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MATH 205

LECTURE 14

LINEAR ALGEBRA

P.215 (8TH ED.) OR P.226 (7TH ED.)

DEFINITION: A VECTOR $\underline{w}$ IS CALLED A LINEAR COMBINATION OF THE VECTORS $\underline{V}_1, \underline{V}_2, \ldots, \underline{V}_r$ IF IT CAN BE EXPRESSED IN THE FORM

$$\underline{w} = k_1 \underline{V}_1 + k_2 \underline{V}_2 + \cdots + k_r \underline{V}_r$$

WHERE $k_1, k_2, \ldots, k_r$ ARE SCALARS.

EXAMPLES:

① ANY VECTOR IN 3-DIMENSIONAL SPACE CAN BE EXPRESSED AS A LINEAR COMBINATION OF THE VECTORS $\underline{e}_1$, $\underline{e}_2$ AND $\underline{e}_3$.

$$\because \underline{u} = u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3$$
$$= u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1) = (u_1, u_2, u_3)$$

② ANY POLYNOMIAL OF DEGREE N CAN BE WRITTEN AS A LINEAR COMBINATION OF THE FOLLOWING $\boxed{N+1}$ ELEMENTS

$$\{1, x, x^2, \ldots, x^n\} \text{ AS}$$

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

③ ANY $\boxed{2 \times 2}$ MATRIX CAN BE WRITTEN AS A LINEAR COMBINATION OF

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ AND } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

DEFINITION: (P. 222 8TH ED.) OR (P. 232 7TH ED.)

IF $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r\}$ IS A NONEMPTY SET OF VECTORS, THEN THE VECTOR EQUATION

$$k_1\underline{v}_1 + k_2\underline{v}_2 + \cdots + k_r\underline{v}_r = \underline{0}$$

HAS $\boxed{\text{ATLEAST}}$ ONE SOLUTION, NAMELY $\boxed{k_1 = k_2 = \cdots = k_r = 0}$

IF THIS IS THE ONLY SOLUTION, THEN $\boxed{S}$ IS CALLED A LINEARLY INDEPENDENT SET AND THE VECTORS IN SET $\boxed{S}$ ARE CALLED LINEARLY INDEPENDENT VECTORS.

EXAMPLES: ① THE SET S GIVEN BY $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ IS LINEARLY INDEPENDENT AND THE VECTORS

$\underline{e_1}$, $\underline{e_2}$ AND $\underline{e_3}$ ARE LINEARLY INDEPENDENT VECTORS.

CHECK:

CONSIDER

$$k_1 \underline{e_1} + k_2 \underline{e_2} + k_3 \underline{e_3} = \underline{0} = (0,0,0)$$

$$\Rightarrow k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) = (0,0,0)$$

$$\Rightarrow (k_1, k_2, k_3) = (0,0,0)$$

$$\Rightarrow \boxed{k_1 = 0,\ k_2 = 0,\ k_3 = 0}$$

② $S = \{1, x, x^2, \ldots\ldots, x^n\}$ IS LINEARLY INDEPENDENT SINCE FOR

$$\boxed{a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n = 0} \quad \text{—①}$$

$\boxed{\forall x}$ $\Rightarrow \boxed{a_0 = a_1 = \cdots\cdots = a_n = 0}$

SINCE ① IS SATISFIED BY INFINITE VALUES OF $\boxed{x}$, OTHERWISE ① HAS AT MOST $\boxed{n}$ DISTINCT ROOTS IF ALL OR OF THE COEFFICIENTS $\neq 0$.

(SOME)

③ $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

IS LINEARLY INDEPENDENT (CHECK)

SOLUTION:

$$k_1\begin{bmatrix}1&0\\0&0\end{bmatrix}+k_2\begin{bmatrix}0&1\\0&0\end{bmatrix}+k_3\begin{bmatrix}0&0\\1&0\end{bmatrix}$$

$$+k_4\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}=\underline{0}$$

$$\Rightarrow\begin{bmatrix}k_1&k_2\\k_3&k_4\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\Rightarrow \boxed{k_1=k_2=k_3=k_4=0} \quad \therefore \boxed{S}$$

IS LINEARLY INDEPENDENT.

DEFINITION: P. 217 (8TH ED.) OR P. 228 7TH ED.

IF $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ ARE VECTORS IN A VECTOR SPACE $\boxed{V}$ AND IF $\boxed{\text{EVERY}}$ VECTOR IN $\boxed{V}$ IS EXPRESSIBLE AS A LINEAR COMBINATION OF THESE VECTORS, THEN WE SAY THAT $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ $\boxed{\text{SPAN}}$ $\boxed{V}$.

LET US CONSIDER SOME EXAMPLES.

EXAMPLES:

① $\{\underline{e}_1, \underline{e}_2\ \underline{e}_3\}$ SPANS $\underset{\text{3 DIMENSIONAL SPACE}}{R^3}$ SINCE

$$(u_1, u_2, u_3) = \underline{u} = u_1\underline{e}_1 + u_2\underline{e}_2 + u_3\underline{e}_3$$

$\forall\ \underline{u} \in R^3$

FOR ALL

② $\{1, x, x^2, \ldots\ldots, x^n\}$ SPANS THE VECTOR SPACE $\boxed{P_n}$ SINCE EACH POLYNOMIAL $\boxed{p(x)}$ IN $\boxed{P_n}$ CAN BE WRITTEN AS

$$\boxed{p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n}$$

③ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

SPAN $\boxed{M_{22}}$ (ALL MATRICES OF ORDER 2) SINCE

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}$$

$\forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}$

P.233 (8TH ED.)/P.244 (7TH ED.)

DEFINITION:

IF $\boxed{V}$ IS ANY VECTOR SPACE AND $\boxed{S} = \{v_1, v_2, \ldots\ldots, v_n\}$ IS A SET OF VECTORS IN $\boxed{V}$, THEN $\boxed{S}$ IS CALLED A BASIS FOR $\boxed{V}$ IF THE FOLLOWING TWO CONDITIONS HOLD:

(a) $\boxed{S}$ IS LINEARLY INDEPENDENT

(b) $\boxed{S}$ SPANS $\boxed{V}$

LET US CONSIDER SOME EXAMPLES OF SETS WHICH ARE BASES i.e. THEY ARE LINEARLY INDEPENDENT AS WELL AS SPAN DIFFERENT VECTOR SPACES.

EXAMPLES: (OF BASES)

BASES → PLURAL OF BASIS

① $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ IS A $\boxed{\text{BASIS}}$ FOR $\boxed{R^3}$ BECAUSE IT'S LINEARLY INDEPENDENT AS WELL AS SPANS $\boxed{R^3}$,

SIMILARLY

② $\{1, x, x^2, \ldots\ldots, x^n\}$ IS A BASIS FOR $\boxed{P_n}$ AND

③ $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

IS THE $\boxed{\text{BASIS}}$ FOR $\boxed{M_{22}}$.

$\boxed{\text{DIMENSION:}}$ P.239 (8th ED.) → P.239

P.251 ← P.251 (7th ED.)

THE $\boxed{\text{DIMENSION}}$ OF A VECTOR SPACE $\boxed{V}$ IS DEFINED TO BE THE $\boxed{\text{NUMBER}}$ OF VECTORS IN A BASIS FOR $\boxed{V}$.

REMARKS: ① DIMENSION OF $R^3 = 3$ SINCE THERE ARE $\boxed{\text{THREE}}$ VECTORS IN $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

② DIMENSION OF $P_n = n+1$

③ DIMENSION OF $M_{22} = 4$

TRY THE FOLLOWING:

CHECK WHETHER

$$\left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

IS A BASIS FOR M_{22}?

HINT: (i) FIRST CHECK THAT THE GIVEN MATRICES ARE LINEARLY INDEPENDENT

PUT $a_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} +$

$a_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

AND SEE IF $\boxed{a_1 = a_2 = a_3 = a_4 = 0}$

(ii) TAKE AN ARBITRARY ELEMENT OF M_{22} AS $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

AND CHECK IF IT CAN BE WRITTEN AS A LINEAR COMBINATION OF THE GIVEN MATRICES. FOR THIS

9) PUT $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$= k_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$+ k_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ AND TRY TO FIND

$\boxed{k_1, k_2, k_3, k_4}$ IN TERMS OF

$\boxed{a, b, c}$ AND $\boxed{d}$.

$\left.\begin{matrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{matrix}\right\}$ UNKNOWNS $\left.\begin{matrix} a \\ b \\ c \\ d \end{matrix}\right\}$ KNOWNS

ANSWER: $\boxed{\text{YES}}$ IT IS A BASIS

$\because$ $a_1 = a_2 = a_3 = a_4 = 0$ AND

$$\boxed{k_1 = \frac{b-a}{2}, \quad k_2 = \frac{a+b}{2}, \quad k_3 = c, \quad k_4 = d}$$

REMARK: A VECTOR SPACE MAY HAVE MORE THAN ONE BASIS. BUT IN ALL THE BASES (PLURAL) THE NUMBER OF ELEMENTS (VECTORS) ARE SAME. AS WE

SAW THAT

10 $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

AND $\left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

ARE BASES FOR M_{22} AND BOTH CONTAIN $\boxed{4}$ VECTORS = DIMENSION OF M_{22}.

NOTE: ① $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ IS ALSO CALLED STANDARD BASIS FOR $\boxed{M_{22}}$

AND SIMILARLY $\boxed{\{e_1, e_2, e_3\}}$ AND $\boxed{\{1, x, x^2, \ldots, x^n\}}$ ARE STANDARD BASES FOR $\boxed{R^3}$ AND $\boxed{P_n}$ RESPECTIVELY.

② $\left\{ \boxed{\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}}, \boxed{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ IS A BASIS BUT NOT A STANDARD BASIS FOR $\boxed{M_{22}}$.