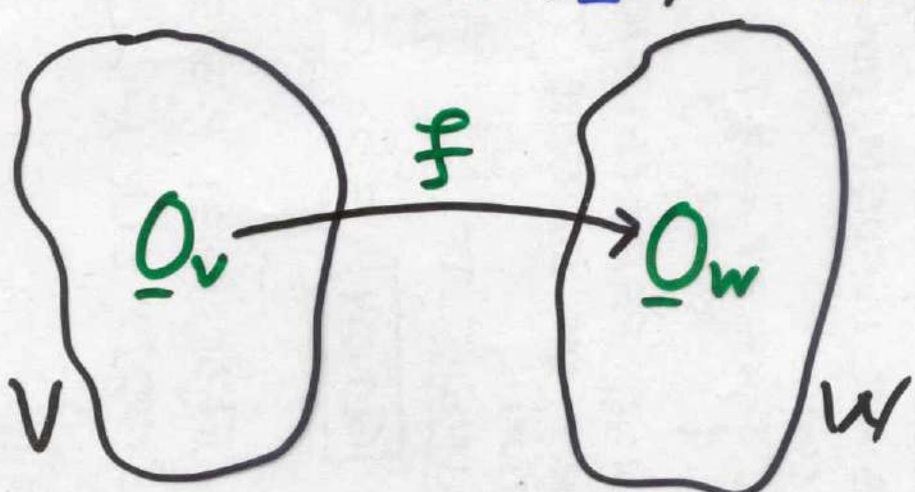
LINEAR ALGEBRA

LECTURE 27

MATH 205

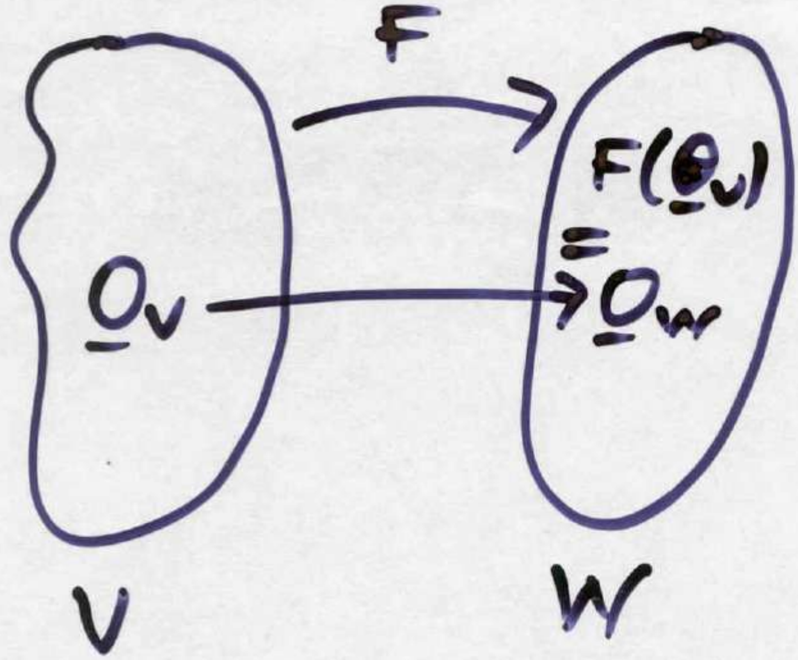
RESULT: IF $\boxed{f}$ IS A LINEAR TRANSFORMATION FROM $V \rightarrow W$, THEN THE $\boxed{\text{ZERO}}$ VECTOR OF THE VECTOR SPACE $\boxed{V}$ IS ALWAYS GOING TO $\boxed{\text{MAP}}$ ON THE $\boxed{\text{ZERO VECTOR}}$ OF THE VECTOR SPACE $\boxed{W}$ AS SHOWN BELOW:

$$f(\underline{0}_V) = \underline{0}_W$$



$\underline{0}_V \rightarrow$ $\boxed{\text{ZERO}}$ VECTOR OF $\boxed{V}$

$\underline{0}_W \rightarrow$ $\boxed{\text{ZERO}}$ VECTOR OF $\boxed{W}$



$F(\underline{0}_V) = \underline{0}_W$

NOTE: IN FUTURE WE SHALL USE $\underline{0}$ INSTEAD OF $\underline{0}_V$ OR $\underline{0}_W$.

EXAMPLES:

① $T: R^n \longrightarrow R^m$

$T(\underline{x}) = A\underline{x}$ IS LINEAR.

$T(\underline{0}) = A\underline{0} = \underline{0}$

$\Rightarrow \boxed{T(\underline{0}) = \underline{0}}$

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$$\boxed{\text{WE ALREADY PROVED THE FOLLOWING TRANSFORMATIONS AS LINEAR.}}$$

② $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \\ 5x \end{bmatrix}$

IT IS EASILY SEEN THAT

$$\boxed{F\left(\underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\text{ZERO OF } R^2}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \longrightarrow \text{ZERO OF } R^3$$

③ $J : V \longrightarrow R \qquad V = C[0,1]$

$$J(f) = \int_0^1 f(x)\,dx$$

$$\boxed{J(0) = \int_0^1 0\,dx = 0}$$

④ $D : W \longrightarrow V$

$$D(f) = f'(x) = \frac{d}{dx}(f(x))$$

$$\boxed{D(0) = \frac{d}{dx}(0) = 0}$$

BUT HOW TO PROVE IN GENERAL?

PROOF :-

4 IF $T: V \longrightarrow W$ IS A LINEAR TRANSFORMATION, THEN

$$T(\underline{0}) = \underline{0}$$

PROOF: $T(\underline{0}) = T(0\underline{v}) = 0T(\underline{v}) = \underline{0}$ (VECTOR, SCALAR)

$\underline{v} \in V$ $\qquad \because T(k\underline{v}) = kT(\underline{v})$

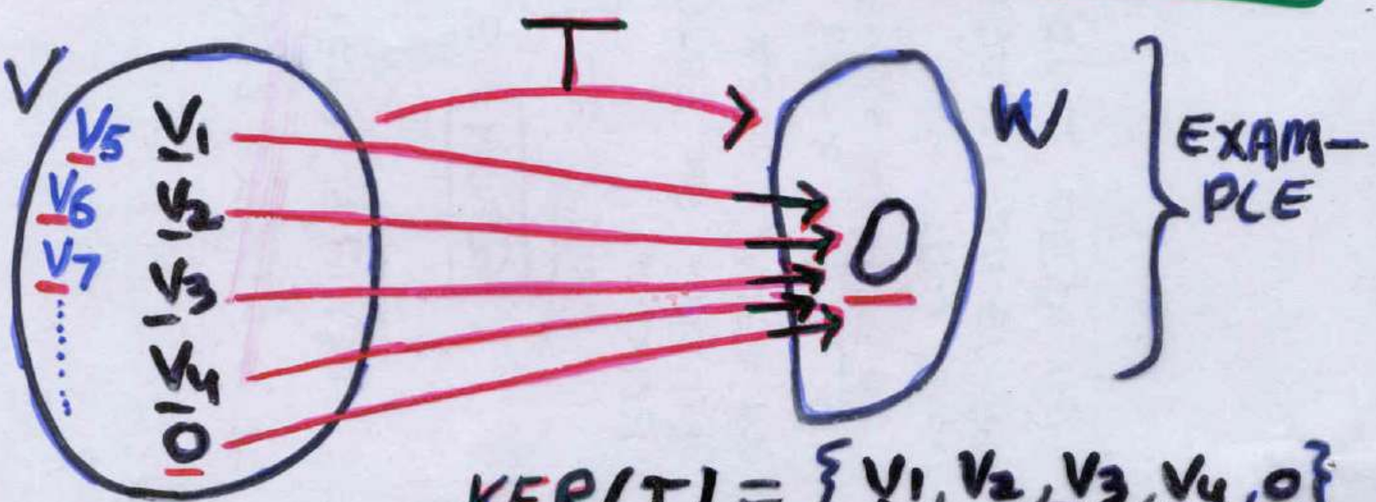
OR FOR ANY $\underline{v} \in V$

$$T(\underline{0}) = T(\underline{v} - \underline{v}) = T(\underline{v} + (-\underline{v}))$$

$$= T(\underline{v}) + T(-\underline{v}) = T(\underline{v}) - T(\underline{v}) = \underline{0}.$$

DEFINITION: P.316 (6th ED.) P.395 (7th ED.)

IF $T: V \rightarrow W$ IS A LINEAR TRANSFORMATION, THEN THE SET OF VECTORS IN V THAT $\boxed{\text{MAPS}}$ INTO $\underline{0}$ IS CALLED THE KERNEL (OR NULLSPACE) OF $\boxed{T}$; IT IS DENOTED BY $\boxed{\text{KER}(T)}$.



$$\text{KER}(T) = \{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4, \underline{0}\}$$

$$\therefore T(\underline{v}_1) = T(\underline{v}_2) = T(\underline{v}_3) = T(\underline{v}_4) = T(\underline{0}) = \underline{0}$$

EXAMPLES: ① $D: V \to W$

$\Rightarrow \quad D(f) = f'(x)$

$KER(D)$ = SET OF ALL FUNCTIONS S.t. $D(f) = 0$

$\Rightarrow \quad f'(x) = 0 \Rightarrow \boxed{f(x) = k}$

$\boxed{k \to \text{CONSTANT}}$

$\therefore$ $KER(D)$ = SET OF ALL CONSTANT FUNCTIONS IN $\boxed{V}$.

② $J: P_1 \to R$ $\quad P_1(x)$ } POLYNOMIALS OF DEGREE ≤ 1

FIND $KER(J)$, WHERE

$$\boxed{J(P) = \int_{-1}^{1} p(x)\,dx}$$

ANSWER: $KER(J)$ CONSISTS OF ALL POLYNOMIALS OF THE FORM $\boxed{P(x) = kx}$

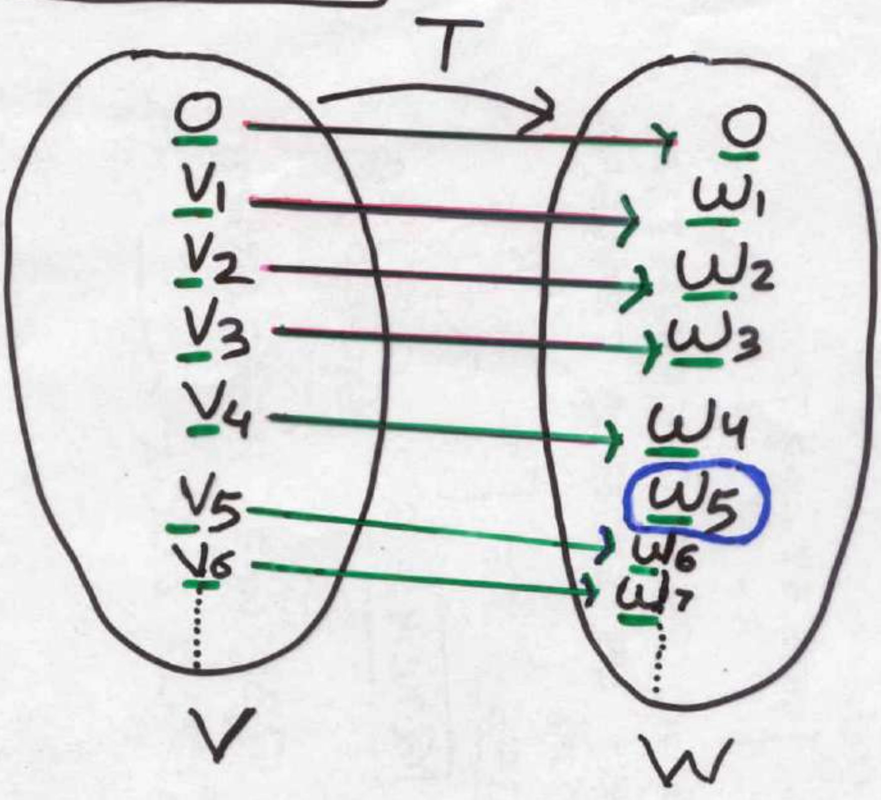
$k \to$ CONSTANT

6 RANGE OF LINEAR TRANSFORMATION:

(P. 376 8TH ED.)/(P. 395 7TH ED.)

IF $\boxed{T: V \to W}$ IS LINEAR, THEN THE SET OF ALL VECTORS IN $\boxed{W}$ THAT ARE IMAGES UNDER $\boxed{T}$ OF ATLEAST ONE VECTOR IN $\boxed{V}$ IS CALLED THE RANGE OF T; IT IS DENOTED BY $\boxed{R(T)}$.

EXAMPLE:



$\nearrow \underline{w}_5$

$\underline{w}_5$ HAS NO PRE-IMAGE $\therefore \underline{w}_5 \notin R(T)$

$$R(T) = \{\underline{0}, \underline{w}_1, \underline{w}_2, \underline{w}_3, \underline{w}_4, \underline{w}_6, \underline{w}_7, \ldots\ldots\}$$

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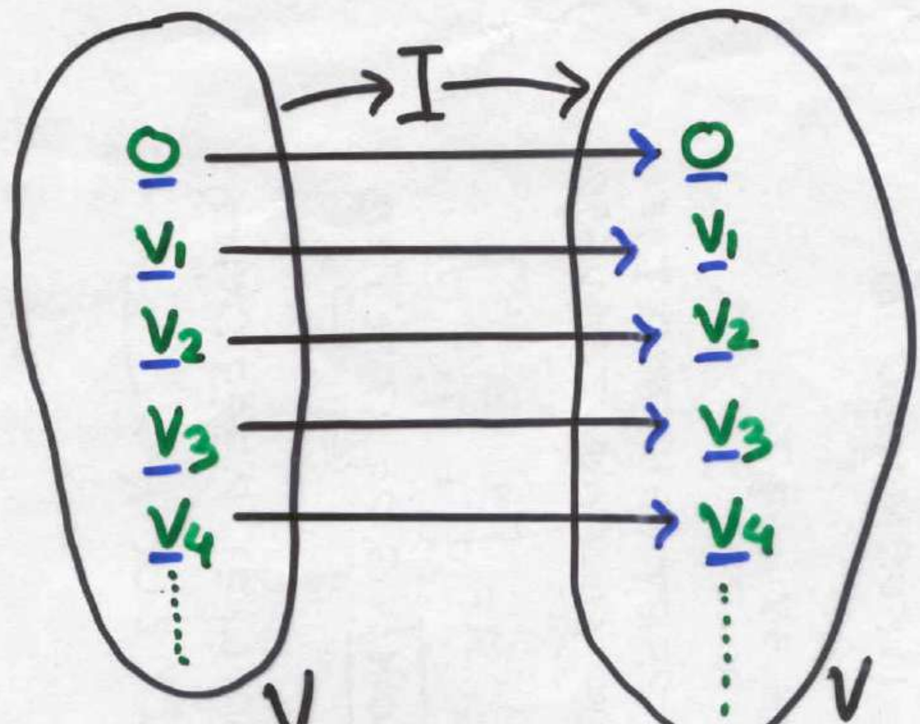
IDENTITY TRANSFORMATION:

(P. 366 8TH ED.)/(P. 384 7TH ED.)

LET $\boxed{V}$ BE ANY VECTOR SPACE

THE MAPPING $I: V \longrightarrow V$ DEFINED BY

$\boxed{I(\underline{v}) = \underline{v}}$ IS CALLED THE IDENTITY OPERATOR AS SHOWN IN THE FIGURE:



$\Rightarrow I(\underline{v}_1) = \underline{v}_1, \quad I(\underline{v}_2) = \underline{v}_2,$

$I(\underline{v}_3) = \underline{v}_3, \quad I(\underline{v}_4) = \underline{v}_4, \ldots\ldots$

$I(\underline{0}) = \underline{0}$ ETC.

TRY THE FOLLOWING:

IF $I : V \to V$ IS AN IDENTITY TRANSFORMATION i.e. $I(\underline{v}) = \underline{v} \ \forall \underline{v} \in V$ THEN

(1) I IS LINEAR

(2) FIND $R(I)$, (3) $KER(I)$

SOLUTION:

$k\underline{u} \in V$,

(1) LET $\underline{u}, \underline{v} \in \boxed{V} \Rightarrow \underline{u} + \underline{v} \in \boxed{V}$

$$\Rightarrow I(\underline{u}+\underline{v}) = \underline{u}+\underline{v} = I(\underline{u}) + I(\underline{v})$$

ALSO $I(k\underline{u}) = k\underline{u} = kI(\underline{u})$

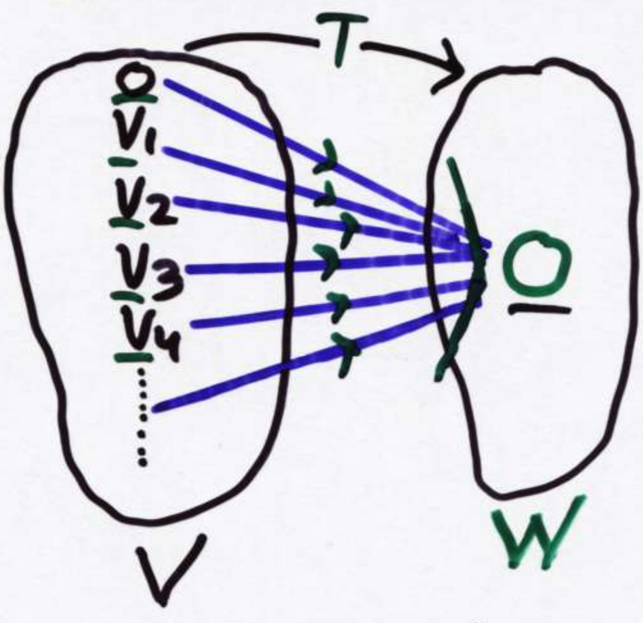
(2) $R(I) = V$ $\because$ EVERY VECTOR IN $\boxed{V}$ HAS A PREIMAGE

(3) $KER(I) = \underline{0}$ $\because$ $\underline{0}$ IS THE ONLY VECTOR WHICH MAPS INTO $\underline{0}$.

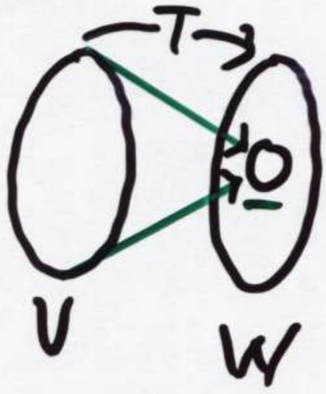
ZERO TRANSFORMATION:

(P. 366 8TH ED.) / (P. 384 7TH ED.)

LET $\boxed{V}$ AND $\boxed{W}$ BE ANY $\boxed{\text{TWO}}$ VECTOR SPACES. THE $\boxed{\text{MAPPING}}$ $\boxed{T : V \rightarrow W}$ SUCH THAT $\boxed{T(\underline{v}) = \underline{0}}$ FOR $\boxed{\text{EVERY } \underline{v}}$ IN V IS CALLED THE ZERO TRANSFORMATION.



OR



$$\Rightarrow T(\underline{v}_1) = T(\underline{v}_2) = \cdots\cdots = \underline{0}$$

DEFINITION: $\begin{cases} \text{P. 382 (8TH ED.)} \\ \text{P. 402 (7TH ED.)} \end{cases}$

A LINEAR TRANSFORMATION $T: V \longrightarrow W$ IS SAID TO BE ONE-TO-ONE IF $\boxed{T}$ MAPS DISTINCT VECTORS IN $\boxed{V}$ INTO DISTINCT VECTORS IN $\boxed{W}$.

EXAMPLE:

$\boxed{\text{IDENTITY}}$ TRANSFORMATION IS ONE-TO-ONE, $\forall\ \underline{v} \in V$

$\boxed{I(\underline{v}) = \underline{v}}$, $I: V \longrightarrow V$

NOTE: $\boxed{\text{ZERO}}$ TRANSFORMATION IS $\boxed{\text{NOT}}$ ONE-TO-ONE.

FOR DETAIL SEE SLIDES $\boxed{7}$ AND $\boxed{9}$.

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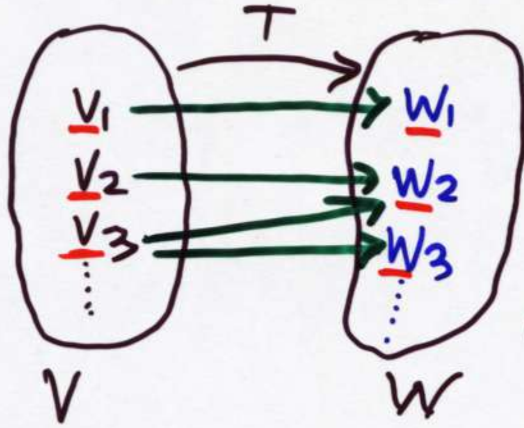
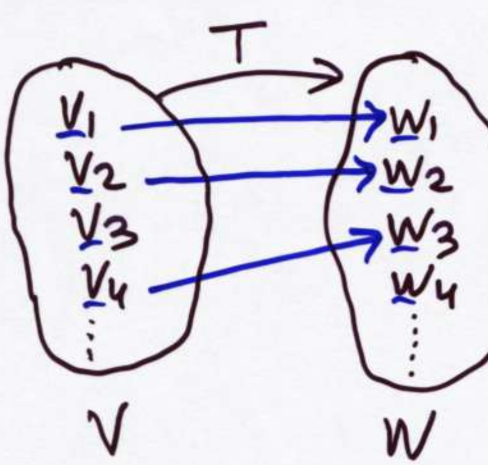
NOTE: IN A MAPPING EVERY ELEMENT HAS ONLY ONE IMAGE, SEE BELOW



T IS NOT A MAPPING $\because T(v_3) = w_2$ AND $T(v_3) = w_3$

v_3 HAS TWO IMAGES.

IN ADDITION EVERY ELEMENT IN DOMAIN MUST HAVE AN IMAGE.



T IS NOT A MAPPING $\because$ v_3 HAS NO IMAGE.

INVERSE LINEAR TRANSFORMATION

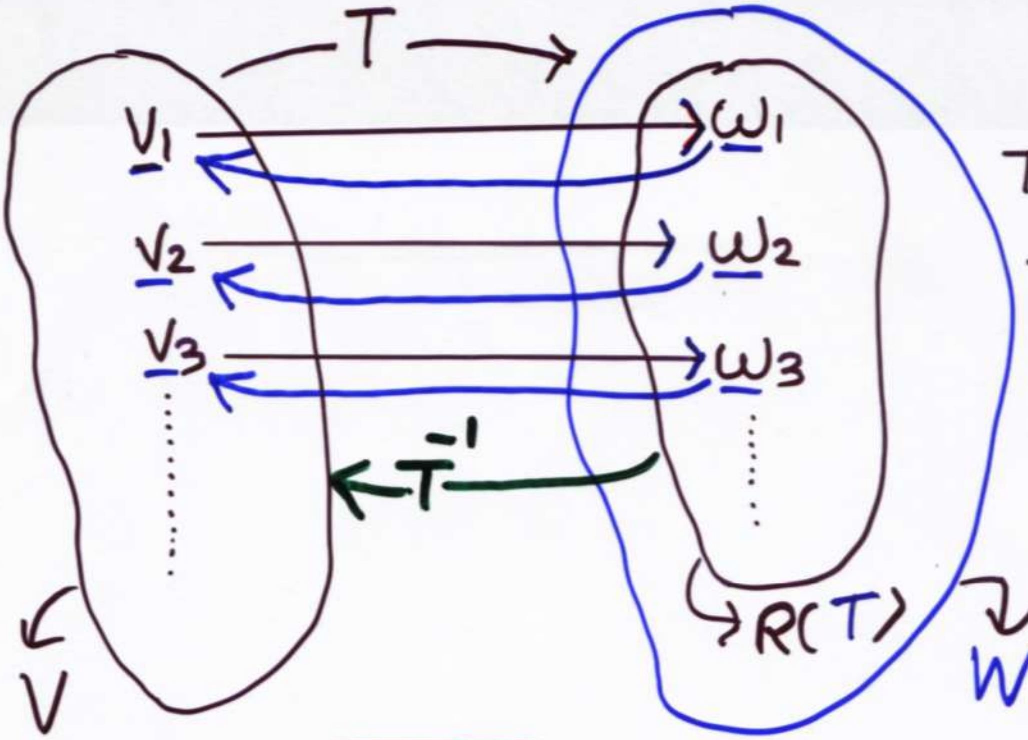
8TH ED. P.385
P.402 7TH ED.

IF $T: V \to W$ IS LINEAR AND ONE-TO-ONE THEN THE INVERSE LINEAR TRANSFORMATION IS GIVEN BY

$T^{-1}: R(T) \longrightarrow V$ WHICH MAPS

$w \in R(T)$ BACK INTO $v \in V$.

SEE THE FOLLOWING FIGURE



$T(\underline{v}_1)=\underline{w}_1$
$T^{-1}(\underline{w}_1)=\underline{v}_1$
ETC.

$\because$ T IS ONE-TO-ONE $\therefore$ EACH VECTOR IN R(T) IS THE IMAGE OF A UNIQUE VECTOR IN V. R(T) MAY OR MAY NOT BE ALL OF W. FOR MORE DETAIL SEE Q.no.7 ASSIGNMENT 6(b).

RESULT: IF $T: R^n \to R^n$ IS MULTIPLICATION BY AN INVERTIBLE MATRIX A THEN THE INVERSE $T^{-1}: R^n \to R^n$ IS MULTIPLICATION BY A^{-1}. (T IS LINEAR, T^{-1} IS INVERSE LINEAR)

EXAMPLE: $T: R^2 \to R^2$ BE THE LINEAR OPERATOR THAT ROTATES EACH VECTOR IN R^2 THROUGH AN ANGLE θ GIVEN BY

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}, \text{ ALSO}$$

$T^{-1}: R^2 \to R^2$ IS $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

THAT ROTATES EACH VECTOR THROUGH AN ANGLE $-\theta$.