LECTURE 3

MATH 205

REVISION:

THEOREM (P.42) → (P.42)/8TH ED. (P.43)/7TH ED.

IF $\boxed{B}$ AND $\boxed{C}$ ARE BOTH INVERSES OF THE MATRIX $\boxed{A}$, THEN $\boxed{B=C}$

PROOF: ∵ $\boxed{B}$ IS AN INVERSE OF $\boxed{A}$, WE HAVE $\boxed{BA=I} \rightarrow (1)$

MULTIPLYING BOTH SIDES (OF (1)) ON THE RIGHT BY $\boxed{C}$ GIVES

$$(BA)C = IC = C. \rightarrow (2)$$

BUT $(BA)C = B(AC)$, ∵ $\boxed{AC=I}$

$$= B(I) = B, \rightarrow (3)$$

SO THAT $C=B$ FROM (2) AND (3).

RESULT: INVERSE OF A NONSINGULAR MATRIX IS $\boxed{\text{UNIQUE}}$.

THEOREM (P.43)

IF $\boxed{A}$ AND $\boxed{B}$ ARE INVERTIBLE MATRICES OF THE SAME SIZE, THEN $\boxed{(AB)^{-1} = B^{-1}A^{-1}}$

PROOF:

CONSIDER

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1}$$

$$= AA^{-1} = I \quad \text{--- (1)}$$

ALSO

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB$$

$$= B^{-1}B = I \quad \text{--- (2)}$$

FROM (1) AND (2)

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

RESULT: IF $A_1, A_2, \ldots\ldots, A_n$ ARE INVERTIBLE MATRICES OF SAME SIZE THEN

$$\boxed{\begin{aligned}&(A_1A_2\ldots\ldots A_n)^{-1}\\ &= A_n^{-1}A_{n-1}^{-1}\ldots\ldots A_2^{-1}A_1^{-1}\end{aligned}}$$

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IN SHORT

(1) A PRODUCT OF ANY NUMBER OF INVERTIBLE MATRICES IS INVERTIBLE.

(2) THE INVERSE OF THE PRODUCT IS THE PRODUCT OF THE INVERSES IN THE $\boxed{\text{REVERSE ORDER.}}$

NOTATION: $A^n = \underbrace{A \cdot A \cdot A \cdot \ldots \cdot A}_{n \text{ TIMES}}$ $(n>0)$

P.44 $A \to$ SQUARE MATRIX

(P.44) 8TH ED.

TRY THE FOLLOWING: (P.45) 7TH ED.

(1) IF $\boxed{A}$ IS AN INVERTIBLE MATRIX, THEN:

$$\boxed{(A^{-1})^{-1} = A}$$

PROOF: $\because AA^{-1} = A^{-1}A = I$

$$\Rightarrow \boxed{(A^{-1})^{-1} = A}$$

(2) $(A^n)^{-1} = (A^{-1})^n$ FOR $n = 0, 1, 2, \ldots$

HINT: $A^n = \underbrace{AA \ldots A}$ $\to$ n TIMES

→ P.46, 7TH ED.

TRANSPOSE OF A MATRIX, P.45 (8TH ED.)

IF $\boxed{A}$ IS $\boxed{m \times n}$ MATRIX THEN TRANSPOSE OF $\boxed{A}$ (DENOTED BY A^T) IS OBTAINED BY INTERCHANGING ROWS AND COLUMNS OF $\boxed{A}$. ORDER (SIZE) OF $\boxed{A^T}$ $= \boxed{n \times m}$.

EXAMPLE:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \leftarrow 2 \times 3$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix} \rightarrow 3 \times 2$$

NOTICE THAT FIRST ROW OF $\boxed{A}$ BECOMES FIRST COLUMN OF $\boxed{A^T}$ ETC.

ALSO $(A^T)^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$
$= A$.

RESULT: $\boxed{(A^T)^T = A}$ i.e. INTERCHANGING ROWS AND

5) COLUMNS $\boxed{\text{TWICE}}$ LEAVES A MATRIX UNCHANGED.

RESULT: IF $\boxed{A}$ AND $\boxed{B}$ ARE MATRICES OF SAME SIZE THEN

$$\boxed{(A+B)^T = A^T + B^T}$$

EXAMPLE: LET $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$$\boxed{(A+B)^T} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}^T$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix} \text{ — } ①$$

ALSO $\boxed{A^T + B^T}$

$$= \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix} \text{ — } ②$$

$\therefore (A+B)^T = A^T + B^T$ FROM ① AND ②

(P.49)/7TH ED.

DEFINITION: (P.49)/8TH ED.

A $\boxed{\text{SQUARE}}$ MATRIX $\boxed{A}$ IS CALLED SYMMETRIC IF $\boxed{A^T = A.}$

EXAMPLES:

(1) IDENTITY MATRIX IS SYMMETRIC $\because I^T = I$ (ALWAYS)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) EVERY DIAGONAL MATRIX IS SYMMETRIC e.g.

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}^T = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

(3) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix} = A$ IS SYMMETRIC SINCE

$A^T = \begin{bmatrix} 0 & 1 & \boxed{2} \\ 1 & 4 & 3 \\ \boxed{2} & 3 & 5 \end{bmatrix} = A$, HERE $\boxed{a_{12} = a_{21} = 1}$, $a_{13} = a_{31} = 2$

AND $a_{23} = a_{32} = 3$.

NOTE: WHILE TAKING THE TRANSPOSE, DIAGONAL ENTRIES OF A SQUARE MATRIX DON'T CHANGE THEIR POSITIONS.

$$\text{IF } \boxed{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\boxed{A^t} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

DEFINITION: (P.48)

A SQUARE MATRIX $\boxed{A}$ IS CALLED SKEW-SYMMETRIC IF $\boxed{A^T = -A}$

EXAMPLE:

$$B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$ IS SKEW-SYMMETRIC

$$\because B^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} = -B$$

HERE $a_{12} = +1 = -a_{21}$, $a_{21} = \underbrace{1}_{=-1}$ ETC.

DIAGONAL ENTRIES IN A SKEW-SYMMETRIC MATRIX ARE ALWAYS = $\boxed{\text{ZERO}}$, $\because a_{11} = -a_{11} \Rightarrow a_{11} = 0$, ETC.

$\boxed{\text{ASSIGNMENT NO. 1}}$

$\boxed{\text{Q.no.1}}$

(a) LET $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix}$, FIND $\boxed{A^2}$ AND $\boxed{A^3}$.

(b) SHOW THAT $A^3 = A^2 + A - 5I$

(c) USING (b) WITHOUT DOING ANY MORE MULTIPLICATION PROVE THAT

(i) $A^4 = 2A^2 - 4A - 5I$,

(ii) $A^{-1} = \frac{1}{5}(I + A - A^2)$

$\boxed{\text{Q.no.2}}$

SHOW THAT THE MOST GENERAL MATRIX THAT $\boxed{\text{COMMUTES}}$ WITH

$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ IS OF THE FORM

$$\begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}.$$

$\boxed{\text{Q.no.3}}$

DON'T USE MATRIX ENTRIES.

(a) IF $\boxed{A}$ BE A SQUARE MATRIX THEN $\boxed{A + A^T}$ IS SYMMETRIC AND $\boxed{A - A^T}$ IS SKEW SYMMETRIC.

(b) IF $\boxed{A}$ IS $\boxed{m \times n}$ MATRIX THEN PROVE THAT $\boxed{AA^T}$ AND $\boxed{A^T A}$ ARE BOTH SYMMETRIC. (SEE Q.no.7)

(c) IF $\boxed{A^2 = A}$, $\boxed{A^{-1}}$ EXISTS THEN $\boxed{A = I}$.

(d) IF $\boxed{A}$ IS INVERTIBLE THEN PROVE THAT $\boxed{(A^{-1})^T = (A^T)^{-1}}$

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$\boxed{\text{Q. no. 4}}$

Q. no. 2 (P. 34)/8TH ED.,

OR (P. 351)/7TH ED.,

$\boxed{\text{Q. no. 5}}$

LET $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ AND $B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$,

SHOW THAT IF THE MATRIX

$X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ SATISFIES THE

EQUATION $\boxed{AX = XB}$ THEN $\boxed{X}$ IS

A SCALAR MULTIPLE OF $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

$\boxed{\text{Q. no. 6}}$

IF $\boxed{A}$ IS A SQUARE MATRIX OF

ORDER 3 s.t. $\boxed{A^t = -A}$ THEN

PROVE THAT THE DIAGONAL ENTRIES OF $\boxed{A = 0}$.

Q.no.7

IMPORTANT RESULT:

IF $\boxed{A}$, $\boxed{B}$ ARE MATRICES S.T. AB IS DEFINED THEN

$$\boxed{(AB)^T = B^T A^T}$$

CHECK THIS RESULT FOR $\boxed{2\times 2}$ MATRICES BY TAKING GENERAL ENTRIES.

NOTE: THE TRANSPOSE OF A PRODUCT OF ANY NUMBER OF MATRICES IS EQUAL TO THE PRODUCT OF THEIR TRANSPOSES IN THE REVERSE ORDER.

i.e.

$$\boxed{(A_1 A_2 \ldots\ldots A_n)^T = A_n^T \ldots\ldots A_2^T A_1^T}$$