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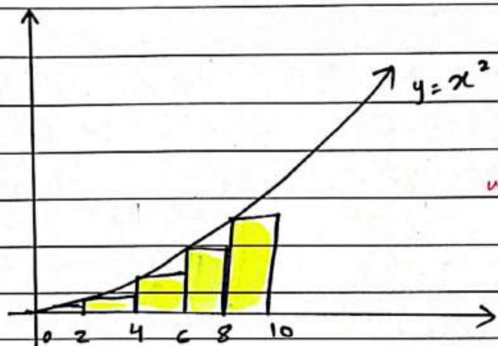
Riemann Sum



To calculate area under graph we draw 4 rectangle we are calculating left-hand sum.

$n = 4$ n = no of rectangles

Area $\displaystyle\sum_{i=1}^{n=4} \underbrace{\Delta x}_{\text{width}} \underbrace{f(x_i)}_{\text{height}}$

$\Delta x = \dfrac{b-a}{n}$ b = last x coordinate

a = first " "

$\dfrac{8-0}{4} = 2$ $[a,b]\,[0,8]$

$\Delta x\,[\underbrace{f(x_1) + f(x_2) + f(x_3) + f(x_4)}_{\text{sum of all y left value.}}]$

$\Delta x\,[0^2 + 2^2 + 4^2 + 6^2] \rightarrow 2[0+4+16+36]$

underestimated area.

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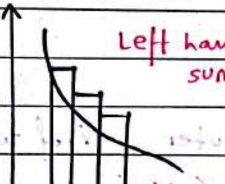

(overestimate sum)

$A = f(x_0)\Delta x + f(x_{n-1})\Delta x$

(underestimate sum)

$A = f(x_1)\Delta x + \dots f(x_n)\Delta x$

→ we make Δx (width of rectangle) very small to find accurate area under the curve.

$\lim\limits_{x \to 0} = \lim\limits_{n \to \infty}$ → infinite no of rectangles.

VOLUME UNDER CURVE

$$V = \int_a^b \pi [f(x)]^2 \, dx$$ (about x-axis) ($y=0$)

$$V = \int_c^d \pi [f(y)]^2 \, dy$$ (about y-axis) ($x=0$)

Axis of symmetry is same as line of rotation

⇒ considers only half part of it

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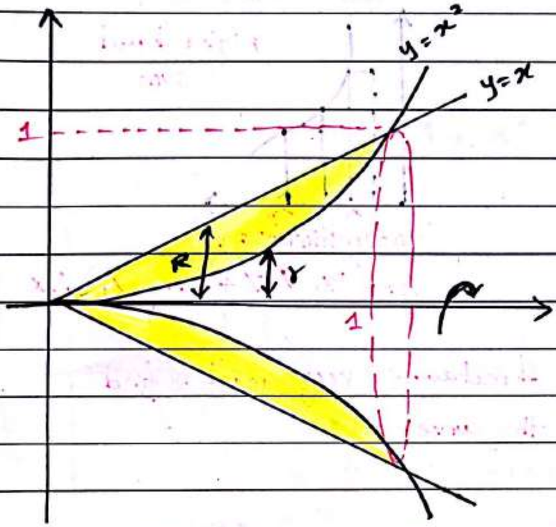


V. of outer − V. of inner

$$x - x^2$$

$$\int_0^1 \pi [x]^2 dx - \int_0^1 \pi [x^2]^2 dx$$

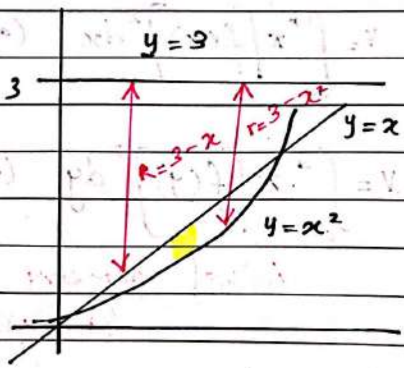
Axis Of Rotation

about $y=3$.

V. of outer − V. of inner.

$$\int \pi (3-x^2)^2 - \pi (3-x)^2 \, dx$$

outer inner



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RECALL :- **RECAP**

Vol of revolution :-

(i) About x-axis $\pi\int_a^b (f(x))^2\,dx$

(ii) About y-axis $\int_c^d (f(y))^2\,dy$

(iii) Between two curves $\pi\int_a^b [f(x)]^2 - [g(x)]^2\,dx$

$f(x) > g(x)$ (revolved around $x=0$)

outer inner (iv) Between two curves $\pi\int_c^d \left[(3-f(x))^2 - (3-g(x))^2\right]dx$

(revolved around $y=3$)

If the axis of rotation $y=a$ $f(x) > g(x)$

i) where $a > 0$

then $R = a - f(x)$

and $r = a - g(x)$

ii) where $a < 0$

then $R = a + f(x)$

and $r = a + g(x)$

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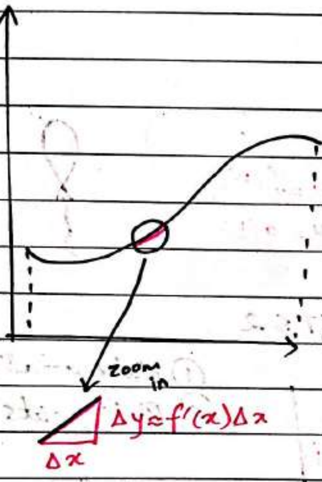
PARAMETRIC EQUATIONS

Finding arc length.



$$f'(x) = \frac{\Delta y}{\Delta x}$$

$$\Delta y \approx f'(x)\,\Delta x$$

$$= \sqrt{(\Delta x)^2 + (f'(x)\Delta x)^2}$$

taking $(\Delta x)^2$ common:

$$= \sqrt{(\Delta x)^2 (1 + (f'(x))^2)}$$

$$l = \Delta x \sqrt{1 + (f'(x))^2}$$

$$\lim_{x \to 0} \sum_{i=1}^{n} \sqrt{1 + [f(x_i)]^2}\, \Delta x$$

arc length — $\boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx}$

* Parametric Equations:

→ Used to explain motion of any object in a plane.

→ There are certain problems in defining a plane curve like a "normal" function, (x, y).

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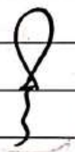
(i) Many times ? they're **not** functions

(ii) Many times we can not explicitly define in terms of a single variable (doesn't define the position at a given time)

(iii) No direction is given.

$(x,y) = (f(t), g(t))$

parametric equation $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$



Example $X = T^2 - 4$, $Y = 2T$: $-1 \leq T \leq 2$

① Make a table

② Plot points

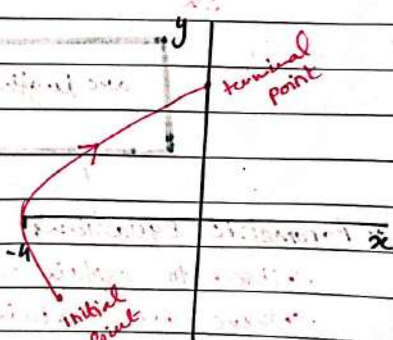
T	-1	-1/2	0	1/2	1	2
(x, y)	(-3, -2)	($\frac{-15}{4}$, -1)	(-4, 0)	($\frac{-15}{4}$, 1)	(-3, 2)	(0, 4)

OR

$T = \frac{y}{2}$

$x = \left(\frac{y}{2}\right)^2 - 4 \rightarrow x = \frac{y}{4} - 4$

parametric equation



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* Parametric equation of a circle $x = r\cos t$, $y = r\sin t$
* Parametric equation of a straight line.

$$\left.\begin{array}{l} x = x_0 + at \\ y = y_0 + bt \end{array}\right\} \quad a = \frac{dx}{dt}, \quad b = \frac{dy}{dt}$$

* Instantaneous speed & velocity:

$$\vec{v} = v_x \vec{i} + v_y \vec{j} \quad \text{(velocity)}$$

$$|\vec{v}| = \sqrt{(v_x)^2 + (v_y)^2} \quad \text{(speed)}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

* When a particle is STOPPED,

$$\frac{dy}{dx} = 0$$

* put a negative in front of first coordinate to show counterclockwise.

* When a particle is moving parallel to x-axis,

$$\frac{dy}{dt} = 0$$

* When a particle is moving parallel to y-axis,

$$\frac{dx}{dt} = 0$$

$(\cos t)^2 + (\sin t)^2 = 1$

$x^2 + y^2 = 1$

* if your coefficients/amplitudes are SAME then it is a circle

$$\underbrace{x = \underline{a}\cos t \text{ and } y = \underline{a}\sin t}_{\text{radius.}}$$

$x = \cos t$
$y = \sin t$

equation of circle

LBA

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★ Parametric Equations. **RECAP**

$x = f(t) \quad y = g(t)$

• If a circle: $x = r\cos t,\ y = r\sin t$

• If a line: $x = x_0 + at,\ y = y_0 + bt$

$a = dx/dt,\ b = dy/dt$

★ Inst. velocity: $\vec{V} = V_x^2 + V_y^2$

$V_x = \frac{dx}{dt} \quad V_y = dy/dt$

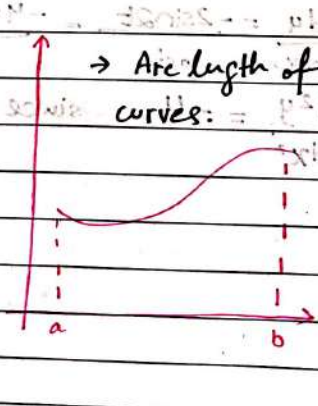
★ Inst. speed: $|\vec{V}| = \sqrt{(dx/dt)^2 + (dy/dt)^2}$

$L = \int_a^b \sqrt{1 + (f'(t))^2}\, dt$

cartesian plane

Arc length ①

→ Arc length of parametric curves:



$\frac{d}{dt}(\text{distance}) = \text{speed}$ $\quad$ distance $= \int \text{speed}$

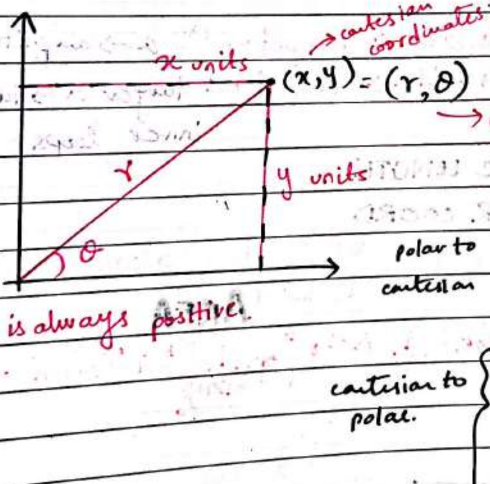
Distance covered by particle from a to b $\quad \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = L$

Arc length ②

parametric equation.

POLAR COORDINATES

(~~special case of~~ parametric curve)



→ r is always positive.

polar to cartesian
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ \sin\theta = \frac{y}{r} \end{cases}$$

cartesian to polar.
$$r = \sqrt{x^2+y^2}$$

1) $\theta = \sin^{-1}(y/r)$

2) $\theta = \cos^{-1}(x/r)$

we will not use ← 3) $\theta = \tan^{-1}(y/x)$

* $r = 1 - \sin\theta$ with $0 \le \theta \le n\pi$
  ↳ graph will begin to draw over itself

* $r = 1 - n\sin\theta$
  ↳ larger n increases inner loops size

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Polar to Cartesian	Cartesian to Polar
a) $x = r\cos\theta$	c) $r = \sqrt{x^2 + y^2}$
∴ b) $y = r\sin\theta$	d) $\theta = \sin^{-1}(y/r)$
	e) $\theta = \cos^{-1}(x/r)$
	f) $\theta = \tan^{-1}(y/x)$

$r = a\sin(n\theta)$
$r = a\cos(n\theta)$

n is even → 2n petal
n is odd → n petals

a → amplitude
larger a → larger inner loops.

AREA & ARC LENGTH IN POLAR COORD

$A = \int_a^b f(x)\,dx$

AREA

Area of Sector $= \frac{1}{2}\theta r^2$ OR $\frac{1}{2}\Delta\theta r^2$ or $\frac{1}{2}\int_a^b r^2(\theta)\,d\theta$

ARC LENGTH

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$x = r\cos\theta$
$y = r\sin\theta$.

WORKING IN 3D

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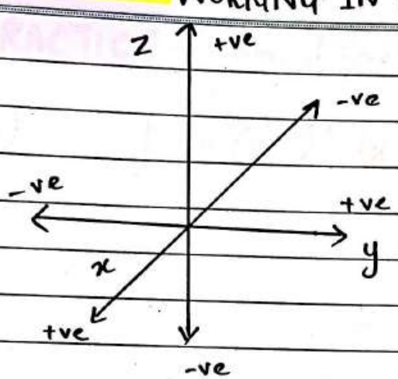


$z = f(x \cdot y)$ → independent

↓ dependent

$y \cdot z$ plane ← $x = 0$

a plane parallel to $y \cdot z$ plane & passing through $x = 3$ ← $x = 3$

(x, y, z) → general form.

* In xy-plane $z = 0$ (horizontal)
* In yz-plane $x = 0$ (vertical)
* In xz-plane $y = 0$ (diagnol)

distances

$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$\boxed{x^2 + y^2 + z^2 = r^2}$ •) SPHERE

$\boxed{z = x^2 + y^2}$ •) PARABOLA

$\boxed{z = \sqrt{x^2 + y^2}}$ •) CONE different radii of circles.

$y^2 + z^2 = x^2$ •) CYLINDER

Date:

FUNCTIONS IN TWO VARIABLES

The function can be written $z = f(x, y)$

→ In 2-space, the formula for distance between two points (x, y) and (a, b) is given by

$$\text{distance} = \sqrt{(x-a)^2 + (y-b)^2}$$

→ The distance b/w (x, y, z) and (a, b, c) in 3-space

$$\text{distance} = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

$\boxed{x^2 + y^2 + z^2 = 1}$ SPHERE

$\boxed{z = x^2 + y^2}$ PARABOLA

$\boxed{z = \sqrt{x^2 + y^2}}$ CONE

derivation.

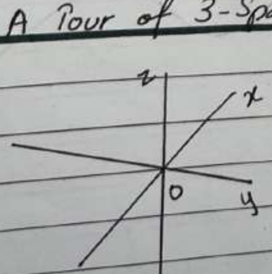
$\sqrt{x^2 + y^2 + z^2} = 1$ → radius

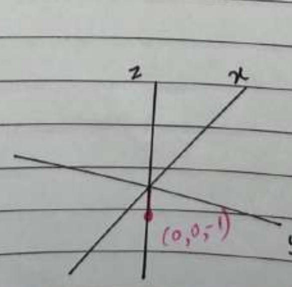
$x^2 + y^2 + z^2 = 1$ (surface of sphere)

$x^2 + y^2 + z^2 \leq 1$

★ The curves on a weather map is called isotherms. The weather map is called contour map.

A Tour of 3-Space.





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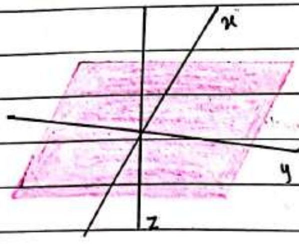
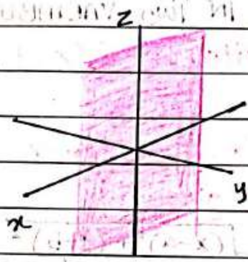
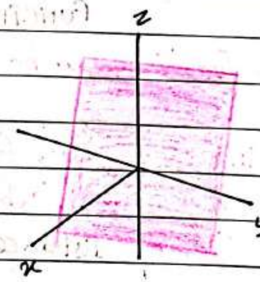


xy-plane
$z=0$

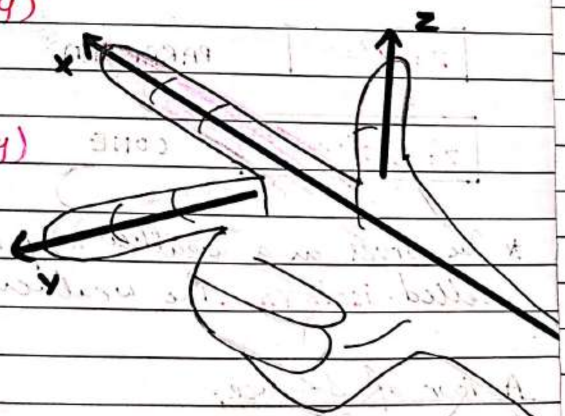


xz-plane
$y=0$



yz-plane
$x=0$

The graph of two variables, f, is the set of all points (x, y, z) such that $z=f(x,y)$



→ To sketch a graph of $f(x,y)$ we first make a table of values of $f(x,y)$.

* Countor lines or countour curves are obtained from a surface by slicing it with horizontal planes.

* Linear functions are equally spaced & parallel

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IMPORTANT

↳ the larger the contour lines → flatter surface.
↳ closer contour lines → more steeper.
↳ The distance b/w contour lines are the z-values.

Example 7: which of the points $A = (1, -1, 0)$, $B = (0, 3, 4)$, $C = (2, 2, 1)$ and $D = (0, -4, 0)$ lies closest to the xz-plane? Which points lies on y-axis?

Solution. The magnitude of the y-coordinate gives the distance to the xz-plane. The point A lies closest to that plane because it has the smallest y-coordinate in magnitude. To get a point on y-axis, we move along the y-axis, but we don't move at all in the x- or the z-direction. Thus D point lies on y-axis.

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LINEAR FUNCTIONS.

A linear function can be recognised by:
- each row & column is linear
- all rows have same slope
- all columns have same slope.

Linear function in 2D- (straight line)

* $y = mx + c$
* $y - y_1 = m(x - x_1)$

Linear function in 3D- (plane)

* if a plane has slope m in x-direction, has slope n in y-direction & passes through point (x_0, y_0, z_0).

$z = z_0 + m(x - x_0) + n(y - y_0)$

* if we write $c = z_0 - mx_0 - my_0$ then;

$f(x,y) = c + mx + my$

* $m = \frac{\Delta z}{\Delta x}$ y-same $n = \frac{\Delta z}{\Delta y}$ x same

$z - z_1 = m(x - x_0) + n(y - y_1)$

$z = mx - mx_0 + ny - ny_1 + z_0$

$\boxed{z = mx + ny + c}$

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e.g Find equation of the plane through the points $(1,0,1)$, $(1,-1,3)$ and $(3,0,-1)$

$AB \rightarrow x$ · coordinates same

$AC \rightarrow y$ · coordinates same $\rightarrow$ $m = \dfrac{\Delta z}{\Delta x} = \dfrac{-1-1}{3-1}$

colinear

$$n = \frac{\Delta z}{\Delta y} = \frac{3-1}{-1-0} \Rightarrow -2$$

$$= \frac{-2}{2} = -1$$

$m = -1$

$n = -2$

$$z = 1 - 1(x-1) - 2(y-0)$$

$$z = 1 - x + 1 - 2y$$

$$x + 2y + z = 2 \quad — \quad (*)$$

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PARTIAL DERIVATIVES

$f(x,y) = x^2y^2 + x + y^4 + 5$

$\frac{\partial x}{\partial y} f(x,y) = f_x = 2xy^2 + 1 +$ * y^4 and 5 are constant.

$= f_y = 2x^2y^2 + 4y^3$

$f_x(a,b) =$ Rate of change of f with respect to x at point (a,b) $= \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$

$f_y(a,b) =$ Rate of change of f with respect to y at point (a,b) $= \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$

$$\boxed{f_x(x,y) = \frac{\partial z}{\partial y} \quad \text{and} \quad f_y(x,y) = \frac{\partial z}{\partial y}}$$

EXAMPLES:- (a) $f(x,y) = y^2e^{3x}$

$f_x(x,y) = y^2 \frac{\partial}{\partial x}(e^{3x}) = 3y^2e^{3x}$

$f_y(x,y) = e^{3x}\frac{\partial}{\partial y}(y^2) = 2ye^{3x}$

(b) $z = (3xy+2x)^5$

$\frac{\partial z}{\partial x} = 5(3xy+2x)^4 \cdot (3y+2)$

$\frac{\partial z}{\partial y} = 5(3xy+2x)^4 \cdot (3x)$

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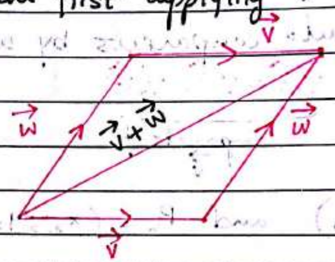
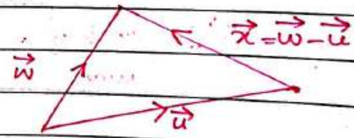
VECTORS.

The sum, $\vec{v}+\vec{w}$, of two vectors $\vec{v}$ and $\vec{w}$ is the combined displacement resulting from first applying $\vec{v}$ and then $\vec{w}$.



The difference $\vec{w}-\vec{v}$ is the displacement vector that, when added to $\vec{v}$, gives $\vec{w}$.



The zero vector, $\vec{0}$, is a displacement vector with zero length.

If λ is a scalar & $\vec{v}$ is a displacement vector, the scalar multiple of $\vec{v}$ by λ, written $\lambda\vec{v}$ is the displacement vector;

* $\lambda\vec{v}$ is parallel to $\vec{v}$, pointing in same direction if $\lambda>0$ and in opposite if $\lambda<0$.
* The magnitude of $\lambda\vec{v}$ is $|\lambda|$ times magnitude of $\vec{v}$, that is $\|\lambda\vec{v}\|=|\lambda|\,\|\vec{v}\|$.

* two parallel vectors have same normal plane equation.

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Parallel Vectors:

Two vectors $\vec{v}$ and $\vec{w}$ are parallel if one is a scalar multiple of the other, if $\vec{w} = \lambda\vec{v}$, for some scalar λ.

We resolve $\vec{v}$ into components by writing $\vec{v}$ in the form

$$\vec{v} = \underbrace{v_1\vec{i} + v_2\vec{j} + v_3\vec{k}}_{\text{components of } \vec{v}}$$

$P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$

$$\boxed{\overrightarrow{P_1P_2} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}}$$ Displacement Vectors.

$$\boxed{\text{Magnitude of } \vec{v} = \|\vec{v}\| = \text{Length of arrow} = \sqrt{v_1^2 + v_2^2 + v_3^2}}$$

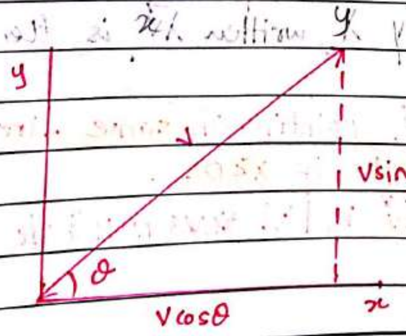
HOW TO RESOLVE A VECTOR INTO COMPONENTS?

$v_1 = v\cos\theta$ and $v_2 = v\sin\theta$

Thus,

length $\|v\|$

$$\vec{v} = (v\cos\theta)\vec{i} + (v\sin\theta)\vec{j}$$



→ Increases $\|\vec{v}\|$ / $\|\vec{u}\|$ increases $\vec{u}\cdot\vec{v}$

→ Increasing θ decreases, $\vec{v}\cdot\vec{u}$ because $\cos\theta$ is decreasing function.

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Unit Vectors.

A unit vector is a vector whose magnitude is 1. The vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors in the directions of coordinate axes.

$$\boxed{\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}}$$

DOT PRODUCT

- Geometric Distribution.

$\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta$ *where θ is the angle b/w $\vec{v}$ and $\vec{w}$.

- Algebraic Definition.

$\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + v_3w_3$

↳ Dot product of two vectors is a number.

Properties of Dot Product:

① $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$

② $\vec{v}\cdot(\lambda\vec{w}) = \lambda(\vec{v}\cdot\vec{w})$

③ $(\vec{v}+\vec{w})\cdot\vec{u} = \vec{v}\vec{u} + \vec{w}\cdot\vec{u}$

→ Two non-zero vectors $\vec{v}$ and $\vec{w}$ are perpendicular →

$$\boxed{\vec{v}\cdot\vec{w} = 0}$$

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Angle b/w two vectors.

$$\boxed{\cos\theta = \frac{u \cdot v}{\|u\|\,\|v\|}}$$

Example:

$\vec{i}+\vec{j}+\vec{k}$ and $\vec{i}-\vec{j}-\vec{k}$

$$\cos\theta = \frac{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-1\end{pmatrix}}{\sqrt{3}\cdot\sqrt{3}} \Rightarrow \frac{-1}{\sqrt{9}}$$

CROSS PRODUCT

- Geometric Definition.

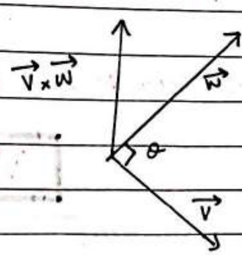
if $\vec{v}$ and $\vec{w}$ are not parallel.

$$\|\vec{v}\times\vec{w}\| = \begin{pmatrix}\text{Area of}\\ \text{Parallelogram}\end{pmatrix}\vec{n} = \left(\|\vec{u}\|\,\|\vec{w}\|\sin\theta\right)\vec{n}$$

- Algebraic Definition.

$$\vec{v}\times\vec{w} = (v_2w_3 - v_3w_2)\vec{i} + (v_3w_1 - v_1w_3)\vec{j} + (v_1w_2 - v_2w_1)\vec{k}$$

Properties of Cross Product



(1) $\vec{w}\times\vec{v} = -(\vec{v}\times\vec{w})$ *

(2) $(\lambda\vec{v})\times\vec{w} = \lambda(\vec{v}\times\vec{w}) = \vec{v}\times(\lambda\vec{w})$

(3) $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\vec{v} + \vec{u}\vec{w}$

128 or 2
72°

$(a+b+c)^2$

$\Rightarrow a^2+b^2+c^2+2ab+2bc+2ac.$

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(1)

$$\text{Area} = \|\vec{v}\|\,\|\vec{w}\|\sqrt{1-\left(\frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}\right.}$$

When you multiply two vectors → cross product of two vectors, ~~the~~ you get another vector whose magnitude is area of parallelogram.

$$\Rightarrow \cancel{\|\vec{v}\|\,\|\vec{w}\|}\;\frac{\sqrt{\|\vec{v}\|^2\|\vec{w}\|^2-(\vec{v}}}{\cancel{\|\vec{v}\|\,\|\vec{w}\|}}$$

$$\Rightarrow \sqrt{(v_1^2+v_2^2+v_3^2)(w_1^2+w_2^2+w_3^2)-(}$$

$$\Rightarrow \sqrt{\cancel{v_1^2w_1^2}+v_1^2w_2^2+v_1^2w_3^2+v_2^2w_1^2+v_2^2}$$

$$-\left[\cancel{v_1^2w_1^2}+\cancel{v_2^2w_2^2}+\cancel{v_3^2w_3^2}-2v_1w_1v_2w_2-2v_2w_2v_3w_3-2v_1w_1v_3w_3\right]$$

$$\Rightarrow \sqrt{(v_1w_2)^2+(v_1w_3)^2+(v_2w_1)^2+(v_2w_3)^2+(w_1v_3)^2+(v_3w_2)^2-2(v_1w_2)(v_2w_1)-2(v_2w_3)(v_3w_2)-2(v_1w_3)(v_3w_1)}$$

$$A \Rightarrow \sqrt{(v_1w_2-v_2w_1)^2+(v_1w_3-v_3w_1)^2+(v_2w_3-v_3w_2)^2}$$

$\hat{k}$ $\hat{j}$ $\hat{i}$

(2)

(ignore row & column)

$$\vec{v}\times\vec{w} = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

alternating signs.

also used to find normal vector.

$$= \hat{i}\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \hat{j}\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \hat{k}\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= (v_2w_3-v_3w_2)\,\hat{i} - (v_1w_3-v_3w_1)\,\hat{j} + (v_1w_2-v_2w_1)\,\hat{k}$$

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NORMAL VECTORS & EQUATION OF PLANE

Let $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ be a normal vector to the plane, let $P_0 = (x_0, y_0, z_0)$ be a fixed point in the plane, and let $P = (x, y, z)$ be any other point. Then

$\overrightarrow{P_0P} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}$. Thus, the vectors $\vec{n}$ and $\overrightarrow{P_0P}$ are perpendicular, so $n \cdot \overrightarrow{P_0P} = 0$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$ → Equation of plane with normal vector.

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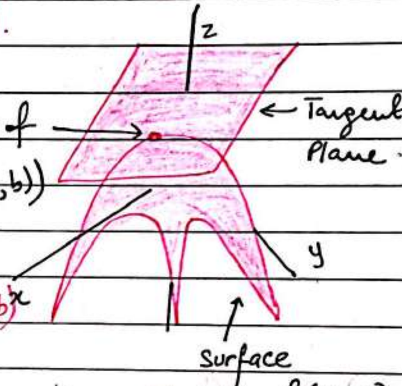
LOCAL LINEARIZATION

→ zooming in on a graph of function until the graph looks like a plane.

→ Zooming in on a contour diagram until the lines look parallel & equally spaced.

① Tangent plane & it's equation.



↳ at Point (a,b)

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

↳ near Point (a,b)

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

(local linearization)

Example

Find an equation for the tangent plane to the surface $z = x^2 + y^2$ at the point $(3,4)$.

(i) $m = f_x(3,4) = 2(x) = 6$

(ii) $n = f_y(3,4) = 2(y) = 8$

(iii) $(x_0, y_0, z_0) \rightarrow (3, 4, 25)$

$z = 3^2 + 4^2 = 25$

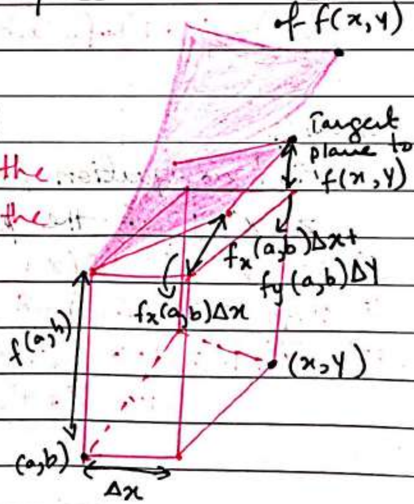
$$\boxed{z = 25 + 6(x-3) + 8(y-4)}$$

Date:

② Local linearity with Three or More Variables.

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

③ Tangent Plane Approx of any surface in 3D near (a,b)



The differential of a function $z = f(x,y)$

The differential df at a point (a,b) is the linear function of dx & dy given by the

$$\boxed{df = f_x(a,b)\,dx + f_y(a,b)\,dy}$$

Date:

Example

compute the differentials of $f(x,y) = x\cos(2x)$

$u = x \quad v = \cos 2x$

$u' = 1 \quad v' = -2\sin 2x$

$= uv' + vu'$

$(x)(-2\sin 2x) + \cos 2x(1)$

$(-2\sin 2x)(x) + \cos 2x$

$df = ((-2x\sin 2x) + \cos 2x)dx + 0.$

PARTIAL DERIVATIVE	DIRECTIONAL DERIVATIVE
→ Derivative in direction of ~~any~~ x and y axes	→ Derivative in the direction of $\hat{u}$ (in any arbitary direction)
→ f_x, f_y	→ $f_{\hat{u}}$ OR $D_{\hat{u}}f$

(i) $= \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

(ii) $f_x(1,0) = 2$

(iii) $f_y(1,0) = 0.$

(iv) $\Delta f = 2\hat{i}$

(v) $\nabla f \cdot \hat{u} = \frac{2}{\sqrt{2}}$

$f_x = 1$

$f_y = e^y = e^1$

$\nabla f = \hat{i} + e^1\hat{j}$

$ye^{x/y}$

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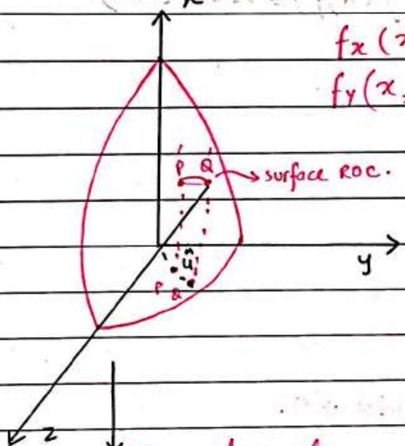
(4) Directional Derivatives

3D



$f_x(x,y)$ $\quad f_{\hat{u}}(x,y)$

$f_y(x,y)$ $\quad D_{\hat{u}} f(x,y)$

$P = (x,y)$

$Q = (x+\Delta x, y+\Delta y$

$\overrightarrow{PQ} = (\cancel{x}+\Delta x-\cancel{x})\hat{i} + (\cancel{y}+\Delta y-\cancel{y})\hat{j}$

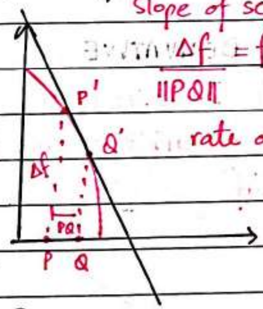
$\overrightarrow{PQ} = \Delta x\hat{i} + \Delta y\hat{j}$

$\overrightarrow{PQ} = h\hat{u}$

$\hat{u} = u_1\hat{i} + u_2\hat{j}$

$h\hat{u} = hu_1\hat{i} + hu_2\hat{j} \rightarrow$ (ii)

$\Delta x = hu_1 \quad \Delta y = hu_2$



Slope of secant

$\dfrac{\Delta f}{\|PQ\|} = f_{\hat{u}}(xy)$

rate of change from P to Q

$f_{\hat{u}}(x,y)$ = slope of secant $= \Delta f / \|\overrightarrow{PQ}\|$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} = \frac{f(x+\Delta x, y+\Delta y)-f(x,y)}{\sqrt{\Delta x^2+\Delta y^2}}$$

$$\lim_{h\to 0} \frac{f(x+hu_1, y+hu_2)-f(x,y)}{h\underbrace{\sqrt{u_1^2+u_2^2}}_{1}}$$

(1)

$$\boxed{f_{\vec{u}}(a,b) = \lim_{h\to 0}\frac{f(a+hu_1, b+hu_2)-f(a,b)}{h}}$$

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$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2$ $\qquad \vec{v} = v_1 \hat{i} + v_2 \hat{j}$

$\vec{w} = w_1 \hat{i} + w_2 \hat{j}$

$f\hat{u} = f_x u_1 + f_y u_2$

$= (f_x \hat{i} + f_y \hat{j}) \cdot (u_1 \hat{i} + u_2 \hat{j})$

$D_{\hat{u}} f = (\text{grad } f) \cdot \hat{u}$ $\qquad \|u\| \, \|f\| \cos\theta$

$\boxed{\Rightarrow \nabla f \cdot \hat{u}}$ (2) $\qquad \boxed{\|\text{grad} f\| \cos\theta}$ (3)

THE DIRECTIONAL DERIVATIVE

if f is differentiable at (a, b) and $\vec{u} = u_1 \hat{i} + u_2 \hat{j}$ is a unit vector, then

$f_{\vec{u}}(a,b) = f_x(a,b) u_1 + f_y(a,b) u_2$

$= \text{grad } f(a,b) \cdot \vec{u}$

$\text{grad } f(a,b) = f_x(a,b)\vec{i} + f_y(a,b)\vec{j}$ THE GRADIENT VECTOR.

$\nabla = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j}$ $\qquad$ (GRAD f)

(i) $\|V\| \longrightarrow \hat{u} = \frac{\vec{v}}{\|\vec{v}\|}$

(ii) $f_x(1,0)$

(iii) $f_y(1,0)$

(iv) ∇f

(v) $\nabla f \cdot \hat{u} = f\hat{u}$

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$f_{\vec{u}}(a,b)$ = Rate of change of f in direction of $\vec{u}$ at (a,b) $= \lim_{h\to 0} \dfrac{f(a+hu_1, b+hu_2)-f(a,b)}{h}$

Example

Calculate the directional derivative of $f(x,y)=x^2+y^2$ at $(1,0)$ in the direction of the vector $\vec{i}+\vec{j}$

Solution Find unit vector $\vec{u} = \frac{1}{\sqrt{2}}(\vec{i}+\vec{j})$

(i) Using first principle: $f_{\vec{u}}(1,0) = \lim_{h\to 0} \dfrac{f(1+h/\sqrt{2}, h/\sqrt{2})-f(1,0)}{h}$

$$= \lim_{h\to 0} \frac{(1+h/\sqrt{2})^2 + (h/\sqrt{2})^2 - 1}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{2}h + h^2}{h} = \lim_{h\to 0}(\sqrt{2}+h) = \sqrt{2}$$

(ii) Using grad f:

$$f_{\vec{u}}(1,0) = f_x(1,0)u_1 + f_y(1,0)u_2 = (2)\left(\frac{1}{\sqrt{2}}\right) + 0 = \sqrt{2}$$

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Example

Find the gradient vector of $f(x,y) = x + e^y$ at point $(1,1)$.

Solution

Using definition, we have

$$grad\, f = f_x \vec{i} + f_y \vec{j} = \vec{i} + e^y \vec{j}$$

so at point $(1,1)$

$$grad\, f(1,1) = \vec{i} + e\vec{j}$$

① $f_{\vec{u}}(a,b) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$

② $f_{\vec{u}}(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2 = grad\, f(a,b) \cdot \vec{u}$

③ $f_{\vec{u}} = grad\, f \cdot \vec{u} = \|grad\, f\| \underbrace{\|\vec{u}\|}_{1} \cos\theta = \|grad\, f\| \cos\theta$

* when more flatness, less magnitude of grad f.

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We calculate directional derivatives of function of three variables, 4D

$$\boxed{f_{\vec{u}}(a,b,c) = f_x(a,b,c)u_1 + f_y(a,b,c)u_2 + f_z(a,b,c)u_3}$$

GRADf IN 4D $\quad \text{grad} f(a,b,c) = f_x(a,b,c)\vec{i} + f_y(a,b,c)\vec{j} + f_z(a,b,c)\vec{k}$

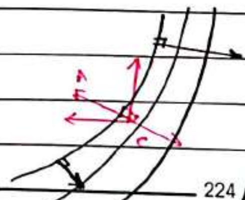
4D → super surface
3D → contour surface.

GEOMETRIC PROPERTIES OF THE GRADIENT VECTOR IN PLANE

If f is a differentiable function at the point (a,b) & $\text{grad} f(a,b) \neq \vec{0}$ then:

* rate of change zero is when ved

- The direction of grad $f(a,b)$ is
  - → perpendicular to the contour graph
  - → in direction of maximum rate of increase of f.
- magnitude of the gradient vector $\|\text{grad} f\|$, is.
  aximum rate of change of f.
  rge when contours are close together & small
  hey are far apart.



Date:

THE CHAIN RULE

Example)

Corn production, C, depends on annual rainfall, R, and average temperature, T, so $C = f(R, T)$. Global warming predicts that both rainfall & temperature depend on time. Suppose that according to a particular model of global warming rainfall is decreasing at 0.2 cm per year & temperature is increasing at 0.1 °C per year. Use the fact that at current levels of production $f_R = 3.3$ and $f_T = -5$ to estimate the current rate of change, dC/dt.

$C = f(R, T) \Rightarrow C = f(g(t), h(t))$

$R = g(t)$, $T = h(t)$

$$\frac{dC}{dt} = \frac{\partial C}{\partial R} \times \frac{dR}{dt} + \frac{\partial C}{\partial T} \times \frac{dT}{dt}$$

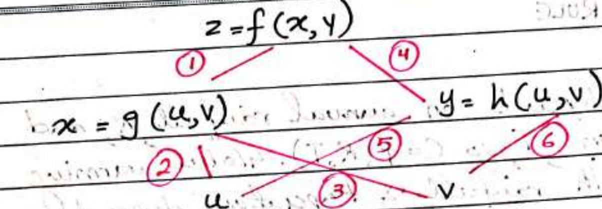
$$\frac{dC}{dt} = f_R \frac{dR}{dt} + f_T \frac{dT}{dt}$$

$$= (3.3)(-0.2) + (-5)(0.1)$$

$$= -1.16 \text{ units of prod/yr.}$$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$ "CHAIN RULE"

Date:



$$\frac{dz}{du} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{dz}{dv} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Second-Order Partial Derivatives of $z = f(x,y)$.

$$\frac{\partial^2 z}{\partial x^2} = f_{xx} = (f_x)_x \qquad \frac{\partial^2 z}{\partial x \partial y} = f_{yx} = (f_y)_x$$

$$\frac{\partial^2 z}{\partial y \partial x} = f_{xy} = (f_x)_y \qquad \frac{\partial^2 z}{\partial y^2} = f_{yy} = (f_y)_y$$

Laplace's Equation $\quad F_{xx} + F_{yy} = 0$

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$\left.\begin{matrix} f_x \\ f_y \end{matrix}\right\}$ Rate of Change / Slope

$y = f(x)$
$y' = f'(x)$
$y'' = f''(x)$

$\left.\begin{matrix} f_{xx} \\ f_{yy} \end{matrix}\right\}$ concavity

$\left.\begin{matrix} f_{xy} \\ f_{yx} \end{matrix}\right\}$ Rate of Change of x in direction of y "twistness"

In 2D:

- Stationary Points ① $f' > 0$, f is increasing
  ② $f' < 0$, f is decreasing.

- Turning points ① $f'' > 0$, concave up
  ② $f'' < 0$, concave down

In 3D:

① $f_x = 0$, $f_y = 0$ $\Rightarrow \nabla f = 0$ } critical points.

② f_{xx} ~~as~~ and D sign will tell nature ~~of~~ derivative.

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to L} f(a)$$

$\rightarrow$ if all true then continuous.

$\hat{u} = a\vec{i} + b\vec{j}$

$\nabla f = f_x\vec{i} + f_y\vec{j}$

Date: $z = f(x, y)$

$$D\hat{u}f = \nabla f \cdot \hat{u} = \underbrace{f_x a}_{z_x} + \underbrace{f_y b}_{z_y} = z^*$$ $\rightarrow$ directional derivative. (1st derivative)

$$D\hat{u}\, z^* = D^2\hat{u}f$$

$$D^2\hat{u}f = \frac{\partial z^*}{\partial x}a + \frac{\partial z^*}{\partial y}b$$

$$= \frac{a\partial}{\partial x}(f_x a + f_y b) + \frac{b\partial}{\partial y}\underbrace{(f_x a + f_y b)}_{\text{replacing } z^*}$$

$$= a^2 f_{xx} + abf_{yx} + abf_{xy} + b^2 f_{yy}$$

$$= a^2 f_{xx} + \underbrace{2abf_{xy}}_{} + b^2 f_{yy}$$

$(a+b)^2 = a^2 + 2ab + b^2$

$f_{yx} = f_{xy}$
(using Clairo's theorem)

$$= f_{xx}\left(a^2 + 2ab\frac{f_{xy}}{f_{xx}} + b^2\frac{f_{yy}}{f_{xx}}\right)$$

$\rightarrow$ completing square.

$$= f_{xx}\left[\underbrace{(a)^2}_{a^2} + \underbrace{2(a)\left(\frac{bf_{xy}}{f_{xx}}\right)}_{2(a)(b)} + \underbrace{\left(\frac{bf_{xy}}{f_{xx}}\right)^2}_{b^2}\right] - \frac{b^2(f_{xy})^2}{(f_{xx})^2} + \frac{b^2 f_{yy}}{f_{xx}}$$

$$= f_{xx}\overbrace{\left[a + \frac{bf_{xy}}{f_{xx}}\right]^2}^{(a+b)^2} + f_{xx}\left(\frac{b^2 f_{yy}}{f_{xx}} - \frac{b^2(f_{xy})^2}{(f_{xx})^2}\right) \leftarrow \text{LCM}$$

$$= f_{xx}\underbrace{\left[a + \frac{bf_{xy}}{f_{xx}}\right]^2}_{>0} + \underbrace{\frac{b^2}{f_{xx}^2}}_{>0}\left(\underbrace{f_{xx}f_{yy} - (f_{xy})^2}_{D}\right)$$

$\uparrow$ determinant

$ad - bc$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

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$\Rightarrow f_{xx}$ and D sign will tell if $D^2_{\hat{u}} f$ is >0 or <0.

OPTIMIZATION IN 3D.. (unconstrained)

→ if $D>0$ and $f_{xx}(x_0,y_0)>0$, then f has a local minimum at (x_0,y_0)

→ if $D>0$ and $f_{xx}(x_0,y_0)<0$, then f has a local maximum at (x_0,y_0).

→ if $D<0$, then f has a saddle point at (x_0,y_0).

→ if $D=0$, anything can happen: f can have a local maximum, or a local minimum, or a saddle point or none of these at (x_0,y_0).

(i) find $f_x=0$ and $f_y=0$ → critical points.

(ii) f_{xx}, f_{xy} and f_{yy}

(iii) $D=f_{xx}f_{yy}-(f_{xy})^2$

(iv)

$D>0$	$D>0$	$D<0$
(a) $f_{xx}>0$	(b) $f_{xx}<0$	(saddle point)
(local minimum)	(local maximum)	

$$\boxed{D=f_{xx}f_{yy}-(f_{xy})^2}$$

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OPTIMIZATION IN 3D

Uncanstrained
(2nd derivative test)

$\Delta f = \vec{0}$

Canstrained
(Langrange multiplier)

(i) find $f_x = 0$ & $f_y = 0$
(ii) find $f_{xx}, f_{yy}, f_{yx}, f_{xy}$
(iii) $D = f_{xx} f_{yy} - (f_{xy})^2$
(iv) ① $D > 0$ and $f > 0$
(local min)
② $D > 0$ and $f < 0$
(local max)
③ $D < 0$ (saddle pt)

(i) $\nabla g = \lambda f$ OR $\nabla f = \lambda \nabla g$ (local extrema on canstraint)
(ii) $f_x = \lambda g_x$, $f_y = \lambda g_y$
(iii) $\frac{f_x}{g_x} = \frac{f_y}{g_y}$ → equation in terms of x and y
(iv) Subs eq ① in the canstraint equation
↳ points (optimum)

close & bounded (→ start & end pt same)
* $\nabla f = \vec{0}$ (inside the region)

not close bounded
* $\nabla f = 0$ X

(v) Substitute points in $f(x,y)$.
(vi) Then insert points to $f(x,y)$ & know nature.

The Meaning of λ

THE LANGRANGIAN FUNCTION

Date:..............................

$$L(x,y,\lambda)$$

$$= f(x,y) - \lambda[g(x,y) - c]$$

→ applicable to not-closed & closed bounded

↳ $g(x,y) = c$

↳ $z = f(x,y)$

↳ $\lambda[g(x,y) - c]$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial \lambda} = 0$$

Example 4: A campany has a production function with three inputs x, y, z given by

$$f(x,y,z) = 50x^{2/5}y^{1/5}z^{1/5}$$

The total budget is \$24,000 and the campany can buy x, y and z at \$80, \$12, and \$10 per unit, respectively. What canbination of inputs will maximise production.

$$L(x,y,\lambda) = 50x^{2/5}y^{1/5}z^{1/5} - \lambda[80x + 12y + 10z - 24000]$$

$$\frac{\partial L}{\partial x} = 20x^{-3/5}y^{1/5}z^{1/5} - 80\lambda = 0$$

$$\frac{\partial L}{\partial y} = 10x^{2/5}y^{1/5}z^{1/5} - 12\lambda = 0$$

$$\frac{\partial L}{\partial z} = 10x^{2/5}y^{1/5}z^{-4/5} - 10\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = -(80x + 12y + 10z - 24000)$$

$$\lambda = \frac{1}{4}x^{-3/5}y^{1/5}z^{1/5},$$

$$\lambda = \frac{5}{6}x^{2/5}y^{-4/5}z^{1/5}$$

$$\lambda = x^{2/5}y^{1/5}z^{-4/5}$$

$$x = 150, \; y = 500, \; z = 600.$$

Date:

DOUBLE INTEGRATION

$$A = \sum_{i=1}^{n} f(x_i^*)\Delta x \qquad 1 \to n \text{ OR } 0 \to n-1$$

$$= \lim_{\substack{\Delta x \to 0 \\ n \to 0}} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx. \leftarrow \text{Area under curve}$$

in 2D

Integral gives Area under the curve.

in 3D

integral gives volume under surface.

Volume of Single Cuboid:

$$V = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

Volume of solid under surface:

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

Date:...............................

Suppose the function f is continuos on R, the rectangle $a \le x \le b$, $c \le y \le d$. If (u_{ij}, v_{ij}) is any point in ij-th subrectangle, we define the definite integral of f over R.

DOUBLE INTEGRAL $\longrightarrow$ $$\int_R f \, dA = \lim_{\Delta x, \Delta y \to 0} \sum_{i,j} f(u_{ij}, v_{ij}) \Delta x \Delta y$$

$$\int_R f \, dA = \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y) \, dx \, dy$$

Example 1 Evaluate $\int_0^{2\pi} \int_0^{2\pi} (\sin x + \cos y + 3) \, dx \, dy$

Date:...............................

FUBINI'S THEOREM

If f is continuous on the rectangle $R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$ then

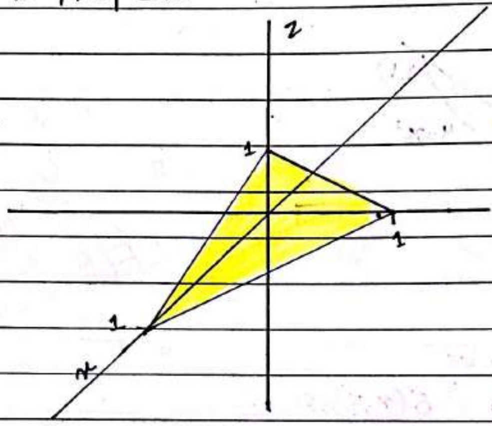
$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Example 2) Evaluate $\int_1^2 \int_0^\pi y\sin(xy)\,dy\,dx$

Date:

Example 3. Sketch the region of the integration for

$$\int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$



Limits on Iterated Integrals.

* the limits on the outer integral must be constants
* the limits on the inner integral can involve only variable in the outer integral. For example, if inner integral is with respect to x, it's limits can be functions of y.

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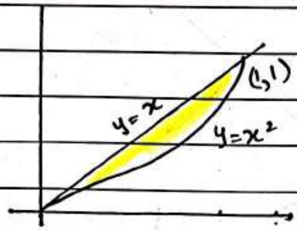
<u>Example 4</u>. Find the mass M of a metal plate R bounded by $y = x$ and $y = x^2$, with density given by $\delta(x,y) = 1 + xy$



Solution

The mass is given by

$$M = \int_R \delta(x,y)\,dA$$

We integrate along vertical strips first; this means we do the y integral first.

$$M = \int_0^1 \int_{x^2}^{x} \delta(x,y)\,dy\,dx = \int_0^1 \int_{x^2}^{x} (1+xy)\,dy\,dx$$

Calculating the inner integral first gives.

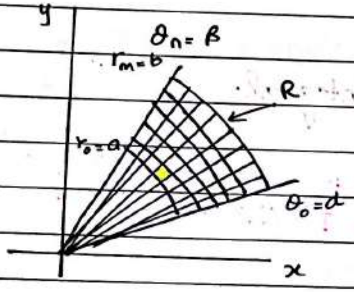
$$M = \int_0^1 \int_{x^2}^{x} (1+xy)\,dy\,dx = \int_0^1 \left(y + \frac{xy^2}{2}\right)\Bigg|_{y=x^2}^{y=x} dx$$

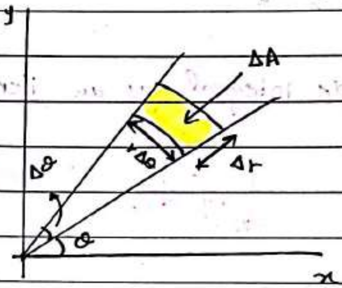
$$\int_0^1 \left(x - x^2 + \frac{x^3}{2} - \frac{x^5}{2}\right)dx = \left(\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{8} - \frac{x^6}{12}\right)\Bigg|_0^1 = \frac{5}{24}\text{ kg.}$$

POOLAR COORDINATES FOR DOUBLE INTEGRALS.

When caputing integrals in polar coordinates, use $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$

Put $dA = r\,dr\,d\theta$ or $dA = r\,d\theta\,dr$

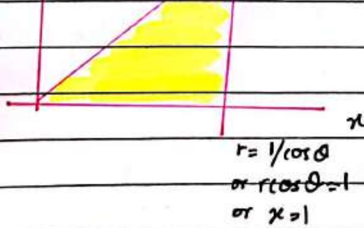




$$\int_R f\,dA = \int_\alpha^\beta \int_a^b f(r,\theta)\,r\,dr\,d\theta$$

Example Sketch the regian of integration $\int_0^{\pi/4} \int_0^{1/\cos\theta} f(r,\theta)\,r\,dr\,d\theta$

* when converting the integrals to polar
↳ first sketch the integrals
↳ then convert the integrals to polar.



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$\int_R f(x,y,z)\,dV$ → Region = 3D

$\int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx$ → Region = cuboid.

Triple Integral as an iterated integral.

$$\int_W f\,dV = \int_p^q \left(\int_c^d \left(\int_a^b f(x,y,z)\,dx \right) dy \right) dz$$

$dA \nearrow dx\,dy$ (cartesian)
$\searrow r\,dr\,d\theta$ (polar).

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$dV \nearrow dx\,dy\,dz$ (cartesian)
$\searrow r\,dr\,d\theta\,dz$ (cylindrical)

* when we have interval of r, we have disk
* when r is fixed it's circle.

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Triple Integrals.

$$\int_W f\, dV = \int_p^q \left(\int_c^d \left(\int_a^b f(x,y,z)\, dx \right) dy \right) dz$$

Limits on Triple Integrals:

- The limits of outer integral are constant
- The limit for the middle integral can involve only one variable (that in outer integral)
- The limits for the inner integral can involve two variables (those on two outer integrals)

$dV \rightarrow dx\,dy\,dz$ (cartesian)

$\rightarrow r\,dr\,d\theta\,dz$ (cylindrical)

Example 1 A cube C has sides of length 4cm and is made of material of variable density. If one corner is at the origin and the adjacent corners are on the positive x, y and z axes, then the density at the point (x,y,z) is $\delta(x,y,z) = 1 + xyz$ gm/cm³. Find the mass of the cube.

Solution: Consider a small piece ΔV of the cube, small enough so that the density remains close to constant over the piece. Then

Mass of small piece = Density · Volume $\approx \delta(x,y,z)\Delta V$

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To get the total mass we add the masses of small pieces & take the limit as $\Delta V \to 0$. Thus, the mass is the triple integral.

$$M = \int_C \delta\, dV = \int_0^4 \int_0^4 \int_0^4 (1 + xyz)\, dx\, dy\, dz$$

$$= \int_0^4 \int_0^4 \left(x + \frac{1}{2}x^2yz\right)\Big|_{x=0}^{x=4} dy\, dz$$

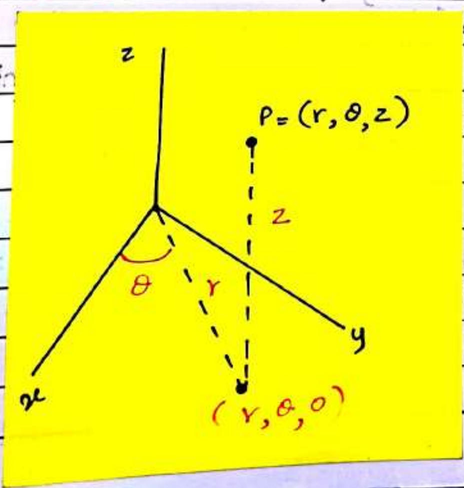
$$= \int_0^4 \int_0^4 (4 + 8yz)\, dy\, dz$$

$$= \int_0^4 4y + 4y^2z \Big|_{y=0}^{y=4} dz \Rightarrow \int_0^4 (16 + 64z)\, dz = 576\text{ gm}$$

CYLINDRICAL

Relation b/w Cartesian & Cylindrical Coordinates

Each point in 3-space using $0 \le r \le \infty$, $0 \le \theta \le 2\pi$, $-\infty \le z \le \infty$



$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$x^2 + y^2 = r^2$

* when [illegible] interval
we have disk, filled cylin

* when r is fixed, we have circle, hollow

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1) Radius r is fixed (θ and z are changing)
↳ hollow cylinder

2) Angle θ is fixed (r and z are changing)
↳ half plane (vertical)

3) Height z is fixed (θ and r are changing)
↳ infinite disk / a plane parallel to xy-plane.

$dV \rightarrow dx\,dy\,dz$ in Cartesian
$\rightarrow r\,dr\,d\theta\,dz$ in Cylindrical.

SPHERICAL

We define spherical coordinates ρ, ϕ and θ for P as follows: $\rho = \sqrt{x^2+y^2+z^2}$ is distance of P from origin; ϕ is angle b/w positive z-axis & line through origin and point P; and θ is the same as cylindrical coordinates.

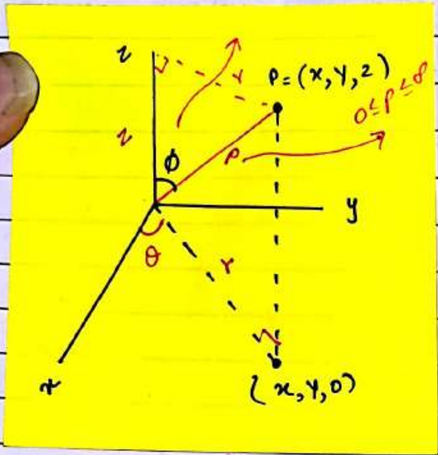
measured from +ve z-axis to -ve z-axis.

$0 \le \phi \le \pi$

$\sin\phi = \frac{r}{\rho}$

$r = \rho\sin\phi$

$\cos\phi = \frac{z}{\rho}$ $\quad z = \rho\cos\phi$



~~$x = r\cos\theta$~~

$y = \rho\sin\phi\sin\theta$

$x = \rho\sin\phi\cos\theta$

$z = \rho\cos\phi$

$r = \rho\sin\phi$

$\rho^2 = x^2 + y^2 + z^2$

(ϕ, θ, ρ)

θ = in xy plane always

1) Distance ρ is fixed

↳ sphere of radius k centred at origin.

2) Angle θ is fixed

↳ half plane with it's edges along z-axis.

3) Angle ϕ is fixed

↳ cone if $k \neq \pi/2$ and xy plane if $k = \pi/2$

$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$x^2 + y^2 + z^2 \le \rho^2$
(ball)

$x^2 + y^2 + z^2 = \rho^2$
(hollow sphere)

parametrization → process

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PARAMETRIC CURVES IN 3D

$\cos(x)$ is → even:

$f(-x) = f(x)$

$\sin x$ is → odd:

$f(-x) = -f(x)$

* Spiral Equation:

$x = \cos t$

$y = \sin t$

$z = t$

* Linear Equation.

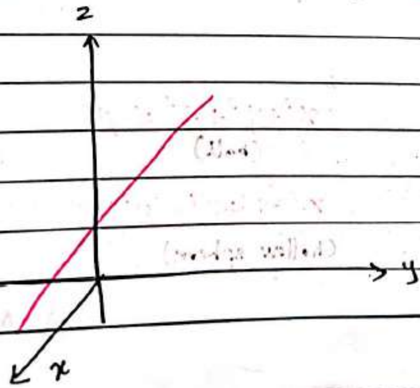
$x = x_0 + at$

$y = y_0 + bt$

$z = z_0 + ct$

$\hat{v} = a\hat{i} + b\hat{j} + c\hat{k}$

* where (x_0, y_0, z_0) are initial points
* a, b. and c, the coefficients of t are partial derivatives/ change in variable



$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

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Parametrization:

position vector

vector-valued function:

$x = \cos t$

$y = \sin t$

$z = t$

parametric equation → $\boxed{\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}}$

→ also requires inter

$\vec{r}(t) = \vec{r_0} + t\vec{v}$

, where $\vec{r_0} = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$

in direction of $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ has parametric equation.

Parametric Equation of Line in Vec

* Parametric equations for a curve e

where a curve intersects a surface.

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VELOCITY

① magnitude of $\vec{v}$ is speed of object

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Example Find parametric equations for the line parallel to vector $2\vec{i}+3\vec{j}+4\vec{k}$ & through point $(1,5,7)$.

Solution

$(1,5,7)$ $\qquad\qquad$ $2\vec{i}+3\vec{j}+4\vec{k}$

$x = x_0 + at \qquad y = y_0 + at \qquad z = z_0 + at$

$x = 1 + 2t \qquad y = 5 + 3t \qquad z = 7 + 4t$

Example Are the lines $x=-1+t$, $y=1+2t$, $z=5-t$ & $x=2+2t$, $y=4+t$, $z=3+t$ parallel? Do they intersect?

$\vec{r} = -\vec{i}+\vec{j}+5\vec{k} + t(\vec{i}+2\vec{j}-\vec{k})$

$\vec{r} = 2\vec{i}+4\vec{j}+3\vec{k} + t(2\vec{i}+\vec{j}+\vec{k})$

Their direction vectors $\vec{i}+2\vec{j}-\vec{k}$ and $2\vec{i}+\vec{j}+\vec{k}$ are not multiples of each other, so lines are not parallel.

To find if they intersect we see if they pass through same point at two different times, t_1 and t_2

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$-1+t_1 = 2+2t_2 \qquad 1+2t_1 = 4+t_2 \qquad 5+t_1 = 3+t_2$

The first two equations give $t_1 = 1$, $t_2 = -1$. Since these values do not satisfy the third equation, the paths do not cross.

VECTOR FIELD

A vector-field in 2-space is a function $\vec{F}(x,y)$ whose value at a point is (x,y). Similarly, a vector field in 3-space is a function $\vec{F}(x,y,z)$ whose values are 3-D vectors

$$\vec{F}(x,y) = -y\vec{i} + x\vec{j} \quad \text{OR} \quad \vec{F}(x,y) = \langle -y, x \rangle$$

Gradient Vector Field.

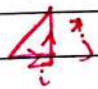
$$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2xy, x^2 - 3y^2 \rangle$$

* plug in (x,y)
  - ↳ this gives a vector
  - ↳ plot the vector

* the length of vector tells strength of field / magnitude

1. we will notice pattern
2. when x increases check the vector magnitude
3. make closed triangle to decide sign.

↳ when vectors are unit length & point outward $\dfrac{\vec{r}}{\|\vec{r}\|}$



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* in level curves, when constantly spaced then there are 'no dots' in gradient fields. (Ref Q27)