Second Test Cal2

Friday, March 10, 2023 3:14 PM

HABIB UNIVERSITY

Name:
HU ID:
Section:

Math 102 Test 2

Spring Semester 2023

INSTRUCTIONS:

Please show all your work wherever possible and attempt all questions. You may use a calculator, unless stated otherwise in the question. Show the work and explain your thinking wherever possible/applicable. You have **60 minutes**. Good luck!

1. Let $\vec{v} = 3\vec{i} + 2\vec{j} - 2\vec{k}$ and $\vec{w} = 4\vec{i} - 3\vec{j} + \vec{k}$. Find: [2]

   a) $\vec{v} \cdot \vec{w}$ $= (3, 2, -2) \cdot (4, -3, 1)$

$\vec{v} \cdot \vec{w} = 3 \times 4 - 2 \times 3 - 2 \times 1 = 12 - 6 - 2 = 4$

   b) The angle between vectors $\vec{v}$ and $\vec{w}$

$\theta = \cos^{-1} \dfrac{\vec{u} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|} = \cos^{-1}\left[\dfrac{4}{\sqrt{17} \times \sqrt{26}}\right] = \cos^{-1}(0.190) = 79^{0}$

2. Let $P = (0, 1, 0), Q = (-1, 1, 2), R = (2, 1, -1)$. Find: [3+2]

   a) The area of the triangle PQR

$\vec{PQ} = -\hat{i} + 2\hat{k}$, $\vec{PR} = 2\hat{i} - \hat{k}$

Area of a triagle $= \dfrac{1}{2} \|\vec{PQ} \times \vec{PR}\|$

$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 2 \\ 2 & 0 & -1 \end{vmatrix} = 3\hat{j}$

Area $= \dfrac{1}{2} \|3\hat{j}\| = \dfrac{1}{2} \sqrt{3^2} = \dfrac{3}{2}$

   b) The equation for a plane that contains P, Q, and R.

normal vector $\vec{n}$ to $\vec{PQ}$ $\vec{PR}$ is $3\hat{j}$

so, $\hat{n} = 0\hat{i} + 3\hat{j} + 0\hat{k}$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

at $(0, 1, 0)$

$3y - 3 = 0$

OR

$3y = 3$ or $y = 1$

3. a) Find the equation of the plane tangent to the graph of $f(x, y) = x^2 e^{xy}$ at (1, 0). [3]

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$f(x,y) = x^2 e^{xy}$ at $(1,0) = (a,b)$

$f_x(x,y) = 2x e^{xy} + x^2 y e^{xy}$ at $f_x(1,0) = 2$

$f_y(x,y) = x^3 e^{xy}$ at $f_y(1,0) = 1$

$z = 1 + 2(x-1) + y$ or $z = 2x + y - 1$

b) Find the differential of f at the point (1, 0). [1]

diff $f(a,b) = f_x(a,b)\,dx + f_y(a,b)\,dy = 2\,dx + dy$

4. Suppose that the height of a hill above sea level is given by $z = 1000 - 0.01\,x^2 - 0.02\,y^2$. If you are at the point (60,100) in what direction is the elevation changing fastest? What is the maximum rate of change of the elevation at this point? [3]

$\nabla f(x,y) = \nabla z$

$\nabla f(x,y) = \langle \partial z/\partial x, \partial z/\partial y \rangle$

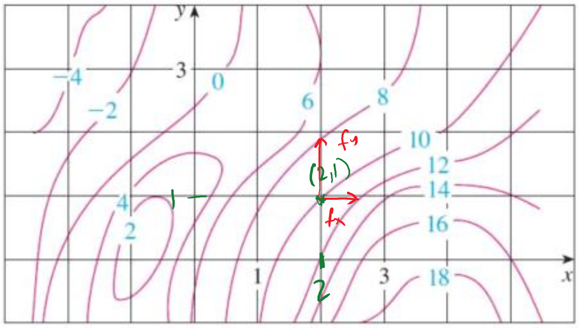
$\nabla f(x,y) = \langle -0.02x, -0.04y \rangle$

$\nabla f(60,100) = \langle -1.2, -4 \rangle$ ← Max rate of change in the direction of this vector

And Maximum rate of change

$\| \nabla f(60,100) \| = \sqrt{(-1.2)^2 + (-4)^2} = \sqrt{17.44} = 4.176$

5. Use the given contour map for a function f to estimate $f_x(2,1)$ and $f_y(2,1)$: [2]



$$f_x \approx \frac{12-10}{2.6-2} = \frac{2}{0.4} = 5$$

$$f_y \approx \frac{8-10}{1.9-1} = \frac{-2}{0.9} = -2.22$$

6. Verify whether the directional derivative of the function $f(x,y,z) = 3\,x^2y^2 + 2yz$ at (-1, 0, 4) in the direction of $\underbrace{\vec{\imath} - \vec{k}}_{\vec{v}}$ is zero or non-zero. [4]

$$\nabla f(x,y,z) = \langle 6xy^2,\ 6x^2y + 2z,\ 2y \rangle$$

$$\nabla f(-1,0,4) = \langle 0,\ 8,\ 0 \rangle \ ; \ \vec{v} = \langle 1, 0, -1 \rangle$$

Now, one can see it $\nabla f(-1,0,4) \cdot \dfrac{\vec{v}}{\|\vec{v}\|} = 0$

7. Critical points of the function $f(x,y) = 8xy - \frac{1}{4}(x+y)^4$ are (0, 0), (1 1) & (-1,-1). Classify them as local maxima, local minima, saddle points, or none of these. [5]

$$D = f_{xx} f_{yy} - f_{xy}^2$$

$$f_{xx} = -\frac{\cancel{12}^{\,3}}{\cancel{4}_{\,1}}(x+y)^2 \quad , \quad f_{yy} = -\frac{\cancel{12}^{\,3}}{\cancel{4}_{\,1}}(x+y)^2$$

$$f_{xy} = 8 - \frac{\cancel{12}^{\,3}}{\cancel{4}_{\,1}}(x+y)^2$$

$$D(x,y) = \left(-3(x+y)^2\right)\left(-3(x+y)^2\right) - \left(8 - 3(x+y)^2\right)^2$$

$$D(x,y) = \cancel{9(x+y)^4} - 64 + 48(x+y)^2 - \cancel{9(x+y)^4}$$

$$D(x,y) = 48(x+y)^2 - 64$$

At $(0,0)$

$D(0,0) = -64 < 0$ saddle

At $(1,1)$

$D(1,1) = 48(2)^2 - 64 > 0$, $f_{xx}(1,1) = -3(1+1)^2 < 0$ local max

At $(-1,-1)$

$D(-1,-1) = 48(-2)^2 - 64 > 0$, $f_{xx}(-1,-1) = -3(-1-1)^2 < 0$ local max