LECTURE 26

ORTHOGONAL DIAGONALIZATION

P. 375 (7TH ED.) / P. 357 (8TH ED.)

THE ORTHONORMAL EIGENVECTOR PROBLEM:

GIVEN AN $\boxed{n \times n}$ MATRIX $\boxed{A}$, DOES THERE EXIST AN ORTHONORMAL BASIS FOR $\boxed{R^n}$ WITH THE EUCLIDEAN INNER PRODUCT CONSISTING OF EIGENVECTORS OF $\boxed{A}$?

DEFINITION: A SQUARE MATRIX $\boxed{A}$ IS CALLED ORTHOGONALLY DIAGONALIZABLE IF THERE IS AN ORTHOGONAL MATRIX P SUCH THAT

$$\boxed{P^{-1}AP = P^{t}AP}$$

IS A DIAGONAL MATRIX, THE MATRIX $\boxed{P}$ IS SAID TO ORTHOGONALLY DIAGONALIZE $\boxed{A}$.

NOTE: RECALL THAT FOR AN ORTHOGONAL MATRIX P WE HAVE

$$\boxed{P^t = P^{-1}} \text{ or } \boxed{P^t P = P P^t = I}$$

TRY THE FOLLOWING:

IF $\boxed{A}$ IS ORTHOGONALLY DIAGONALIZABLE THEN PROVE THAT $\boxed{A}$ IS A SYMMETRIC MATRIX.

PROOF: $\because \boxed{P^t A P = D}$, WHERE $\boxed{D}$ IS A DIAGONAL MATRIX AND $\boxed{P^t P = I.}$

$$\therefore \underbrace{P P^t}_{I} A \underbrace{P P^t}_{I} = P D P^t$$

$$\Rightarrow A = P D P^t \text{ — } (1)$$

$$\Rightarrow A^t = (P D P^t)^t = (P^t)^t D^t P^t \qquad D^t = D$$

$$= P D P^t = \boxed{A} \text{ FROM } (1)$$

$$\Rightarrow \boxed{A^t = A}$$

NOTE: SYMMETRIC MATRIX IS ALWAYS DIAGONALIZABLE.

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PROBLEM: FIND AN ORTHOGONAL MATRIX $\boxed{P}$ THAT DIAGONALIZES

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}, \quad A^t = A$$

STEPS: ① FIND THE EIGENVALUES OF $\boxed{A}$, THEY ARE GIVEN BY $\boxed{\lambda_1 = \lambda_2 = 2}$, AND $\boxed{\lambda_3 = 8}$

(OBTAINED ALREADY)

② FIND THE $\boxed{\text{BASIS}}$ FOR THE EIGENSPACE CORRESPONDING TO $\boxed{\lambda = 2}$, AND IS GIVEN BY

$$\{\underline{u}_1, \underline{u}_2\} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(OBTAINED ALREADY)

$\boxed{\text{NOTE:}}$ $\underline{u}_1 \cdot \underline{u}_2 = 1 \neq 0$, SO $\underline{u}_1$ IS $\boxed{\text{NOT}}$ ORTHOGONAL TO $\underline{u}_2$.

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(3) APPLY THE GRAM-SCHMIDT PROCESS TO $\{\underline{u}_1, \underline{u}_2\}$ TO GET AN ORTHONORMAL BASIS, i.e. $\left\{\frac{\underline{v}_1}{\|\underline{v}_1\|}, \frac{\underline{v}_2}{\|\underline{v}_2\|}\right\}$

$$\underline{v}_1 = \underline{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{\underline{v}_1}{\|\underline{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \underline{w}_1 \text{ (SAY)}$$

$$\underline{v}_2 = \underline{u}_2 - \frac{(\underline{u}_2 \cdot \underline{v}_1)\underline{v}_1}{\|\underline{v}_1\|^2} = \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \text{ (CHECK)}$$

$$\|\underline{v}_2\| = \frac{\sqrt{6}}{2}, \quad \therefore \boxed{\frac{\underline{v}_2}{\|\underline{v}_2\|} = \frac{1}{\sqrt{6}}(-1,-1,2) = \underline{w}_2}$$

[4] FIND THE BASIS FOR THE EIGENSPACE CORRESPONDING TO $\lambda = 8$. IN THIS CASE

$$\text{BASIS} = \{\underline{u}_3\} = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} \text{ (OBTAINED ALREADY)}$$

[5] APPLY THE GRAM-SCHMIDT PROCESS TO $\boxed{\underline{u}_3}$ TO GET $\boxed{\underline{w}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$

$\boxed{\underline{v}_3 = \underline{u}_3}$.

5 NOTE: NO NEED TO FIND $\underline{v_3}$ BY USING $\underline{u_1}$ AND $\underline{u_2}$ IN STEP 3, ∵ EIGENSPACES ARE DIFFERENT.

6 FINALLY USING $\underline{w_1}$, $\underline{w_2}$ AND $\underline{w_3}$ AS COLUMN VECTORS WE OBTAIN

$$P = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{+1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$\boxed{\text{FOR } \lambda = 2}$

$\boxed{\text{FOR } \lambda = 8}$

WHICH ORTHOGONALLY DIAGONALIZES A.

CHECK:

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$$PP^t$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $\boxed{P}$ IS AN ORTHOGONAL MATRIX

FURTHER

$$P^t A P$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} = D$$

7 FOR THE SYMMETRIC MATRIX

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix},$$

BASIS FOR THE EIGENSPACE WHICH CORRESPONDS TO $\boxed{\lambda = 2}$ IS GIVEN BY

$$\boxed{\{\underline{u}_1, \underline{u}_2\}} = \{(-1,1,0), (-1,0,1)\}$$

AND THE BASIS FOR THE EIGENSPACE CORRESPONDING TO $\boxed{\lambda = 8} = \boxed{\{\underline{u}_3\}} = \{(1,1,1)\}$

NOTICE THAT

$$\boxed{\underline{u}_1 \cdot \underline{u}_3 = (-1,1,0) \cdot (1,1,1) = 0}$$

AND $\boxed{\underline{u}_2 \cdot \underline{u}_3 = (-1,0,1) \cdot (1,1,1) = 0}$

THEOREM: 7.3.2 (6th ED.) P. 358
OR 7.3.2 (7th ED.) P. 376

IF $\boxed{A}$ IS A SYMMETRIC MATRIX THEN EIGENVECTORS FROM **DIFFERENT** EIGENSPACES ARE ORTHOGONAL.

DEFINITION: IF $\boxed{A}$ AND $\boxed{B}$ ARE SQUARE MATRICES, WE SAY $\boxed{B}$ IS SIMILAR TO $\boxed{A}$ IF THERE IS AN INVERTIBLE MATRIX $\boxed{P}$ SUCH THAT $B = P^{-1}AP$.

LINEAR TRANSFORMATIONS:

DEPENDENT VARIABLE $y = f(x)$ INDEPENDENT VARIABLE

BOTH ARE SCALARS

WE SHALL BEGIN THE STUDY OF FUNCTIONS OF THE FORM $\boxed{\underline{w} = f(\underline{v})}$ WHERE THE INDEPENDENT VARIABLE $\boxed{\underline{v}}$ AND THE DEPENDENT VARIABLE $\boxed{\underline{w}}$ ARE BOTH VECTORS.

WE SHALL STUDY FUNCTIONS WHICH ARE CALLED LINEAR TRANSFORMATIONS. CONSIDER THE FOLLOWING DEFINITION.

DEFINITION: P. 366 (8th ED.)
P. 383 (7th ED.)

IF $f: V \to W$ IS A FUNCTION FROM THE VECTOR SPACE $\boxed{V}$ INTO THE VECTOR SPACE $\boxed{W}$, THEN f IS CALLED A **LINEAR TRANSFORMATION** IF

(a) $\boxed{f(\underline{u}+\underline{v}) = f(\underline{u}) + f(\underline{v})}$

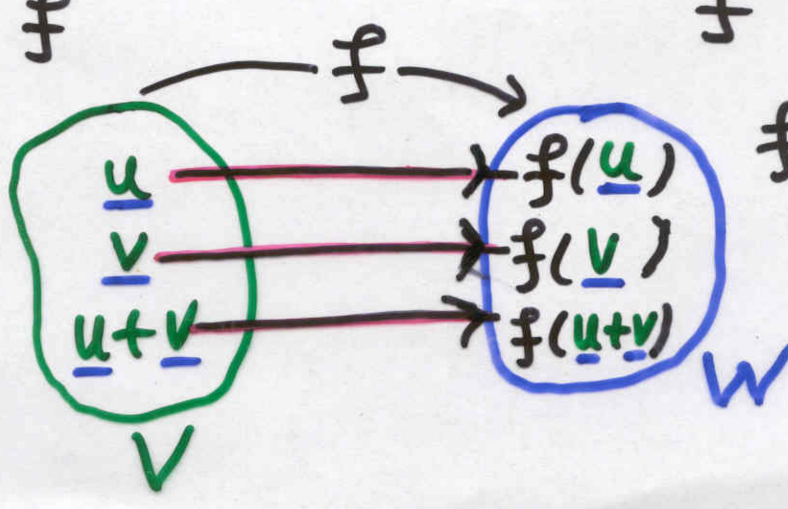
FOR ALL $\underline{u}, \underline{v} \in V$

(b) $\boxed{f(k\underline{u}) = k f(\underline{u})}$ FOR ALL $\underline{u} \in V$ AND ALL SCALARS $\boxed{k}$.

$$f : V \longrightarrow W$$

DOMAIN OF f (V)

IMAGE SPACE OF f (W)



$f: V \to W$

EXAMPLE:

LET $f : R^2 \longrightarrow R^3$ BE GIVEN

BY $\boxed{f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \\ 5x \end{bmatrix}} \longrightarrow (*)$

IS $\boxed{f}$ LINEAR ? OR IS $\boxed{f}$ A LINEAR TRANSFORMATION/MAPPING FROM $R^2 \longrightarrow R^3$?

SOLUTION: LET $\underline{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \underline{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in R^2$

$$f(\underline{u}+\underline{v}) = f\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

$$= f\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = \begin{bmatrix} (x_1+x_2)-(y_1+y_2) \\ (x_1+x_2)+(y_1+y_2) \\ 5(x_1+x_2) \end{bmatrix}$$

USING $(*)$

$$= \begin{bmatrix} x_1 - y_1 \\ x_1 + y_1 \\ 5x_1 \end{bmatrix} + \begin{bmatrix} x_2 - y_2 \\ x_2 + y_2 \\ 5x_2 \end{bmatrix}$$

$$= f(\underline{u}) + f(\underline{v})$$ USING $(*)$

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$$\therefore f(\underline{u}+\underline{v}) = f(\underline{u}) + f(\underline{v}) \text{ --- } (1)$$

NOW CONSIDER

$$f(k\underline{u}) = f\left(k\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$= f\left(\begin{bmatrix} kx \\ ky \end{bmatrix}\right)$$

$$= \begin{bmatrix} kx - ky \\ kx + ky \\ 5kx \end{bmatrix} \quad \because f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \\ 5x \end{bmatrix}$$

$$= k\begin{bmatrix} x-y \\ x+y \\ 5x \end{bmatrix} = k f(\underline{u})$$

$$\therefore f(k\underline{u}) = k f(\underline{u}) \longrightarrow (2)$$

FROM (1) AND (2) f IS A LINEAR TRANSFORMATION FROM $R^2 \longrightarrow R^3$

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TRY THE FOLLOWING:

LET $D: W \longrightarrow V$ BE THE TRANSFORMATION THAT MAPS $f = f(x)$ INTO ITS DERIVATIVE, THAT IS,

$$\boxed{D(f) = f'(x)}.$$

IS $\boxed{D}$ LINEAR ?

SOLUTION:

$$D(f+g) = \left(f(x) + g(x)\right)'$$

$$= \frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

$$= f'(x) + g'(x)$$

$$= D(f) + D(g) \longrightarrow ①$$

$$D(Kf) = \left(Kf(x)\right)' = \frac{d}{dx}\left(Kf(x)\right)$$

$$= K\frac{d}{dx}\left(f(x)\right) = K\, D(f)$$

$$\Rightarrow D(Kf) = K\,D(f) \longrightarrow ②$$

THEREFORE FROM ① AND ② WE SEE THAT $\boxed{D}$ IS LINEAR FROM $\boxed{W}$ TO $\boxed{V}$.

TRY THE FOLLOWING:

LET $V = C[0,1]$ (CONTINUOUS FUNCTIONS FROM 0 TO 1),

LET $J : V \longrightarrow \mathbb{R}$ BE DEFINED BY

REAL NUMBERS SPACE

$$J(f) = \int_0^1 f(x)\,dx$$

PROVE THAT $\boxed{J}$ IS A LINEAR TRANSFORMATION FROM $\boxed{V}$ TO $\boxed{R}$.

$\boxed{\text{SOLUTION:}}$ LET $f, g \in V$

$$J(f+g) = \int_0^1 (f+g)(x)\,dx$$

$$= \int_0^1 (f(x) + g(x))\,dx = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx$$

$$= J(f) + J(g)$$

$$\Rightarrow J(f+g) = J(f) + J(g) \text{ — ①}$$

ALSO $J(Kf) = \int_0^1 Kf(x)\,dx = K\int_0^1 f(x)\,dx$

$$\Rightarrow J(Kf) = K\,J(f) \text{ — ②}$$

AND HENCE THE PROOF FROM ① AND ②

TRY THE FOLLOWING:

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LET $T: R^n \longrightarrow R^m$ GIVEN BY

$$T(\underline{x}) = A\underline{x} = \underline{b}$$

$\boxed{A} \rightarrow \underline{m \times n}$ MATRIX

$\boxed{\underline{x}} = \boxed{\underline{x}} \rightarrow n \times 1$ MATRIX (COLUMN VECTOR) $\rightarrow \underline{x}$

$\boxed{\underline{b}} \rightarrow m \times 1$ MATRIX (COLUMN VECTOR)

CHECK WHETHER $\boxed{T}$ IS LINEAR ?

NOTE:

$$A\underline{x} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m \times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n \times 1} = \underbrace{\underline{b}}_{m \times 1}$$

$$\underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \quad \underline{x} \in R^n, \quad \underline{b} \in R^m$$

$T: R^n \to R^m$

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T.P. $T(\underline{x}) = A\underline{x}$ IS $\boxed{\text{LINEAR.}}$

Pf: LET $\underline{x}_1, \underline{x}_2 \in R^n$

$$T(\underline{x}_1 + \underline{x}_2) = A(\underline{x}_1 + \underline{x}_2) \quad ①$$

$$= A\underline{x}_1 + A\underline{x}_2 = T(\underline{x}_1) + T(\underline{x}_2)$$

ALSO $T(k\underline{x}_1) = A(k\underline{x}_1) = kA\underline{x}_1$

$$= kT(\underline{x}_1) \to ②$$

$\therefore T(\underline{x}) = A\underline{x}$ IS LINEAR FROM ① AND ②.

DEF: $T(\underline{x}) = A\underline{x}$ IS LINEAR AND IS ALSO CALLED MATRIX TRANSFORMATION OR A LINEAR TRANSFORMATION CALLED MULTIPLICATION BY $\boxed{A.}$

HERE $\boxed{A}$ in $T(\underline{x}) = A\underline{x}$ IS CALLED MATRIX OF LINEAR TRANSFORMATION.