REVISION

(a) FOR $A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 5 & -1 \\ 7 & -1 & 5 & 8 \end{bmatrix}$

RANK (A) = 2, WHERE RANK (A) MEANS DIMENSION OF COLUMN SPACE OR ROW SPACE OF $\boxed{A}$,

OR

$\boxed{\text{NO.}}$ OF ELEMENTS IN THE BASES OF ROW SPACE $\boxed{\text{OR}}$ COLUMN SPACE. RECALL THAT BASIS OF ROW SPACE OF A

$= \{(2,-1,0,3),(1,2,5,-1)\}$,

WHERE $(2,-1,0,3)$, $(1,2,5,-1)$ ARE TWO LINEARLY INDEPENDENT (ROW) VECTORS. ALSO BASIS OF COLUMN SPACE OF $\boxed{A}$

$= \left\{ \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}$, CONSISTING

OF TWO LINEARLY INDEPENDENT COLUMN VECTORS.

2 (b) FIND NULLITY (A)

SOLUTION: FOR THIS WE HAVE TO SOLVE $A\underline{x} = \underline{0}$

$$\Rightarrow \underbrace{\begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 5 & -1 \\ 7 & -1 & 5 & 8 \end{bmatrix}}_{3\times 4} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{4\times 1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{3\times 1} \quad (1)$$

AUGMENTED MATRIX IS GIVEN BY

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 3 & 0 \\ 1 & 2 & 5 & -1 & 0 \\ 7 & -1 & 5 & 8 & 0 \end{array}\right]$$

NOW REDUCE THIS TO REDUCED ECHELON FORM

$$\sim \begin{bmatrix} 1 & 2 & 5 & -1 & 0 \\ 2 & -1 & 0 & 3 & 0 \\ 7 & -1 & 5 & 8 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

NOTE: IN (1) NO. OF COLUMNS = NO. OF UNKNOWNS

$\boxed{3}$ $\sim \begin{bmatrix} 1 & 2 & 5 & -1 & 0 \\ 0 & -5 & -10 & 5 & 0 \\ 0 & -15 & -30 & 15 & 0 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 7R_1$

$\sim \begin{bmatrix} 1 & 2 & 5 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{bmatrix}$ $R_2 \to -\frac{1}{5} R_2$, $R_3 \to -\frac{1}{15} R_3$

$\sim \begin{bmatrix} 1 & 2 & 5 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - R_2$

$\sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $R_1 \to R_1 - 2R_2$

REDUCED ECHELON FORM

$\Rightarrow \begin{cases} x_1 + x_3 + x_4 = 0 & — (2) \\ x_2 + 2x_3 - x_4 = 0 & — (3) \end{cases}$

→ $\boxed{2}$ EQUATIONS AND $\boxed{4}$ UNKNOWNS.

LET $x_3 = t$, $x_4 = s$

$\Rightarrow x_1 = -t - s$ from (2)

(3) $\Rightarrow x_2 = -2x_3 + x_4 = x_4 - 2x_3 = s - 2t$

THEREFORE THE COMPLETE SOLUTION IS GIVEN BY

4

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t-s \\ -2t+s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ BASIS FOR NULLSPACE OF

$A = \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\therefore$ $\boxed{\text{NULLITY}}$ OF $\boxed{A = 2}$

= NUMBER OF FREE VARIABLES = DIMENSION OF NULLSPACE OF $\boxed{A}$.

$\boxed{x_3 = t,\ x_4 = s} \rightarrow$ FREE VARIABLES

$x_1 = -t - s$

$x_2 = -2t + s$

LEADING VARIABLES $\rightarrow$ = RANK(A) = 2

CORRESPOND TO LEADING 1's IN ECHELON FORM.

REDUCED

(DIMENSION THEOREM FOR MATRICES)

P. 262 (6th ED.) OR P. 275 (7th ED.)

IF $\boxed{A}$ HAS $\boxed{n}$ COLUMNS THEN

$\boxed{\text{RANK}(A) + \text{NULLITY}(A) = n}$

$n \rightarrow$ NO. OF COLUMNS = TOTAL NUMBER OF VARIABLES

= LEADING + FREE

= RANK(A) + NULLITY(A) = n

RESULT: (P.18, 8th ED.)//(P.19, 7TH ED)

IN $A\underline{X}=\underline{0}$ → m → EQUATIONS IN $\boxed{n}$ → UNKNOWNS WITH $\boxed{m < n}$, AND IF THERE ARE $\boxed{r}$ NONZERO ROWS IN THE REDUCED ROW-ECHELON FORM OF THE AUGMENTED MATRIX THEN NUMBER OF $\boxed{\text{FREE}}$ VARIABLES ARE $= \boxed{n - r.}$ LEADING VARIABLES $= r$.

EXAMPLE: WE PROVED THAT FOR THE FOLLOWING MATRIX

$$\begin{bmatrix} 2 & -1 & 0 & 3 & 0 \\ 1 & 2 & 5 & -1 & 0 \\ 7 & -1 & 5 & 8 & 0 \end{bmatrix}$$

REDUCED ROW-ECHELON FORM IS

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

WHICH CONTAINS TWO NONZERO ROWS $\Rightarrow \boxed{r = 2}$

HERE $m = 3 < 4 = n$

$\therefore \quad n - r = 4 - 2 = 2$, THEREFORE

6)

NUMBER OF FREE VARIABLES

$= \boxed{n-r} = 4-2=2.$

AS WE SAW THAT

$x_3 = t$ $\boxed{x_3=t}$ AND $\boxed{x_4=s}$ WERE FREE VARIABLES AND THEIR NO. = 2

ALSO $\boxed{x_1}$ AND $\boxed{x_2}$ WERE LEADING VARIABLES GIVEN BY

$$\boxed{x_1 = -t-s}$$
$$\boxed{x_2 = s-2t}$$

AND ARE $= \boxed{r} = 2 =$ RANK

WHERE $\boxed{r}$ INDICATES NONZERO ROWS IN THE REDUCED ROW-ECHELON FORM OF THE AUGMENTED MATRIX OF $\boxed{A\underline{X} = \underline{0}}$.

ANOTHER METHOD TO FIND THE BASIS FOR THE ROW SPACE OF A MATRIX.

IN ORDER TO DO THIS WE REFER TO THE FOLLOWING THEOREM: (5.5.6 P.252 6th ED.) OR (5.5.6 P.265 7th ED.)

(5.5.6 P.252 8TH ED.)

THE NONZERO ROW VECTORS IN ANY ROW-ECHELON FORM OF A MATRIX FORM A BASIS FOR THE ROW SPACE OF THAT MATRIX.

EXAMPLE: ∵ THE REDUCED ROW ECHELON FORM OF

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 5 & -1 \\ 7 & -1 & 5 & 8 \end{bmatrix}$$ IS GIVEN BY

$$R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$ ∴ ACCORDING TO THE ABOVE THEOREM THE TWO NONZERO ROW VECTORS IN $\boxed{R}$ FORM THE BASIS FOR THE ROW SPACE OF $\boxed{A}$ AND $\boxed{R}$, THEREFORE THE BASIS FOR THE ROW

SPACE OF $\boxed{R}$ AND $\boxed{A}$ IS

8 GIVEN BY

$$\{(1,0,1,1),\ (0,1,2,-1)\}$$

BECAUSE ACCORDING TO THEOREM 5.5.4 (P.251 8th ED.)
OR THEOREM (5.5.4) (P.263 7th ED.)
ELEMENTARY ROW OPERATIONS DO NOT CHANGE THE ROW SPACE OF A MATRIX, BECAUSE

IF $\boxed{A}$ IS AN $m \times n$ MATRIX, AND $\boxed{B}$ IS ROW EQUIVALENT TO A → $\boxed{A}$

THEN (i) ROW SPACE IS A SUBSPACE OF $\boxed{R^n}$

(ii) IF THE ROW OPERATION IS A ROW INTERCHANGE, THEN $\boxed{B}$ AND $\boxed{A}$ HAVE SAME ROW VECTORS

(iii) IF THE ROW OPERATION IS MULTIPLICATION OF A ROW BY A NONZERO SCALAR $\boxed{\text{OR}}$ THE ADDITION OF A MULTIPLE OF ONE ROW TO ANOTHER, THEN THE ROW VECTORS

$r_1', r_2', \ldots, r_m'$ OF $\boxed{B}$ ARE LINEAR COMBINATIONS OF $r_1, r_2, \ldots, r_m$; THUS, THEY LIE IN THE ROW SPACE OF $\boxed{A}$, $\because$ ROW SPACE (SUBSPACE OF R^n) IS CLOSED UNDER ADDITION AND SCALAR MULTIPLICATION.

METHOD TO FIND THE BASIS FOR THE COLUMN SPACE OF AN $\boxed{m \times n}$ MATRIX $\boxed{A}$.

FOR THIS WE REFER TO TH. 5.5.6 (P. 252 8th ED.) OR (TH. 5.5.6. P. 265 7th ED.) WHICH STATES THAT

IF $\boxed{A}$ IS AN $\boxed{m \times n}$ MATRIX AND $\boxed{R}$ IS ITS ROW-ECHELON FORM THEN THE COLUMN VECTORS WITH THE LEADING $\boxed{1\text{'s}}$ OF THE ROW VECTORS FORM A BASIS FOR THE COLUMN SPACE OF $\boxed{R}$, AND THE CORRESPONDINE COLUMN VECTORS IN $\boxed{A}$ FORM THE BASIS

FOR THE COLUMN SPACE OF A.

10 EXAMPLE:

FOR $A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 5 & -1 \\ 7 & -1 & 5 & 8 \end{bmatrix}$

THE REDUCED ROW ECHELON FORM IS GIVEN BY

$R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$c_1 + 2c_2 = c_3$

$c_1 - c_2 = c_4$

→ THE FIRST TWO COLUMN VECTORS IN R WHICH CONTAIN THE LEADING 1'S FORM THE BASIS FOR THE COLUMN SPACE OF R (BUT NOT A) AND THE CORRESPONDING COLUMN VECTORS IN A FORM THE BASIS FOR THE COLUMN SPACE OF A (BUT NOT R).

BECAUSE ELEMENTARY ROW OPERATIONS USUALLY CHANGE THE COLUMN SPACE.

11

REMARKS:

① BASIS FOR THE COLUMN SPACE FOR $\boxed{R} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ AND

② BASIS FOR THE COLUMN SPACE FOR $\boxed{A} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\}$

③ ELEMENTARY $\boxed{\text{ROW}}$ OPERATIONS CAN CHANGE THE COLUMN SPACE:

LET $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, $\boxed{3C_1 = C_2}$

$\therefore$ THE COLUMN SPACE OF $\boxed{A}$ CONSISTS OF ALL SCALAR MULTIPLES OF THE FIRST COLUMN VECTOR $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

NOW $A \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \quad R_2 \rightarrow R_2 - 2R_1$

AGAIN $\longrightarrow \boxed{3C_1 = C_2}$

$\therefore$ THE COLUMN SPACE OF $B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ CONSISTS OF ALL SCALAR MULTIPLES OF THE FIRST COLUMN VECTOR i.e. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. THIS IS NOT THE SAME AS THE COLUMN SPACE OF A.

REMARK:

IF A IS A MATRIX AND R IS ITS ECHELON FORM THEN THE NUMBER OF NONZERO ROWS OR NUMBER OF THE COLUMN VECTORS THAT CONTAIN THE LEADING 1'S IN R IS THE RANK OF THE MATRIX A $\therefore$ RANK OF

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 5 & -1 \\ 7 & -1 & 5 & 8 \end{bmatrix} = 2 \quad \because$$

$$\text{IN } R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{TWO NONZERO ROWS}$$

AND NO. OF THE COLUMN VECTORS WITH LEADING 1'S IN R = 2.

ANOTHER DEFINITION OF RANK OF A MATRIX.

RANK (A) = HIGHEST ORDER OF THE NONZERO DETERMINANT OR SUBDETERMINANT OF A.

EXAMPLE:

RANK OF $A = \begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix} = ?$

$$DET(A) = 2(2+1) - 1(-1) + 7(-1)$$
$$= 6 + 1 - 7 = 7 - 7 = 0$$

$\therefore$ $\boxed{RANK(A) \neq 3}$

CONSIDER THE SUBDETERMINANT $\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 4 + 1 = 5 \neq 0$

$\therefore$ $\boxed{RANK(A) = 2}$

NOTE: LAST TIME WE PROVED THAT NULLITY (A) = 1, AND RANK (A) + NULLITY (A) = 2 + 1 = 3 = NO. OF COLUMNS ACCORDING TO THE DIMENSION THEOREM.