1) LINEAR ALGEBRA

# LECTURE 20

MATH 205

$\boxed{\text{PROBLEM:}}$ GRAM-SCHMIDT PROCESS; P. 298 (8TH ED.), P. 312 (7TH ED.)

V → INNER PRODUCT SPACE.

GIVEN: $\{u_1, u_2, \ldots\ldots, u_n\}$ BE ANY BASIS FOR V THEN HOW TO PRODUCE AN ORTHOGONAL BASIS $\{v_1, v_2, \ldots\ldots, v_n\}$ FOR V?

i.e. $\langle v_i, v_j \rangle = 0,\ i \neq j,$

$1 \leq j \leq n,\ 1 \leq i \leq n$

WHICH CAN BE NORMALISED TO PRODUCE AN ORTHONORMAL BASIS i.e.

$$\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \ldots\ldots, \frac{v_n}{\|v_n\|} \right\}$$

i.e. NORM OF EACH VECTOR = 1 (IN ADDITION TO ORTHOGONALITY PROPERTY)

2] RECALL THAT FOR EUCLIDEAN INNER PRODUCT (DOT PRODUCT)

$$\text{Proj}_{\underline{a}}\underline{u} = \frac{(\underline{u}\cdot\underline{a})\underline{a}}{\|\underline{a}\|^2} \quad \text{AND}$$

VECTOR PROJECTION (COMPONENT) OF $\boxed{\underline{u}}$ PEPENDICULAR TO $\boxed{\underline{a}}$ IS GIVEN BY

$$\boxed{\underline{u} - \text{Proj}_{\underline{a}}\underline{u} = \underline{u} - \frac{(\underline{u}\cdot\underline{a})\underline{a}}{\|\underline{a}\|^2}}$$

SIMILARLY IF $\underline{v}_1$ AND $\underline{v}_2$ ARE ORTHOGONAL VECTORS IN AN INNER PRODUCT SPACE $\boxed{V}$ AND $\underline{u} \in V$ SUCH THAT HORIZONTAL (OP) PROJECTION OF $\underline{u}$ LIES ALONG $\underline{v}_1$ THEN

$$\boxed{\underline{v}_2 = \underline{u} - \frac{\langle \underline{u}, \underline{v}_1 \rangle \underline{v}_1}{\|\underline{v}_1\|^2}}$$

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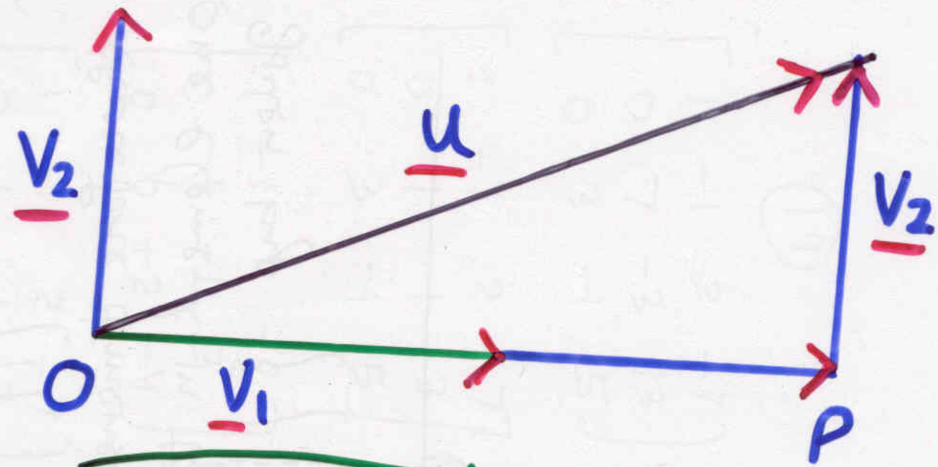
$$\underline{u} = \underline{v}_2 + \vec{OP}$$

**PROOF:**

$\underline{v}_2 = ?$




$$\because \quad \underline{u} = k\underline{v}_1 + \underline{v}_2 = \vec{OP} + \underline{v}_2$$

$$\because \quad \vec{OP} = k\underline{v}_1$$

$$\Rightarrow \quad \underline{v}_2 = \underline{u} - \vec{OP} = \underline{u} - k\underline{v}_1 \quad \text{— (1)}$$

TAKING INNER PRODUCT ON BOTH SIDES BY $\underline{v}_1$, WE GET

$$\cancelto{0}{\langle \underline{v}_2, \underline{v}_1 \rangle} = \langle \underline{u}, \underline{v}_1 \rangle - \langle k\underline{v}_1, \underline{v}_1 \rangle$$

$$\Rightarrow \langle \underline{u}, \underline{v}_1 \rangle = \langle k\underline{v}_1, \underline{v}_1 \rangle$$

$$\Rightarrow \langle \underline{u}, \underline{v}_1 \rangle = k \langle \underline{v}_1, \underline{v}_1 \rangle$$

$$\Rightarrow \boxed{k = \frac{\langle \underline{u}, \underline{v}_1 \rangle}{\|\underline{v}_1\|^2}}$$

PUTTING IN (1), WE GET

$$\boxed{\underline{v}_2 = \underline{u} - \overbrace{\frac{\langle \underline{u}, \underline{v}_1 \rangle}{\|\underline{v}_1\|^2}\underline{v}_1}^{\text{Proj}_{\underline{v}_1}\underline{u}}}$$

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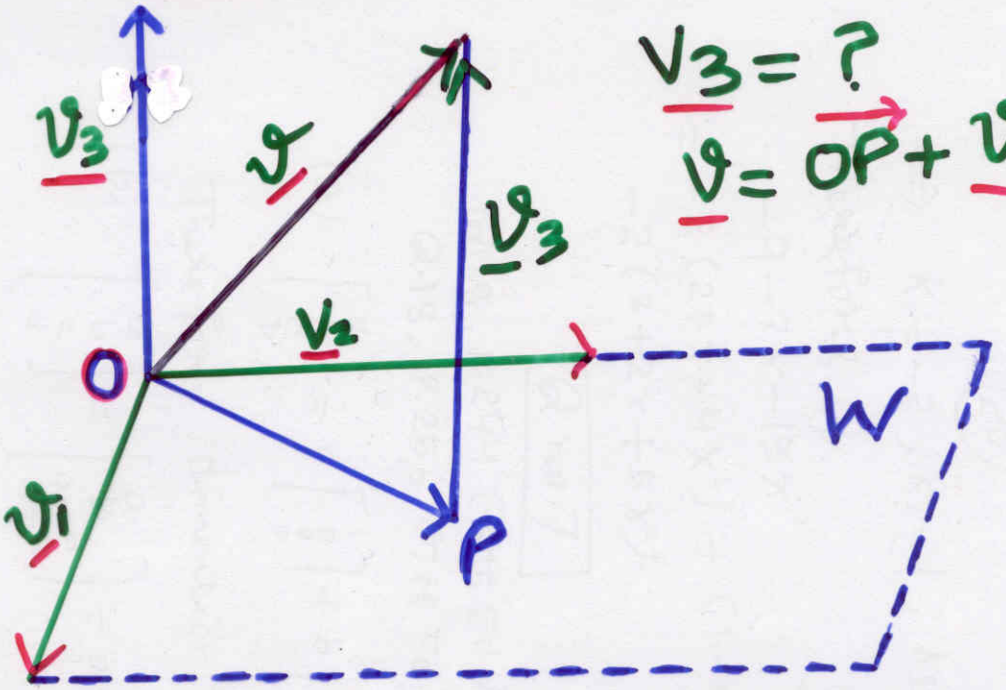
**NOTE:** VECTORS $\underline{e}_1 = (1,0,0)$ AND $\underline{e}_2 = (0,1,0)$ SPAN THE XY-PLANE BECAUSE ANY VECTOR IN THE XY-PLANE CAN BE WRITTEN AS THEIR LINEAR COMBINATION

$$(x,y,0) = x(1,0,0) + y(0,1,0)$$
$$= x\underline{e}_1 + y\underline{e}_2$$

SIMILARLY IF $\underline{v}_1$ AND $\underline{v}_2$ ARE ORTHOGONAL VECTORS SPANING A PLANE $\boxed{W}$ AS SHOWN BELOW




$\underline{v}_3 = ?$

$\underline{v} = \overrightarrow{OP} + \underline{v}_3$

$\underline{v}_3$ IS ORTHOGONAL TO BOTH $\underline{v}_1$ AND $\underline{v}_2$. $\overrightarrow{OP}$ IS THE PROJECTION OR COMPONENT OF $\underline{v}$ IN $\boxed{W}$. $\because$ $\overrightarrow{OP}$ LIES IN $\boxed{W}$ (SPANNED BY $\underline{v}_1$ AND $\underline{v}_2$) THEREFORE

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$\vec{OP}$ IS A LINEAR COMBINATION OF $\underline{v}_1$ AND $\underline{v}_2$ $\therefore \vec{OP} = k_1\underline{v}_1 + k_2\underline{v}_2$

BUT $\underline{v} = \vec{OP} + \underline{v}_3 \Rightarrow \underline{v}_3 = \underline{v} - \vec{OP}$

$$\Rightarrow \underline{v}_3 = \underline{v} - k_1\underline{v}_1 - k_2\underline{v}_2 \quad \text{— (1)}$$

TO FIND $\boxed{k_1}$ TAKE INNER PRODUCT WITH $\underline{v}_1$

$$\Rightarrow \cancelto{0}{\langle \underline{v}_3, \underline{v}_1 \rangle} = \langle \underline{v}, \underline{v}_1 \rangle - k_1 \langle \underline{v}_1, \underline{v}_1 \rangle - k_2 \cancelto{0}{\langle \underline{v}_2, \underline{v}_1 \rangle}$$

$$\langle \underline{v}_1, \underline{v}_1 \rangle \to \|\underline{v}_1\|^2$$

$$\Rightarrow \boxed{k_1 = \frac{\langle \underline{v}, \underline{v}_1 \rangle}{\|\underline{v}_1\|^2}}$$

SIMILARLY TO FIND $\boxed{k_2}$ TAKE INNER PRODUCT WITH $\underline{v}_2$

$$\therefore (1) \Rightarrow \cancelto{0}{\langle \underline{v}_3, \underline{v}_2 \rangle} = \langle \underline{v}, \underline{v}_2 \rangle - k_1 \cancelto{0}{\langle \underline{v}_1, \underline{v}_2 \rangle} - k_2 \langle \underline{v}_2, \underline{v}_2 \rangle$$

$$\Rightarrow \boxed{k_2 = \frac{\langle \underline{v}, \underline{v}_2 \rangle}{\|\underline{v}_2\|^2}}, \quad \therefore \text{FROM (1)}$$

$$\boxed{\underline{v}_3 = \underline{v} - \frac{\langle \underline{v}, \underline{v}_1 \rangle \underline{v}_1}{\|\underline{v}_1\|^2} - \frac{\langle \underline{v}, \underline{v}_2 \rangle \underline{v}_2}{\|\underline{v}_2\|^2}}$$

TO BE CONTINUED

# ASSIGNMENT NO. 5(a)

## Q. no. 1 (a)

CHECK WHETHER $\underline{V}_1 = (1,1,2)$, $\underline{V}_2 = (1,0,1)$, AND $\underline{V}_3 = (2,1,3)$ SPAN THE VECTOR SPACE $R^3$

## Q. no. 1 (b)

ARE THE FOLLOWING TRUE OR FALSE?

(I) A SET THAT CONTAINS THE ZERO VECTOR IS LINEARLY DEPENDENT.

(II) IF W IS A SUBSPACE OF V, THEN $\dim(W) \leq \dim(V)$; MOREOVER, $\dim(W) = \dim(V)$; IF AND ONLY IF $W = V$

(III) EVERY NONZERO FINITE-DIMENSIONAL INNER PRODUCT SPACE HAS AN ORTHONORMAL BASIS.

(IV) THE PRODUCT $A\underline{x}$ IN A LINEAR SYSTEM IS A LINEAR COMBINATION OF THE COLUMN VECTORS OF $\boxed{A}$.

(V) A SYSTEM OF LINEAR EQUATIONS $A\underline{x}=\underline{b}$ IS CONSISTENT IF AND ONLY IF $\boxed{\underline{b}}$ IS IN THE COLUMN SPACE OF A.

(VI) $A\underline{x}=\underline{b}$ IS CONSISTENT IF AND ONLY IF THE RANK OF THE COEFFICIENT MATRIX $\boxed{A}$ IS THE SAME AS THE RANK OF THE AUGMENTED MATRIX $[A/\underline{b}]$.

## $\boxed{\text{Q.no.2}}$

LET $A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$

FIND:

(a) ECHELON FORM OF $\boxed{A}$.

(b) BASIS FOR THE COLUMN SPACE OF $\boxed{A}$ BY WATCHING THE

COLUMN VECTORS IN $\boxed{A}$ WHICH CORRESPOND TO THE COLUMN VECTORS IN ECHELON FORM ((CONTAINING THE LEADING 1'S).

(C) CAN WE ALSO FIND THE BASIS FOR THE ROW SPACE OF $\boxed{A}$ (CONSISTING ENTIRELY OF ROW VECTORS FROM $\boxed{A}$) BY THE METHOD (USED IN (b))? EXPLAIN YOUR ANSWER?

**Q. no. 3**

EXAMPLE (1) P. 261 (8th ED.) OR EXAMPLE (1) P. 274 (7th ED.)

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IS THERE ANY SHORTER METHOD FOR THIS EXAMPLE? SHORTER

**Q. no. 4** (INNER PRODUCT SPACE)

(a) SEE THE DEFINITION OF WEIGHTED EUCLIDEAN INNER PRODUCT. (P. 277 8th ED. OR P. 298 7th ED.)

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(b) LET $\underline{u} = (u_1, u_2)$ AND $\underline{v} = (v_1, v_2)$ BE VECTORS IN $R^2$. VERIFY THAT THE WEIGHTED EUCLIDEAN INN-

(9) $<\underline{u},\underline{v}> = 3u_1v_1 + 2u_2v_2$

SATISFIES THE FOUR INNER PRODUCT AXIOMS.

**Q.no.5**

(a) IF $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ AND $V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$

ARE 2X2 MATRICES, THEN PROVE THAT THE FOLLOWING FORMULA DEFINES AN INNER PRODUCT ON $M_{22}$ $(\underline{u} = U, \underline{v} = V)$

$$<\underline{u},\underline{v}> = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

(b) If $P = a_0 + a_1x + a_2x^2$ AND $Q = b_0 + b_1x + b_2x^2$ ARE ANY TWO VECTORS IN $P_2$, THEN PROVE THAT THE FOLLOWING FORMULA DEFINES AN INNER PRODUCT ON $P_2$

$$<P,Q> = a_0b_0 + a_1b_1 + a_2b_2$$

**Q.no.6**

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(a) Q.no.7 (P.284 6th ED.) OR
Q.no.7 (P.297 7th ED.)

(b) Q.no.17 (P. 285 6th ED.) OR

Q.no.17 (P.298 7TH ED.)

(C) Q.no.28 (P.286 8TH ED.) OR (P.299 7TH ED.)

$\boxed{\text{Q.no.7}}$

FOR ANY INNER PRODUCT SPACE PROVE THAT

(a) $\|\underline{u}+\underline{v}\|^2 + \|\underline{u}-\underline{v}\|^2 = 2\|\underline{u}\|^2 + 2\|\underline{v}\|^2$

(b) $\langle \underline{u}, \underline{v} \rangle = \frac{1}{4}\|\underline{u}+\underline{v}\|^2 - \frac{1}{4}\|\underline{u}-\underline{v}\|^2$

$\boxed{\text{Q.no.8}}$

Q. 17, 20 P.296 8TH ED. OR

Q.17, 20 P. 310 7TH ED.