UNIFORM RANDOM VARIABLE

$S_X = \{1, 2, \dots, L\}$

$p_k = \frac{1}{L} \quad k = 1, 2, \dots, L$

$E[X] = \frac{L+1}{2}$

$VAR[X] = \frac{L^2 - 1}{12}$

BERNOULLI RANDOM VARIABLE

$S_X = \{0, 1\}$

$p_0 = q = 1 - p \quad p_1 = p$

$E[X] = p$

$VAR[X] = p(1-p)$

→ has only two possible outcomes: success or failure

BINOMIAL RANDOM VARIABLE

$S_X = \{0, 1, \dots, n\}$

$p_k = \binom{n}{k} p^k (1-p)^{n-k}$

$E[X] = np$

$VAR[X] = np(1-p)$

→ describes probability of k successes out of n trials.

GEOMETRIC RANDOM VARIABLE

① Version $S_X = \{0, 1, 2, \dots\}$

$p_k = p(1-p)^k \quad k = 0, 1, \dots$

$E[X] = \frac{1-p}{p}$

$VAR[X] = \frac{1-p}{p^2}$

② Version $S_X = \{1, 2, \dots\}$

$p_k = p(1-p)^{k-1}$

$E[X'] = \frac{1}{p}$

$VAR[X'] = \frac{1-p}{p^2}$

GEOMETRIC CONTD.

→ $P(X = n)$: $(1-p)^{n-1} p$

→ $P(X < n)$: $1 - q^{n-1}$

→ $P(X \le n)$: $1 - (1-p)^n$

→ $P(X > n)$: $(1-p)^n$

→ $P(X \ge n)$: $(q)^{n-1}$

POISSON RANDOM VARIABLE

$S_X = \{0, 1, 2, \dots\}$

$p_k = \frac{\lambda^k}{k!} e^{-\lambda}$ ∴ where $\lambda = np$

$E[X] = \lambda$

$VAR[X] = \lambda$

→ n is very LARGE

→ p is very SMALL

MEAN/EXPECTATION

$E[X] = \sum_x x \, P(X = x)$

→ $E(aX + b) = aE(X) + b$

→ $E(aX^2 + bX + c) = aE[X^2] + bE[X] + c$

→ $Var(aX) = a^2 Var(X)$ → $Var(X + b) = Var(X)$

VARIANCE

$VAR(X) = E[X^2] - (E[X])^2$

MEMORY LESS PROPERTY

→ what is $P(X > a+b \mid X > a)$?

$= \frac{P(X > a+b)}{P(X > a)} = \frac{(1-p)^{a+b}}{(1-p)^a}$

$= (1-p)^b$

$= P(X > b)$

→ what is $P(X \le a+b \mid X > a)$?

$= \frac{P(a < X \le a+b)}{P(X > a)}$

$= \frac{P(X > a) - P(X > a+b)}{(1-p)^a}$

$= \frac{(1-p)^a - (1-p)^{a+b}}{(1-p)^a}$

$= 1 - (1-p)^b = P(X \le b)$

CONDITIONAL PMF

e.g. $P(X = k \mid X > 1) = \frac{P(X = k)}{P(X > 1)}$

$P_{X|A}(x_i) = \frac{P(X = x_i \mid A)}{P(A)}$

CONDITIONAL EXPECTATION.

$E[X \mid A] = \sum_x x \, P(X = x \mid A)$

TOTAL EXPECTATION THEOREM

$E[X] = \sum_i E[X \mid A_i] \, P(A_i)$

MARGINAL PMF

$P_X(x) = \sum_j p(x, y_j)$

$P_Y(y) = \sum_i p(x_i, y)$

• $\sum_x P(X = x) = 1$

JOINT PMF

$p(x, y) = P(X = x$ and $Y = y$

① $0 \le p(x, y) \le 1$

② $\sum\sum_{(x,y)} p(x, y) = 1$

③ $P((X, Y) \in A) = \sum\sum_{x,y \in A} p($

→ condition ① and ② are required for $p(x, y)$ to be VALID JOINT PMF

JOINT CDF

$F_{X,Y}(x, y) = P(X \le x, Y \le y)$

$= \sum_{X \le x} \sum_{Y \le y} p(x, y)$

EXPECTATION of JOINT DRV.

$E[g(X, Y)] = \sum\sum_{(x,y)} g(x, y) \, p(x, y)$

SUMMATION FORMULAS.

→ $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

→ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

→ $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

→ $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$

→ $\sum_{i=1}^{\infty} a r^{i-1} = \frac{a}{1-r}$

→ $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$

DERIVATION OF POISSON MEAN & VARIANCE

$$E[X] = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} = \lambda \underbrace{\sum_{k'=0}^{\infty} \frac{\lambda^{k'}}{k'!} e^{-\lambda}}_{1} = \lambda$$

$$E[X^2] = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} = \lambda \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda}$$

$$= \lambda \sum_{k'=0}^{\infty} (k'+1) \frac{\lambda^{k'}}{k'!} e^{-\lambda} = \lambda \{\lambda + 1\}$$

$$VAR(X) = E[X^2] - (E[X])^2$$

$$= \lambda \{\lambda + 1 - \lambda\} = \lambda$$

DERIVATION OF MEAN & VARIANCE OF UNIFORM VARIABLE

$$E[X] = \sum_{k=1}^{n} k\, P[X=k] = \sum_{k=1}^{n} \frac{k}{n} = \frac{1}{n} \sum_{k=1}^{n} k = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

$$Var(X) = \sum_{k=1}^{n} \frac{k^2}{n} - \left(\frac{n+1}{2}\right)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}$$

INDEPENDENT RANDOM VARIABLE

→ if $X_1, X_2, \ldots X_n$ are independent

$$P(x_1, x_2, \ldots, x_n) = P_{X_1}(x_1) \times \cdots \times P_{X_n}(x_n)$$

→ $P(X=x \mid Y=y) = P(X=x)$

→ $E[XY] = E[X]\,E[Y]$

→ if $P(X,Y) \neq P_X(X)\,P_Y(Y)$

then they are NOT INDEPENDENT

→ $VAR(X+Y) = VAR(X) + VAR(Y)$

→ $var(X_1 + X_2 + X_3 + \ldots + X_n) = var(X_1) + \ldots + var(X_n)$

Q) The loose minute hand of clock is spun hard. The coordinates (x,y) of point whee the tip of hand caues to rest is noted. z is defined as the sgn function of the product of x and y, where $\text{sgn}(t)$ is 1 if $t>0$, 0 if $t=0$ and -1 if $t<0$

(a) Describe underlying space S.

$S = \{(x,y) : x^2+y^2 = r^2\}$ where r = radius of circle.

The set S has uncountably infinitely many points. All points in S are equally likely

$P[(i,j)=0]$ for all $(i,j) \in S$.

(b) Show mapping from S to S_x, the range of X.

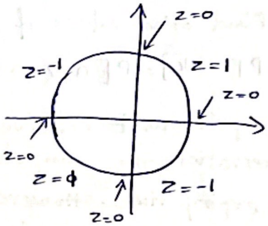
$S_z = \{-1, 0, 1\}$

$S \longrightarrow S_z$

$x>0, y<0$ or $x<0, y>0 \longrightarrow -1$

$x=0$ or $y=0 \longrightarrow 0$

$x>0, y>0$ or $x<0, y<0 \longrightarrow 1$



'Quadrant'
S=4.

(c) find probabilities of various values of X.

$P(Z=-1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$P(Z=0) = 0$

$P(Z=1) = \frac{1}{4} + \frac{1}{4} = 1/2$

Q) Two transmitters send messages through bursts of radio signals to an antena. During each time slot each transmitter sends a message with probability 1/2. Simultaneous transmissions result in loss of messages. Let X be number of time slots until first message gets through.

(a) " "

$S = \{S, FS, FFS, FFFS, FFFFS, \dots\}$

$P(S) = 1/2$ OR $(1/2 \times 1/2) + (1/2 \times 1/2)$

$P(F) = 1 - 1/2 = 1/2$

$P[F \dots FS] = \underbrace{P(F) \dots P(F)}_{k-1} P(S)$

$= (1/2)^k$

(b) " "

$S_x = \{1, 2, 3, \dots\} = \mathbb{N}$

$S \longrightarrow 1$

$FS \longrightarrow 2$

$FFS \longrightarrow 3$

$FFFS \longrightarrow 4$

$\vdots \qquad \vdots$

(c) $P(X=1) = 1/2$

$P(X=2) = 1/2 \times 1/2 = 1/2^2$

$P(X=3) = 1/2^3$

$P(X=k) = 1/2^k$

• Suppose terminal 1 transmits with $p = 1/2$ in given time slot but terminal 2 with p.

(a) find pmf.

$P_{success} = \frac{1}{2} q + \frac{1}{2} p = \frac{1}{2}$

		term 2: p	1-p
term 1	1/2	1/2 p	1/2 q
	1/2	1/2 p	1/2 q

Q) A coin is tossed n times. Let the random variable Y be the difference b/w number of heads and number of tails in the n tosses of coin. Assume $P[\text{heads}] = p$.

(a) Describe " "

let m be number of tails $0 \le m \le n$

the number of heads is $n-m$ and difference is $Y = n-m-m = n-2m$

$$\therefore S_Y = \{-n, -n+2, \dots, n-2, n\}.$$

(b) Find probability of event $\{Y=0\}$

$$P[Y=0] = P[n=2m] = P[m = n/2] \text{ for } n \text{ even.}$$

(c) Find other values of Y.

$$P[Y=k] = P[n-2m=k] = P\left[m = \frac{n-k}{2}\right]$$

Q) A computer must reserves a path in a network for 10 minutes. To extend the reservation the computer must successfully send a "refresh" message before expiry time. However, messages are lost with probability 1/2. Suppose that it takes 10 seconds to send a refresh request and receive acknowledgement. When the should the computer start sending refresh messages in order to have 99% chance of successfully extending reservation time?

$X \sim$ be number of transmission until success.

$$P[X \le k] = \sum_{j=1}^{k} (1/2)^j = 1/2 \sum_{j=0}^{k-1} (1/2)^j = \frac{1}{2} \cdot \frac{1-(1/2)^k}{1/2} = 1-(1/2)^k$$

$$1-(1/2)^k = 0.99$$

$$(1/2)^k = 0.01$$

$$k = \frac{\ln(100)}{\ln(2)} = 6.64 \approx 7$$

start sending refresh messages

7×10 seconds before expiry time.

(b) Find the mean and variance of time that it takes to renew the reservation.

$$P[X=j] = (1/2)^j$$

$$E[X] = \sum_{j=1}^{\infty} j\,(1/2)^j$$

$$= \frac{1}{2}\sum_{j=1}^{\infty} j\,(1/2)^{j-1}$$

$$= \frac{1}{2}\sum_{j=0}^{\infty} j\,(1/2)^j$$

from geometric series we have $\sum_{j=0}^{\infty} \alpha^j = \frac{1}{1-\alpha}$

$$\therefore \frac{d}{dx}\sum_{j=0}^{\infty} \alpha^j = \sum_{j=0}^{\infty} j\alpha^{j-1}$$

$$= \frac{1}{(1-\alpha)^2}$$

$$\therefore \sum_{j=0}^{\infty} j\,(1/2)^{j-1} = \frac{1}{(1-1/2)^2} = 4$$

and $E[X] = \frac{1}{2} \times 4 = 2.$

4) (a) Suppose a fair coin is tossed n times. Each coin toss costs d dollars and the reward in obtaining X heads is aX^2+bX. Find the expected value of net reward.

$$P[X=k] = \binom{n}{k}\left(\frac{1}{2}\right)^n$$

$$E[aX^2+bX] = aE[X^2] + bE[X]$$

$$E[X] = \sum_{j=0}^{n} j\binom{n}{j}\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^n \sum_{j=0}^{n} j\,\frac{n!}{j!\,(n-j)!}$$

$$= \left(\frac{1}{2}\right)^n \sum_{j=1}^{n} \frac{n!}{(j-1)!\,(n-j)!} \qquad \text{let } j' = j-1$$

$$= \left(\frac{1}{2}\right)^n n \sum_{j'=0}^{n-1} \frac{(n-1)!}{j'!\,(n-1-j')!} = n\left(\frac{1}{2}\right)^n \sum_{j'=0}^{n-1}\binom{n-1}{j'}$$

$$= n\left(\frac{1}{2}\right)^n 2^{n-1} = \frac{n}{2}$$

$$E[X^2] = \sum_{j=0}^{n} j^2\binom{n}{j}\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^n n \sum_{j=1}^{n} j\,\frac{(n-1)!}{(j-1)!\,(n-j)!}$$

$$= n\left(\frac{1}{2}\right)^n \sum_{j'=0}^{n-1} (j'+1)\binom{n-1}{j'}$$

$$= \frac{n}{2}\left[\frac{n}{2}+1\right] \Rightarrow E[aX^2+bX] = a\frac{n}{2}\left(\frac{n}{2}+1\right) + b\frac{n}{2}$$

(b) Suppose that reward in obtaining X heads is a^X where $a>0$. Find expected value of reward.

$$E[a^X] = \sum_{j=0}^{n} a^j\binom{n}{j}\left(\frac{1}{2}\right)^j = \sum_{j=0}^{n}\binom{n}{j}\left(\frac{a}{2}\right)^j$$

$$= \left(1+\frac{a}{2}\right)^n$$

5) Let M be a geometric random variable. Show that M satisfies memoryless property: $P[M\geq k+j \mid M\geq j+1] = P[M\geq k]$.

$$P[M\geq k+j \mid M>j] = \frac{P[M\geq k+j,\, M>j]}{P[M>j]} = \frac{P[M\geq k+j]}{P[M>j]}$$

$$= \frac{\sum_{i=k+j}^{\infty} p(1-p)^{i-1}}{\sum_{i'=j+1}^{\infty} p(1-p)^{i'-1}} = \frac{(1-p)^{k+j-1}}{(1-p)^j} = (1-p)^{k-1}$$

$$= P[M\geq k]$$

Q) Let X be binomial random variable.

(a) Show that

$$\frac{P_X(k+1)}{P_X(k)} = \frac{n-k}{k+1}\,\frac{p}{1-p} \qquad \text{where } P_X(0) = (1-p)^n$$

$$\frac{P_k}{P_{k-1}} = \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}} = \frac{\dfrac{n!}{k!\,(n-k)!}\,p}{\dfrac{n!}{(k-1)!}\,q} = \frac{(n-k+1)p}{kq}$$

$$\frac{P_k}{P_{k-1}} = \frac{(n+1)p - k(1-q)}{kq} = 1 + \frac{(n+1)p-k}{kq}$$

(b) Show that part a implies that: (1) $P[X=k]$ is maximum at $k_{max} = [(n+1)p]$, where $[x]$ denotes largest integer that is smaller than or equal to x: and (2) when $(n+1)p$ is an integer then maximum is achieved at k_{max} and $k_{max}-1$.

Suppose $(n+1)p$ is not an integer, then

for $0 \le k \le [(n+1)p] < (n+1)p$

$$(n+1)p - k > 0$$

so $$\frac{P_k}{P_{k-1}} = 1 + \frac{(n+1)p-k}{kq} > 1 \Rightarrow P_k \text{ increases as } k \text{ increases from } 0 \text{ to } [(n+1)p]$$

$$(n+1)p - k < 0$$

so $$\frac{P_k}{P_{k-1}} = 1 + \frac{(n+1)p-k}{kq} < 1 \Rightarrow P_k \text{ decreases as } k \text{ increases}$$

P_k attains it's maximum at $k_{MAX} = [(n+1)p]$

if $(n+1)p = k_{MAX}$ then above implies that

$$\frac{P_{k_{MAX}}}{P_{k_{MAX}-1}} = 1 \Rightarrow P_{k_{MAX}} = P_{k_{MAX}-1}.$$

Q) Show $E[X \mid X>1] = E[X]+1$.

$$P(X=k \mid X>1) = (1/2)^{k-1}$$

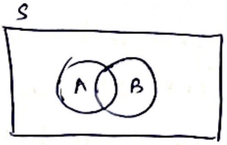
$$E[X \mid X>1] = \sum_{k=2}^{\infty} k\,(1/2)^{k-1} = \sum_{k'=1}^{\infty} (k'+1)(1/2)^{k'}$$

$$= \sum_{k'=0}^{\infty} k'\,(1/2)^{k'} + \sum_{k'=1}^{\infty} (1/2)^{k'}$$

$$= E[X] + 1 \quad \text{(proven)}$$

Q) Let A and B be events for random experiment with sample space S. Show that the Bernoulli Random Variable satisfies following properties:

(a) $I_S = 1$ and $I_\phi = 0$



$I_S = 1$ iff $\xi \in S \Rightarrow I_S = 1$ all ξ

$I_\phi = 0$ iff $\nexists\, \xi \in \phi \Rightarrow I_\phi = 0$ all ξ

(b) $I_{A\cap B} = I_A I_B$ and $I_{A\cup B} = I_A + I_B$

$I_{A\cap B}(\xi) = 1$ iff $\xi \in A$ and $\xi \cap B \Leftrightarrow I_A(\xi)=1$ and $I_B(\xi)=1$

$\Leftrightarrow I_{A\cap B}(\xi) = I_A(\xi) I_B(\xi)$

$I_{A\cup B}(\xi) = 0$ iff $\xi \notin A\cup B \Leftrightarrow \xi \in A^c \cap B^c \Leftrightarrow I_{A^c}(\xi)\, I_{B^c}(\xi) = 1$

$\Leftrightarrow (1 - I_A(\xi))(1 - I_B(\xi)) = 1$

$\Leftrightarrow 1 - I_A(\xi) - I_B(\xi) + I_A(\xi) I_B(\xi) = 1$

$\Leftrightarrow I_A(\xi) + I_B(\xi) - I_A(\xi) I_B(\xi) = 0$

$\Leftrightarrow I_A(\xi) + I_B(\xi) - I_{A\cap B}(\xi) = 0 = I_{A\cup B}(\xi)$

(c) find expected value of I_A

$E[I_S] = 1 \cdot P[S] = 1$

$E[I_\phi] = 1 \cdot P[\phi] = 0$

$E[I_{A\cap B}] = 1 \cdot P[A\cap B]$

$E[I_{A\cup B}] = E[I_A] + E[I_B] - E[I_{A\cap B}]$

$= P[A] + P[B] - P[A\cap B]$

Q) Let X be a binomial random variable that results from performance of n Bernoulli trials with probability of success p.

(a) Suppose that X = 1. Find probability that single event occured in k-th Bernoulli trial.

$$P[I_k=1 \mid X=1] = \frac{P[I_k=1 \text{ and } I_j=0 \text{ for all } j\neq k]}{P[X=1]}$$

$$= \frac{P[0\ 0..1\ 0...0]}{P[X=1]} = \frac{p(1-p)^{n-1}}{\binom{n}{1}p(1-p)^{n-1}} = \frac{1}{n}$$

(b) Suppose X = 2. Find probability that two events occured in jth and kth Bernoulli trials where j<k.

$$P[I_j=1, I_k=1 \mid X=2] = \frac{P[I_j=1, I_k=1, I_m=0 \text{ for all } m\neq j,k]}{P[X=2]}$$

Q) Consider the expression $(a+b+c)^n$.

(a) Use binomial expansion for (a+b) and c to obtain an expression for $(a+b+c)^n$.

$$\sum_{k=0}^{n} \binom{n}{k} (a+b)^k c^{n-k}$$

(b) Now expand all terms of $(a+b)^k$ and obtain an expression that involves the multinomial coefficient for M=3 mutually exclusive A_1, A_2, A_3.

$$= \sum_{k=0}^{n} \binom{n}{k} c^{n-k} \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{n!\,k!}{k!\,(n-k)!\,j!\,(k-j)!} a^j b^{k-j} c^{n-k}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{n!}{j!\,(n-k)!\,(k-j)!} a^j b^{k-j} c^{n-k}$$

$$= \sum_{j_1} \sum_{j_2} \sum_{j_3} \frac{n!}{j_1!\,j_2!\,j_3!} a^{j_1} b^{j_2} c^{j_3}$$

(c) Let $P_1 = P[A_1]$ $P_2 = P[A_2]$, $P_3 = P[A_3]$. Use result from part b to show that multinomial probabilities add to one.

$$1 = (P_1+P_2+P_3)^n = \sum_{j_1 j_2 j_3} \frac{n!}{j_1!\,j_2!\,j_3!} P_1^{j_1} P_2^{j_2} P_3^{j_3}$$

Q) Let X be discrete random variable that assumes only nonnegative integer values and that satisfies memoryless property. Show that X must be geometric random variable.

$$P[M \geq k] = P[M \geq k+j \mid M > j]$$

$$= \frac{P[M \geq k+j]}{P[M > j]} = \frac{P[M \geq k+j]}{P[M \geq j+1]}$$

$\Rightarrow$

$$P[M \geq k+j] = P[M \geq k] P[M \geq j+1]$$

let $a_k = P[M \geq k]$ then we have

$$(*) \quad a_{k+j} = a_k a_{j+1} \quad j \geq 1, k \geq 1$$

where $a_1 = 1$ and $a_2 = 1 - P[M=1] = 1-p$

$$a_{k+1} = a_2 a_k \rightarrow k \geq 1$$

$$\Rightarrow a_k = a_2^{k-1}$$

$$\Rightarrow P[M \geq k] = (1-p)^{k-1} \rightarrow k \geq 1$$

$$P[M=k] = P[M \geq k] - P[M \geq k+1]$$

$$= (1-p)^{k-1} - (1-p)^k$$

$$= (1-p)^{k-1}(1-(1-p))$$

$$= (1-p)^{k-1} p.$$