Ver 1

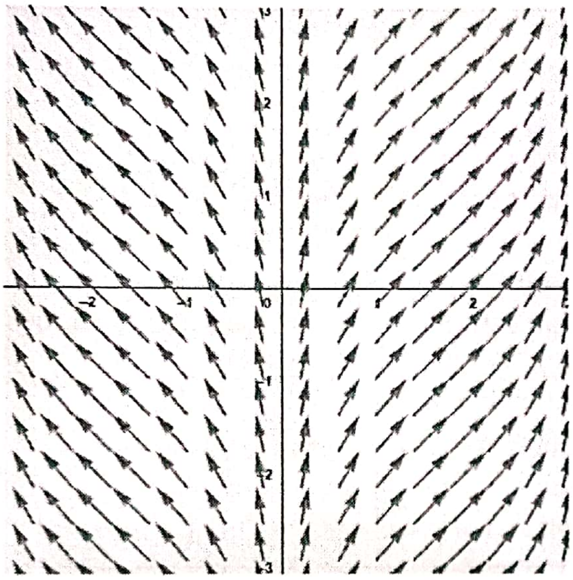
Problem 1. Match each of the given vector fields, with one of the plots. For each vector field, just write the number of the corresponding plot. No need to redraw the plots in the answer book. (4 points)

a. $\vec{F}(x,y) = x\hat{\imath}$

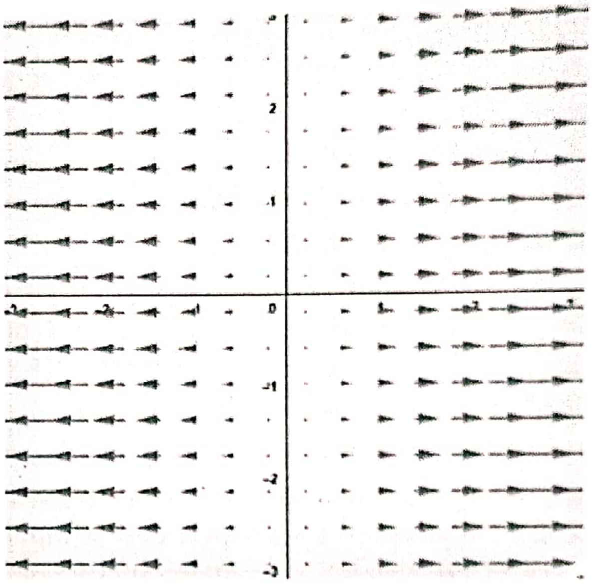
b. $\vec{F}(x,y) = \hat{\imath} + \hat{\jmath}$

c. $\vec{F}(x,y) = -y\hat{\imath} + x\hat{\jmath}$

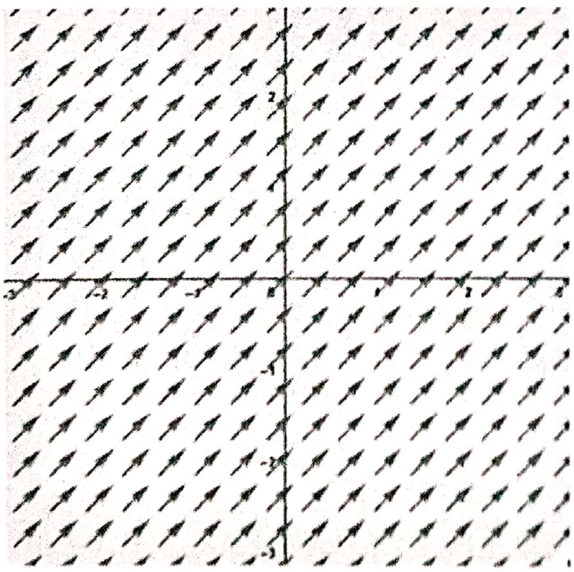
d. $\vec{F}(x,y) = \sin x\,\hat{\imath} + \hat{\jmath}$



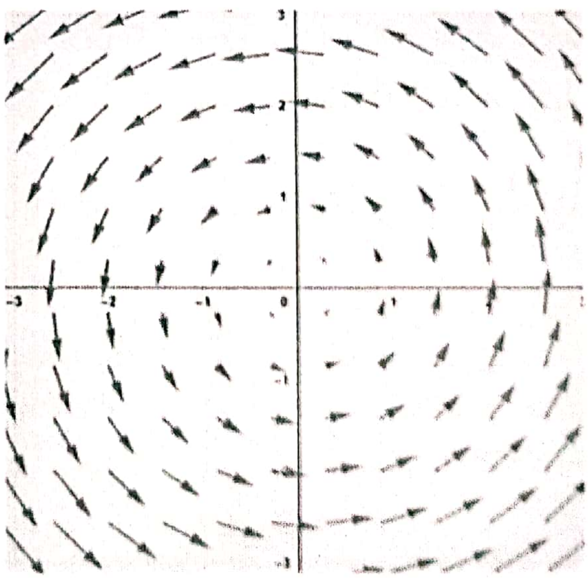
I. c ✗ d



III. a ✓



II. b ✓



IV. d ✗ c

Problem 2. Find the work done due to the force field (4 points)

$$\vec{F}(x,y,z) = (x+y)\hat{\imath} + (y+z)\hat{\jmath} + (z+x)\hat{k}$$

along the curve $C: \overrightarrow{r(t)} = t\hat{\imath} + 5t\hat{\jmath} + 2t\hat{k}, -1 \le t \le 1.$

Problem 3. (2+4 points)

a. Show that the function

$$f(x,y,z) = \frac{GmM}{\sqrt{x^2+y^2+z^2}}$$

is a potential function for the gravitational field

$$\vec{F}(x,y,z) = -GmM\frac{(x\hat{\imath} + y\hat{\jmath} + z\hat{k})}{(x^2+y^2+z^2)^{3/2}}$$

Note: G, m and M are constants.

b. Let P_1 and P_2 be points at a distance s_1 and s_2 from the origin. Show that the work done by the gravitational field in part (a) in moving a particle from P_1 to P_2 is

$$GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right).$$

Problem 4. For scalar functions f and g, where $g \neq 0$, show that (4 points)

$$\nabla\left(\frac{f}{g}\right) = \frac{1}{g^2}(g\nabla f - f\nabla g).$$

Problem 5. Determine whether the following statements are true or false. In either case, explain your answer. (3+3 points)

a. If for a vector field $\vec{F}$, there exists a closed curve C such that the line integral $\int_C \vec{F} \cdot \overrightarrow{dr} = 0$, then $\vec{F}$ is conservative.

b. Let a and b be constants. If $\vec{F}(x,y) = ax\hat{\imath} + by\hat{\jmath}$ is a conservative vector field then $a = b$.

Problem 6. Show that a conservative vector field $\vec{F}(x,y)$ is constant if and only if its associated potential function represents a plane in $\mathbb{R}^3$. (6 points)

Hint 1: A constant vector field is of the form $\vec{F}(x,y) = \alpha\hat{\imath} + \beta\hat{\jmath}$ where α and β are constant real numbers. General equation of a plane: $z = f(x,y) = ax + by + c$, where a, b and c are constants.

Hint 2: "If and only if" means that you have to prove implications in both directions.

(0' (-2' (0' Ver 2

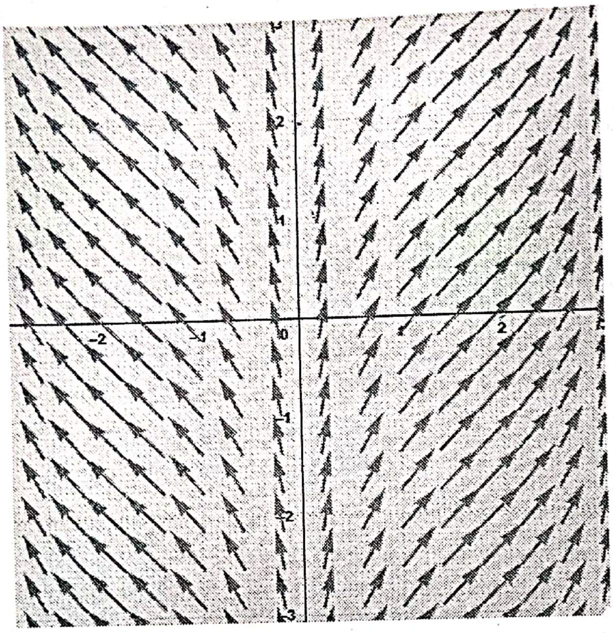
Problem 1. Match each of the given vector fields, with one of the plots. For each vector field, just write the number of the corresponding plot. No need to redraw the plots in the answer book. (4 points)

a. $\vec{F}(x,y) == -y\hat{\imath} + x\hat{\jmath}$

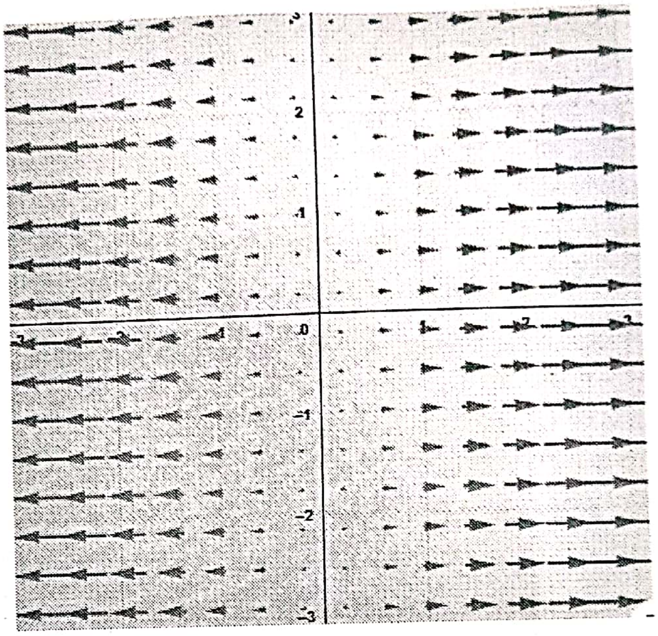
b. $\vec{F}(x,y) = \sin x\,\hat{\imath} + \hat{\jmath}$

c. $\vec{F}(x,y) = \hat{\imath} + \hat{\jmath}$

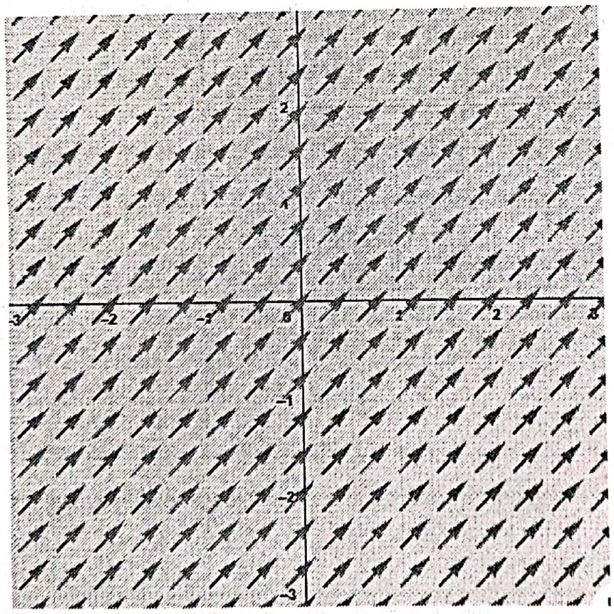
d. $\vec{F}(x,y) = x\hat{\imath}$



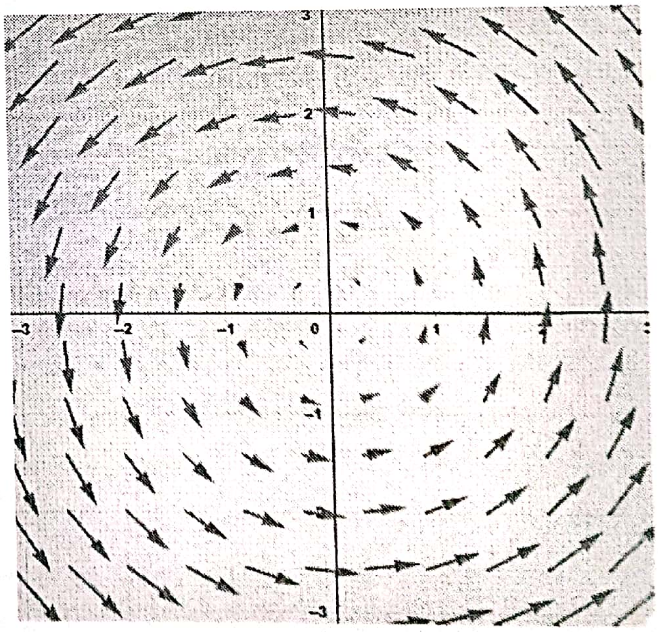
~~III.~~ I



~~III.~~ III



~~IV.~~ II



IV.

(2,0)

(2,1)

(0,2)

Problem 2. Find the flow due to the velocity field (4 points)

$$\vec{F} = (x-y)\hat{\imath} + (y-z)\hat{\jmath} + (z-x)\hat{k}$$

along the curve $C: \overrightarrow{r(t)} = 2t\hat{\imath} + 5t\hat{\jmath} + t\hat{k}, 0 \le t \le 1.$

Problem 3. (2+4 points)

a. Show that the function

$$f(x,y,z) = \frac{GmM}{\sqrt{x^2+y^2+z^2}}$$

$\int \vec{F} \cdot dr$

is a potential function for the gravitational field

$f(B) -$

$$\vec{F}(x,y,z) = -GmM\frac{(x\hat{\imath} + y\hat{\jmath} + z\hat{k})}{(x^2+y^2+z^2)^{3/2}}$$

Note: G, m and M are constants.

b. Let P_1 and P_2 be points at a distance s_1 and s_2 from the origin. Show that the work done by the gravitational field in part (a) in moving a particle from P_1 to P_2 is

$$GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right).$$

Problem 4. For scalar functions f and g, show that (4 points)

$$\nabla(fg) = f\nabla g + g\nabla f.$$

Problem 5. Determine whether the following statements are true or false. In either case, explain your answer. (3+3 points)

a. Let a and b be constants. If $\vec{F}(x,y) = ax\hat{\imath} + by\hat{\jmath}$ is a conservative vector field then $a = b$.

b. If for a vector field $\vec{F}$, there exists a closed curve C such that the line integral $\int_C \vec{F} \cdot \overrightarrow{dr} = 0$, then $\vec{F}$ is conservative.

Problem 6. Show that a conservative vector field $\vec{F}(x,y)$ is constant if and only if its associated potential function represents a plane in $\mathbb{R}^3$. (6 points)

Hint 1: A constant vector field is of the form $\vec{F}(x,y) = \alpha\hat{\imath} + \beta\hat{\jmath}$ where α and β are constant real numbers. General equation of a plane: $z = f(x,y) = ax + by + c$, where a, b and c are constants.

Hint 2: "If and only if" means that you have to prove implications in both directions.