VECTOR FIELDS

→ is a function which assigns ... t in

In R-2 : $P\hat{i} + Q\hat{j}$

In R-3 : $P\hat{i} + Q\hat{j} + R\hat{k}$ where P, Q, R are define...

* plug in (x,y)
- this gives a vector
- plot the vector **AT THE POINT**

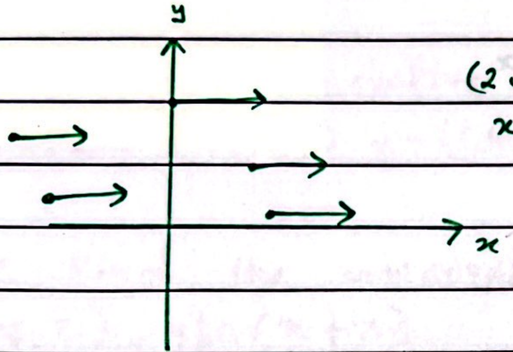
$\hat{i} = x$

$\hat{j} = y$

$\hat{k} = z$

IMPORTANT

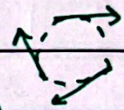
Example 1 $F(x,y) = 2\hat{i}$, $(x,y) \to$ R-2



★ NOTE

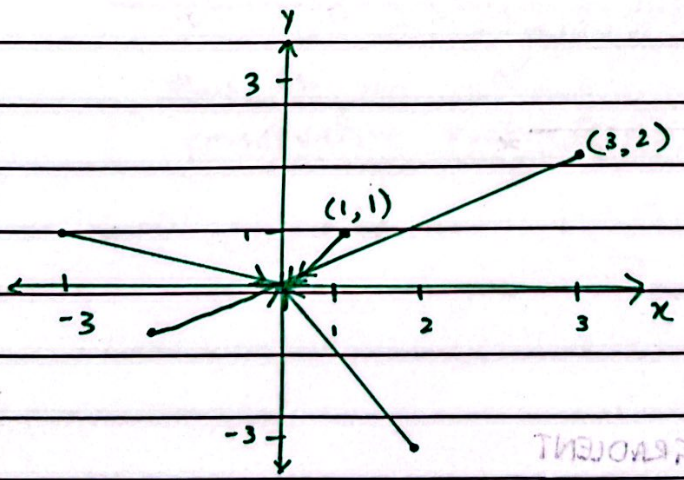
$y\hat{i} - x\hat{j}$

(tangent vector)



Example 2 $F(x,y) = -x\hat{i} - y\hat{j}$ R-2

(1,1) (3,2) (−3,1) (2,−3) (−2,−1)



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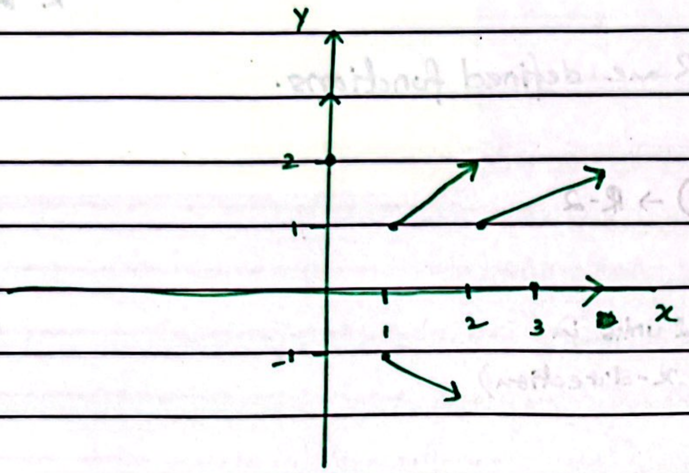
Example 3 $F(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\hat{i} + \dfrac{y}{\sqrt{x^2+y^2}}\hat{j}$



$(1,1) \rightarrow \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

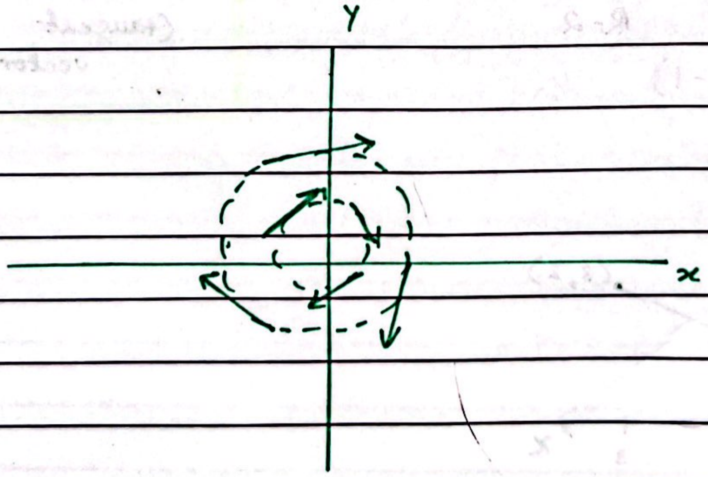
$(1,-1) \rightarrow \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

$(2,1) \rightarrow \frac{2}{\sqrt{5}}\hat{i} + \frac{1}{\sqrt{5}}\hat{j}$

$(0,2) \rightarrow +\hat{j}$

$(3,0) \rightarrow \hat{i}$

Example 4 $F(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}\hat{i} - \dfrac{x}{\sqrt{x^2+y^2}}\hat{j}$



$\nabla f(x,y)$ OR $\nabla f(x,y,z)$ GRADIENT

→ check next page.

Example 1 $f(x,y) = x^2 y - y^3$

$\nabla f(x,y) = f_x \hat{i} + f_y \hat{j}$ → GRAD F

$\nabla f(x,y) = 2xy\hat{i} + (x^2 - 3y^2)\hat{j}$

* gradient is a type of vectors field!

CONSERVATIVE VECTOR FIELD:

$F(x,y)$ is a ~~cond~~ conservative vector field if $F(x,y) = \nabla f(x,y)$ for some $f(x,y)$ ← (CALLED A POTENTIAL FUNCTION)

Example Find the conservative vector field, $F(x,y,z)$ for $f(x,y,z) = y\ln(x+z)$.

$$\underbrace{\nabla f(x,y,z) = \left(\frac{y}{x+z}\right)\hat{i} + \ln(x+z)\hat{j} + \left(\frac{y}{x+z}\right)\hat{k}}_{\text{this is a conservative vector field}}$$

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A general vector field can also provide useful information,

→ Local direction of flow / movement: The vector at each point indicates the direction of flow of some quantity.

→ Tendency for convergence / divergence: Divergence tells you if a given region acts as source or sink of flow. Positive divergence indicates spreading out, negative indicates converging.

→ Curl reveals local rotation: Non-zero curl at a point means the vectors are rotating in a clockwise or counterclockwise manner locally.

→ Streamline / Pathline: Streamlines show integral curves that are everywhere tangent to the vectors & indicate likely paths of particles in flow over time.

→ Critical points: Points where vectors is zero act as singularities in field.

GRADIENT FIELDS

We define the gradient field of a differentiable function $f(x,y,z)$ to be the field of gradient vectors.

$$\nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$$

* grad f gives you F or gradient field.

LINE INTEGRALS

$\int_a^b f(x)\,dx \rightarrow$ gives area under the function.

* oriented: (direction of movement is fixed)

* integrals gives the area under the curve
* double integrals give the volume under surface

We develop the line integral the way we develop all integrals, in this case by first slicing up the curve C into n small, approximately straight pieces along which $\vec{F}$ is approximately constant.

Two type of 'changes' that can occur at a point in a vector field accelerates/decelerates?

$t_s \leq t \leq t_{end}$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \rightarrow$ position vector.

$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

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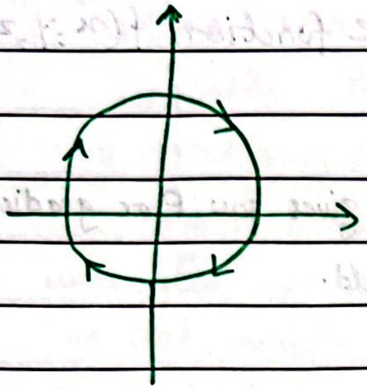
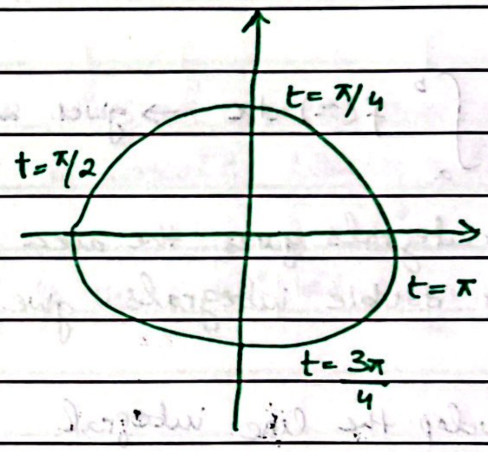
Example :



$\vec{r}(t) = \langle \cos t, \sin t \rangle \Rightarrow \cos t\,\hat{i} + \sin t\,\hat{j}$

$0 \leq t \leq 2\pi$

$0 \leq t \leq 4\pi$ → now the curve is tracing the circle twice.

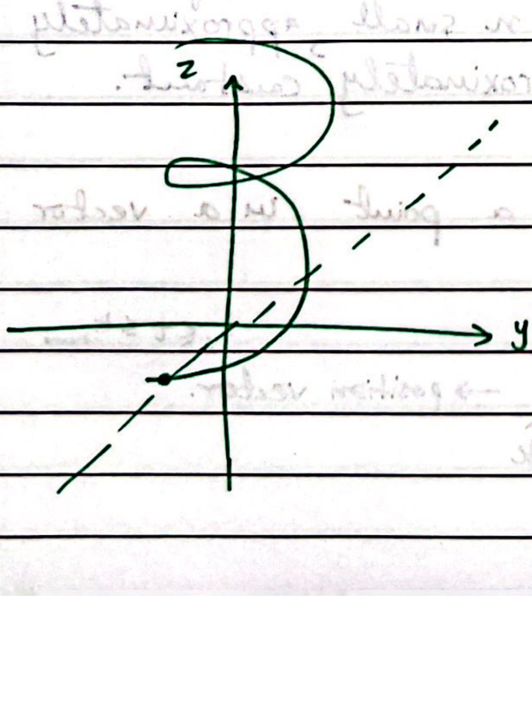


$\vec{r}(t) = \langle \cos 2t, \sin 2t \rangle$

$0 \leq t \leq \pi$

orginally $\vec{r}(t) = \langle \cos t, \sin t \rangle$

* a curve can have multiple parametrization.



$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$0 \leq t \leq 4\pi$

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PARAMETRIZATION

$$r(t) = g(t)i + h(t)j + k(t)k \qquad a \le t \le b$$

$$f(x,y) = f(x(t), y(t)) \qquad ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$
$$\vec{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j}$$

$$\|\vec{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$\|\vec{r}'(t)\| = ds$$

$$\int_c f(x,y)\, \underline{ds} = \int_{t=a}^{t=b} f(x(t), y(t)) \underline{\|\vec{r}'(t)\|}\, dt$$

$$L = \int_a^b \|\vec{r}'(t)\|\, dt$$ (Arc length)

$$A = \int_a^b f(x(t), y(t)) \|\vec{r}'(t)\|\, dt$$ (Area) (Mass)

Example; $\int_c y\, ds$, $C: \vec{r}(t) = 2t\hat{i} + t^3\hat{j}$ $\quad 0 \le t \le 1$

$\hookrightarrow \int_a^b f(x(t), y(t))\, \|\vec{r}\,'(t)\|\, dt$

$x = 2t \qquad y = t^3$

$x' = 2 \qquad y' = 3t^2$

1) $\vec{r}\,'(t) = 2\hat{i} + 3t^2\hat{j}$

2) $\|\vec{r}\,'(t)\| = \sqrt{2^2 + (3t^2)^2}$

$\|\vec{r}\,'(t)\| = \sqrt{4 + 9t^4}$

$$\int_{t=0}^{t=1} t^3 \sqrt{4 + 9t^4}\, dt = \frac{1}{54}(13\sqrt{13} - 8)$$

(Mass)

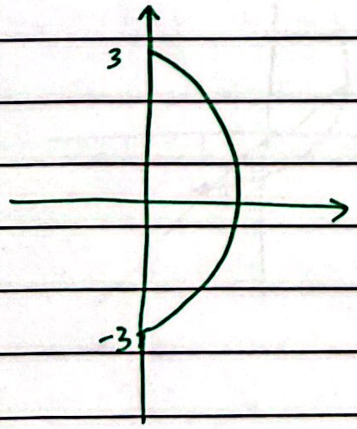
Example; $\int_c (x^2+y^2)\, ds$ $\quad C$: The right 1/2 circle $x^2+y^2 = 9$

For CIRCLES

$x = r\cos\theta$, $y = r\sin\theta$

So, $x = 3\cos\theta$ and $y = 3\sin\theta$

$x = 3\cos t$ and $y = 3\sin t$



$\vec{r}(t) = x\hat{i} + y\hat{j}$

$\vec{r}(t) = 3\cos t\,\hat{i} + 3\sin t\,\hat{j}$

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1) $\vec{r}\,'(t) = -3\sin t\,\hat{i} + 3\cos t\,\hat{j}$

2) $\|\vec{r}\,'(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t}$

$\|\vec{r}\,'(t)\| = 3$

$$\int_{t=-\pi/2}^{t=\pi/2} (9\cos^2 t + 9\sin^2 t)(3)\, dt \Rightarrow$$

VECTOR FIELD $\rightarrow$ Spin field.

VECTOR FIELD $\rightarrow$ ~~Gradient~~ Radial Field. (attractive)

LINE INTEGRAL $\rightarrow$

LINE INTEGRAL $\rightarrow$

Eample: $r(t) = ti + t^2 j + t^4 k$

$x = t \quad x'(t) = 1$

$F = \sqrt{z}\, i - 2xj + \sqrt{y}\, k$

$y = t^2 \quad y'(t) = 2t$

$f(r(t)) = t^2 i - 2tj + tk$

$z = t^4 \quad z'(t) = 4t^3$

$r'(t) = i + 2tj + 4t^3 k$

$$\int_0^1 F \cdot \frac{dr}{dt} = \int_0^1 (t^2 - 4t^2 + 4t^4)\, dt$$

$$\int_0^1 (-3t^2 + 4t^4)\, dt$$

$$-t^3 + \frac{4t^5}{4}\Big|_0^1 \Rightarrow -1 + \frac{4}{5} \quad \underline{\text{Ans}}$$

Example: $\int_c 2xy\, ds$, C : segment $(-2,-1)$ to $(1,3)$

Note: FOR <u>SEGMENTS</u> WE WANT $0 \le t \le 1$

$X = C_1 + k_1 t$, $Y = C_2 + k_2 t$

AT $T=0$: $-2 = C_1 + k_1 0 \rightarrow C_1 = -2 \Rightarrow X = -2 + k_1 t$

$1 = -2 + k_1 (1)$

$X = -2 + 3t$

AT $T=0$: $Y = C_2 + k_2 t$ $\quad Y = -1 + k_2 t$

$-1 = C_2 + \cancel{k_2(0)} \longrightarrow 3 = -1 + k_2 \cdot 1$, $k_2 = 4$

$C_2 = -1$

$Y = -1 + 4t$

$\vec{r} = x\hat{i} + y\hat{j}$

$\vec{r}(t) = (-2+3t)\hat{i} + (-1+4t)\hat{j}$

1) $\vec{r}'(t) = 3\hat{i} + 4\hat{j}$

2) $\|\vec{r}\| = \sqrt{9+16} = 5$

$$\int_0^1 2(-2+3t)(-1+4t)\,5\,dt$$

Example: $\int_C (x+y)\,dx + (xy)\,dy + y\,dz$, $C: \vec{r}(t) = e^t\hat{i} + e^{-t}\hat{j} + 2e^{2t}\hat{k}$

$0 \le t \le 1$

$x = e^t$, $y = e^{-t}$, $z = 2e^{2t}$

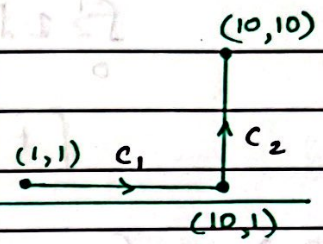
$dx = e^t dt$, $dy = -e^{-t} dt$, $dz = 4e^{2t} dt$

$$\int_0^1 (e^t + e^{-t}) e^t dt + (e^t \cdot e^{-t})(-e^{-t}) dt + e^{-t} \cdot 4e^{2t} dt$$

$$\int_0^1 (e^{2t} + 1 - e^{-t} + 4e^t)\,dt \Rightarrow \frac{1}{2}e^2 + 4e + \frac{1}{e} - \frac{9}{2}$$

Example: Find the line integral of constant vector field $\vec{F} = \vec{i} + 2\vec{j}$ along the path from $(1,1)$ to $(10,0)$

Let C_1 be horizontal segment of the path going from $(1,1)$ to $(10,1)$.



$$\Delta\vec{r} = \Delta x\vec{i} \quad \text{so} \quad \vec{F}\cdot\Delta\vec{r} = (\vec{i}+2\vec{j})\cdot\Delta x\vec{i} = \Delta x$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_1^{10} dx = 9$$

$$\Delta\vec{r} = \Delta y\vec{j} \quad \text{so} \quad \vec{F}\cdot\Delta\vec{r} = (\vec{i}+2\vec{j})\cdot\Delta y\vec{j} = 2\Delta y$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_1^{10} 2dy = 18$$

Thus, $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = 9+18 = 27$

PROPERTIES OF LINE INTEGRALS

1. $\int_C \lambda\vec{F}\cdot d\vec{r} = \lambda\int_C \vec{F}\cdot d\vec{r}$

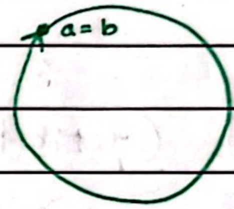
2. $\int_C (\vec{F}+\vec{G})\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_C \vec{G}\cdot d\vec{r}$

3. $\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$

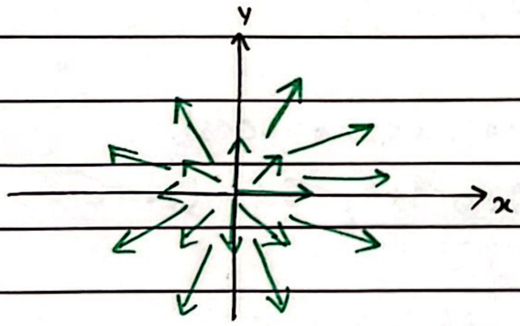
4. $\int_{C_1+C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$

$F \rightarrow$ force field $\rightarrow \int \rightarrow$ work done.

$F \rightarrow$ Fluid field $\rightarrow \int \rightarrow$ flux.

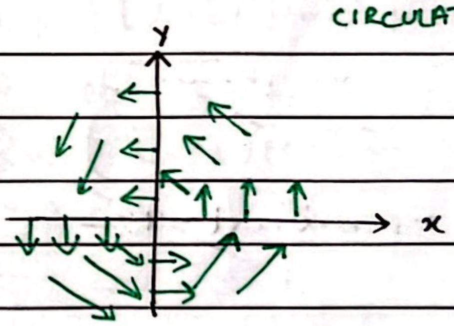


CIRCULATION



radial field

$F = x\mathbf{i} + y\mathbf{j}$



"spin" field / rotational field.

$$F = \frac{-y\mathbf{i} + x\mathbf{j}}{(x^2+y^2)^{1/2}}$$

$F = \nabla f$ → for a function to be "CONTINUOS" it needs to be differentiable.

↙

conservative.

cotinuos: F conservative $\rightleftharpoons F = \nabla f$

differentials

F

Δf → differentiable

conservative vector field

→ path independent.

Example:

$u = xy \quad v = \cos z$

$u' = x \quad v' = \cancel{\sin z}\ 0$

$$F = \underbrace{yz}_{M}i + \underbrace{xz}_{N}j + \underbrace{xy}_{P}k$$

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

$x = x$ ✓ $\qquad y = y$ ✓ $\qquad z = z$ ✓

Example:

$$F = \underbrace{(y\sin z)}_{M}i + \underbrace{(x\sin z)}_{N}j + \underbrace{(xy\cos z)}_{P}k$$

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

$x\cos z = x \qquad y = y\cos z \qquad \sin z = \sin z$

'F' is conservative if and only if $F = \nabla f$ for some 'f'.

$\int_C F \cdot dr$ The line $\int$ is independent of path!

- so the $\int$ will be the same NO MATTER WHAT CURVE we CHOOSE As long AS END POINTS STAYS THE SAME.

- FOR CONS V.F WE DON'T EVEN DEFINE A CURVE

$F = \nabla f$

→ potential function for F.

→ for function to be CONTINUOS f needs to be differentiable

→ F is conservative.

FUNDAMENTAL THEOREM OF CALCULUS.

→ to find conservative vector field

$$\int_C F \cdot dr = f(B) - f(A)$$

Proof of Theorem.

$$\int_C F \cdot dr = \int_A^B \nabla f \cdot dr \qquad F = \nabla f$$

$$\int_{t=a}^{t=b} \nabla f(r(t)) \cdot r'(t)\, dt$$ chain rule.

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$= \int_a^b \frac{d}{dt} f(r(t))\, dt$ fundamental theorem of calculus.

$= f(r(b)) - f(r(a))$, $\quad r(b) = B,\ r(a) = A$

$= f(B) - f(A)$

CONSERVATIVE FIELDS ARE VECTOR FIELDS

$$F = Mi + Nj + Pk$$

Component Test for Conservative Fields.

$$F = M(x,y,z)i + N(x,y,z)j + P(x,y,z)k$$

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Example 2. Find the work done by the conservative field
$F = yzi + xzj + xyk = \nabla f$, where $f(x,y,z) = xyz$.
along any smooth curve C joining the point $A(-1,3,9)$ to $B(1,6,-4)$

$$\int_C F \cdot dr = \int_A^B \nabla f \cdot dr$$

$$= f(B) - f(A)$$

$$= xyz\Big|_{(1,6,-4)} - xyz\Big|_{(-1,3,9)}$$

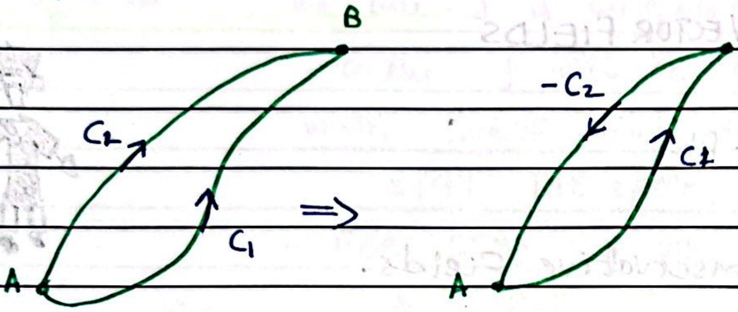
$$= (1)(6)(-4) - (-1)(3)(9)$$

$$= -24 + 27 = 3$$

LOOP PROPERTY OF CONSERVATIVE FIELDS

1. $\oint_C F \cdot dr = 0$ around every loop (that is closed curve C) in D.
2. The field F is conservative on D.

Proof that Part 1 ⇒ Part 2. $\oint \rightarrow$ closed curve



$F = \nabla f$ on D

$\Updownarrow$

F conservative on D.

$\Updownarrow$

$\oint_C F \cdot dr = 0$ over any loop in D.

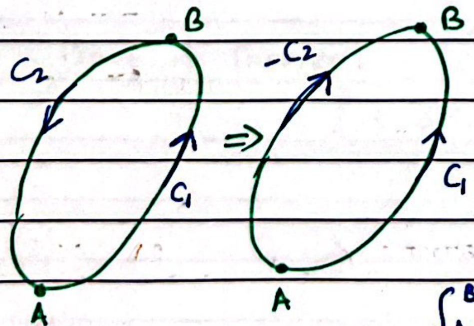
$$\oint_C F \cdot dr = 0 \longrightarrow F \text{ is conservative}$$

$$= \int_{C_1} F \cdot dr + \int_{-C_2} F \cdot dr = 0$$

$$= \int_{C_1} F \cdot dr - \int_{C_2} F \cdot dr = 0$$

$$= \int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr \longrightarrow \text{Conservative Vector field.}$$

Proof that Part 2 ⇒ Part 1



$$\int_{A \, C_1}^{B} F \cdot dr = \int_{C_2 \, A}^{B} F \cdot dr$$

$$\int_A^B F \cdot dr - \int_A^B F \cdot dr = 0$$

$$\int_A^B F \cdot dr + \int_B^A F \cdot dr = 0 \Rightarrow \oint_C F \cdot dr = 0$$

conservative VF.

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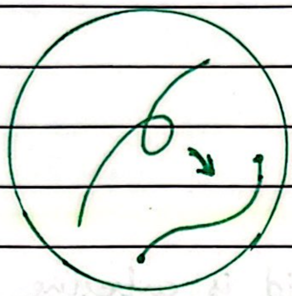
Simply connected: Smooth curve can be drawn in the region

$\oint$ => closed siple curve 



→ simply connected.

Smooth Curve: No overlap.
can be converted into smoth line/point.

not simply connected ←

Magnetic Flux

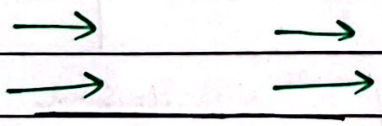
The flux calculates the rate at which a fluid is entering or leaving a region enclosed by closed curve.

Divergence.

→ if $F(x_0, y_0) > 0$: a gas is EXPANDING. positive divergence

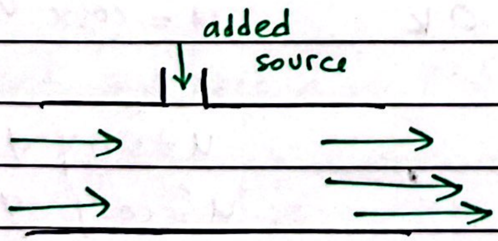
→ if $F(x_0, y_0) < 0$: a gas is COMPRESSING. negative divergence.

→ if $F(x_0, y_0) = 0$: neither expanding nor compressing.

1. 

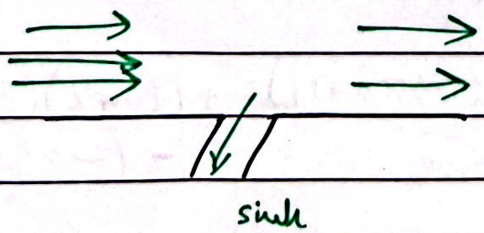
flow in = flow out

⇒ 0 divergence.

2. 

flow in < flow out

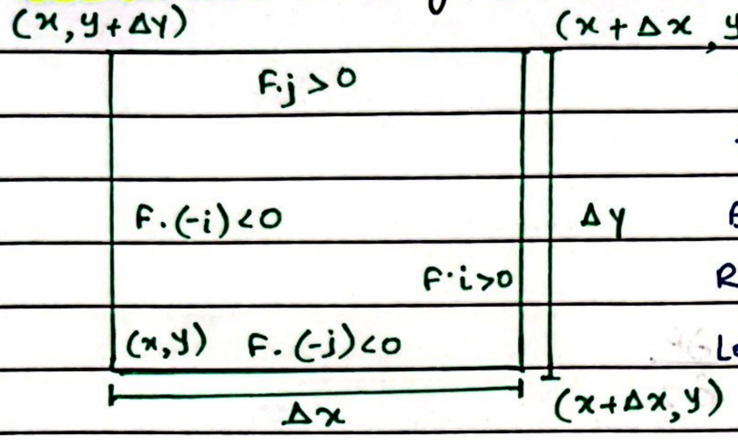
⇒ + divergence.



flow in > flow out

⇒ −ve divergence.

Derivation of Divergence formula;



Fluid flow rates :

Top :- $F(x,y+\Delta y)\cdot j\Delta x = N(x,y+\Delta y)\Delta x$

Bottom :- $F(x,y)\cdot(-j)\Delta x = -N(x,y)\Delta x$

Right :- $F(x+\Delta x, y)\cdot i\Delta y = M(x+\Delta x, y)\Delta y$

Left :- $F(x,y)\cdot(-i)\Delta y = -M(x,y)\Delta y$

Summing opposite pairs gives,

Top and bottom : $(N(x,y+\Delta y) - N(x,y))\Delta x \approx \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$

Right and left : $(M(x+\Delta x, y) - M(x,y))\Delta y \approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y.$ +

Adding these last two equations gives the net effect of flow rate,

Flux across rectangle boundary $\approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x\Delta y$

We now divide by $\Delta x\Delta y$ to estimate total flux per unit area or flux density for rectangle,

$$\approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

The divergence (flux density) of a vector field $F = Mi + Nj$ at point (x, y) is.

$$\text{div } F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Divergence in 3D.

$$\text{div } F = \nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

Example 1

(a) Expansion: $F(x, y, z) = xi + yj + zk$

$$\text{div } F = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3 \quad \text{(Expansion)}$$

(b) Compression: $F(x, y, z) = -xi - yj - zk$

$$\text{div } F = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) + \left(\frac{\partial}{\partial z}\right)(-z) = -3 \quad \text{(compression)}$$

PRACTICE NOW:

17. $\vec{F}(x,y,z) = x^2z\hat{i} + y^2x\hat{j} + (y+2z)\hat{k}$

$$\text{div } F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= 2xz + 2yx + 2$$

18. $\vec{F}(x,y,z) = 3xyz^2\hat{i} + y^2\sin z\hat{j} + xe^{2z}\hat{k}$

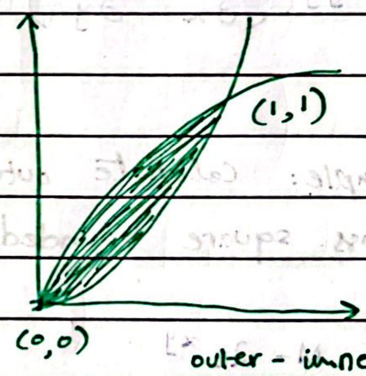
$$\text{div } F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= 3yz^2 + 2y\sin z + 2xe^{2z}$$

Curl $\rightarrow$ circulation density / Area.

divergence $\rightarrow$ $\frac{\text{flux}}{\text{Area}}$

x Area $\rightarrow$



① $\iint \text{div } dydx = \text{flux}$

$$\int_c P dx + Q dy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) dx\, dy \qquad \vec{F} = \langle P, Q \rangle$$

$$= \oint F \cdot \vec{r} \qquad \text{OR} \qquad \oint M dy - N dx$$

② $\iint_R \text{curl } dydx = \text{Circulation}$

$$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dydx = \oint F \cdot dr \quad \text{OR} \quad \oint_c M dx + N dy$$

→ when walking, region should be on left (+ve orientation / counter)
→ " ", region when on right (−ve orientation / clockwise)

$\oint$ negative positive oriented $\oint$ positive oriented.

Flux * not coming in exam.

$$\iint \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dy\,dx = \oint M\,dy - N\,dx$$

Circulation.

$$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy\,dx = \oint M\,dx + N\,dy$$

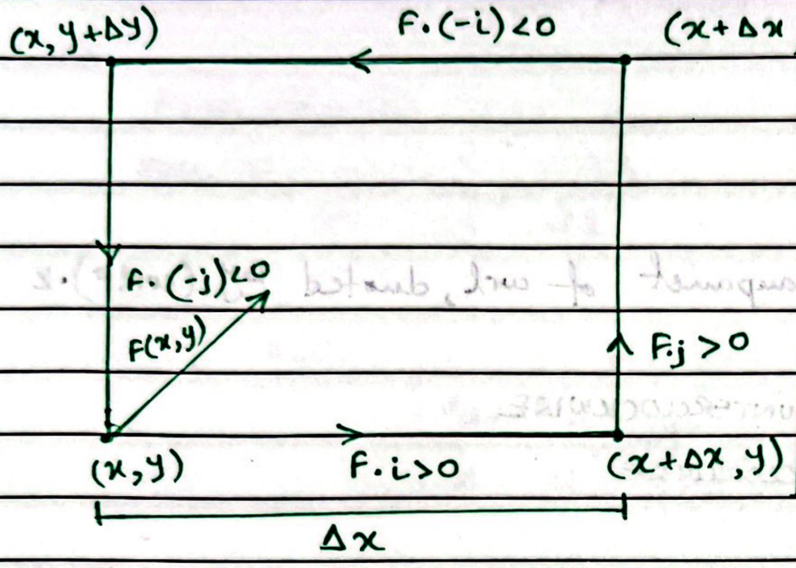
Example: Calculate outward flux of vector field $F(x, y) = 2e^{xy}i + y^3 j$ across square bounded by lines. $x = \pm 1$ and $y = \pm 1$.

$M = 2e^{xy}$ $\qquad \partial M / \partial x = 2ye^{xy}$

$N = y^3$ $\qquad \partial N / \partial y = 3y^2$

$$\int_1^{-1} \int_1^{-1} (2ye^{xy} + 3y^2)\, dy\,dx = \oint 2e^{xy} dy - y^3 dx.$$

CURL



$F(x,y) = M(x,y)\,i + N(x,y)\,j$

$F(x,y)\cdot i\,\Delta x = M(x,y)\,\Delta x$

Δy

Top: $F(x,y+\Delta y)\cdot(-i)\Delta x = -M(x,y+\Delta y)\Delta x$

Bottom: $F(x,y)\cdot i\,\Delta x = M(x,y)\Delta x$

Right: $F(x+\Delta x,y)\cdot j\,\Delta y = N(x+\Delta x,y)\Delta y$

Left: $F(x,y)\cdot(-j)\Delta y = -N(x,y)\Delta y$

we sum opposite pairs

Top & Bottom: $-(M(x,y+\Delta y)-M(x,y))\Delta x \approx -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$

Right & left: $(N(x+\Delta x,y)-N(x,y))\Delta y \approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y$

Adding these last two equations.

Circulation rate around rectangle $\approx \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x\Delta y$

We now divide by $\Delta x\Delta y$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

→ orientation is really important
→ We use right hand rule to find axis (direction) of rotation.

* Put the fingers of right hand in the direction of rotation.
* Your thumb will point towards direction of curl.

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The circulation density of vector field $F = Mi + Nj$ at point (x, y)

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

The expression is also called k-component of curl, denoted by $(\text{curl} F) \cdot k$

→ if $c > 0$: +ve rotation COUNTERCLOCKWISE
→ if $c < 0$: −ve rotation. CLOCKWISE
→ if $c = 0$: irrotational field

EXAMPLE 1

(b) $F(x, y) = -cyi + cxj$

$$\frac{\partial}{\partial x}(cx) - \frac{\partial}{\partial y}(-cy) = c + c = 2c$$

(a) $F(x, y) = cxi + cyj$

$$\frac{\partial}{\partial x}(cy) - \frac{\partial}{\partial y}(cx) \Rightarrow 0 \quad \text{no rotation}$$

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Curl in 3D

$F = Mi + Nj + Pk$

Here $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k.$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) i + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) j + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) k$$

curl $F = \nabla \times F$

ZERO CURL MEANING.

$$\Rightarrow \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) i + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) j + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) k$$

$\Rightarrow 0i + 0j + 0k \rightarrow$ zero vector.

Comparing all components give:

$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$; $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$; $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

irrotational field $\rightleftharpoons$ conservative vector field.

First ODES

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DIFFERENTIAL EQUATIONS.

(1) ordinary Differential Equations.

e.g $y'' + 2y' = 3y$

$f''(x) + 2f'(x) = 3f(x)$ Solution: functions

$\frac{d^2y}{dx^2} + \frac{2dy}{dx} = 3y$

Example : $y = e^{-3x}$ | $9e^{-3x} + 2(-3e^{-3x}) = 3(e^{-3x})$

$y' = -3e^{-3x}$ | $3e^{-3x} = 3e^{-3x}$ verified

$y'' = 9e^{-3x}$ | $\underbrace{y = e^{cx}}_{\text{family of solutions.}}$ c : is abitary constant

Solution has 'c' then → General sol

else → Particular Sol.

Variable Seperating Method →

$y' = 0.2y$

$\frac{dy}{dx} = 0.2y$

$\int \frac{dy}{y} = \int 0.2 dx$

$\ln y = 0.2x + c$

$e^{\ln y} = e^{(0.2x+c)}$

$e^{\ln y} = e^{0.2x} \cdot e^{c}$ → constant.

$y = ce^{0.2x}$

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LAPLACE PARTIAL DIFFERENTIAL EQ.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

* Order will be highest ~~power~~ ~~degree of independent~~ ~~variable~~ derivative.

* $y^{(4)} \to$ derivative 4

$y^4 \to$ power of 4

* Degree will be ~~highest~~ power of highest order derivative.

algebraic linear eq:-

$y = mx + c$, $ax_1 + ax_2 = 0$

differential equation

$a(x)y' + b(x)y = c(x)$

where $a(x), b(x), c(x) \to$ known function of x.

Non linear algebraic eq:-

$y_1 y_2 = x$, $a(x)y_1^2 + b(x)y_1^{1/2} = 0$, $\sqrt{y_1/y_2} = \cos xy$

Non linear D.E:-

$y'y = e^x$, $y' + \cos y = e^{xy}$

VARIABLE ~~SEPERATOR~~ SEPERABLE :-

→ for first order L/NL DE

Step (1) $g(y)dy = f(x)$

Step (2) Integrate both side $\int g(y)\,y'\,dx = \int f(x)dx + c$

Step (3) on left we can switch to y as variable of integ.

$y'dx = dy$ so that,

$$\int g(y)dy = \int f(x)\,dx + C.$$

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EXAMPLE:-

Q4) $y' \sin 2\pi x = \pi y \cos 2\pi x$

$\frac{dy}{dx}$

$$\int \frac{\cancel{y'}1}{y} dy = \int \frac{\pi \cos 2\pi x}{\sin 2\pi x} dx$$

$$\int \frac{1 \cancel{y}}{y} dy = \pi \int \frac{1}{\tan 2\pi x} dx$$

$$\int \frac{1}{y} dy = \pi \int \frac{1}{\tan 2\pi x} dx$$

$$\ln|y| = \pi \int \qquad u = \sin 2\pi x, \quad du = \cos 2\pi x \cdot 2\pi dx$$

$$= \cancel{\pi} \int \frac{\cancel{\cos 2\pi x}}{u} \cdot \frac{du}{\cancel{\cos 2\pi x} \cdot 2\cancel{\pi}}$$

$$\ln y = \frac{1}{2} \int \frac{1}{u} du$$

$$\ln y = \frac{1}{2} \ln u + c \Rightarrow \ln y = \ln(\sin 2\pi x)^{1/2} + c \quad \text{Ans.}$$

Q5) $yy' + 36x = 0$ $\qquad y' = dy/dx$

$$yy' = -36x$$

$$\int yy' = \int -36x \, dx$$

$$\int y \, dy = -36 \int x \, dx$$

$$\frac{y^2}{2} = \frac{-36x^2}{2}$$

$$y^2 = -36x^2 + C \qquad y' = \frac{1}{2}(C - 36x^2)^{-1/2}(-72x)$$

$$y = (C - 36x^2)^{1/2}$$

$$\frac{-36x}{(C - 36x^2)^{1/2}} \cdot (C - 36x^2)^{1/2} + 36x = 0$$

$$-36x + 36x = 0$$

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EXACT ODE

A first order ODE $M(x,y)dx + N(x,y)dy = 0$ is called an exact differential equation if differential form $M(x,y)dx + N(x,y)dy$ is EXACT.

if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then it is exact

Example:

(1) $\overbrace{2xy}^{M}dx + \overbrace{x^2}^{N}dy = 0$ $\qquad M_y = N_x$ ✓ EXACT

$M_y = N_x$ for exactness

$2x = 2x$

$u_x = M$ $\qquad u_y = N$

$u_x = 2xy$ —integrating with respect to x. $\qquad u_y = x^2$

$u = \int 2xy\,dx + k(y)$ $\qquad$ ~~$u = \int x^2 dy + k$.~~

(1) — $u = x^2y + k(y)$ → differentiating ~~$x^2y + k(y) = x^2y +$~~ with respect to y.

$u_y = x^2 + k'(y)$

$x^2 + k'(y) = x^2$ → inserting u_y

$k'(y) = 0$

$\int k'(y)\,dy = \int 0\,dy$

$k(y) = c$

$u = x^2y + c = 0$

$x^2y = c$

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(2) $M_y = N_x$

$0 = 0$

$u_x = x^3$ $\qquad$ $u_y = y^3$

$u = \int x^3 \, dx + k(y)$

$u = \frac{x^4}{4} + k(y)$ $\qquad$ $u = \frac{x^4}{4} + \frac{y^4}{4}$

$u_y = 0 + k'(y) = y^3$

$k'(y) = y^3$

$k(y) = \frac{y^4}{4}$

Method 2

$\int x^3 dx = -\int y^3 dy$

$\frac{x^4}{4} = -\frac{y^4}{4} + c$

$\frac{x^4}{4} + \frac{y^4}{4} = c$

(3) $\sin x \cos y \, dx + \cos x \sin y \, dy = 0$

$M = \sin x \cos y$ $\qquad$ $N = \cos x \sin y$

$M_y = -\sin x \sin y$ $\qquad$ $N_x = -\sin x \sin y$

$u_x = \sin x \cos y$ $\qquad$ $u_y = \cos x \sin y$

$u = \int \sin x \cos y \, dx + k(y)$

$u = \cos x \cos y + k(y)$

$u_y = -\cos x \sin y + k'(y) = \cos x \sin y$

$k'(y) = 2\cos x \sin y$

$k(y) = -2\cos x \cos y$

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NON-EXACT ODE

$$P(x,y)dx + Q(x,y)dy = 0 \qquad P_y \neq Q_x$$

→ use integrating factor. $F(x)$

$$F(x) = \exp\int R(x)dx \rightarrow e^{\int R(x)dx}$$

where $R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ * should only depend on x.

→ if not in x-terms then find $F^*(y)$

$P_y \neq Q_x$

$$F^*(y) = \exp\int R^*(y)\,dy$$

where $R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ * should only depend on y.

EXAMPLE

$$\underbrace{(e^{x+y} + ye^y)}_{P}dx + \underbrace{(xe^y - 1)}_{Q}dy = 0 \qquad y(0) = -1$$

$P_y : e^{x+y} + ye^y$ $P_y : e^{x+y} + e^y + ye^y$

$Q : xe^y - 1$ $Q_x : e^y$

$$R = \frac{1}{Q}(P_y - Q_x) = \frac{e^{x+y} + ye^y}{xe^y - 1}$$ is not depending on x only.

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$Q_x - P_y$: $\dfrac{e^y - e^{x+y} - e^y - ye^y}{e^{x+y} + ye^y} = \dfrac{-e^{x+y} - ye^y}{e^{x+y} + ye^y}$

, $R^*(y) = -1$ → only on y.

Integrating factor : $F^*(y) = e^{\int -1\, dy}$

$= e^{-y}$

→ now multiply this 'IF' with equation.

$$\underbrace{(e^x + y)}_{M} dx + \underbrace{(x - e^{-y})}_{N} dy = 0$$

$M_y = N_x$

$1 = 1$ ⇐ exact.

$u_x = M$, $u_y = N$

$u_x = e^x + y$, $u_y = x - e^{-y}$

$u = \int (x - e^{-y}) dy + h(x)$

$^*u = xy + e^{-y} + h(x)$

$u_x = y + h'(x)$

$e^x + y = y + h'(x)$

$h'(x) = e^x$

$\int h'(x) = \int e^x$

$h(x) = e^x + C$

$u = xy + e^{-y} + e^x + C = 0$

SOLUTION $\boxed{xy + e^{-y} + e^x = c}$

$y(0) = -1$

$0 + e^1 + e^0 = c$

$c = e + 1$

$xy + e^{-y} + e^x = e + 1$

Ans

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Linear ODE

Q5) e.g. $y' + ky = e^{-kx}$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$P(x) = k \quad Q(x) = e^{-kx}$$

Integrating factors $= e^{\int P(x)dx} \Rightarrow e^{\int k\,dx}$

$$I.F = e^{kx}$$

Multiplying 'IF' with equation.

$$e^{kx}y' + e^{kx}ky = 1$$

$$\frac{d}{dx}\left(e^{kx}y\right) = 1$$

taking integration $e^{kx}y = \int 1\,dx + C$

$$ye^{kx} = x + C$$

$$y = xe^{-kx} + Ce^{-kx}.$$ Ans

$$\frac{d}{dx}\left((x^2-9)^{1/2}y\right)$$

$u = (x^2-9)^{1/2}$ $\quad v = y$

$u' = \frac{1}{2}(2x)(x^2-9)^{-1/2}$ $\quad v' = 1$

$$u' = \frac{x}{\sqrt{x^2-9}}$$

$$y' + p(x)y = r(x)$$ — polynomial in derivatives of y

$$\frac{dy}{dx} + Py = Q \quad \dots (i)$$

Multiplying both sides of (i) by $e^{\int P dx}$, we get.

$$e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int P dx} \Rightarrow \frac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx}$$

On integrating both sides

$$ye^{\int P dx} = \int Qe^{\int P dx} + C \quad \dots (ii)$$

Q12) $xy' + 4y = 8x^4$, $y(1) = 2$.

$$y' + \frac{4}{x}y = 8x^3$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$P(x) = 4/x \qquad Q(x) = 8x^3$

$$\mu = e^{\int P(x) dx} = e^{\int 4/x \, dx} = e^{4\ln(x)}$$

$$\mu = x^4$$

$$x^4 y = \int x^4 \, 8x^3 \, dx + C$$

$$x^4 y = 8\int x^7 dx + C$$

$$x^4 y = 8\left(\frac{x^8}{8}\right) + C$$

$$y = x^4 + x^{-4}$$

$$y = x^4 + Cx^{-4}$$

$$y(1) = 2$$

$2 = 1 + C \qquad C = 1$

Q11) $y' = (y-2)\cot x$

$y' - (y-2)\cot x = 0$

$y' - y\cot x + 2\cot x = 0$

$y' - y\cot x = -2\cot x$

$P(x) = -\cot x \qquad Q(x) = -2\cot x$

$e^{\int P(x)dx} = e^{\int -\cot x dx} = e^{-\ln|\sin x|}$

$\sin x \left(c + \int \left(\frac{-2\cot x}{\sin x}\right) dx\right.$

$= \sin x \,(c + 2\csc x)$

$= c\sin x + \cancel{\sin x} \cdot \frac{2}{\cancel{\sin x}} \quad = c\sin x + 2$

Q10) $y'\cos x + (3y-1)\sec x = 0 \qquad y(\pi/4) = 4/3$

$y'\cos x + 3y\left(\frac{1}{\cos x}\right) = \frac{1}{\cos x}$

$y' + \frac{3y}{\cos^2 x} = \frac{1}{\cos^2 x}$

$y' + 3\sec^2 x \cdot y = \sec^2 x$

$P(x) = 3\sec^2 x \qquad Q(x) = \sec^2 x$

$e^{3\int \sec^2 x dx} = e^{3\tan x}$

$= e^{-3\tan x}\left(c + \int e^{3\tan x}\sec^2 x \, dx\right)$

$= e^{-3\tan x}\left(c + 1/3\, e^{3\tan x}\right)$

$y = ce^{-3\tan x} + \frac{1}{3}$

$\frac{4}{3} = y(\pi/4)$

$\frac{4}{3} = ce^{-3\tan(\pi/4)} + \frac{1}{3}$

$c = e^3$

$y = e^{3 - 3\tan x} + \frac{1}{3}$

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Data Structures

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CASE I:-

Distinct real λ_1, λ_2 — **Roots**

$e^{\lambda_1 x}, e^{\lambda_2 x}$ — **Basis**

$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ — **General Solution**

CASE II:-

Real double root $\lambda = -\frac{1}{2}a$ — **Roots**

$e^{-ax/2}, xe^{-ax/2}$ — **Basis**

$y = (c_1 + c_2 x)e^{-ax/2}$ — **General Solution**

CASE III:-

Complex conjugate

$\lambda_1 = -\frac{1}{2}a + i\omega$ — **Roots**

$\lambda_2 = -\frac{1}{2}a - i\omega$

$e^{-ax/2}\cos\omega x$, $e^{-ax/2}\sin\omega x$ — **Basis**

$y = e^{-ax/2}(A\cos\omega x + B\sin\omega x)$

General Solution

$y' =$

$y'' =$

$y'' =$

y

$+ a_m)x^m = 0$

$m_{1,2} = \alpha \pm \beta i$

$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$

Second ODES

$ay'' + by' + cy = 0$

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Linear ODES of Second Order

A second-order ODE is called linear if it can be written

$$y'' + p(x)y' + q(x)y = r(x) \quad (1)$$ * p and q are coefficients

and non linear if it cannot be written in this form. If $r(x) \equiv 0$ (that is, $r(x) = 0$ for all x considered; read "$r(x)$ is identically zero"), then (1) reduces to

(2) $y'' + p(x)y' + q(x)y = 0$ and is called homogeneous. If $r(x) \not\equiv 0$, then (1) is called non-homogeneous. Finally, an example of a nonlinear ODE is

$$y''y + y'^2 = 0$$

1

Homogeneous Linear ODEs: Superposition Principle.

Example ① The functions $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE

$y'' + y = 0$ for all x. We verify this by differentiating & substitution. We obtain $(\cos x)'' = -\cos x$; hence $y'' + y = -\cos x + \cos x = 0$

If we multiply $\cos x$ by constant for instant 4.7 and $\sin x$ by -2 and take sum of results claiming that it is a solution.

$$(4.7\cos x - 2\sin x)'' + (4.7\cos x - 2\sin x) = 0$$

In this example we've obtained $y_1 = \cos x$ and $y_2 = \sin x$

(3) $y = c_1y_1 + c_2y_2$ (c_1, c_2 arbitrary constants) This is called LINEAR COMBINATION of y_1 and y_2. Often called superposition principle or linearity principle.

(2) A Nonhomogeneous Linear ODE

Verify by substitution that the function $y = 1+\cos x$ and $y = 1+\sin x$ are solutions of nonhomogeneous linear ODE.

$y'' + y = 1$ but their sum is not a solution. Neither is, for instance, $2(1+\cos x)$ or $5(1+\sin x)$

(3) Non linear ODE

Verify by substitution that the functions $y = x^2$ and $y = 1$ are solutions of non linear ODE

$y''y - xy' = 0$ but their sum is not a solution - Neither is $-x^2$, so you cannot even multiply -1.

Homogeneous Linear ODE with constant coefficients.

$$y'' + ay' + by = 0 \qquad (1)$$

To solve (1), we recall Sec 1.5 that the solution of first-order linear ODE with constant coefficient k.

$$y' + ky = 0$$

is an exponential function $y = ce^{-kx}$. This gives us the idea to try as a solution of (1) the function

(2) $y = e^{\lambda x}$ $\quad y' = \lambda e^{\lambda x} \quad y'' = \lambda^2 e^{\lambda x}$

Substituting in eq (1) $(\lambda^2 + a\lambda + b)e^{\lambda x} = 0$

$\lambda^2 + a\lambda + b = 0$ (characteristic / auxilary equation)

CASES

(Case I) Two real roots if $a^2 - 4b > 0$

(Case II) A real double root if $a^2 - 4b = 0$

(Case III) complex conjugate roots if $a^2 - 4b < 0$

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CASE I :- In this case, a basis of solutions of (1) on any interval is $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ because y_1 and y_2 are defined for all x. The general solution is $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

EX ① $y'' - y = 0 \rightarrow m^2 - 1 = 0 \rightarrow m = 1, -1 \rightarrow y = c_1 e^{x} + c_2 e^{x}$

EX ② $y'' + y' - 2y = 0 \rightarrow m^2 + m - 2 = 0$ $\quad y_0 = 4,\ x_0 = 0$

$y(0) = 4$ $\quad m = 1, -2$ $\quad 4 = C_1 + C_2$

$y'(0) = -5$ $\quad y' = c_1 e^{x} - 2c_2 e^{-2x}$

$y = e^{x} + 3e^{-2x}$ Ans $\quad y_0' = -5,\ x_0 = 0$

$c_1 = 1,\ c_2 = 3$

CASE II :- If the discriminant $a^2 - 4b$ is zero, we see directly that we get only one root $\lambda = \lambda_1 = \lambda_2 = -a/2$

$$y_1 = e^{-(a/2)x}$$

To obtain second independent solution y_2 (needed for basis), we use method of reduction of order, setting $y_2 = uy_1$. Substituting this and it's derivatives $y_2' = u'y_1 + uy_1'$ and y_2'' into (1)

$(u''y_1 + 2u'y_1' + uy_1'') + a(u'y_1 + uy_1') + buy_1 = 0$ (Refer to book)

The general solution is $y = (c_1 + c_2 x)e^{-ax/2}$ (7)

CASE III :- If discriminant $a^2 - 4b < 0$. The general solution is

$y = e^{-ax/2}(A\cos wx + B\sin wx)$ (9)

$y'' + 0.4y' + 9.04 = 0$

$m^2 + 0.4m + 9.04 = 0$

$m_1 = \frac{-1}{5} + 3i$; $m_2 = \frac{-1}{5} - 3i$ $\quad y = e^{-3/5}(c_1 \sin 3x + c_2 \cos 3x)$

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Method of Undetermined Coefficients

→ To solve nonhomogeneous linear differential eq.

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x)$

step ① find complementary function y_c and

step ② find any particular solution y_p of non homogeneous

★ → $(y = y_c + y_p)$

The method of undetermined coefficients is not applicable to eqs

$g(x) = \ln x \qquad g(x) = 1/x \qquad g(x) = \tan x \qquad g(x) = \sin^{-1} x$

Example ① General solution using undetermined coefficients.

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$

we first solve

$$y'' + 4y' - 2y = 0$$

$$m^2 + 4m - 2 = 0$$

$$m_1 = -2 - \sqrt{6} \quad \text{and} \quad m_2 = -2 + \sqrt{6}$$

Hence the complementary function is

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}$$

Assume

$y_p = Ax^2 + Bx + C, \quad y_p' = 2Ax + B \quad y_p'' = 2A$

$$2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6$$

$$x^2(-2A) + x(8A - 2B) + (2A + 4B - 2C) = 2x^2 - 3x + 6$$

$$A = -1 \qquad B = -5/2 \qquad C = -9$$

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The general solution is

$$y = y_c + y_p = c_1 e^{-(2+\sqrt{6})x} + c_1 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9.$$

POWER SERIES

Power Series

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Example 1: Familiar Power Series are Maclaurin series.

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \dots \quad (|x|<1; \text{ geometric series})$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots$$

Example 2: Power series Solution. Solve $y' - y = 0$ ①

let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots = \sum_{m=0}^{\infty} a_m x^m$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots = \sum_{m=1}^{\infty} m a_m x^{m-1}$

Now insert them to eq ① $(a_1 + 2a_2 x + 3a_3 x^2 + \dots) - (a_0 + a_1 x + a_2 x^2$

$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \dots = 0$

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Equating each coefficient to zero

$a_1 - a_0 = 0$, $2a_2 - a_1 = 0$, $3a_3 - a_2 = 0$

$a_1 = a_0$, $a_2 = \frac{a_1}{2} = \frac{a_0}{2!}$, $a_3 = \frac{a_2}{3} = \frac{a_0}{3!}$, ...

$$y = a_0 + a_0 x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \dots = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)$$

$$\Rightarrow a_0 e^x$$

$y = \sum_{n=0}^{\infty} c_n x^n$ $\qquad y' = \sum_{n=1}^{\infty} c_n n x^n$

we take $n=1$ because $n=0$ will make it $(0)x^0$

e.g. $y' = y$

$$\sum_{n=1}^{\infty} c_n \underbrace{n x^{n-1}} = \sum_{n=0}^{\infty} c_n \underbrace{x^n}$$

the power are not same

Shift : $n \longrightarrow n+1$

$$\sum_{n=0}^{\infty} \underbrace{c_{n+1}(n+1)}x^n = \sum_{n=0}^{\infty} \underbrace{c_n}x^n$$

$c_{n+1}(n+1) = c_n$

$c_{n+1} = \frac{c_n}{(n+1)}$ (recurrence relation)

$y = \sum_{n=0}^{\infty} c_n x^n$ $\qquad c_n = \frac{c_0}{n!}$

$c_0 =$ constant

$c_1 = c_0$

$c_2 = \frac{c_0}{(2)}$

$c_3 = \frac{c_0}{2} \div 3 = \frac{c_0}{6}$

$\star\ c_n = \frac{c_0}{n!}$

$$y = c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow \boxed{c_0 e^x}$$

RATIO TEST

$$\lim_{n\to\infty} \left| \frac{c_0 x^{n+1}/(n+1)!}{c_0 x^n/(n)!} \right| = \lim_{n\to\infty} \left| \frac{x}{(n+1)} \right| = 0$$

Divergence Theorem

DIVERGENCE THEOREM.

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$$\underbrace{\iint F\cdot n\, d\sigma}_{\text{outward flux}} = \underbrace{\iiint_D \nabla\cdot F\, dV}_{\text{divergence integral.}}$$

$$\operatorname{div} F = \nabla\cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Example: Evaluate both sides of Equation (2) for expanding vector field $F = xi + yj + zk$ over sphere $x^2+y^2+z^2 = a^2$

$$n = \frac{\nabla g}{|\nabla g|}$$

$$n = \frac{2(xi+yj+zk)}{\sqrt{4(x^2+y^2+z^2)}} \Rightarrow \frac{xi+yj+zk}{a}$$

It follows

$$F\cdot n\, d\sigma = \frac{x^2+y^2+z^2}{a}\, d\sigma = \frac{a^2}{a}\, d\sigma = a\, d\sigma$$

Therefore outward flux is

$$\iint_S F\cdot n\, d\sigma = \iint_S a\, d\sigma = a\underbrace{\iint_S d\sigma}_{\text{surface area.}} = a(4\pi a^2) = 4\pi a^3$$

For right hand side of Equation (2), divergence of F is,

$$\nabla\cdot F = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

So we obtain

$$\iiint_D \nabla\cdot F\, dV = \iiint 3\, dV = 3\left(\underbrace{\frac{4}{3}\pi a^3}_{\text{volume of solid.}}\right) = 4\pi a^3$$

$\iiint \nabla \cdot F \, dV$ $\qquad$ $F = xi + yj + zk$

$\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$

$\iiint 3 \, dx\,dy\,dz$

$3 \iiint dx\,dy\,dz$

→ volume of cylinder

$3(\pi r^2 h)$

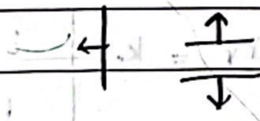
$3(\pi a^2 b)$ Ans

n Normal

(1) $r_u \times r_v$ normal.

(2) $\frac{r_u \times r_v}{\sqrt{r_u \times r_v}}$ unit normal

(3) $f(x,y,z) = \frac{\nabla f}{|\nabla f|}$

(4)  $k, -k$

$i, -i$

$j, -j$

not convex

simple bodies/simple surface $\qquad$ CONVEX

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TYPE ① CONVEX

TYPE ② NON-CONVEX → split

simple regions

TYPE ①

$F = Mi + Nj + Pk$

$$\iint_S F\cdot n\, d\sigma = \iint_S (Mi + Nj + Pk)\cdot \vec{n}\, d\sigma$$

$$= \iint_S \big(M(i\cdot n) + N(j\cdot n) + P(k\cdot n)\big)\, d\sigma$$

$$= \iiint_R M_x\, dV + \iiint_R N_y\, dV + \iiint_R P_z\, dV$$

$\operatorname{div} F = M_x + N_y + P_z$

$$\iint_S P(k\cdot n)\, d\sigma = \iiint_R P_z\, dV$$

*if perpendicular crosses more than two points ~~CONVEX~~ NON CONVEX

$$R = \{(x,y,z) \mid (x,y) \in \underbrace{R_{xy}}_{\text{Domain}},\ f_1 \le z \le f_2(x,y)\}$$

for S_2, $xi + yj + f_2(x,y)k$

$$k\cdot n = k\cdot r_x \times r_y = k\cdot \begin{vmatrix} i & j & k \\ 1 & 0 & \partial f_2/\partial x \\ 0 & 1 & \partial f_2/\partial y \end{vmatrix}$$

$$= k\cdot\left(\frac{\partial f_2}{\partial x} i - \frac{\partial f_2}{\partial y} j + k\right)$$

$$= 1$$

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for S_1, $xi + yj + f_1(x,y)k$

$$k \cdot n = k \cdot r_y \times r_x$$

$$= k \cdot \begin{vmatrix} i & j & k \\ 0 & 1 & \partial f_1/\partial y \\ 1 & 0 & \partial f_1/\partial x \end{vmatrix}$$

$$= k \cdot \left(\partial f_1/\partial x \, i - \partial f_1/\partial y \, j \mp k \right)$$

$$= -1$$

$$\iint_S P(k \cdot n)\, d\sigma = \iint_{S_1} P(x,y,f_1)(-1)\, d\sigma + \iint_{S_2} P(x,y,f_2)(1)\, d\sigma$$

$$= \iint \left(P(x,y,f_2) - P(x,y,f_1)\right) d\sigma$$

$$= \iiint_{(x,y,f_1)}^{(x,y,f_2)} f(B) - f(A)\, d\sigma$$

fundamental theorem of calculus. ←

$$= \iiint_{f_1}^{f_2} \frac{\partial P}{\partial z}\, dz\,dx\,dy \quad \text{(proven)}$$

$$= \iiint P_z\, dV \longrightarrow \text{Proved}$$

TYPE 2

Incomplete

Stokes Theorem

Green's Theorem

Curl $\longleftrightarrow$ Circulation

$\downarrow$

Stokes Theorem.

Flux $\longleftrightarrow$ Divergence.

$\downarrow$

Divergence Theorem

Stokes Theorem.

$$\underbrace{\oint_C \vec{F}\cdot d\vec{r}}_{\text{CCW Circulation}} = \iint_S \underbrace{(\nabla \times F)}_{\text{Curl. Integral.}} \cdot \vec{n} \, d\sigma \qquad d\sigma = |r_u \times r_v| \, du \, dv$$

Line integral or surface integral $\rightarrow$ Parametrizat

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EXAMPLE:- Evaluate eq. (4) for hemisphere $S: x^2+y^2+z^2=9 \quad z \geq 0$ it's bounding circle $C: x^2+y^2=9, z=0$ and field $F = yi - xj$.

$r^2 = 9 \qquad r(\theta) = (3\cos\theta)i + (3\sin\theta)j$

$r = 3 \qquad dr = (-3\sin\theta)i + (3\cos\theta)j$

$F = (3\sin\theta)i - (3\cos\theta)j$

$F \cdot dr = -9\sin^2\theta d\theta - 9\cos^2\theta d\theta = -9d\theta \qquad d\sigma = |r_u \times r_v| du dv$

$\oint_C F \cdot dr = \int_0^{2\pi} -9d\theta = -18\pi$

$\nabla \times F = -2k$

$n = r_u \times r_v = \dfrac{xi + yj + zk}{3}$

$d\sigma = \dfrac{3}{z} dA \qquad 0 \leq r \leq 3 \; ; \; 0 \leq \theta \leq 2\pi$

$\nabla \times F \cdot n \, d\sigma = \dfrac{-2z}{3} \times \dfrac{3}{z} dA = -2dA$

$\iint_{x^2+y^2 \leq 9} -2 \, dA = -18\pi$

Q1) $F = x^2 i + 2xj + z^2 k \qquad C: 4x^2 + y^2 = 4$ counterclockwise

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 2x & z^2 \end{vmatrix} = 0i + 0j + (2-0)k = 2k$$

$n = k$

$\text{curl } F \cdot n = 2 \Rightarrow \oint_C F \cdot dr = \iint 2dA = 4\pi$

$d\sigma = dx dy$

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Q2) $F = 2yi + 3xj - z^2 k \quad C: x^2+y^2=9$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y & 3x & -z^2 \end{vmatrix} = 0i + 0j + (3-2)k = k$$

$n = k$

$\text{curl } F \cdot n = 1 \qquad \oint_C F \cdot dr = \iint_R dx\,dy = 9\pi$

$d\sigma = dx\,dy$

Q3) $F = yi + xzj + x^2 k \qquad x+y+z=1$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & xz & x^2 \end{vmatrix} = -xi - 2xj + (z-1)k$$

$n = \dfrac{i+j+k}{\sqrt{3}}$

$\text{curl } F \cdot n = \dfrac{1}{\sqrt{3}}(-x - 2x + z - 1)$

$d\sigma = \sqrt{3}\, dA$

$\iint_R \dfrac{1}{\sqrt{3}}(-3x+z-1)\sqrt{3}\, dA$

$\int_0^1 \int_0^{1-x} [-3x + (1-x-y) - 1]\, dy\,dx$

$\int_0^1 \int_0^{1-x} (-4x - y)\, dy\,dx = \int_0^1 -\left[4x(1-x) + \tfrac{1}{2}(1-x)^2\right] dx$

$= -\int_0^1 \left(\dfrac{1}{2} + 3x - \dfrac{7}{2}x^2\right) dx = \dfrac{-5}{6}$