1 LINEAR ALGEBRA

LECTURE 7

MATH 205

REVISION:

IF $\boxed{A}$ IS AN SQUARE MATRIX → INVERTIBLE
THEN $\boxed{A^{-1}}$ COULD BE EVALUATED BY THE FOLLOWING METHOD:

$$[A \,|\, I] \xrightarrow{\text{E.R.O.S.}} [I \,|\, A^{-1}]$$

I → IDENTITY MATRIX

$\boxed{\text{E.R.O.S}}$ → ELEMENTARY ROW OPERATIONS

EXAMPLE:

LET $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ THEN USING THE TECHNIQUE

$$[A \,|\, I] \xrightarrow{\text{e.r.o.s}} [I \,|\, A^{-1}]$$

PROVE THAT

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad \boxed{ad-bc \neq 0}$$

SOLUTION:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & b/a & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right] \quad \boxed{R_1 \to \frac{R_1}{a}}, \; a \neq 0$$

$$\sim \left[\begin{array}{cc|cc} 1 & b/a & \frac{1}{a} & 0 \\ 0 & d - \frac{cb}{a} & -\frac{c}{a} & 1 \end{array}\right] \quad \boxed{R_2 \to R_2 - cR_1} \quad \left(\frac{ad-bc}{a}\right)$$

$$= \left[\begin{array}{cc|cc} 1 & b/a & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & b/a & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\boxed{R_2 \to \frac{a}{(ad-bc)} R_2}, \; ad-bc \neq 0$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\boxed{R_1 \to R_1 - \frac{b}{a} R_2}$$

WHICH COMPLETES THE PROOF.

RESULT: FOR $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

PROVIDED $ad-bc \neq 0$

THIS NUMBER $ad-bc$ IS CALLED THE DETERMINANT OF MATRIX A AND IS WRITTEN AS

$$\boxed{ad-bc} = \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$
$$= |A|$$

HERE A IS INVERTIBLE,
$\therefore$ A IS A SQUARE MATRIX

SO WE HAVE THE FOLLOWING RESULT: (P. 82 8th ED. OR P. 79 7th ED.)

DETERMINANT OF A NONSQUARE MATRIX IS NOT DEFINED.

ALSO IF A^{-1} EXISTS THEN $\det(A) \neq 0$

4]

RESULT: IF $A\underline{X}=\underline{B}$ IS THE SYSTEM OF n LINEAR EQUATIONS IN n UNKNOWNS THEN:

(1) UNIQUE SOLUTION IF A IS INVERTIBLE ($\det(A) \neq 0$) AND

$$\underline{X} = A^{-1}\underline{B}$$

(2) IF A^{-1} DOESN'T EXIST

NO SOLUTION

INFINITE SOLUTIONS

LET US CONSIDER EXAMPLES WHEN $A\underline{X}=\underline{B}$ IS THE SYSTEM OF TWO LINEAR EQUATIONS IN TWO UNKNOWNS

(1) FOR UNIQUE SOLUTION.

CONSIDER THE FOLLOWING SYSTEM: $(\underline{X} = A^{-1}\underline{B})$

5

CONSIDER

$$3x_1 + 4x_2 = 7 \quad \text{— } ①$$

$$x_1 - x_2 = 0 \quad \text{— } ②$$

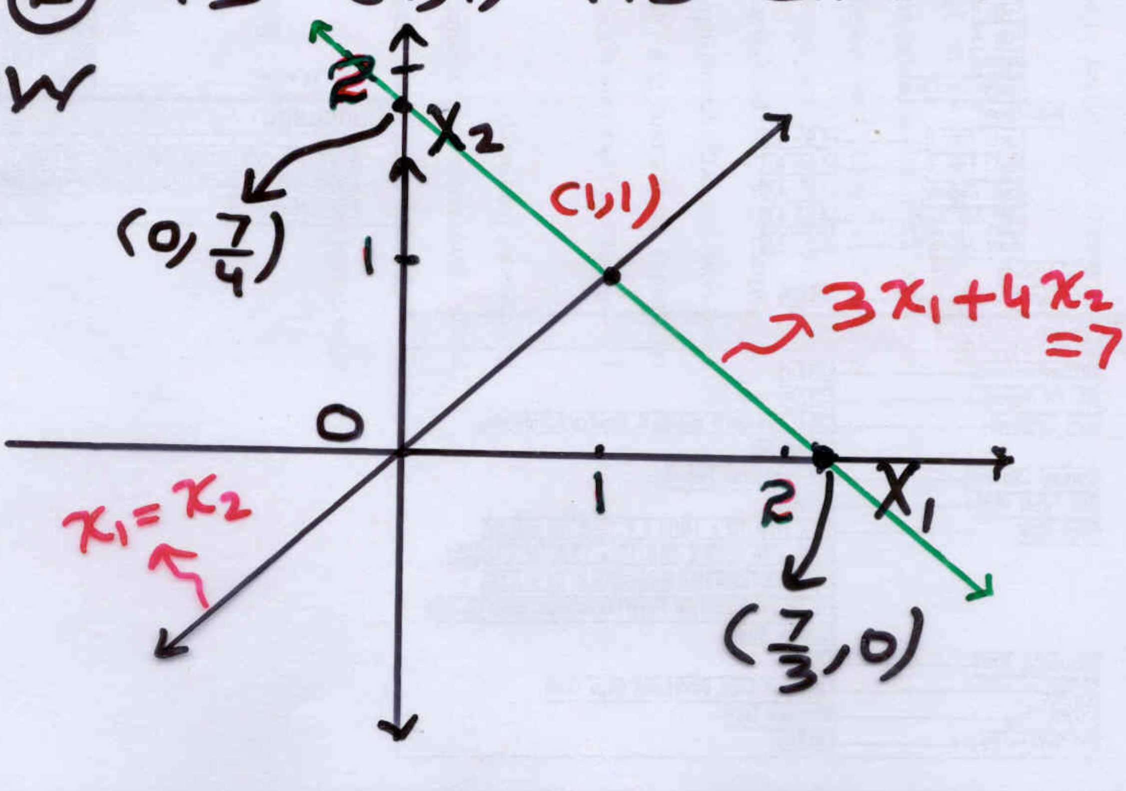
$$\Rightarrow \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

NOW $\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} = -3 - 4 = -7 \neq 0$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 0 \end{bmatrix} \qquad (A^{-1}\underline{B} = \underline{X})$$

$$= -\frac{1}{7} \begin{bmatrix} -1 & -4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \end{bmatrix} = - \begin{bmatrix} -1 & -4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ i.e. POINT OF INTERSECTION OF LINES ① AND ② IS (1,1) AS SHOWN BELOW



(2) IF A^{-1} DOESN'T EXIST

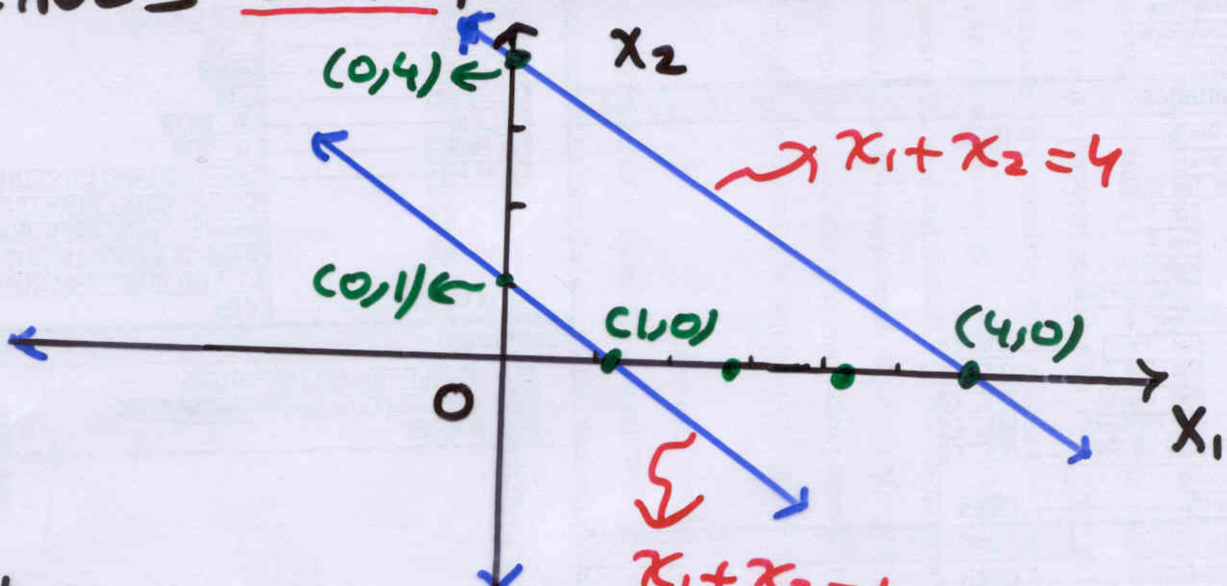
(a) NO SOLUTION

(b) INFINITE SOLUTIONS

(a) CONSIDER $\boxed{\begin{matrix} x_1 + x_2 = 4 \\ x_1 + x_2 = 1 \end{matrix}}$ $4 \neq 1$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$ (NO SOLUTION)

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0$$ i.e. PARALLEL LINES DON'T INTERSECT



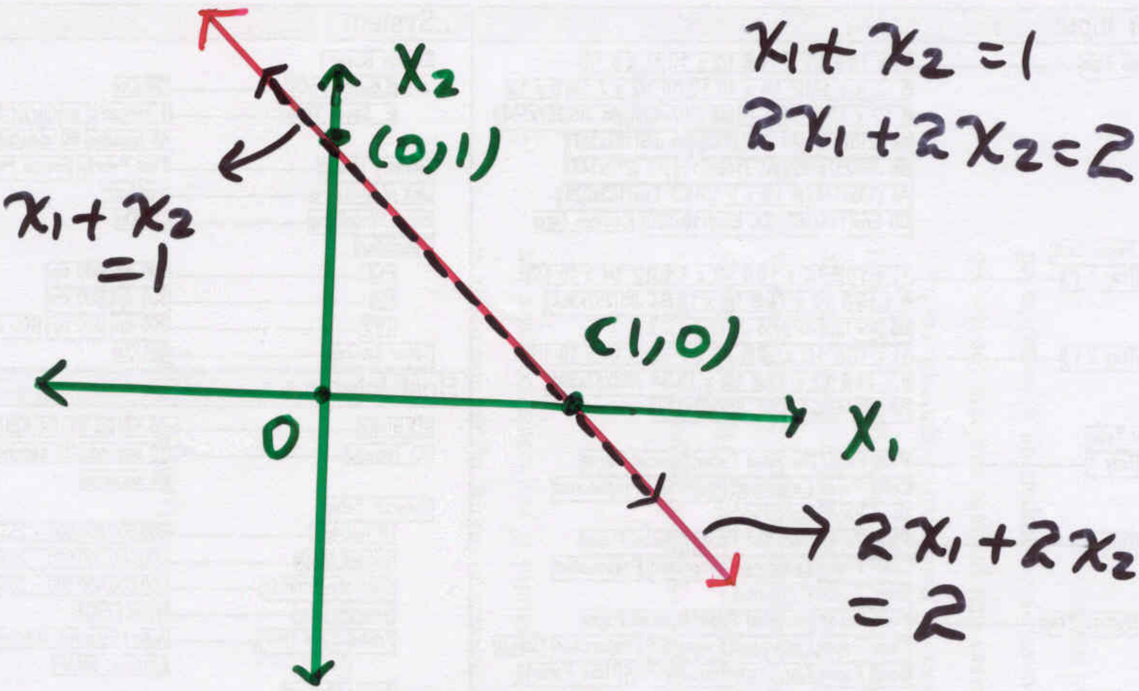
(b) $x_1 + x_2 = 1$, $2x_1 + 2x_2 = 2$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ (INFINITE SOLUTIONS)

$$\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2 - 2 = 0$$

7) BOTH LINES ARE $\boxed{\text{COINCIDENT.}}$

$$x_1 + x_2 = 1$$
$$2x_1 + 2x_2 = 2$$



$\boxed{\text{ONE}}$ EQUATION AND $\boxed{\text{TWO}}$ UNKNOWNS.

$x_1 + x_2 = 1$, LET $\boxed{x_2 = t}$

$\Rightarrow \boxed{x_1 = 1 - t}$, INFINITE SOLUTIONS FOR DIFFERENT VALUES OF $\boxed{t}$.

NOTE: HERE
$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right] R_2 \to R_2 - 2R_1$$

$$\nsim \left[\begin{array}{cc|c} 1 & 0 & A^{-1} \\ 0 & 1 & \end{array}\right],$$ A^{-1} DOESN'T EXIST

PROPERTIES OF DETERMINANTS (OF ORDER n) OR OF MATRICES OF ORDER n.

① IF $\boxed{A}$ AND $\boxed{B}$ ARE SQUARE MATRICES OF THE SAME SIZE, THEN $\boxed{\det(AB) = \det(A)\det(B)}$

(P. 97 8th ED.) OR (P. 95 7th ED.)

CHECK IT FOR $\boxed{2\times 2}$ MATRICES.

FOR $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, $\det(B) = b_{11}b_{22} - b_{12}b_{21}$

PROVE THAT

$$\boxed{\det(AB)} = \boxed{\det(A)\det(B)}$$

$$= a_{11}a_{22}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21}$$

② IF $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\boxed{\det(I) = 1}$

WHICH IS TRUE FOR IDENTITY MATRIX OF ANY ORDER.

9]

PROVE THE FOLLOWING:

IF A SQUARE MATRIX $\boxed{A}$ IS INVERTIBLE THEN $\det(A) \neq 0$. $\boxed{\text{P.(98-99) 8th ED.}}$

OR $\boxed{\text{(P. 96 7th ED.)}}$

RESULT: IF A^{-1} EXISTS THEN

$$A^{-1}A = I \Rightarrow \det(A^{-1}A) = 1$$

$\det(A) \neq 0$

$$\Rightarrow \det(A^{-1})\det(A) = 1 \Rightarrow \det(A^{-1}) = 1/\det(A) \Rightarrow \boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

③ $\boxed{\det(A) = \det(A^T)}$ $\boxed{\text{P. 89 8TH ED. P. 86 7th ED.}}$

FOR $\boxed{2 \times 2}$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$\det(A) = \det(A^T) = a_{11}a_{22} - a_{21}a_{12}$$

④ IF TWO ROWS OR TWO COLUMNS OF A MATRIX ARE IDENTICAL THEN $\boxed{\det(A) = 0}$

CHECK FOR $\boxed{2 \times 2}$ MATRIX

$$\begin{vmatrix} a & a \\ b & b \end{vmatrix} = ab - ba = 0$$

OR

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0$$

(5) P.90 (8th ED.) OR P.88 (7th ED.)

ADDING ROWS (OR COLUMNS) TOGETHER MAKES NO DIFFERENCE TO THE DETERMINANT.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

COLUMN 2

CONSIDER $(C_1 \to C_1 + C_2)$ $\begin{vmatrix} a_{11}+a_{12} & a_{12} \\ a_{21}+a_{22} & a_{22} \end{vmatrix}$

COLUMN 1

$$= (a_{11}+a_{12})a_{22} - a_{12}(a_{21}+a_{22})$$

$$= a_{11}a_{22} + \cancel{a_{12}a_{22}} - a_{12}a_{21} - \cancel{a_{12}a_{22}} = a_{11}a_{22} - a_{12}a_{21}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

ALSO

$$R_1 \rightarrow R_1 + R_2 \quad \begin{vmatrix} a_{11}+a_{21} & a_{12}+a_{22} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= (a_{11}+a_{21})a_{22} - (a_{12}+a_{22})a_{21}$$

$$= a_{11}a_{22} + \cancel{a_{21}a_{22}} - a_{12}a_{21} - \cancel{a_{22}a_{21}} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

⑥ $\boxed{\text{P. 90 (8th ED.)}}$ OR $\boxed{\text{P. 88 (7th ED.)}}$

IF $\boxed{B}$ IS THE MATRIX THAT RESULTS WHEN A SINGLE ROW OR SINGLE COLUMN OF $\boxed{A}$ IS MULTIPLIED BY A SCALAR (k), THEN $\boxed{\det(B) = k\det(A)}$

LET $\det(B) = \begin{vmatrix} ka_{11} & a_{12} \\ ka_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & ka_{12} \\ a_{21} & ka_{22} \end{vmatrix}$

$$= \begin{vmatrix} ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ ka_{21} & ka_{22} \end{vmatrix}$$

$$= k(a_{11}a_{22} - a_{21}a_{12}) = k \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= k\det(A) = k\det(A).$$

TRY THE FOLLOWING:

FIND det(A) WHERE

$$A = \begin{bmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix}$$

ANS: $\boxed{\det(A) = 0}$

(7) **RESULT:** (P. 92 8th ED.) OR (P. 89 7th ED.)

IF $\boxed{A}$ IS A SQUARE MATRIX WITH $\boxed{\text{TWO}}$ PROPORTIONAL ROWS OR $\boxed{\text{TWO}}$ PROPORTIONAL COLUMNS, THEN $\boxed{\det(A) = 0}$.

EXAMPLE: $\boxed{C_2 = 2C_3}$

$$\det\left(\begin{bmatrix} -2 & 8 & 4 \\ 3 & 2 & 1 \\ 1 & 10 & 5 \end{bmatrix}\right) = 0$$

$$\because \begin{vmatrix} -2 & 8 & 4 \\ 3 & 2 & 1 \\ 1 & 10 & 5 \end{vmatrix} = 2\begin{vmatrix} -2 & 4 & 4 \\ 3 & 1 & 1 \\ 1 & 5 & 5 \end{vmatrix} = 0$$

$\because C_2$ IS IDENTICAL TO C_3.

(8) P. 89 (8th ED.) OR P. 86 (7th ED.)

IF $\boxed{A}$ IS A SQUARE MATRIX SUCH THAT $\boxed{A}$ HAS A ROW OF $\boxed{\text{ZEROS}}$ OR A COLUMN OF ZEROS THEN $\det(A) = 0$.

CONSIDER FOR 2X2 CASE

$$\begin{vmatrix} a & 0 \\ b & 0 \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix}$$

$$= a(0) - b(0) = 0$$

NOTE: IF $\boxed{A}$ IS ANY SQUARE MATRIX THAT CONTAINS A ROW OF ZEROS OR A COLUMN OF ZEROS THEN $\boxed{A}$ IS SINGULAR.