MATH202-ENGINEERING MATHEMATICS

FALL2022

FINAL EXAM

TOTAL MARKS: 55 DATE:07.12.2022

NAME:________________________ TIME: 12:45 – 15:15

STUDENT ID:___________________ SECTION:___________________

INSTRUCTIONS:

1. This is a closed books/notes exam. Any notes, books, bags should be kept at the front of the classroom before the start of the test.
2. Use of cellphones, laptops or any other communicating device is prohibited. Your cellphones should be switched off and submitted to the invigilator, or kept at the front of the class room before the start of the test.
3. You are to attempt all questions on the answer sheet provided.
4. Your solutions should be comprehendible. Explain what you are doing, and if you use any results that you have studied in this course or Calculus 1 and 2, mention them explicitly.
5. Use of calculators is allowed for this exam.
6. Any form of communication/sharing among your peers is not allowed during the exam.
7. If you want to communicate with the invigilator, raise your hand.
8. You are not allowed to leave your desk without permission of the invigilator during the exam.
9. Failure to abide by the instructions above will lead to immediate cancellation of the exam.

PROBLEM 1. (08 Points) Solve the following in-exact ODE

$$\cos(y)\,dx - (2x\sin(y) + \cos(y))dy = 0$$

for $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Hint: Make it exact by computing the integrating factor and make use of the following identity

$$\int \tan x = -\ln|\cos x| + C$$

PROBLEM 2. **(09 Points)** Our goal is to solve the initial value problem:

$$y'' + 2y' = 5 - e^{-2x} \tag{1}$$

a. (3 Points) Find the general solution, y_c, of the associated homogeneous equation $y'' + 2y' = 0$.
b. (2 Point) Suggest a guess function y_p for the non-homogeneous part $r(x) = 5 - e^{-2x}$. Justify your answer.
c. (3 Points) Solve for y_p using the Method of Undetermined Coefficients.
d. (1 Point) Write down the general solution of the second-order non-homogeneous equation (1).

PROBLEM 3. (11 Points + 2 Bonus Points) Answer the questions below for the following initial value problem

$$y'' + xy' + y = 0, \qquad y(0) = 1, y'(0) = 1$$

a. (1.5 Points) What is the order of the equations? Is the equation linear or non-linear? Does it have constant coefficients or variable coefficients?
b. (1 Point) Write down the guess for y as a power series. Also determine y' and y'' as power series.
c. (1 Point) Use the initial conditions to find the first two coefficients of the power serries for y.
d. (4 Points) Substitute your guess for y, y', and y'' into the ODE and simplify.
e. (2 Points) Find the recurrence relation for the coefficients, i.e., write a_{m+2} in terms of a_{m+1} and a_m, $m = 0, 1, 2, \ldots$
f. (1.5 Points) Write down the general solution $y(x)$ as a power series with at least six terms.
g. (Bonus question, 2 marks): Write the solution $y(x)$ using the summation notation as

$$y(x) = a_0 \sum_{i=0}^{\infty} (\cdots) + a_1 \sum_{i=0}^{\infty} (\cdots)$$

PROBLEM 4. (07 Points) Use the divergence theorem to compute the flux

$$\oint_S \vec{F} \cdot \hat{n}\, d\sigma$$

where $\vec{F}(x, y, z) = (x^3 - e^y)\,\hat{\boldsymbol{\imath}} + (y^3 + \sin z)\,\hat{\boldsymbol{\jmath}} + (z^3 + \mathrm{xy})\hat{\boldsymbol{k}}$

and S is the boundary surface of the solid bounded by $z = \sqrt{4 - x^2 - y^2}$ and the xy-plane.

PROBLEM 5. (02 Points each) For each of the statements below, indicate whether they are TRUE or FALSE. In each case provide a brief explanation or a counter example for your choice.

a. A first-order ODE can never be linear, separable and exact at the same time.
b. The function $y = 0$ is always a solution of a linear homogeneous ODE.
c. The ODE $y' = e^{3x-5y}$ is separable.
d. For any linear ODEs, the sum of any two solutions is also a solution.
e. Every polynomial has a convergent power series representation centered at 0.
f. The vector field $\vec{F}(x, y, z) = 5xy\,\hat{\boldsymbol{\imath}} + 5yz\,\hat{\boldsymbol{\jmath}} + 5xz\,\hat{\boldsymbol{k}}$ is a gradient field.
g. The set of points (x, y, z) in 3-dimensional space such that $x^2 + y^2 = 4$ form a circle of radius 2.
h. If $\vec{F}$ and $\vec{G}$ are vector fields such that $\vec{\nabla} \times \vec{F} = \vec{\nabla} \times \vec{G}$ then $\vec{F} = \vec{G}$.
i. If $\vec{F}$ is conservative then $\vec{\nabla} \cdot \vec{F} = 0$.
j. The curl of a conservative vector field is always zero.

END OF EXAM

Important Formulae

- Assume sufficient differentiability of the functions and that the conditions for curves and surfaces are met, then:

Del Operator $\vec{\nabla}$	$\vec{\nabla}=\frac{\partial}{\partial x}\boldsymbol{\hat{\imath}}+\frac{\partial}{\partial y}\boldsymbol{\hat{\jmath}}+\frac{\partial}{\partial z}\boldsymbol{\hat{k}}$
Gradient of a scalar function $\phi = \phi(x,y,z)$	$\vec{\nabla}\phi$
Divergence of a vector field $\vec{F}$	$\vec{\nabla}\cdot\vec{F}$
Curl of a vector field $\vec{F}$	$\vec{\nabla}\times\vec{F}$
Line integral of a vector field $\vec{F}$	$\int_C \vec{F}.d\vec{r}$
Closed line integral	$\oint_C \vec{F}.d\vec{r}$
Surface integral of vector field $\vec{F}$ over a surface S, where (u,v) are the parameters on the surface.	$\oiint_S \vec{F}.\hat{n}\,dS = \iint_S \vec{F}.\vec{N}dudv$

- For a point $P = (x,y,z)$ in Cartesian Coordinates, a representation in other coordinate systems *can be*

Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
$\boldsymbol{x=x}$ $\boldsymbol{y=y}$ $\boldsymbol{z=z}$	$x=rcos\theta$ $y=rsin\theta$ $z=z$ where $r=\sqrt{x^2+y^2}>0$ $0\le\theta\le 2\pi$	$x=\rho sin\phi cos\theta$ $y=\rho sin\phi sin\theta$ $z=\rho cos\phi$ where $\rho=\sqrt{x^2+y^2+z^2}>0$ $0\le\theta\le 2\pi$ $0\le\phi\le\pi$
$\boldsymbol{dV=dxdydz}$	$dV=rdrd\theta dz$	$dV=\rho^2 sin\phi d\rho d\theta d\phi$

- Divergence theorem for a vector field $\vec{F}$,

$$\iiint_D \vec{\nabla}\cdot\vec{F}\,dV = \oiint_S \vec{F}\cdot\hat{n}\,d\sigma$$

Here, D is the region enclosed by the closed surface S and V is its volume.

- Stoke's Theorem for a vector field $\vec{F}$,

$$\iint_{S} (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma = \oint_{C} \vec{F} \cdot d\vec{r}$$

Here the closed curve C bounds the surface S in 3-dimensional space.

- A first order ODE $M(x,y)dx + N(x,y)dy = 0$ is exact iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- A first order non-exact ODE, $P(x,y)\, dx + Q(x,y)\, dy = 0$, can be made exact by multiplying it with an integrating function $F(x)$:

$$F(x) = e^{\int R(x)\, dx}, \text{ where } R(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right).$$

if such a function exists.
Or, multiplying it with an integrating function $G(y)$:

$$G(y) = e^{\int R^*(y) dy}, \text{ where } R^*(y) = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

if such a function exists.

- Function guesses for Method of Undetermined Coefficient,

$S(x)$	**Guess for $y_p(x)$**
$k = \text{Constant}$	K (Some other constant)
$ke^{\gamma x}$	$Ke^{\gamma x}$
kx^n, (n = positive integer)	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$k \cos \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

Table 1: Short list of guesses of particular solutions for some forms of non-homogeneous part.

- The power series of $y(x)$ about $x_0 = 0$,

$$y(x) = a_0 + a_1(x) + a_2(x)^2 + a_3(x)^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$
