MATH 205

LECTURE 19 LINEAR ALGEBRA

REVISION:

IF $\underline{u}$ AND $\underline{v}$ ARE VECTORS IN 2-SPACE OR 3-SPACE AND $\boxed{\theta}$ IS THE ANGLE BETWEEN $\underline{u}$ AND $\underline{v}$, THEN THE DOT PRODUCT OR EUCLIDEAN INNER PRODUCT $\underline{u}\cdot\underline{v}$ IS DEFINED BY

$$\underline{u}\cdot\underline{v} = \begin{cases} \|\underline{u}\|\,\|\underline{v}\|\cos\theta & \text{IF } \underline{u}\neq\underline{0} \text{ AND } \underline{v}\neq\underline{0} \\ 0 & \text{IF } \underline{u}=\underline{0} \text{ OR } \underline{v}=\underline{0} \end{cases}$$

WHICH ALSO SATISFIES THE FOLLOWING PROPERTIES

(i) $\underline{u}\cdot\underline{v} = \underline{v}\cdot\underline{u}$

(ii) $\underline{u}\cdot(\underline{v}+\underline{w}) = \underline{u}\cdot\underline{v} + \underline{u}\cdot\underline{w}$

(iii) $k(\underline{u}\cdot\underline{v}) = (k\underline{u})\cdot\underline{v} = \underline{u}\cdot(k\underline{v})$

WHERE k IS ANY $\boxed{\text{SCALAR}}$

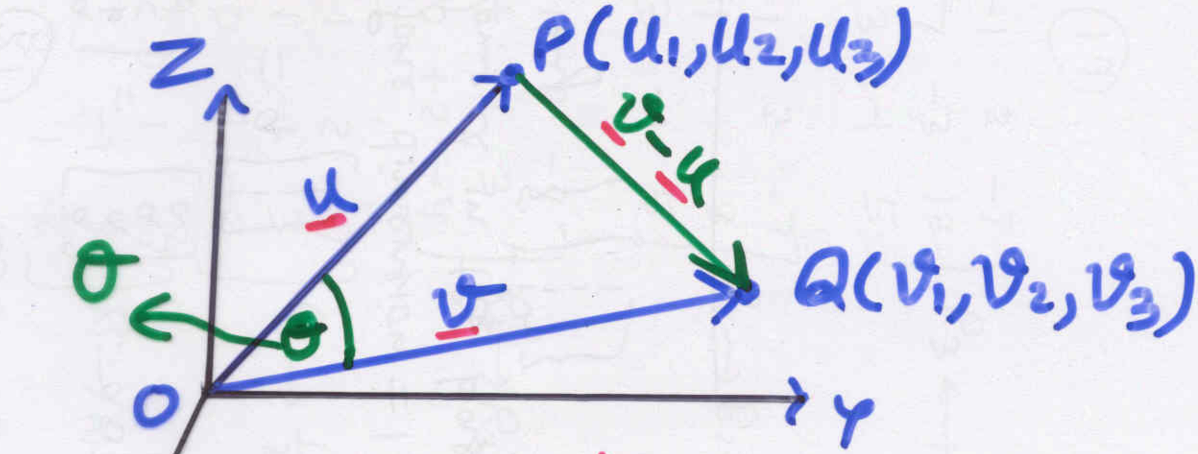
(iv) $\underline{v}\cdot\underline{v} > 0$ IF $\underline{v}\neq\underline{0}$ AND

$\underline{v}\cdot\underline{v} = 0$ IF $\underline{v}=\underline{0}$

NOTE: CHECK (ii), (iii), (iv) BY

2 TAKING $\underline{u} = (u_1, u_2, u_3)$, $\underline{v} = (v_1, v_2, v_3)$ AND $\underline{w} = (w_1, w_2, w_3)$, ALSO

$$\boxed{\underline{v} \cdot \underline{v} = \|\underline{v}\|^2 > 0}, \text{ IF } \underline{v} \neq \underline{0}$$



$$\underline{v} = \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$$

$$\underline{v} = \underline{u} + \overrightarrow{PQ}$$

$$\overrightarrow{PQ} = \underline{v} - \underline{u}$$

$$= (v_1 - u_1, v_2 - u_2, v_3 - u_3)$$

$d(\underline{u}, \underline{v})$ = THE DISTANCE BETWEEN TWO POINTS (VECTORS) $\underline{u}$ AND $\underline{v}$ IS DEFINED BY

$$\|\underline{u} - \underline{v}\| = \|\underline{v} - \underline{u}\|$$

$$= \left[(\underline{v} - \underline{u}) \cdot (\underline{v} - \underline{u})\right]^{\frac{1}{2}}$$

$$= \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}$$

DEFINITION: (P. 275 8th ED.
P. 287 7th ED.)

AN INNER PRODUCT ON A REAL VECTOR SPACE V IS A REAL NUMBER $\langle u, v \rangle$ WHICH SATISFIES THE FOLLOWING AXIOMS FOR ALL VECTORS u, v AND $w \in V$ AND ALL SCALARS k

	EXAMPLE
(1) $\langle u, v \rangle = \langle v, u \rangle$	$u \cdot v = v \cdot u$
(2) $\langle u + v, w \rangle = \langle w, u + v \rangle$ $= \langle u, w \rangle + \langle v, w \rangle$ $= \langle w, u \rangle + \langle w, v \rangle$	$u \cdot (v + w)$ $= u \cdot v + u \cdot w$
(3) $\langle k u, v \rangle = k \langle u, v \rangle$	$k(u \cdot v)$ $= (k u) \cdot v$
(4) $\langle v, v \rangle \geq 0$ WHERE $\langle v, v \rangle = 0$ IF AND ONLY IF $v = 0$	$v \cdot v > 0, v \neq 0$ $v \cdot v = 0$ IF AND ONLY IF $v = 0$

DEFINITION:

A REAL VECTOR SPACE WITH AN INNER PRODUCT IS CALLED A REAL INNER PRODUCT SPACE.

EXAMPLE:

IF $\underline{u} = (u_1, u_2, u_3)$ AND $\underline{v} = (v_1, v_2, v_3)$ ARE VECTORS IN R^3, THEN THE FORMULA

$$\boxed{\langle \underline{u}, \underline{v} \rangle = \underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3}$$

DEFINES $\langle \underline{u}, \underline{v} \rangle$ TO BE THE INNER PRODUCT ON R^3. (PROVED ALREADY)

DEFINITION: P. 278 (6th ED.) OR P. 290 (7th ED.)

IF $\boxed{V}$ IS AN INNER PRODUCT SPACE, THEN THE NORM (OR LENGTH) OF A VECTOR $\underline{u}$ IN $\boxed{V}$ IS DENOTED BY $\|\underline{u}\|$ AND IS DEFINED BY

$$\boxed{\|\underline{u}\| = \langle \underline{u}, \underline{u} \rangle^{\frac{1}{2}}}$$

ANOTHER NOTATION OF EUCLIDEAN INNER PRODUCT:

FOR $\underline{u}, \underline{v} \in R^3$

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$\langle \underline{u}, \underline{v} \rangle = \underline{u} \cdot \underline{v} = \underline{V}^t \underline{u} \quad \text{WHY?}$$

$$\because \underline{V}^t \underline{u} = [V_1 \ V_2 \ V_3] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= u_1 V_1 + u_2 V_2 + u_3 V_3 = \underline{u} \cdot \underline{V}$$

$\underline{v}^t \longrightarrow$ TRANSPOSE OF $\underline{V}$

TRY THE FOLLOWING:

LET $f = f(x)$, $g = g(x)$ BE TWO FUNCTIONS (CONTINUOUS) THEN CHECK WHETHER

$$\boxed{\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx}$$ DEFINES AN INNER PRODUCT ON $C[a,b]$

HINT: CHECK (1) $\langle f, g \rangle = \langle g, f \rangle$

(2) $\langle f+g, s \rangle = \langle f, s \rangle + \langle g, s \rangle$

(3) $\langle kf, g \rangle = k \langle f, g \rangle$

(4) $\langle f, f \rangle \geq 0$ AND $\langle f, f \rangle = 0$ IF AND ONLY IF $f = 0$.

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SOLUTION:

LET $\boxed{f(x)}$ AND $\boxed{g(x)}$ BE $\boxed{\text{TWO}}$ FUNCTIONS SUCH THAT $f, g \in C[a,b]$, $\{f = f(x), g = g(x)\}$

$C[a,b] \longrightarrow$ ALL CONTINUOUS FUNCTIONS DEFINED ON THE INTERVAL $[a,b]$, CONSIDER

(1) $$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx$$

$$= \int_a^b g(x) f(x)\, dx = \langle g, f \rangle$$

(2) $$\langle f+g, s \rangle = \int_a^b [f(x)+g(x)]\, s(x)\, dx$$

$$= \int_a^b f(x) s(x)\, dx + \int_a^b g(x) s(x)\, dx$$

$$= \langle f, s \rangle + \langle g, s \rangle$$

(3) $$\boxed{\langle kf, g \rangle} = \int_a^b k f(x) g(x)\, dx$$

$$= k \int_a^b f(x) g(x)\, dx = \boxed{k \langle f, g \rangle}$$

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(4) $\langle f, f \rangle = \int_a^b f(x)f(x)\,dx$

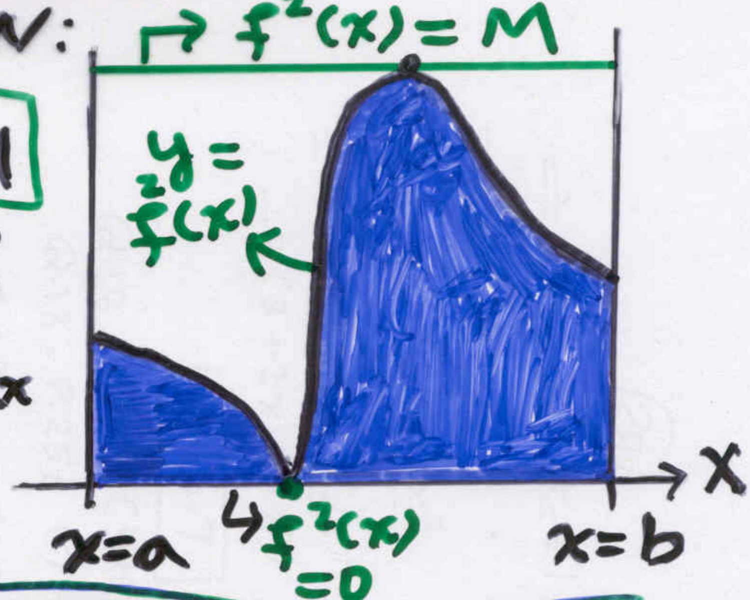
$= \int_a^b f^2(x)\,dx$, $\because f \in C[a,b]$

$\Rightarrow \boxed{f^2 \in C[a,b]}$

$\therefore f^2$ IS BOUNDED , $\because f^2(x) \geq 0$

$\Rightarrow \boxed{\min.\ f^2(x) = 0}$ AND

$\boxed{\max.\ f^2(x) = M}$ (SAY) AS SHOWN BELOW:



$\Rightarrow \boxed{0 \leq f^2(x) \leq M}$

INTEGRATING

$\int_a^b 0\,dx \leq \int_a^b f^2(x)\,dx$

$\leq \int_a^b M\,dx$

$\Rightarrow \boxed{0 \leq \int_a^b f^2(x)\,dx \leq M(b-a)}$

$\because \int_a^b 0\,dx = c\Big|_a^b = c - c = 0$

NOTICE THAT

(1) $\int_a^b 0\,dx$ IS THE AREA BETWEEN X-AXIS AND X-AXIS AND $= 0$,

(2) $\int_a^b f^2(x)\,dx$ IS THE AREA BETWEEN $y=f^2(x)$ AND X-AXIS (SHOWN BY SHADED PORTION)

(3) $\int_a^b M\,dx = M(b-a)$ IS THE AREA BETWEEN $f^2(x)=M$ AND X-AXIS ON $[a,b]$, FURTHER

$M>0,\ b-a>0 \Rightarrow M(b-a)>0$

$\therefore \langle f,f\rangle \geq 0$, $\langle f,f\rangle = 0$ (IF AND ONLY IF) $\boxed{f=0}$.

$$\therefore \langle f,g\rangle = \int_a^b f(x)g(x)\,dx$$

DEFINES AN INNER PRODUCT ON $C[a,b]$.

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DEFINITION: IF $\boxed{V}$ IS AN INNER PRODUCT SPACE AND $\underline{u}, \underline{v} \in \boxed{V}$ THEN $\underline{u}$ AND $\underline{v}$ ARE CALLED ORTHOGONAL VECTORS IF $\langle \underline{u}, \underline{v} \rangle = 0$, FURTHER IF $\|\underline{u}\| = \|\underline{v}\| = 1$ THEN $\underline{u}$ AND $\underline{v}$ ARE CALLED ORTHONORMAL VECTORS.

TRY THE FOLLOWING:

IF $f(x) = \frac{1}{\sqrt{2}}$ AND $g(x) = \sqrt{\frac{3}{2}}\, x$

(a) THEN ACCORDING TO THE INNER PRODUCT DEFINED IN THE LAST EXAMPLE

$$\boxed{\langle f, g \rangle = \int_a^b f(x) g(x)\, dx}$$

$f(x)$ AND $g(x)$ ARE ORTHOGONAL ON $[-1, 1]$ AS WELL AS

(b) ORTHONORMAL ON $[-1, 1]$

IMPORTANT NOTES:

① FOR $\boxed{\langle f, g \rangle = \int_a^b f(x)g(x)\,dx}$

$$\langle f, f \rangle^{\frac{1}{2}} = \| f \| = \sqrt{\int_a^b f^2(x)\,dx}$$

② IF $\underline{u}$ AND $\underline{v}$ ARE $\boxed{\text{VECTORS}}$ FROM $\boxed{R^2}$ OR $\boxed{R^3}$ AND $\langle \underline{u}, \underline{v} \rangle = \boxed{\underline{u} \cdot \underline{v} = 0}$ THEN $\underline{u}, \underline{v}$ ARE ORTHOGONAL AS WELL AS PERPENDICULAR.

③ IF $\boxed{V}$ IS ANY INNER PRODUCT SPACE AND $\boxed{\langle \underline{u}, \underline{v} \rangle \neq \underline{u} \cdot \underline{v}}$ THEN $\langle \underline{u}, \underline{v} \rangle = 0$ MEANS THAT $\underline{u}, \underline{v}$ ARE ORTHOGONAL BUT NOT PERPENDICULAR.

EXAMPLE: SEE $\boxed{\text{SLIDE NO. 9}}$

$\boxed{f(x) = \frac{1}{\sqrt{2}}}$ AND $\boxed{g(x) = \sqrt{\frac{3}{2}}\,x}$ ARE ORTHOGONAL BUT $\boxed{\text{NOT}}$ PERPENDICULAR.