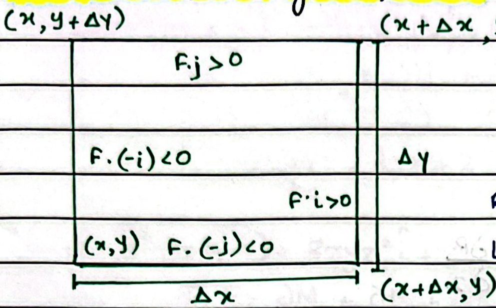
⇒ + divergence.



flow in > flow out

⇒ -ve divergence.

Derivation of Divergence formula:



Fluid flow rates :

Top :- $F(x, y+\Delta y) \cdot j \Delta x = N(x, y+\Delta y)\Delta x$

Bottom :- $F(x,y) \cdot (-j)\Delta x = -N(x,y)\Delta x$

Right :- $F(x+\Delta x, y) \cdot i \Delta y = M(x+\Delta x, y)\Delta y$

Left :- $F(x,y) \cdot (-i)\Delta y = -M(x,y)\Delta y$

Summing opposite pairs gives,

Top and bottom : $(N(x, y+\Delta y) - N(x,y))\Delta x \approx \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$

Right and left : $(M(x+\Delta x, y) - M(x,y))\Delta y \approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y.$ $+$

Adding these last two equations gives the net effect of flow rate,

Flux across rectangle boundary $\approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y$

We now divide by $\Delta x \Delta y$ to estimate total flux per unit area or flux density for rectangle,

$$\approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

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The divergence (flux density) of a vector field $F = Mi + Nj$ at point (x, y) is.

$$\boxed{\text{div } F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}}$$

Divergence in 3D.

$$\text{div } F = \nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

Example (1)

(a) Expansion: $F(x, y, z) = xi + yj + zk$

$$\text{div } F = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3 \quad \text{(Expansion)}$$

(b) Compression: $F(x, y, z) = -xi - yj - zk$

$$\text{div } F = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) + \left(\frac{\partial}{\partial z}\right)(-z) = -3 \quad \text{(compression)}$$

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PRACTICE NOW:

17. $\vec{F}(x,y,z) = x^2z\hat{i} + y^2x\hat{j} + (y+2z)\hat{k}$

$$\text{div } F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= 2xz + 2yx + 2$$

18. $\vec{F}(x,y,z) = 3xyz^2\hat{i} + y^2\sin z\hat{j} + xe^{2z}\hat{k}$

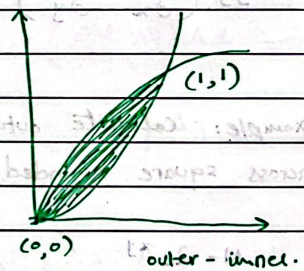
$$\text{div } F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= 3yz^2 + 2y\sin z + 2xe^{2z}$$

Curl $\rightarrow$ circulation density / Area.

divergence $\rightarrow$ $\frac{\text{flux}}{\text{Area}}$

$\times$ Area $\longrightarrow$



① $\iint \text{div } dydx = \text{flux}$

$$\int_c P dx + Q dy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} \right) dx\, dy \qquad \vec{F} = \langle P, Q \rangle$$

$$= \oint F \cdot \vec{r} \qquad \text{OR} \qquad \oint M dy - N dx$$

② $\iint_R \text{curl } dydx = \text{Circulation}$

$$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dydx = \oint F \cdot dr \qquad \text{OR} \qquad \oint_c M dx + N dy$$

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→ when walking, region should be on left (+ve orientation / counter)
→ " ", region when on right (-ve orientation / clockwise)

$\oint$ negative positive oriented $\oint$ positive oriented.

Flux * not coming in exam.

$$\iint \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dy\,dx = \oint M dy - N dx$$

Circulation.

$$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = \oint M dx + N dy$$

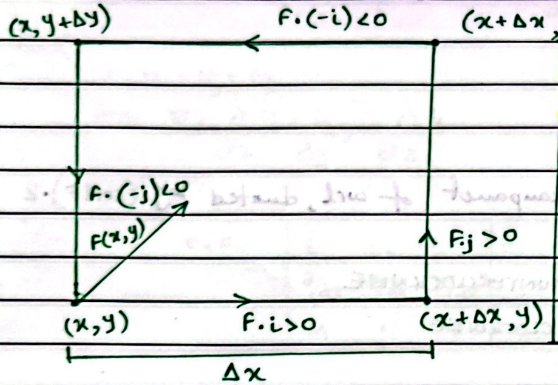
Example: Calculate outward flux of vector field $F(x, y) = 2e^{xy} i + y^3 j$ across square bounded by lines. $x = \pm 1$ and $y = \pm 1$.

$M = 2e^{xy}$ $\quad \partial M / \partial x = 2y e^{xy}$

$N = y^3$ $\quad \partial N / \partial y = 3y^2$

$$\int_1^{-1} \int_1^{-1} (2y e^{xy} + 3y^2)\, dy\,dx = \oint 2e^{xy} dy - y^3 dx.$$

CURL



$F(x,y) = M(x,y)\,i + N(x,y)\,j$

$F(x,y)\cdot i\,\Delta x = M(x,y)\,\Delta x$

Top: $F(x,y+\Delta y)\cdot(-i)\Delta x = -M(x,y+\Delta y)\Delta x$

Bottom: $F(x,y)\cdot i\,\Delta x = M(x,y)\Delta x$

Right: $F(x+\Delta x,y)\cdot j\,\Delta y = N(x+\Delta x,y)\Delta y$

Left: $F(x,y)\cdot(-j)\Delta y = -N(x,y)\Delta y$

we sum opposite pairs

Top & Bottom: $-(M(x,y+\Delta y)-M(x,y))\Delta x \approx -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$

Right & left: $(N(x+\Delta x, y) - N(x,y))\Delta y \approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y$

Adding these last two equations.

Circulation rate around rectangle $\approx \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x\Delta y$

We now divide by $\Delta x\Delta y$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

→ Orientation is really important
→ We use right hand rule to find axis (direction) of rotation.

* Put the fingers of right hand in the direction of rotation.
* Your thumb will point towards direction of curl.

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The circulation density of vector field $F = Mi + Nj$ at point (x,y)

$$\boxed{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}$$

The expression is also called k-component of curl, denoted by $(\text{curl} F)\cdot k$

→ if $c>0$: +ve rotation COUNTERCLOCKWISE
→ if $c<0$: −ve rotation. CLOCKWISE
→ if $c=0$: irrotational field

EXAMPLE 1

(b) $F(x,y) = -cyi + cxj$

$$\frac{\partial}{\partial x}(cx) - \frac{\partial}{\partial y}(-cy) = c + c = 2c$$

(a) $F(x,y) = cxi + cyj$

$$\frac{\partial}{\partial x}(cy) - \frac{\partial}{\partial y}(cx) \Rightarrow 0 \quad \text{no rotation}$$

Signature

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Curl in 3D

important

* $\text{div}(\text{curl } F) = \nabla \cdot (\nabla \times F) = 0$

$F = Mi + Nj + Pk$

Here $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k.$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) i + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) j + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) k$$

$\text{curl } F = \nabla \times F$

ZERO CURL MEANING.

$$\Rightarrow \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) i + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) j + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) k$$

$\Rightarrow 0i + 0j + 0k \rightarrow$ zero vector.

Comparing all components give:

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \; ; \; \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \; ; \; \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

→ no curl ⇒ irrotational field.

irrotational field $\rightleftharpoons$ conservative vector field.

GREEN'S THEOREM AREA FORMULA.

$$\text{Area of } R = \frac{1}{2} \oint_C x\,dy - y\,dx$$