Quiz 4

CS/CE 412/471 Algorithms: Design and Analysis, Spring 2025

26 Mar, 2025. 4 questions, 25 points, 3 printed sides

Reference

Definition 1 (Flow Network)**.** A *flow network* $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative *capacity* $c(u, v) \geq 0$.
Each flow network contains two distinguished vertices: a *source* s and a *sink* t. Each vertex lies on some path from the source to the sink. That is, for each vertex $v \in V$, the flow network contains a path $s \rightsquigarrow v \rightsquigarrow t$.

Definition 2 (Flow)**.** Let $G = (V, E)$ be a flow network with a capacity function c. Let s be the source of the network, and let t be the sink. A *flow* in G is a real-valued function $f : V \times V \to \mathbb{R}$ that satisfies the following two properties:
<u>Capacity constraint:</u> For all $u, v \in V,\ 0 \leq f(u, v) \leq c(u, v)$.
<u>Flow conservation:</u> For all $u \in V - \{s, t\},\ \sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$.
The *value* $|f|$ of a flow f is defined as $|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$.

Definition 3 (Residual Capacity)**.** For a flow network $G = (V, E)$ with source s, sink t, and a flow f, consider a pair of vertices $u, v \in V$. We define the *residual capacity* $c_f(u, v)$ by

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E, \\ f(v, u) & \text{if } (v, u) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4 (Residual Network)**.** Given a flow network $G = (V, E)$ and a flow f, the *residual network* of G induced by f is $G_f = (V, E_f)$, where $E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$.

Definition 5 (Flow Augmentation)**.** If f is a flow in G and f' is a flow in the corresponding residual network G_f, we define $f \uparrow f'$, the *augmentation* of flow f by f', to be a function from $V \times V$ to $\mathbb{R}$, defined by

$$(f \uparrow f')(u, v) = \begin{cases} f(u, v) + f'(u, v) - f'(v, u) & \text{if } (u, v) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 6 (Cut of a Flow Network)**.** A *cut* (S, T) of flow network $G = (V, E)$ is a partition of V into S and $T = V - S$ such that $s \in S$ and $t \in T$. If f is a flow, then the *net flow* $f(S, T)$ across the cut (S, T) is defined to be

$$f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u).$$

The *capacity* of the cut (S, T) is

$$c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v).$$

A *minimum cut* of a network is a cut whose capacity is minimum over all cuts of the network.

FORD-FULKERSON-METHOD(G, s, t)

initialize flow f to 0
while there exists an augmenting path p in the residual network G_f
  augment flow f along p
return f

Lemma 24.1. *Let $G = (V, E)$ be a flow network with source s and sink t, and let f be a flow in G. Let G_f be the residual network of G induced by f, and let f' be a flow in G_f. Then the function $f \uparrow f'$ defined in equation (24.4) is a flow in G with value $|f \uparrow f'| = |f| + |f'|$.*

Lemma 24.2. *Let $G = (V, E)$ be a flow network, let f be a flow in G, and let p be an augmenting path in G_f. Define a function $f_p : V \times V \to \mathbb{R}$ by*

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u, v) \text{ is on } p, \\ 0 & \text{otherwise.} \end{cases}$$

Then, f_p is a flow in G_f with value $|f_p| = c_f(p) > 0$.

Corollary 24.3. *Let $G = (V, E)$ be a flow network, let f be a flow in G, and let p be an augmenting path in G_f. Let f_p be defined as in equation (24.7), and suppose that f is augmented by f_p. Then the function $f \uparrow f_p$ is a flow in G with value $|f \uparrow f_p| = |f| + |f_p| > |f|$.*

Lemma 24.4. *Let f be a flow in a flow network G with source s and sink t, and let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$.*

Corollary 24.5. *The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G.*

Theorem 24.6 (Max-flow min-cut theorem)**.** *If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:*

1. *f is a maximum flow in G.*
2. *The residual network G_f contains no augmenting paths.*
3. *$|f| = c(S, T)$ for some cut (S, T) of G.*

Problems

1. (5 points) Suppose that, in addition to edge capacities, a flow network has *vertex capacities*. That is, each vertex v has a limit $l(v)$ on how much flow can pass through v. Show how to transform a flow network $G = (V, E)$ with vertex capacities into an equivalent flow network $G' = (V', E')$ without vertex capacities, such that a maximum flow in G' has the same value as a maximum flow in G.

   Solution: $G' = (V', E')$ can be obtained from $G = (V, E)$ as follows. For every vertex v in V, add a vertex v' and an edge (v, v') such that $c(v, v') = l(v)$. Furthermore, change the starting vertex of all outgoing edges from v to v'. When $v = t$, then the newly added vertex becomes the new sink.

2. (5 points) Prove or disprove: Augmenting a flow f in a flow network G by another flow f', also in G, yields a valid flow.

   Solution: We use a counterexample to show that the claim is false.

Proof. The claim is false.

Consider the following flow, f. $(s) \xrightarrow{4/5} (t)$

Augmenting f by itself will result in a flow of 8 along the edge (s,t) which violates the capacity constraint. $\square$

3. (5 points) Prove or disprove: Every iteration of the Ford-Fulkerson method increases f.

Solution: We refer to a corollary above to prove that the claim is true.

Proof. The claim is true.

This is exactly the claim in Corollary 24.3. $\square$

4. (10 points) We are given 2 flows f and f^* in a flow network G where $|f^*|$ is maximal and $|f| < |f^*|$. Prove whether (S,T) cuts of G exist such that: (do any 5)

(a) $c(S,T) < |f|$ (b) $c(S,T) = |f|$ (c) $c(S,T) > |f|$ (d) $c(S,T) < |f^*|$ (e) $c(S,T) = |f^*|$ (f) $c(S,T) > |f^*|$

Solution: We use results from above to prove or disprove the existence of cuts for each of the parts. In one instance, we use a counterexample.

Proof. The claims in (a) and (d) are False. That is, no cut (S,T) exists for which they are true.

The claims contradict Corollary 24.5 $\square$

Proof. The claim in (b) is False. That is, no cut (S,T) exists for which it is true.

As f is not the maximum flow, the claim contradicts Theorem 24.6. $\square$

Proof. The claim in (c) is True. That is, some cut (S,T) exists for which it is true.

From Corollary 24.5, this claim is true for every cut of G. $\square$

Proof. The claim in (e) is True. That is, some cut (S,T) exists for which it is true.

The claim follows from Theorem 24.6. $\square$

Proof. The claim in (f) is False. That is, there are networks in which no cut (S,T) exists for which the claim is true.

Consider the following maximum flow, $f*$. $(s) \xrightarrow{5/5} (t)$

There is no cut (S,T) in this network such that $c(S,T) > |f^*|$. $\square$

<u>Note</u>: As the question only asks for existence, and not universality, an existence proof for (f) will also be considered correct.