Algorithms: Design and Analysis - CS 412

Problem Set 01: Asymptotic Analysis

1. Let

$$p(n) = \sum_{i=0}^{d} a_i n^i$$

where $a_d > 0$, be a degree-d polynomial in n and let k be a constant. Use the definition of the asymptotic notations to prove the following properties:

(a) If $k \geq d$, then $p(n) = O(n^k)$.

Definition of Big-Oh: $f(n) = O(g(n))$ if there exists positive constants c and n_0 such that $0 \leq f(n) \leq c.g(n) \;\; \forall n \geq n_0$

Proof. Choose $c = \sum_{i=0}^{d} |a_i|$ and $n_0 = 1$. Then $\forall n \geq n_0$:

$$p(n) = \sum_{i=0}^{d} a_i n^i \leq \sum_{i=0}^{d} |a_i| n^d \leq \left(\sum_{i=0}^{d} |a_i| \right) n^k = cn^k$$

Since $k \geq d, n^i \leq n^d \leq n^k \;\; \forall i \leq d$, thus $p(n) = O(n^k)$ $\square$

(b) If $k \leq d$, then $p(n) = \Omega(n^k)$.

Definition of Big-Omega: $f(n) = \Omega(g(n))$ if there exists positive constants c and n_0 such that $0 \leq c.g(n) \leq f(n) \;\; \forall n \geq n_0$

Proof. Choose $c = a_d$ and $n_0 = 1$. Then $\forall n \geq n_0$:

$$p(n) = \sum_{i=0}^{d} a_i n^i \geq a^d n^d \geq a_d n^k = cn^k$$

Since $a_d > 0$ and $k \leq d, n^d \geq n^k \;\; \forall n$, thus cn^k is a lower bound for $p(n)$, and $p(n) = \Omega(n^k)$. $\square$

(c) If $k = d$, then $p(n) = \Theta(n^k)$.

Definition of Big-Theta: $f(n) = \Theta(g(n))$ if there exists positive constants c_1, c_2 and n_0 such that $0 \leq c_1.g(n) \leq f(n) \leq c_2.g(n) \;\; \forall n \geq n_0$. Or in other words, $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Proof. From parts (a) and (b), we have shown that if $k \geq d$, then $p(n) = O(n^k)$ and if $k \leq d$, then $p(n) = \Omega(n^k)$. When $k = d$, both conditions are satisfied, which means $p(n)$ is both upper and lower bounded by n^k, hence is both $O(n^k)$ and $\Omega(n^k)$, and therefore $p(n) = \Theta(n^k)$. □

(d) If $k > d$, then $p(n) = o(n^k)$.

Definition of Little-Oh: $f(n) = o(g(n))$ if for every positive constant c, there exists a constant n_0 such that $0 \leq f(n) < c.g(n) \;\; \forall n \geq n_0$

Proof. Given any $c > 0$, choose n_0 such that $n_0^k > \sum_{i=0}^{d} |a_i| n_0^i$. This is possible since $k > d$, and n^k grows faster than any n^i for $i < d$ as n approaches infinity. Then $\forall n \geq n_0$:

$$p(n) = \sum_{i=0}^{d} a_i n^i < \sum_{i=0}^{d} |a_i| n^i < \left(\sum_{i=0}^{d} |a_i| \right) n^k < cn^k$$

The above inequality holds because we can always find an n_0 such that the polynomial sum is less than cn^k for any c, thus $p(n) = o(n^k)$. □

(e) If $k < d$, then $p(n) = \omega(n^k)$.

Definition of Little-Omega: $f(n) = \omega(g(n))$ if for all constants $c > 0$, there exists some constant n_0 such that $0 \leq c.g(n) < f(n) \;\; \forall n \geq n_0$, or $p(n) > cn^k$.

Proof. Let $p(n) = a_d n^d + a_{d-1} n^{d-1} + ... + a_1 n + a_0$, with $a_d > 0$ and $k < d$. Consider the leading term $a_d n^d$, which dominates $p(n)$ as n grows large. For any $c > 0$, we can choose n_0 such that for all $n > n_0$, $a_d n^d > cn^k$. This is because the degree of n^d is higher than n^k, and $a_d > 0$.

Thus, as n approaches infinity, the ratio $p(n)/n^k$ approaches infinity which implies that $p(n)$ grows strictly faster than cn^k for any constant c, proving that $p(n) = \omega(n^k)$. □

2. Indicate for each pair of expressions (A, B) in the table below, whether A is O, o, Ω, ω, or Θ of B. Assume that $k \geq 1$, $\epsilon > 0$, and $c > 1$ are constants. Write your answer in the form of the table with "yes" or "no" written in each box.

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ϵ	yes	yes	no	no	no
b.	n^k	c^n	yes	yes	no	no	no
c.	$\sqrt{n}$	$n^{\sin n}$	no	no	no	no	no
d.	2^n	$2^{n/2}$	no	no	yes	yes	no
e.	$n^{\lg c}$	$c^{\lg n}$	yes	no	yes	no	yes
f.	$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes

3. Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove or disprove each of the following conjectures.

(a) $f(n) = O(g(n))$ implies $g(n) = O(f(n))$.

False. Consider $f(n) = n$ and $g(n) = n^2$. Then $f(n) = O(g(n))$ but $g(n) \neq O(f(n))$.

(b) $f(n) + g(n) = \Theta(\min\{f(n), g(n)\})$.

False. Consider $f(n) = n$ and $g(n) = n^2$. Then $f(n) + g(n) = n + n^2 = O(n^2)$ but $\min\{f(n), g(n)\} = n$, and $n^2 \neq O(n)$.

(c) $f(n) = O(g(n))$ implies $\lg f(n) = O(\lg g(n))$, where $\lg g(n) \geq 1$ and $f(n) \geq 1$ for all sufficiently large n.

True. Suppose that $f(n) = O(g(n))$. Let c and n_0 be positive constants such that $1 \leq f(n) \leq cg(n)$ and $\lg g(n) \geq 1$ for all $n \geq n_0$. Then,

$$\begin{aligned} \lg f(n) &\leq \lg c + \lg g(n) \\ &\leq \lg c \cdot \lg g(n) + \lg g(n) \\ &= (\lg c + 1) \lg g(n) \\ &= O(\lg g(n)). \end{aligned}$$

(d) $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$

False. Consider $f(n) = 2n = O(n)$, and $g(n) = n = O(n)$. It holds that $f(n) = O(g(n))$, but $2^{2n} \neq O(2^n)$. If it were, there would exist n_0 and c such that $n \geq n_0$ implies $2^n.2^n = 2^{2n} \leq c2^n$, so $2^n \leq c$ for $n \geq n_0$ which is clearly impossible since c is a constant.

(e) $f(n) = O((f(n))^2)$.

False. If $f(n) = 1/n$, then $f^2(n) = 1/n^2$. Since there doesn't exist any positive constant c such that $1/n \leq c/n^2$ for arbitrarily large n, then $f(n) \neq O(f^2(n))$.

(f) $f(n) = O(g(n))$ implies $g(n) = \Omega(f(n))$.

True. Suppose that $f(n) = O(g(n))$. Let c and n_0 be positive constants such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$. Dividing all parts of the inequality by c yields $0 \leq f(n)/c \leq g(n)$, and since $1/c > 0$, then $g(n) = \Omega(f(n))$.

(g) $f(n) = \Theta(f(\frac{n}{2}))$

False. Let $f(n) = 2^n$, then $f(n/2) = 2^{n/2} = \sqrt{2^n}$. Suppose that $f(n) = O(f(n/2))$. Then for a positive constant c and for sufficiently large n, it holds $2^n \leq c\sqrt{2^n}$. But then $c \geq \sqrt{2^n}$ and c cannot be a constant. Therefore, $f(n) \neq O(f(n/2))$, which implies $f(n) \neq \Theta(f(n/2))$.

(h) $f(n) + o(f(n)) = \Theta(f(n))$

True. Let $h(n) = o(f(n))$ Then, for any positive constant c there exists a positive constant n_0 such that $0 \leq h(n) < cf(n)$ for all $n \geq n_0$. This implies that

$$\begin{aligned} f(n) &\leq f(n) + o(f(n)) \\ &= f(n) + h(n) \\ &< (c+1)f(n) \\ &< 2f(n), \end{aligned}$$

so $f(n) + o(f(n)) = \Theta(f(n))$

4. Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove the following identities.

(a) $\Theta(\Theta(f(n))) = \Theta(f(n))$

Let $p(n) = \Theta(f(n))$, and let c_1, c_2, and n_p be positive constants such that

$$0 \le c_1 f(n) \le p(n) \le c_2 f(n)$$

for all $n \ge n_p$. Also, let $q(n) = \Theta(p(n))$ and let d_1, d_2, and n_q be positive constants such that

$$0 \le d_1 p(n) \le q(n) \le d_2 q(n)$$

for all $n \ge n_q$. Then, for all $n \ge \max\{n_p, n_q\}$,

$$\begin{aligned} 0 &\le c_1 d_1 f(n) \\ &\le d_1 p(n) \\ &\le q(n) \\ &\le d_2 p(n) \\ &\le c_2 d_2 f(n), \end{aligned}$$

which implies that $q(n) = \Theta(f(n))$.

(b) $\Theta(f(n)) + O(f(n)) = \Theta(f(n))$

Let $p(n) = \Theta(f(n))$ and $q(n) = O(f(n))$. Then there exist positive constants c_1, c_2, d, n_p, and n_q such that

$$0 \le c_1 f(n) \le p(n) \le c_2 f(n)$$

for all $n \ge n_p$, and

$$0 \le q(n) \le df(n)$$

for all $n \ge n_q$. Then, for all $n \ge \max n_p, n_q$,

$$\begin{aligned} 0 &\le c_1 f(n) \\ &\le p(n) \\ &\le p(n) + q(n) \\ &\le c_2 f(n) + df(n) \\ &= (c_2 + d) f(n), \end{aligned}$$

which implies that $p(n) + q(n) = \Theta(f(n))$.

(c) $\Theta(f(n)) + \Theta(g(n)) = \Theta(f(n) + g(n))$

Let $p(n) = \Theta(f(n))$ and $q(n) = \Theta(g(n))$. Then there exist positive constants c_1, c_2, d_1, d_2, n_p, and n_q such that

$$0 \le c_1 f(n) \le p(n) \le c_2 f(n)$$

for all $n \geq n_p$, and

$$0 \leq d_1 g(n) \leq q(n) \leq d_2 g(n)$$

for all $n \geq n_q$. Then, for all $n \geq \max n_p, n_q$,

$$\begin{aligned}
0 &\leq \min\{c_1, d_1\}(f(n) + g(n)) \\
&\leq c_1 f(n) + d_1 g(n) \\
&\leq p(n) + q(n) \\
&\leq c_2 f(n) + d_2 g(n) \\
&\leq \max c_2, d_2(f(n) + g(n)),
\end{aligned}$$

which implies that $p(n) + q(n) = \Theta(f(n) + g(n))$.

(d) $\Theta(f(n)).\Theta(g(n)) = \Theta(f(n).g(n))$

For the same functions and constants as in the previous part, it is true that for all $n \geq \max\{n_p, n_q\}$,

$$\begin{aligned}
0 &\leq \min\{c_1, d_1\}^2(f(n) \cdot g(n)) \\
&\leq c_1 f(n) \cdot d_1 g(n) \\
&\leq p(n) \cdot q(n) \\
&\leq c_2 f(n) \cdot d_2 g(n) \\
&\leq \max c_2, {d_2}^2(f(n) \cdot g(n)),
\end{aligned}$$

which implies that $p(n) \cdot q(n) = \Theta(f(n) \cdot g(n))$.