Name:
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Section:

HABIB UNIVERSITY

Math 102 Test 3

Spring Semester 2023

INSTRUCTIONS:
Please show all your work wherever possible and attempt all questions. You may use a calculator, unless stated otherwise in the question. Show the work and explain your thinking wherever possible/applicable. You have **60 minutes**. Good luck!

1. Given $z = u^2 - ue^v,\ u = x + 2y,\ v = 2x - y$, use the chain rule to find $\left.\dfrac{\partial z}{\partial x}\right|_{(x,y)=(1,2)}$ [2]

The chain rule gives

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = (2u - e^v)1 + (-ue^v)2.$$

At $(x, y) = (1, 2)$, we have $u = 1 + 2 \cdot 2 = 5$ and $v = 2 \cdot 1 - 2 = 0$, so

$$\left.\frac{\partial z}{\partial y}\right|_{(x,y)=(1,2)} = (2 \cdot 5 - e^0)1 - 5e^0 \cdot 2 = -1.$$

2. Arrange the followings steps for unconstrained optimization in ascending order (1 – 6) : [1.5]

__(3)__ Put $grad\ f = \vec{0}$ to find all critical points.

__(4)__ Find 2nd order partial derivatives f_{xx}, f_{yy}, f_{xy}.

__(2)__ Find $grad\ f$.

__(6)__ Classify critical points into saddle, local maxima, local minima or none of these using the 2nd derivative test.

__(5)__ Find $D = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$ for all critical points.

__(1)__ Identify the function that you want to optimize.

3. Explain what is wrong with the following statement. Also, state the correct expression. [2]

If R is the region $x^2+y^2 \le 4$, then $\int_R (x^2+y^2)dA = \int_0^{2\pi}\int_0^2 r^2\, dr\, d\theta$.

When converting to polar coordinates, we need an extra factor of r, because $dA = r\,dr\,d\theta$. Thus, we should have:

$$\int_R (x^2+y^2)\,dA = \int_0^{2\pi}\int_0^2 r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta$$

4. Find the volume under the graph of the function $f(x,y) = 6x^2y$ over the region shown below:

[4]

The region is bounded by $x=1$, $x=4$, $y=2$, and $y=2x$. Thus

$$\text{Volume} = \int_1^4\int_2^{2x} (6x^2y)\,dydx.$$

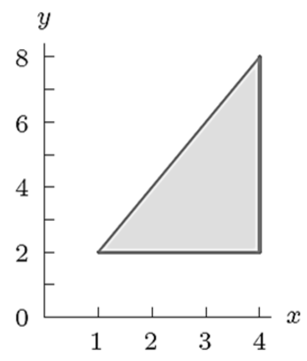
To evaluate this integral, we evaluate the inside integral first:

$$\int_2^{2x} (6x^2y)\,dy = (3x^2y^2)\Big|_2^{2x} = 3x^2(2x)^2 - 3x^2(2^2) = 12x^4 - 12x^2.$$

Therefore, we have

$$\int_1^4\int_2^{2x} (6x^2y)\,dydx = \int_1^4 (12x^4 - 12x^2)\,dx = \left(\frac{12}{5}x^5 - 4x^3\right)\Big|_1^4 = 2203.2.$$

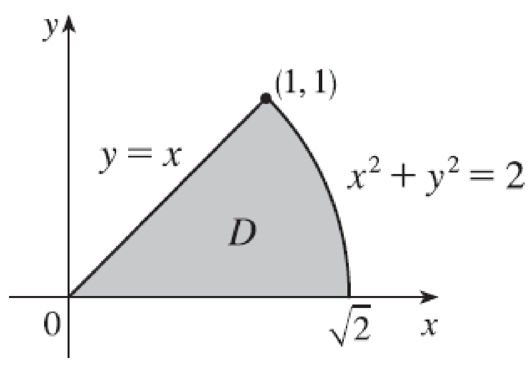
The volume of this object is 2203.2.



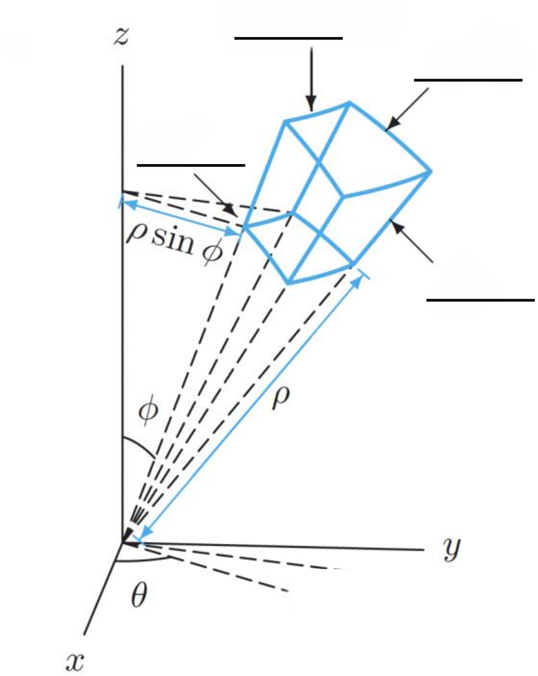
5. Sketch the region of integration for the given integral and convert it into polar coordinates *(you DO NOT need to evaluate the integral).* [3]

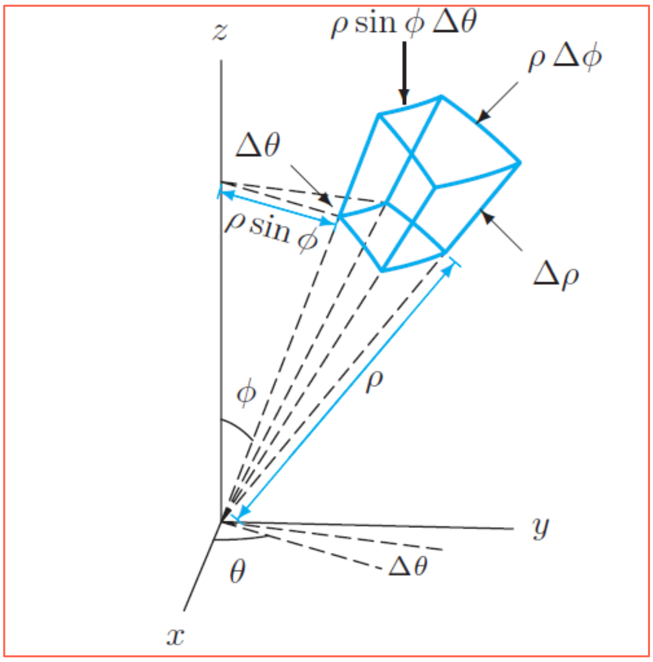
$$\int_0^1\int_y^{\sqrt{2-y^2}} (x+y)\,dx\,dy$$

$$\int_0^{\pi/4}\int_0^{\sqrt{2}} (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$



6. Label the missing parts in the following diagram. Hence, derive the expression for the volume element dV for spherical coordinates. [2.5]

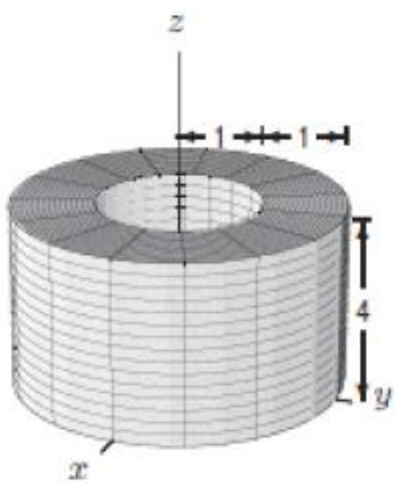




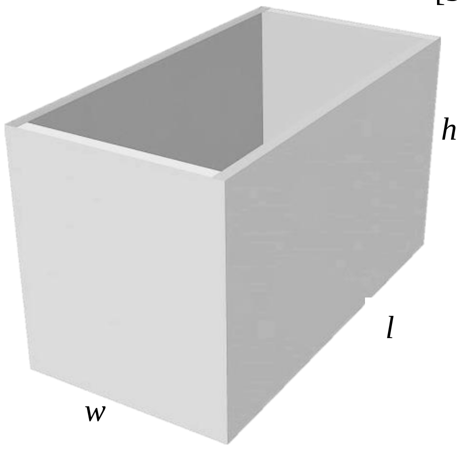
7. Sketch the region of integration W where $1 \le x^2 + y^2 \le 4$ and $0 \le z \le 4$ and then evaluate the integral below by converting to either spherical or cylindrical coordinates. [5]

$$\int_W \frac{z}{\left(x^2+y^2\right)^{3/2}} dV$$

$$\begin{aligned}
\int_W \frac{z}{(x^2+y^2)^{3/2}}\, dV &= \int_0^4 \int_0^{2\pi} \int_1^2 \frac{z}{r^3}\, r dr\, d\theta\, dz \\
&= \int_0^4 \int_0^{2\pi} \int_1^2 \frac{z}{r^2}\, dr\, d\theta\, dz \\
&= \int_0^4 \int_0^{2\pi} \left(-\frac{z}{r}\right)\Big|_1^2\, d\theta\, dz \\
&= \int_0^4 \int_0^{2\pi} \frac{z}{2}\, d\theta\, dz \\
&= \int_0^4 \frac{z}{2} \cdot 2\pi\, dz = \frac{1}{2}\pi \cdot z^2\Big|_0^4 = 8\pi
\end{aligned}$$



8. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box. (*Hint: Set up the objective and constraint functions for the problem first.)* [5]



SOLUTION As in Example 6 in Section 14.7, we let x, y, and z be the length, width, and height, respectively, of the box in meters. Then we wish to maximize

$$V = xyz$$

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$. This gives the equations

$$V_x = \lambda g_x$$

$$V_y = \lambda g_y$$

$$V_z = \lambda g_z$$

$$2xz + 2yz + xy = 12$$

which become

$$\boxed{2} \qquad yz = \lambda(2z + y)$$

$$\boxed{3} \qquad xz = \lambda(2z + x)$$

$$\boxed{4} \qquad xy = \lambda(2x + 2y)$$

$$\boxed{5} \qquad 2xz + 2yz + xy = 12$$

There are no general rules for solving systems of equations. Sometimes some ingenuity is required. In the present example you might notice that if we multiply $\boxed{2}$ by x, $\boxed{3}$ by y, and $\boxed{4}$ by z, then the left sides of these equations will be identical. Doing this, we have

stem of equa-
ations 2, 3,
e resulting

$$\boxed{6} \qquad xyz = \lambda(2xz + xy)$$

$$\boxed{7} \qquad xyz = \lambda(2yz + xy)$$

$$\boxed{8} \qquad xyz = \lambda(2xz + 2yz)$$

We observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from $\boxed{2}$, $\boxed{3}$ and $\boxed{4}$ and this would contradict $\boxed{5}$. Therefore, from $\boxed{6}$ and $\boxed{7}$, we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$. But $z \neq 0$ (since $z = 0$ would give $V = 0$), so $x = y$. From $\boxed{7}$ and $\boxed{8}$ we have

$$2yz + xy = 2xz + 2yz$$

which gives $2xz = xy$ and so (since $x \neq 0$) $y = 2z$. If we now put $x = y = 2z$ in $\boxed{5}$, we get

$$4z^2 + 4z^2 + 4z^2 = 12$$

Since x, y, and z are all positive, we therefore have $z = 1$ and so $x = 2$ and $y = 2$. This agrees with our answer in Section 14.7.