MATH 205 LECTURE 17 LINEAR ALGEBRA

RESULTS:

(1) IF $\underline{V_1}$ IS A NONZERO VECTOR THEN $\{\underline{V_1}\}$ IS ALWAYS INDEPENDENT.

$\because k\underline{V_1} = \underline{0} \Rightarrow k=0 \quad \because \underline{V_1} \neq \underline{0}$

HERE $\{\underline{V_1}\}$ IS A SET CONTAINING ONLY $\boxed{\text{ONE}}$ NONZERO VECTOR.

(2) IF $\underline{0}$ IS A ZERO VECTOR THEN $\{\underline{0}\}$ IS ALWAYS DEPENDENT $\because k\underline{0} = \underline{0}$ IS SATISFIED BY INFINITE $\boxed{\text{NONZERO}}$ VALUES OF $\boxed{k}$.

DEFINITION: (P.247 8TH ED.) (P.259 7TH ED.) THE SOLUTION SPACE OF THE HOMOGENEOUS SYSTEM OF EQUATIONS $A\underline{X} = \underline{0}$, WHICH IS A SUBSPACE OF R^n, IS CALLED THE NULLSPACE OF $\boxed{A}$.

EXAMPLE: (DONE LAST TIME) (SLIDE 10 LEC. 16)

NULLSPACE OF

$$A = \begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$ IS GIVEN

BY $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$, t IS ANY REAL NUMBER

DEFINITION: (P. 261 8TH ED. P. 273 7TH ED.) THE DIMENSION OF THE NULLSPACE OF $\boxed{A}$ IS CALLED THE NULLITY OF $\boxed{A}$ AND IS DENOTED BY NULLITY(A).

EXAMPLE: IN THE ABOVE EXAMPLE $\boxed{\text{NULLITY}(A) = 1}$ $\because$ THIS IS NUMBER OF ELEMENTS IN THE $\boxed{\text{BASIS}}$ OF NULLSPACE OF $\boxed{A}$ i.e. NUMBER OF ELEMENTS IN $\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$ WHICH IS ONE.

$\hookrightarrow$ IN SIMPLE WORDS THIS IS THE BASIS FOR THE SOLUTION SPACE OF $A\underline{x} = \underline{0}$.

3) <u>DEFINITION:-</u> FOR AN $m \times n$ MATRIX

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \text{ THE}$$

VECTORS $\boxed{(\text{in } R^n)}$

$$\underline{r}_1 = (a_{11}, a_{12}, \ldots, a_{1n})$$
$$\underline{r}_2 = (a_{21}, a_{22}, \ldots, a_{2n})$$
$$\vdots \qquad\qquad \vdots$$
$$\underline{r}_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$$

FORMED FROM THE $\boxed{\text{ROWS}}$ OF $\boxed{A}$ ARE CALLED THE <u>ROW VECTORS</u> OF $\boxed{A}$, AND THE VECTORS

$$\underline{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \underline{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \underline{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

(IN R^m) FORMED FROM THE <u>COLUMNS</u> OF $\boxed{A}$ ARE CALLED THE <u>COLUMN VECTORS</u> OF $\boxed{A}$.

DEF. P.259 → P.259 (7th ED.)

4 IF A IS AN $m \times n$ MATRIX, THEN THE SUBSPACE OF R^n SPANNED BY THE ROW VECTORS OF A IS CALLED THE ROW SPACE OF A, AND THE SUBSPACE OF R^m SPANNED BY THE COLUMN VECTORS IS CALLED THE COLUMN SPACE OF A. (P.247 8TH ED.)

DEF. THE DIMENSION OF THE ROW SPACE OR COLUMN SPACE OF A MATRIX A IS CALLED THE RANK OF A AND IS DENOTED BY RANK (A). {SEE P.261 8th ED. OR P.273 7th ED.}

P.261 8TH ED.

EXAMPLE: CONSIDER THE FOLLOWING MATRIX

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 5 & -1 \\ 7 & -1 & 5 & 8 \end{bmatrix}$$

(a) FIND THE ROW (RANK) OF A?

RANK

ROW (RANK) → DIMENSION OF ROW SPACE OR FIND

$\boxed{\text{NUMBER}}$ OF LINEARLY INDEP-

5 ENDENT ROW VECTORS. SO

LET $k_1 \underline{r}_1 + k_2 \underline{r}_2 + k_3 \underline{r}_3 = \underline{0}$

$\Rightarrow k_1(2,-1,0,3) +$

$k_2(1,2,5,-1) +$

$k_3(7,-1,5,8) = (0,0,0,0)$

BUT LAST TIME IT WAS PROVED THAT $\boxed{k_1 = 3,\ k_2 = 1,\ k_3 = -1}$

$\therefore$ 3 ROW VECTORS ARE LINEARLY DEPENDENT $\therefore$ ROW(RANK) $\neq 3$.

NOW IGNORE $(7,-1,5,8)$, WHICH IS A LINEAR COMBINATION OF THE FIRST TWO.

CONSIDER FIRST TWO ROW VECTORS S.t.

$a_1 \underline{r}_1 + a_2 \underline{r}_2 = \underline{0}$

$\Rightarrow a_1(2,-1,0,3) + a_2(1,2,5,-1) = (0,0,0,0) \Longrightarrow 2a_1 + a_2 = 0,$

$-a_1 + 2a_2 = 0,\ 5a_2 = 0,\ 3a_1 - a_2 = 0$

$\Longrightarrow \boxed{a_1 = a_2 = 0}$ $\therefore$ FIRST TWO

ARE LINEARLY INDEPENDENT $\therefore$

6 ROW(RANK) = 2 OF THE GIVEN MATRIX OR (2, −1, 0, 3) AND (1, 2, 5, −1) ARE LINEARLY INDEPENDENT SINCE NONE OF THEM IS A MULTIPLE OF THE OTHER.

NOTE: ① ROW SPACE OF A IS SPANNED BY $\{(2,-1,0,3), (1,2,5,-1)\}$ WHICH IS A LINEARLY INDEPENDENT SET AND HENCE BASIS FOR THE ROW SPACE

$\therefore$ ROW(RANK) = RANK(A) = 2 = DIMENSION OF ROW SPACE OF A = NO. OF LINEARLY INDEPENDENT ROW VECTORS OF A.

② SIMILARLY WE CAN PROVE THAT COLUMN SPACE OF A IS SPANNED BY $\rightarrow \left\{ \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}$ WHICH IS A LINEARLY INDEPENDENT SET OF VECTORS AND $\begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$

HENCE BASIS FOR THE

COLUMN SPACE

$\therefore$ RANK(A) = COLUMN(RANK)

= 2 = DIMENSION OF COLUMN SPACE OF $\boxed{A}$ = NO. OF LINEARLY INDEPENDENT COLUMN VECTORS OF $\boxed{A}$.

ASSIGNMENT NO. 4

Q.no.1

(a)(i) FIND THE DISTANCE BETWEEN A POINT AND A LINE.

HINT: SEE P.135 (8TH ED.) OR P. 138 (7TH ED.)

(ii) EXAMPLE 7, P. 208 8TH ED. OR P. 219 7TH ED.

(iii) THEOREM 5.1.1, P. 208 8TH ED. OR, P. 220 7TH ED.

(iv) EXAMPLE 3, P. 212 8TH ED. OR P. 223 7TH ED.

(V) FIND THE DIMENSION OF ZERO VECTOR SPACE

(vi) FIND $-\underline{u}$ AND $\underline{0}$ FOR $\underline{u} = (x, y)$ IF $(x,y)+(x',y')=(x+x'+1, y+y'+1)$

8)

(b) ARE THE FOLLOWING TRUE OR FALSE?

(I) IF $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ ARE VECTORS IN A VECTOR SPACE $\boxed{V}$ THEN

(a) THE SET $\boxed{W}$ OF ALL LINEAR COMBINATIONS OF $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ IS A SUBSPACE OF V.

(b) $\boxed{W}$ IS THE SMALLEST SUBSPACE OF $\boxed{V}$ THAT CONTAINS $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ IN THE SENSE THAT EVERY OTHER SUBSPACE OF $\boxed{V}$ THAT CONTAINS $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ MUST CONTAIN $\boxed{W}$.

(II) ZERO VECTOR SPACE HAS DIMENSION ZERO.

(III) IF V IS AN n-DIMENSIONAL VECTOR SPACE AND $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r\}$ IS A LINEARLY INDEPENDENT SET OF VECTORS IN $\boxed{V}$ AND IF $\boxed{r < n}$, THEN $\boxed{S}$ CAN BE ENLARGED TO A BASIS

FOR V, THAT IS, THERE ARE VECTORS $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ SUCH THAT $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ IS A BASIS FOR V.

Q.no.2

(a) PROVE PART (I) OF Q.no.1 (b)

HINT: THIS IS THEOREM 5.2.3. (P.217) (6th ED.) OR THEOREM 5.2.3 (P.227) (7th ED.)

(b) PROVE THAT A SET OF VECTORS IS LINEARLY DEPENDENT IF AND ONLY IF AT LEAST ONE OF THE VECTOR IN S IS EXPRESSIBLE AS A LINEAR COMBINATION OF THE OTHER VECTORS IN S.

HINT: SEE THEOREM 5.3.1. (a) (P.224), 6th ED. OR THEOREM 5.3.1. PART (a) (P.234) (7th ED.)

(c) PROVE THAT IF $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ BE A SET OF VECTORS IN R^n. IF $\boxed{r > n}$, THEN S IS LINEARLY DEPENDENT. **HINT:** SEE P.237 TH.5.3.3 (7th ED.) OR TH.5.3.3 (P.226) (6th ED.)

Q.NO.3 ✓

(a) IF $\boxed{A}$ IS AN $\boxed{m \times n}$ MATRIX, THEN PROVE THAT

(i) ROW SPACE OF $\boxed{A}$ IS A SUBSPACE OF $\boxed{R^n}$ AND

(ii) COLUMN SPACE OF $\boxed{A}$ IS A SUBSPACE OF $\boxed{R^m}$.

(b) DO ELEMENTARY $\boxed{\text{ROW}}$ OPERATIONS CHANGE THE ROW SPACE OF A MATRIX?

✓ Q.no.4 → BASIS

DO QUESTION NO. 3(a,d), 4(a,c) 5, 6, 7(c), 8(a), 9(a), 10 (P.244) (8th ED.) OR (P.256) (7th ED.)

↳P.244

↳8th

Q.no.5 ✓

FIND A BASIS FOR M_{22} WHICH CONTAINS

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

<u>HINTS:</u> (1) START WITH THE FOLLOWING SET AFTER INCLUDING THE STANDARD BASIS

$$\left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

(2) CHECK IF

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ IS A LINEAR COMBINATION OF FIRST 5, IF YES THEN IGNORE IT.

(3) THEN CHECK IF $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ AS A LINEAR COMBINATION OF FIRST 4, IF YES THEN IGNORE IT.

(4) JUST CHECK REST 4 AS LINEARLY INDEPENDENT, IF YES THEN THEY FORM A BASIS BECAUSE M_{22} IS A 4 DIM. SPACE.

$\boxed{\text{Q.no. 6}}$ ✓

Q. 9(a), P.220 8TH ED. OR
Q. 9(a), P. 231 7TH ED.

$\boxed{\text{Q.no.7}}$ ✓

Q.18, P.244 (8TH ED.) OR
Q.18, P.256 (7TH ED.)