1 LINEAR ALGEBRA

LECTURE 13

MATH 205

$\boxed{\text{Q.no.3}}$ P.209 8TH ED. OR P.220-221 (7TH ED.)

DETERMINE WHETHER THE GIVEN (V) SET IS A VECTOR SPACE UNDER THE GIVEN OPERATIONS.

THE SET OF ALL PAIRS OF REAL NUMBERS (x,y) WITH THE OPERATIONS

$(x,y)+(x',y')=(x+x',y+y')$ AND

→ $\boxed{\text{VECTOR ADDITION}}$

$k(x,y)=(2kx,2ky)$

→ $\boxed{\text{SCALAR MULTIPLICATION}}$

SOLUTION:

(1) $(x,y)+(x',y')=(x+x',y+y')\in V$ SINCE THIS IS ALSO AN ORDER PAIR OF REAL NUMBERS

(2) $(x,y)+(x',y')=(x+x',y+y')$

$=(x'+x,y'+y)=(x',y')+(x,y)$

$\Rightarrow \boxed{\underline{u}+\underline{v}=\underline{v}+\underline{u}}$ FOR $\boxed{\underline{u}=(x,y)}$ AND $\boxed{\underline{v}=(x',y')}$.

(3) $(x,y)+[(x',y')+(x'',y'')]$

$=(x,y)+(x'+x'',y'+y'')$

$=\{x+(x'+x''),y+(y'+y'')\}$

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$= \{x+(x'+x''),\ y+(y'+y'')\}$

$= \{(x+x')+x'',\ (y+y')+y''\}$

$= (x+x',\ y+y') + (x'',y'')$

$= [\,(x,y)+(x',y')\,] + (x'',y'')$

(3) $\Rightarrow \boxed{\underline{u}+(\underline{v}+\underline{w}) = (\underline{u}+\underline{v})+\underline{w}}$

FOR $\underline{w} = (x'',y'')$

(4) $(x,y)+(0,0) = (0,0)+(x,y)$

$= (x+0,\ y+0) = (x,y)$

$\Rightarrow \boxed{\underline{0} = (0,0)}$

(5) IF $\boxed{\underline{u} = (x,y)}$, $\boxed{-\underline{u} = (x',y')}$

THEN

$(x,y)+(x',y') = (0,0)$ — ①

BUT $(x,y)+(x',y') = (x+x',\ y+y')$

②

FROM ① AND ②

$x+x' = 0 \Rightarrow \boxed{x' = -x}$

$y+y' = 0 \Rightarrow \boxed{y' = -y}$

$\therefore\ -\underline{u} = (-x,-y)$

$\because\ (x,y)+(-x,-y) = (-x,-y)+(x,y)$

$= (0,0)$

(6) $k(x,y) = (2kx,\ 2ky) \in V$

(OBVIOUS) $\Rightarrow k\underline{u} \in V$

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(7) $k(\underline{u}+\underline{v}) = k[(x,y)+(x',y')]$

$= k[(x+x', y+y')]$

$= \{2k(x+x'), 2k(y+y')\}$

$= (2kx+2kx', 2ky+2ky')$

$= (2kx, 2ky)+(2kx', 2ky')$

$= k(x,y) + k(x',y')$

$= k\underline{u} + k\underline{v}$

(8) $(k+l)\underline{u} = (k+l)(x,y)$

$= [2(k+l)x, 2(k+l)y]$ — ①

ALSO

$k\underline{u} + l\underline{u} = k(x,y) + l(x,y)$

$= (2kx, 2ky) + (2lx, 2ly)$

$= [2(k+l)x, 2(k+l)y]$ — ②

FROM ① AND ②

$\boxed{(k+l)\underline{u} = k\underline{u} + l\underline{u}}$

(9) $k(l\underline{u}) = k[l(x,y)]$

$= k(2lx, 2ly) = (4lkx, 4kly)$ — ①

$(kl)\underline{u} = (kl)(x,y) = (2klx, 2kly)$ — ②

FROM ① AND ② $k(l\underline{u}) \neq (kl)\underline{u}$

SO $\boxed{\text{NOT}}$ A VECTOR SPACE.

(X) $1\underline{u} = 1(x,y) = (2x, 2y) \neq (x,y)$

$$\Rightarrow 1\underline{u} \neq \underline{u}$$

$$\because k(x,y) = (2kx, 2ky)$$

SO GIVEN SET IS NOT A VECTOR SPACE $\because$ AXIOMS (9) AND (10) FAIL.

$\boxed{\text{SUBSPACES}}$

P. 211 8TH ED.
P. 222 7TH ED.

DEFINITION: A SUBSET $\boxed{W}$ OF A VECTOR SPACE $\boxed{V}$ IS CALLED A SUBSPACE OF $\boxed{V}$ IF $\boxed{W}$ IS ITSELF A VECTOR SPACE UNDER THE ADDITION AND SCALAR MULTIPLICATION DEFINED ON $\boxed{V}$.

THEOREM 5.2.1. P. 211 (8TH ED.) OR P. 222 (7TH ED.)

IF $\boxed{W}$ IS A SET OF ONE OR MORE VECTORS FROM A VECTOR SPACE V, THEN $\boxed{W}$ IS A SUBSPACE OF $\boxed{V}$ IF AND ONLY IF THE FOLLOWING CONDITIONS HOLD.

(a) IF u AND v ARE VECTORS IN $\boxed{W}$ THEN $u + v$ IS IN $\boxed{W}$.

(b) IF $\boxed{K}$ IS ANY SCALAR AND u IS ANY VECTOR IN $\boxed{W}$, THEN $Ku \in \boxed{W}$.

PROOF: IF $\boxed{W}$ IS A SUBSPACE OF $\boxed{V}$, THEN ALL THE VECTOR SPACE AXIOMS OR PROPERTIES ARE SATISFIED INCLUDING (1) AND (6) WHICH ARE SAME AS (a) AND (b) ABOVE.

CONVERSELY, ASSUME CONDITIONS $\boxed{(a)}$ AND $\boxed{(b)}$ HOLD. SINCE THEY ARE VECTOR SPACE AXIOMS 1 AND 6, WE NEED ONLY SHOW THAT OTHER 8 AXIOMS ARE SATISFIED.

AXIOMS $\boxed{2, 3, 7, 8, 9}$ AND $\boxed{10}$ ARE AUTOMATICALLY SATISFIED BY THE VECTORS IN $\boxed{W}$ SINCE THEY ARE SATISFIED BY ALL VECTORS IN $\boxed{V}$.

THEREFORE TO COMPLETE THE PROOF, WE NEED ONLY VERIFY

THAT AXIOMS $\boxed{4 \text{ AND } 5}$ ARE SATISFIED BY VECTORS IN $\boxed{W}$.

LET $\underline{u}$ BE ANY VECTOR IN $\boxed{W}$. BY CONDITION (b), $k\underline{u} \in \boxed{W}$ FOR EVERY SCALAR $\boxed{k}$.

SETTING $k=0$, $k\underline{u} = 0\underline{u} = \underline{0}$

BUT $k\underline{u} \in W \Rightarrow \underline{0} \in W$,

AND SETTING $k=-1$, IT FOLLOWS THAT

$$(-1)\underline{u} = -\underline{u} \in W.$$

RESULT. $\boxed{W}$ IS A SUBSPACE OF $\boxed{V}$ IF AND ONLY IF $\boxed{W}$ IS CLOSED UNDER ADDITION AND CLOSED UNDER SCALAR MULTIPLICATION.

EXAMPLE:

SHOW THAT THE SET $\boxed{W}$ OF ALL 2X2 MATRICES HAVING ZEROS ON THE MAIN DIAGONAL IS A SUBSPACE OF THE VECTOR SPACE M_{22} (OF ALL 2X2 MATRICES).

SOLUTION:

LET $\underline{u} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}$

$\underline{u} + \underline{v} = \begin{bmatrix} 0 & a+c \\ b+d & 0 \end{bmatrix} \in \boxed{W}$ AND

$k\underline{u} = \begin{bmatrix} 0 & ka \\ kb & 0 \end{bmatrix} \in \boxed{W}$, SINCE BOTH $\underline{u}+\underline{v}$ AND $k\underline{u}$ CONTAIN ZEROS ON THE MAIN DIAGONAL, $\therefore W$ IS A SUBSPACE OF M_{22}.

TRY THE FOLLOWING:

SHOW THAT THE SET $\boxed{W}$ OF ALL THE POLYNOMIALS OF DEGREE $\leq n$ (INCLUDING THE ZERO POLYNOMIAL) IS A SUBSPACE OF REAL-VALUED FUNCTIONS UNDER ADDITION AND SCALAR MULTIPLICATION.

HINT: CHECK $\underline{u}+\underline{v} \in W$, $k\underline{u} \in W$

TAKE

$\underline{u} = p(x) = a_0 + a_1 x + \cdots + a_n x^n$

AND $\underline{v} = q(x) = b_0 + b_1 x + \cdots + b_n x^n$

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TRY THE FOLLOWING:

CHECK WHETHER THE FOLLOWING SET OF VECTORS GIVEN BY

$$\boxed{W = \{ (a,b,c) \,/\, b = a + c \}}$$

IS A SUBSPACE OF R^3?

ANSWER: $\boxed{\text{YES}}$