LINEAR ALGEBRA | LECTURE 24 | MATH 205

TRACE: IF $\boxed{A}$ IS A SQUARE MATRIX, THEN THE TRACE OF $\boxed{A}$ IS DENOTED BY $\boxed{tr(A)}$ AND IS DEFINED TO BE THE SUM OF THE ENTRIES ON THE MAIN DIAGONAL OF $\boxed{A}$.

EXAMPLE:

$$\text{FOR} \quad A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \rightarrow (1)$$

$$tr(A) = 4+1+1 = 6$$

LAST TIME WE SAW THAT EIGENVALUES OF $\boxed{A} = 1, 2, 3$

$$\lambda_1 \quad \lambda_2 \quad \lambda_3$$

CONSIDER

$$\boxed{\lambda_1 + \lambda_2 + \lambda_3 = 1+2+3 = 6 = tr(A)}$$

RESULT: IF $\boxed{A}$ IS A SQUARE MATRIX THEN $tr(A) =$ SUM OF ITS EIGENVALUES.

ALSO NOTE THAT FROM (1)

$$\boxed{\det(A) = 4 + 1(2) = 6 = \lambda_1 \lambda_2 \lambda_3}$$

2) RESULT: IF $\boxed{A}$ IS A SQUARE MATRIX THEN $\det(A)$ = PRODUCT OF ITS EIGENVALUES.

TRY THE FOLLOWING:

(a) SHOW THAT THE CHARACTERISTIC EQUATION OF A 2×2 MATRIX $\boxed{A}$ CAN BE EXPRESSED AS

$$\boxed{\lambda^2 - tr(A)\lambda + \det(A) = 0} \rightarrow ①$$

(b) IF $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ THEN THE SOLUTIONS OF THE CHARACTERISTIC EQUATION OF $\boxed{A}$ ARE

$$\lambda = \frac{1}{2}\left[(a+d) \pm \sqrt{(a-d)^2 + 4bc}\right]$$

NOTE: IF λ_1, λ_2 ARE ROOTS OF ① THEN

$$\boxed{\lambda_1 + \lambda_2 = tr(A), \ \lambda_1\lambda_2 = \det(A)}$$

RECALL: FOR α, B AS ROOTS OF $ax^2 + bx + c = 0$, $a \neq 0$

$$\alpha + B = -\frac{b}{a} \text{ AND } \alpha B = \frac{c}{a}$$

3)

DIAGONALIZATION P.347

P. 365 (7th ED.) (8th ED.) → P.347

DEFINITION: A SQUARE MATRIX $\boxed{A}$ IS CALLED DIAGONALIZABLE IF THERE IS AN INVERTIBLE MATRIX $\boxed{P}$ SUCH THAT $\boxed{P^{-1}AP}$ IS A DIAGONAL MATRIX; THE MATRIX $\boxed{P}$ IS SAID TO DIAGONALIZE $\boxed{A}$.

EXAMPLE: $\boxed{P^{-1}AP}$

$$= \begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & -2 & 1 \\ -1 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ -3 & -4 & 1 \\ -3 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{P}$ IS THE MATRIX HAVING EIGENVECTORS OF $\boxed{A}$ AS ITS COLUMN VECTORS AND $\boxed{P^{-1}AP}$ IS A DIAGONAL MATRIX HAVING EIGENVALUES ON THE MAIN DIAGONAL.

Application of DIAGONALIZATION

THE EIGENVECTOR PROBLEM.

GIVEN AN $n \times n$ MATRIX A, DOES THERE EXIST A BASIS FOR $\boxed{R^n}$ CONSISTING OF EIGENVECTORS OF $\boxed{A}$?

THEOREM 7.2.1 P. 347 → 8TH ED.
P. 365 → 7TH ED.

IF $\boxed{A}$ IS AN $n \times n$ MATRIX, THEN THE FOLLOWING ARE EQUIVALENT.

(a) $\boxed{A}$ IS DIAGONALIZABLE
(b) $\boxed{A}$ HAS n LINEARLY INDEPENDENT EIGENVECTORS.

PROOF: (b) ⟹ (a)

ASSUME THAT $\boxed{A}$ HAS $\boxed{n}$ LINEARLY INDEPENDENT EIGENVECTORS $\underline{p_1}, \underline{p_2}, \ldots\ldots, \underline{p_n}$, WITH CORRESPONDING EIGENVALUES $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$.

CONSIDER THE MATRIX $\boxed{P}$ WITH $\underline{p_1}, \underline{p_2}, \ldots\ldots, \underline{p_n}$ AS ITS COLUMN VECTORS.

i.e.

$$P = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} \underline{p}_1 & \underline{p}_2 & & \underline{p}_n \\ \downarrow & \downarrow & & \downarrow \end{array}$$

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots\cdots & p_{1n} \\ p_{21} & p_{22} & \cdots\cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots\cdots & p_{nn} \end{bmatrix}$$

NOW THE COLUMNS OF THE PRODUCT AP ARE $A\underline{p}_1, A\underline{p}_2, \dots\dots, A\underline{p}_n$

EXTRA: CONSIDER $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

AND $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, $AP = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$

$$\Rightarrow AP = \begin{bmatrix} a_{11}p_{11} + a_{12}p_{21} & a_{11}p_{12} + a_{12}p_{22} \\ a_{21}\underset{\hookrightarrow p_{11}}{\cancel{p_{12}}} + a_{22}\underset{\hookrightarrow p_{21}}{\cancel{p_{22}}} & a_{21}p_{12} + a_{22}p_{22} \end{bmatrix}$$

FIRST COLUMN OF AP IS $\begin{bmatrix} a_{11}p_{11} + a_{12}p_{21} \\ a_{21}p_{11} + a_{22}p_{21} \end{bmatrix}$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} = \boxed{A\underline{p}_1}$$

BUT $A\underline{P_1} = \lambda_1 \underline{P_1}$, $A\underline{P_2} = \lambda_2 \underline{P_2}$, ...,
$A\underline{P_n} = \lambda_n \underline{P_n}$, SO THAT

$$AP = \begin{bmatrix} \lambda_1 P_{11} & \lambda_2 P_{12} & \cdots & \lambda_n P_{1n} \\ \lambda_1 P_{21} & \lambda_2 P_{22} & \cdots & \lambda_n P_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda_1 P_{n1} & \lambda_2 P_{n2} & \cdots & \lambda_n P_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= PD$$

$\Rightarrow \therefore \boxed{AP = PD}$, WHERE D IS THE DIAGONAL MATRIX HAVING THE EIGENVALUES $\boxed{\lambda_1, \lambda_2, \ldots, \lambda_n}$ ON THE MAIN DIAGONAL. SINCE THE COLUMN VECTORS OF $\boxed{P}$ ARE LINEARLY INDEPENDENT, THEREFORE

$\boxed{\text{RANK}(P) = n}$ SO THAT

$\boxed{\det(P) \neq 0}$,

RECALL THAT $\boxed{\text{RANK}}$ IS ALSO DEFINED AS THE HIGHEST ORDER OF THE NONZERO DETERMINANT.

$\therefore$ P IS INVERTIBLE;

THUS $\boxed{AP = PD}$ CAN BE WRITTEN AS $\boxed{P^{-1}AP = D}$; THAT IS, $\boxed{A}$ IS DIAGONALIZABLE.

$\boxed{\text{CONVERSE}}$ IS ALSO $\boxed{\text{TRUE.}}$

(a) $\Longrightarrow$ (b)

IF $\boxed{A}$ IS DIAGONALIZABLE THEN $\boxed{A}$ HAS n LINEARLY INDEPENDENT EIGENVECTORS

AND THEY FORM A $\boxed{\text{BASIS}}$ FOR $\boxed{R^n.}$