LINEAR ALGEBRA PROOFS.

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MATRICES.

Q) If A and B are two square matrices of same size then find the condition such that

$$(A+B)^2 = A^2 + B^2 + 2AB$$

Solution

$$(A+B)^2 = (A+B)(A+B)$$

$$= A^2 + AB + BA + B^2$$

if A and B commute then $AB = BA$

$$= A^2 + 2AB + B^2$$

(hence proven)

Q) Show that if AB and BA are both defined then AB and BA are square matrices

Solution

$A_{m_1 \times n_1}$ $B_{m_2 \times n_2}$

Since AB defined

$n_1 = m_2 = n$

$A_{m_1 \times n}$ $B_{n \times n_2}$

Now since BA also defined

$n_2 = m_1 = m$

$A_{m \times n}$ $B_{n \times m}$

and $AB = AB_{m \times m}$ (a square matrix)

Q) Show that if A is an $m \times n$ matrix and $A(BA)$ is defined then B is an $n \times m$ matrix

$A_{m \times n}$ $B_{m_1 \times n_1}$

since $A(BA)$ is defined, we first see BA where $n_1 = m = n$ so $A_{n \times n}$ and $B_{m_1 \times n}$ and BA becomes ~~BA~~ $m_1 \times n$ size matrix.

★ NONSINGULAR / INVERTIBLE (inverse exists)

★ SINGULAR (no inverse)

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Now when $A(BA)$ is defined then $n = m$, and size matrix becomes. $m \times n$. So this implies that B is $n \times m$.

Q) Show that if A has a row of zeros and B is any matrix for which AB is defined, then AB also has a row of zeros.

Solution

To see this pick an entry c_{ij} in i-th row of AB. By definition of AB we have,

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

now since i-th row of A is zero, we have

$$a_{i1} = a_{i2} = \dots = a_{in} = 0$$

$$c_{ij} = 0b_{1j} + 0b_{2j} + \dots + 0b_{nj} = 0$$

hence proven.

Q) If B and C are both inverses of matrix A then $B = C$

~~Sin~~

Solution

Since B is an inverse of A,

$$BA = I$$

Now multiplying both sides by C.

$$BAC = IC$$

$$B(AC) = C \quad \because IC = C \text{ and } AC = I$$

$$B(I) = C$$

$$B = C \quad \text{(proven)}$$

Q) If A and B are invertible matrices of same size then $(AB)^{-1} = B^{-1}A^{-1}$

Solution

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$
$$= AIA^{-1} = I \quad \text{— (1)}$$

also

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$$
$$= B^{-1}IB = I \quad \text{— (2)}$$

hence proven

$$(AB)^{-1} = B^{-1}A^{-1}$$

Q) If $Ax = B$ represents a system of n equations in n variables then prove that solution is unique if A is invertible.

Solution

Given $Ax = B$

$$A^{-1}Ax = A^{-1}B$$
$$Ix = A^{-1}B$$
$$x = A^{-1}B \text{ is unique.}$$

Q) Use relationship b/w elementary matrices and ERO to show that EROS that transform A to I will also transform I to A^{-1}.

Solution

Let θ be any ERO and E be elementary matrices corresponding to θ.

$$\theta(A) = EA = B$$

Let $\theta_1, \theta_2, \ldots, \theta_n$ be n number of EROS. and $E_1, E_2, \ldots, E_n$ be n number of elementary matrices

$$\theta_1, \theta_2, \ldots, \theta_n (A) = E_1, E_2, \ldots, E_n (A) = I$$

where $P = E_1, E_2, \ldots, E_n$

$$PA = I \qquad A \sim I$$

$$PAA^{-1} = A^{-1} I$$

$$PI = A^{-1} \qquad I \sim A^{-1}$$

Q) Show that if A is invertible then prove that

$$(A^{-1})^T = (A^T)^{-1}$$

<u>solution</u>

We know that $AA^{-1} = I$

$$(AA^{-1})^T = I^t$$

$$A^t (A^{-1})^T = I$$

$$(A^T)^{-1} A^T (A^{-1})^t = (A^T)^{-1} I$$

$$I (A^{-1})^t = (A^T)^{-1}$$

$$(A^{-1})^t = (A^T)^{-1}$$

Q) Consider A can be rewritten in the row-reduced echelon form and we want to show that A can be expressed as product of elementary matrix.

$$E_k, E_{k-1} \ldots E_2 E_1 A = I$$

$$PA = I$$

$$PAA^{-1} = IA^{-1}$$

$$PI = A^{-1}$$

$$I \sim A^{-1}$$

$$P = E_k^{-1} E_2^{-1} \ldots E_{k-1}^{-1} E_k$$

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Q) Since Ax has only $x=0$ as a solution. Theorem 1.6.4 guarantees that A is invertible. By theorem 1.4.8 (b), A^k is also invertible

$$A^k x = 0$$
$$(A^k)^{-1} A^k x = (A^k)^{-1} 0$$
$$\underbrace{A^{-1} A^{-1} \dots A^{-1}}_{k \text{ factors}} \underbrace{A A \dots A}_{k \text{ factors}} x = 0$$

$$Ix = 0$$
$$x = 0 \quad \text{has only trivial solution.}$$

Q) Let $Ax=0$ be a homogenous solution of n linear equation in n unknown and let Q be an invertible $n \times n$ matrix. Show that $Ax=0$ has just trivial solution.

$$Q(Ax) = Q\,0$$
$$(QA)x = 0$$
$$QQ^{-1}Ax = Q^{-1}0 \quad \because Q \text{ is invertible}$$
$$IAx = 0$$
$$Ax = 0$$

Q) Suppose that x_1 is a fixed matrix which satisfies the equation $Ax_1 = b$. Further let x be any matrix whatsoever which satisfies $Ax=b$. We must then show that there is a matrix x_0 which satisfies both equation $x = x_1 + x_0$. and $Ax_0 = 0$

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Solution

Given $x = x_1 + x_0$

$$x_0 = x - x_1$$

$$A(x_0) = A(x - x_1)$$

$$A(x_0) = A(x) - A(x_1)$$

$$A(x_0) = b - b = 0$$

Now we show that $A(x_1 + x_0)$ has solution

$$A(x_1 + x_0) = A(x_1) + A(x_0)$$

$$= b + 0$$

$$A(x_1 + x_0) = b$$

Q) let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then using the technique $[A \mid I] \longrightarrow [I \mid A^{-1}]$ prove that $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Solution

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \quad R_1 \to \frac{R_1}{a}$$

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right] \quad R_2 \to R_2 - cR_1$$

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d - cb/a & -c/a & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & \frac{ad-bc}{a} & -c/a & 1 \end{array}\right]$$

UNIQUE SOLUTION ...

NO SOLUTION | INFINITELY MANY $\to \det(A)=0 \to$ Parallel | coincident.

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \qquad R_2 \to \frac{a}{ad-bc} R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \qquad R_1 \to R_1 - \frac{b}{a} R_2$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Q) Let A and B be $n \times n$ matrices show that if A is invertible, then $\det(B) = \det(A^{-1}BA)$

<u>Solution</u>

$$\det(A^{-1}BA) = \det(A^{-1})\det(B)\det(A)$$

$$= \frac{1}{\cancel{\det(A)}}\det(B)\cancel{\det(A)}$$

$$\det(A^{-1}BA) = \det(B).$$

Q) Let A and B be $n \times n$ matrices. What can you say about invertibility of AB if are a both of factors are singular?

<u>Solution</u> If either A or B is singular, then either $\det(A)$ or $\det(B)$ will be zero. Hence $\det(AB) = \det(A)\det(B) = 0$. Thus AB is also singular.

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Q) Prove that if $\det(A)=1$ ~~then~~ the entries in A are integer, then all entries in A^{-1} are integers.

not all

Solution

This follows from Theorem 2.1.2 and the fact that the cofactors of A are integer if A has only integer entries since integers are closed under multiplication, addition & subtraction.

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \operatorname{adj}(A)$$

Q) If B is $n \times n$ matrix and E is an elementary matrix of order $n \times n$ then $\det(EB) = \det(E)\det(B)$

Solution

If E is performed by multiplying a row of I_n by scalar of k then this means that,

$$EB = C$$

$$\det(EB) = \det(C)$$

$$\det(EB) = k\det(B)$$

since $\det(E) = k\det(I) = k$

$$\det(EB) = \det(E)\det(B)$$

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Q) Show that a matrix with a row of zeros canot have an inverse OR show that a matrix with column ~~vectors~~ of zeros canot have an inverse.

Solution

Let A denote a matrix which has an entire row or an entire column of zeros. Then if B is any matrix, either AB has an entire row of zeros or BA has entire column of zeros. Neither AB nor BA can be identity matrix therefore A canot have an inverse.

Q) If A and B are square matrices of same size then $\det(AB) = \det(A)\det(B)$

CASE 1 if A is not invertible. $\det(A) = 0$

$$\det(AB) = 0 \cdot \det(B)$$

$$\det(AB) = 0$$

CASE 2 if A is invertible $\det(A) \neq 0$

We take A as product of elementary matrices.

$$A = E_1, E_2, \dots, E_n$$

~~$\det(A) =$~~

$$AB = E_1 E_2, \dots E_n B$$

$$\det(AB) = \det(E_1, E_2, \dots E_n)\det(B)$$

$$\det(AB) = \det(E_1)\det(E_2)\dots\det(E_n)\det(B)$$

$$\det(AB) = \det(A)\det(B).$$

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~~VECTORS~~.

Q) Show that matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is invertible for all values of θ; then find A^{-1} using Theorem 2.1.2.

SOLUTION

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$$\det(A) = 1 \begin{vmatrix} \cos & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$\operatorname{adj} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q) Use raw reduction to show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} =$

$(b-a)(c-a)(c-b)$

Solution

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c^2-a^2)-(c-a)(b+a) \end{vmatrix}$$

$$= (b-a)\left[(c^2-a^2)-(c-a)(b+a)\right]$$

$$= (b-a)(c-a)\left[(c+a)-(b+a)\right]$$

$$= (b-a)(c-a)(c-b).$$

Q) $A^{r+s} = A^r A^s$ is valid for negative integers r, s.

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and let $r=1$ and $s=-1$

$$A^{1-1} = A^1 A^{-1}$$

$$A^0 = A A^{-1}$$

$$I = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^{-1}$$

inverse not exists hence it is not valid for negative integers.

Q) Let A be any $m \times n$ matrix and let 0 be the $m \times n$ matrix each of whose entries is zero. Show that if $kA = 0$, then $k=0$ or $A=0$.

<u>Solution</u> Given $kA=0$ and suppose $k \neq 0$

$$kA = 0$$

$$\frac{1}{k} kA = \frac{1}{k} 0$$

$$A = 0$$

Now suppose $A \neq 0$, if $kA=0$ then only possibility is $k=0$.

VECTORS

Q) Prove that any vector in 3-dimensional e.g. $u = (u_1, u_2, u_3)$ can be written as the linear combination of e_1, e_2 and e_3.

Solution

$$u = u_1 e_1 + u_2 e_2 + u_3 e_3$$

$$e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1)$$

$$\because u_1 e_1 + u_2 e_2 + u_3 e_3$$
$$= u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$$
$$= (u_1,0,0) + (0,u_2,0) + (0,0,u_3)$$
$$= (u_1, u_2, u_3) = u$$

Q) If $u = (u_1, u_2, u_3) = u_1 e_1 + u_2 e_2 + u_3 e_3$

$v = (v_1, v_2, v_3) = v_1 e_1 + v_2 e_2 + v_3 e_3$

then $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$

Solution

$$= u \cdot v$$
$$= (u_1 e_1 + u_2 e_2 + u_3 e_3) \cdot (u_1 e_1 + v_2 e_2 + v_3 e_3)$$
$$= u_1 v_1 e_1 e_1 + u_1 v_2 e_1 \cancel{e_2}^{0} + u_1 v_3 e_1 \cancel{e_3}^{0} + u_2 v_1 e_2 \cancel{e_1}^{0} +$$
$$u_2 v_2 e_2 e_2 + u_2 v_3 e_2 \cancel{e_3}^{0} + u_3 v_3 e_3 e_3 + u_3 v_1 e_1 \cancel{e_3}^{0} +$$
$$u_3 v_2 e_2 \cancel{e_3}^{0}$$

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Q) If u and v are vectors in 2-space or 3-space and if $a \neq 0$ then $\text{proj}_a u = \frac{(u \cdot a)}{\|a\|^2} a$ and

$$u - \text{proj}_a u = u - \frac{(u \cdot a)}{\|a\|^2} a$$

Solution

$u = w_1 + w_2$

$\because w_1 = ka$

$u = ka + w_2$

$w_2 = u - ka$

taking dot product with a:

$w_2 \cdot a = (u - ka) \cdot a = 0$

$= ua - ka \cdot a = 0$

$k = \frac{u \cdot a}{\|a\|^2}$

$\because w_1 = \frac{u \cdot a}{\|a\|^2} a$ and $w_2 = u - \frac{u \cdot a \cdot a}{\|a\|^2}$

Q) Let V be vector space and the zero vector of V is 0. If k is some scalar, then prove that $k0 = 0$.

Solution

$k0 + ku = k(0 + u)$ Axiom 7

$k0 + ku = k(u)$ $\because$ Axiom 4

$(k0 + ku) + (-ku) = k(u) + (-ku)$

$k0 + (ku + (-ku)) = (ku + (-ku))$

$k0 + 0 = 0$

$k0 = 0$ (proven)

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Q) $V = R^n$

$V_1 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ $V_2 = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$V_1 + V_2 = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix} \in R^n$

Q) Prove $\|u+v\| \leq \|u\| + \|v\|$

$\|u+v\|^2 = (u+v)(u+v)$

$= u \cdot u + uv + u \cdot v + v \cdot v$

$= \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$

$\|u+v\|^2 = (\|u\| + \|v\|)^2$

$\|u+v\| \cancel{= \|u\| + \|v\|}$

Q) $(u+v) \cdot w = u \cdot w + v \cdot w$

$= (u_1+v_1, u_2+v_2, \dots, u_n+v_n) \cdot (w_1, \dots w_n)$

$= (u_1+v_1)w_1 + (u_2+v_2)w_2 + (u_n+v_n)w_n$

$= (u_1w_1 + u_2w_2 + \dots u_nw_n) + (v_1w_1 + v_2w_2 + \dots v_nw_n) = u \cdot w + v \cdot w.$

Q) $d(\vec{u}, \vec{v}) \leq d(\vec{u}, \vec{w}) + d(\vec{w}, \vec{v})$

$d(\vec{u}, \vec{v}) = d(\vec{u}, \vec{v})$

$= \|u - v\|$

$= \|u - w\| + \|w - v\|$

$= d(u, w) + d(w, v)$

Q) $u \cdot u \geq 0$. Further $u \cdot u = 0$ if and only if $u = 0$.

We have $u \cdot u = u_1^2 + u_2^2 + \dots + u_n^2 \geq 0$

further equality holds if and if $u_1 = u_2 = \dots = u_n = 0$ that is if and only if $u = 0$.

Q) Prove $\vec{u} \cdot \vec{v} = \frac{1}{4}\|u+v\|^2 -$

$\frac{1}{4}\|u-v\|^2$

$\frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2 = \frac{1}{4}(u+v)$

$(u+v) - \frac{1}{4}(u-v)(u-v)$

$= \frac{1}{4}(u \cdot u + vu + uv + vv)$

$- \frac{1}{4}(u \cdot u - vu - u \cdot v + vv)$

$= \frac{1}{4}(4\, u \cdot v) = u \cdot v$

Q) If $u = (u_1, u_2, \dots, u_n)$ then $ku = (ku_1, \dots, ku_n)$ prove that $\|ku\| = \|k\|\|u\|$

$\|ku\| = \sqrt{(ku_1)^2 + \dots + (ku_n)^2}$

$= \|k\| \sqrt{u_1^2 + \dots + u_n^2}$

$\|ku\| = \|k\| \|u\|$

Q) $(A\vec{u})\cdot v = \vec{u}\cdot(A^T\vec{v})$

$(A\vec{u})\cdot\vec{v} = (A\vec{u})^T\cdot v$

$= u^T A^T \vec{v} \quad \because u^T = u$

$(A\vec{u})\cdot v = u\cdot(A^T v)$

Q) Prove $0\vec{u} = 0$

Ax 8 $\Rightarrow (k+l)\vec{u} = k\vec{u} + l\vec{u}$

$0\vec{u} + 0\vec{u} = (0+0)\vec{u}$

Ax 4 $\Rightarrow u\in V \Leftrightarrow -u\in V$

$0\vec{u} + 0\vec{u} + (-0\vec{u}) = 0\vec{u} + (-0\vec{u})$

Ax 3 $\Rightarrow 0\vec{u} + (0\vec{u} + (-0\vec{u}))$

$= 0\vec{u} + (-0\vec{u})$

Ax 5 $\Rightarrow 0\vec{u} + 0 = 0$

Ax 4 $\Rightarrow 0\vec{u} = 0$

Q) $\|u+v\|^2 - \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$

$\|u+v\|^2 = (u+v)(u+v)$

$= \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$

$\|u-v\|^2 = (u-v)(u-v)$

$= \|u\|^2 - 2\|u\|\|v\| + \|v\|^2$

adding both eqs.

$\|u\|^2 + 2\|u\|\|v\| + \|v\|^2 + \|u\|^2$

$- 2\|u\|\|v\| + \|v\|^2$

$= 2(\|u\|^2 + \|v\|^2)$

(b) The result states theorem about parallelogram. Sum of squares of length of four sides of parallelogram equals sum of squares of length of two diagnol.

Q) Prove $(-1)\vec{u} = -\vec{u}$

$\vec{u} + (-1)\vec{u} = 0$

$1\vec{u} + (-1)\vec{u} = 0$

$(1+(-1))\vec{u} = 0$

$0\vec{u} = 0$

Q) If u and v are vectors in R^n and k is any scalar then $u\cdot(kv) = k(u\cdot v)$

$\vec{u}\cdot(k(\vec{v})) = (u_1, \dots u_n)\cdot(k(v_1, \dots v_n$

$= (u_1, \dots, u_n)\cdot(kv_1, \dots, kv_n)$

$= u_1(kv_1) + \dots u_n(kv_n)$

$= ku_1v_1 + \dots + ku_nv_n$

$= k(u_1v_1 + \dots + u_nv_n)$

$= k(\vec{u}\cdot\vec{v})$

Q) Show that u and v are orthogonal vectors in R^n if $\|u+v\| = \|u-v\|$

$u\cdot v = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$

$u\cdot v = \frac{1}{4}\|u-v\|^2 - \frac{1}{4}\|u-v\|^2$

$u\cdot v = 0$

Q) Show that there are infinitely many ~~zeros~~ vectors in with Euclidean norm 1 whose Euclidean inner product with $(1, -3, 5)$ is zero.

(HW 7)

Q) Prove that if $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for same vector space V then any vector $u \in V$ can be written as lineal combination

Solution

Let u vector has two different representations

$u = c_1 v_1 + \ldots + c_n v_n$ — ①

$v = k_1 v_1 + \ldots + k_n v_n$ — ②

eq ① − eq ②

$u - u = (c_1 v_1 + \ldots + c_n v_n) - (k_1 v_1 + \ldots + k_n v_n)$

$u = (c_1 - k_1) v_1 + \ldots + (c_n - k_n) v_n$

By using linear independent property

$c_1 - k_1 = 0$, $c_2 - k_2 = 0$ $\quad c_i = k_i$

$c_1 = k_1$, $c_2 = k_2$

hence u has exactly one representation.

Q) Show that the set W of all polynomials of degree $\leq n$ is a subspace of real-valued functions under addition and scalar multiplication.

Solution

W is not an empty set since it contains zero polynomial.

$0 = x^0 + 0x^1 + \cdots + 0x^n$

let $u_1, v_1 \in W$

so $u + v \in W$

$u = a_0 + a_1 x + \ldots + a_n x^n$ and $v = b_0 + \ldots + b_n x^n$

so $u + v$ is

$(a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n$

$c_0 + c_1 x + \ldots + c_n x^n \in W$

Next

day / date:

$(u+v)\cdot w = u\cdot w + v\cdot w$

PROOF Let $u = (u_1, u_2, \dots, u_n)$, $v = (v_1, v_2, \dots, v_n)$ and $w = (w_1, w_2, \dots, w_n)$ thus

$$(u+v)\cdot w = (u_1+v_1, u_2+v_2, \dots, u_n+v_n)\cdot(w_1, w_2, \dots, w_n)$$
$$= (u_1+v_1)(w_1) + (u_2+v_2)w_2 + \dots + (u_n+v_n)w_n$$
$$= (u_1w_1 + u_2w_2 + \dots + u_nw_n) + (v_1w_1 + v_2w_2 + \dots + v_nw_n)$$
$$= u\cdot w + v\cdot w$$

$v\cdot v \geq 0$. Further $v\cdot v = 0$ if and only if $v = 0$

PROOF

We have $v\cdot v = v_1^2 + v_2^2 + \dots + v_n^2 \geq 0$. Further equality holds if and if $v_1 = v_2 = \dots = v_n = 0$ – that is, if and only if $v = 0$.

To solve the vector equation $x + u = v$ for x we can add $-u$ to both sides

$$(x+u) + (-u) = v + (-u)$$
$$x + (u-u) = v - u$$
$$x + 0 = v - u$$
$$x = v - u$$

PROPERTIES OF LENGTH IN R^n

(a) $\|u\| \geq 0$

(b) $\|u\| = 0$ if and only if $u = 0$

(c) $\|ku\| = |k|\,\|u\|$

(d) $\|u+v\| \leq \|u\| + \|v\|$

PROOF (c) If $u = (u_1, u_2, \dots, u_n)$ then $ku = (ku_1, ku_2 \dots ku_n)$

so

$$\|ku\| = \sqrt{(ku_1)^2 + (ku_2)^2 + \dots + (ku_n)^2}$$

$$= |k| \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

$$= |k|\,\|u\|$$

PROOF (d)

$$\|u+v\|^2 = (u+v)\cdot(u+v)$$

$$\|u+v\|^2 = (u \cdot u) + 2(u \cdot v) + v \cdot v$$

$$\|u+v\|^2 = \|u\|^2 + 2(u \cdot v) + \|v\|^2$$

$$\|u+v\|^2 = \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$

$$\|u+v\|^2 = (\|u\| + \|v\|)^2$$

$$\|u+v\| = \|u\| + \|v\|$$

day / date:

PROOF

$$d(\vec{u},\vec{v}) \leq d(\vec{u},\vec{w}) + d(\vec{w},\vec{v})$$

$$d(\vec{u},\vec{v}) = d(\vec{u},\vec{v})$$

$$= \|\vec{u}-\vec{v}\|$$

$$= \|(\vec{u}-\vec{w}) + (\vec{w}-\vec{v})\|$$

$$= \|\vec{u}-\vec{w}\| + \|\vec{w}-\vec{v}\|$$

$$d(\vec{u},\vec{v}) = d(\vec{u},\vec{w}) + d(\vec{w},\vec{v})$$

PROOF

$$\vec{u}\cdot\vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2$$

$$\frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2 = \frac{1}{4}(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) - \frac{1}{4}(\vec{u}-\vec{v})(u-v)$$

$$= \frac{1}{4}(\vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v})$$

$$-\frac{1}{4}(\vec{u}\cdot\vec{u} - \vec{v}\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v})$$

$$= \frac{1}{\cancel{4}}(4\vec{u}\cdot\vec{v}) \Rightarrow \vec{u}\cdot\vec{v}$$

day / date:

PROOF :-

$$(A\vec{u})\cdot\vec{v} = \vec{u}\cdot(\vec{A}^{T}\vec{v})$$

$$(A\vec{u})\cdot\vec{v} = (A\vec{u})^{T}\vec{v} \quad \because \vec{u}\cdot\vec{v} = \vec{u}^{T}v$$

$$= u^{T}A^{T}\vec{v}$$

$$= u^{T}(A^{T}\vec{v})$$

$$(A\vec{u})\cdot\vec{v} = u\cdot(A^{T}\vec{v})$$

PROVE:- $0\vec{u} = 0$

PROOF Ax 8 $\Rightarrow (k+l)\vec{u} = k\vec{u}+l\vec{u}$

$$0\vec{u}+0\vec{u} = (0+0)\vec{u}$$

$$0\vec{u}+0\vec{u} = 0\vec{u}$$

Ax 4 $\Rightarrow u\in V \Leftrightarrow -u\in V$

$$0\vec{u}+0\vec{u}+(-0\vec{u}) = 0\vec{u}+(-0\vec{u})$$

Ax 3 $\Rightarrow 0\vec{u}+(0\vec{u}+(-0\vec{u})) = 0\vec{u}+(-0\vec{u})$

Ax 5 $\Rightarrow 0\vec{u}+0 = 0$

Ax 4 $\Rightarrow 0\vec{u} = 0$

PROVE:- $(-1)\vec{u} = -\vec{u} \quad \forall u \in V$

PROOF $(-1)\vec{u} = -\vec{u}$

$$\Rightarrow \vec{u}+(-1)\vec{u} = 0$$

$$\vec{u}+(-1)\vec{u}$$

$$1\vec{u}+(-1)\vec{u}$$

$$(1+(-1))\vec{u}$$

$$0\vec{u} \Rightarrow 0$$

day / date:

TRANSPOSE DOT PRODUCT

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \text{ and } v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$v^T u = [v_1 \; v_2 \; \dots \; v_n] \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = [u_1 v_1 + u_2 v_2 + \dots + u_n v_n]$$

$$= [u \cdot v] = u \cdot v$$

$$\boxed{u \cdot v = v^T u} \text{ — eq } 7$$

— × — × — × —

$$Au \cdot v = u \cdot A^T v \text{ — eq } 8$$

PROOF

$$\underbrace{Au}_{u} \cdot \underbrace{v}_{v} = (\overbrace{v^T A}^{v^T}) \overbrace{u}^{u} \quad \text{using eq } 7$$

$$= \underbrace{(A^T v)^T u}_{v^T \cdot u} = \underbrace{u \cdot (A^T v)}_{u \cdot v}$$

$$u \cdot Av = A^T u \cdot v \text{ — eq } 9$$

PROOF

$$u \cdot Av = (Av)^T u = v^T (A^T u) = A^T u \cdot v$$

Q11) show that if $S : \{v_1, v_2, \ldots, v_r\}$ is a linearly independent set of vectors, then so is every non-empty subset of S.

PROOF

Suppose that S has a linearly dependent subset T. Denote it's vectors by $w_1, \ldots, w_m$. Then

$$k_1 w_1 + \ldots + k_m w_m = 0$$

But if we let $u_1, \ldots, u_{n-m}$ denote the vectors which are in S but not in T, then

$$k_1 w_1 + k_m w_m + 0u_1 + \ldots + 0u_{n-m} = 0$$

Thus we have linear combination of vectors $v_1, \ldots, v_n$ which equals 0. Since not all constants are zero, it follows that S is not linearly independent set of vectors, contrary to the hypothesis. That is, if S is linearly independent set, then so is every non-empty subset, T.

Q13) Show that if $\{v_1, v_2, \dots, v_r\}$ is linearly dependent set of vectors in a vector space V and if $v_{r+1}, \dots, v_n$ are any vectors in V, then $\{v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_n\}$ is also linearly dependent.

PROOF

Since $\{v_1, v_2, \dots, v_r\}$ is linearly dependent set of vectors, there exist constants $c_1, c_2, \dots, c_r$ not all zero such that:

$$c_1v_1 + c_2v_2 + \dots + c_rv_r = 0$$

But then

$$c_1v_1 + c_2v_2 + \dots + c_rv_r + 0v_{r+1} + \dots + 0v_n = 0$$

The above equation implies that vectors $v_1, \dots, v_n$ are linearly dependent.

Q15) Show that if is $\{v_1, v_2\}$ linearly independent and v_3 does not lie in span $\{v_1, v_2\}$ then $\{v_1, v_2, v_3\}$ is linearly independent.

PROOF:

Suppose that $\{v_1, v_2, v_3\}$ is linearly independent then there exists constants a, b and c not all zero such that

(*) $\quad av_1 + bv_2 + cv_3 = 0$

C.

CASE 1 $c=0$. Then $(*)$ becomes

$$av_1 + bv_2 = 0$$

where not both a and b are zero.

CASE 2 $c \neq 0$. Then solving $(*)$ for v_3 yields.

$$v_3 = -\frac{a}{c} v_1 - \frac{b}{c} v_2$$

This equation implies that v_3 is in span $\{v_1, v_2\}$ contrary to hypothesis. Thus $\{v_1, v_2, v_3\}$ is linearly independent.

RESULT: If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V then every vector v in V can be expressed in form

$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

in exactly one way.

PROOF Let $v = c_1v_1 + c_2v_2 + \ldots c_nv_n$ and $v = k_1v_1 + k_2v_2 + \cdots + k_nv_n$ subtracting the second equation from the first gives
$0 = (c_1 - k_1)v_1 + (c_2 - k_2)v_2 + \cdots + (c_n - k_n)v_n$

TRY THE FOLLOWING

If $S = \{v_1, v_2, \dots, v_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

HINT Assume $k_1v_1 + k_2v_2 + \dots + k_nv_n = 0$ and prove

$$k_1 = k_2 = \dots = k_n = 0$$

PROOF Let $k_1v_1 + k_2v_2 + \dots + k_nv_n = 0$ taking inner product with v_i on both sides,

$$\langle k_1v_1 + k_2v_2 + \dots + k_nv_n, v_i \rangle = 0$$

$$\because \langle 0, v_i \rangle = \langle 0+0, v_i \rangle = \langle 0, v_i \rangle + \langle 0, v_i \rangle$$

$$\Longrightarrow \langle 0, v_i \rangle = 0$$

$$k_1 \langle v_1, v_i \rangle + k_2 \langle v_2, v_i \rangle + \dots + k_i \langle v_i, v_i \rangle + \dots + k_n \langle v_n, v_i \rangle = 0 \quad \text{——} \quad (1)$$

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But $S = \{v_1, v_2, \dots, v_n\}$ is an orthogonal set therefore $\langle v_i, v_j \rangle = 0$ when $i \neq j$ so that ① equation reduces to

$k_i \langle v_i, v_i \rangle = 0$ but $v_i \neq 0$

therefore $\langle v_i, v_i \rangle = \|v_i\|^2 > 0$ so that $k_i = 0$. Since the subscript i is arbitrary we have $k_1 = k_2 = \dots = k_n = 0$; thus, S is linearly independent.

LECTURE 22

RESULT:

If A is an orthogonal matrix then

$$\det(A) = \pm 1$$

PROOF:

$$\because A^{-1} = A^T$$

$$\Rightarrow \det(A^{-1}) = \det(A^T)$$

$$\Rightarrow \frac{1}{\det(A)} = \det(A^T) = \det(A)$$

$$\Rightarrow \frac{1}{\det(A)} = \det(A)$$

$$\Rightarrow 1 = [\det(A)]^2$$

$$\Rightarrow \det(A) = \pm 1$$

NOTE: if $A^{-1} = A^T$ and $\det(A) = 1$ then A is called proper orthogonal matrix

TRY THE FOLLOWING

If A is orthogonally diagonalizable then prove that A is a symmetric matrix.

A

PROOF $\because P^t A P = D$ where D is diagnol matrix

and $P^t P = I$

$\therefore \underbrace{PP^t}_{I} A \underbrace{PP^t}_{I} = PDP^t$

$\Rightarrow A = PDP^t$ — (1) $\quad D^t = D$

$\Rightarrow A^t = (PDP^t)^t = (P^t)^t \boxed{D^t} P^t$

$\Rightarrow PDP^t = A$ from (1)

$\Rightarrow A^t = A$

NOTE: SYMMETRIC is always diagonalizable.

PROOF

If $T: V \longrightarrow W$ is a linear tranformation, then
$T(0) = 0$.

PROOF

$$T(\underset{\text{vector}}{0}) = T(\underset{\text{scalar}}{0}v) = 0T(v) = 0$$

$v \in V$ $\qquad \because T(kv) = kT(v)$

OR for any $v \in V$

$$T(0) = T(v - v) = T(v + (-v))$$
$$= T(v) + T(-v) = T(v) - T(v) = 0$$

25. Let S be a basis for an n-dimensional vector space V. Show that if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ form a linearly independent set of vectors in V, then the coordinate vectors $(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, \ldots, (\mathbf{v}_r)_S$ form a linearly independent set in R^n, and conversely.

25. First notice that if $\mathbf{v}$ and $\mathbf{w}$ are vectors in V and a and b are scalars, then $(a\mathbf{v} + b\mathbf{w})_S = a(\mathbf{v})_S + b(\mathbf{w})_S$. This follows from the definition of coordinate vectors. Clearly, this result applies to any finite sum of vectors. Also notice that if $(\mathbf{v})_S = (\mathbf{0})_S$, then $\mathbf{v} = \mathbf{0}$. Why?

Now suppose that $k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$. Then

$$\begin{aligned}(k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r)_S &= k_1(\mathbf{v}_1)_S + \cdots + k_r(\mathbf{v}_r)_S \\ &= (\mathbf{0})_S\end{aligned}$$

Conversely, if $k_1(\mathbf{v}_1)_S + \cdots + k_r(\mathbf{v}_r)_S = (\mathbf{0})_S$, then

$$(k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r)_S = (\mathbf{0})_S, \quad \text{or} \quad k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

Thus the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly independent in V if and only if the coordinate vectors $(\mathbf{v}_1)_S, \ldots, (\mathbf{v}_r)_S$ are linearly independent in R^n.

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26. Using the notation from Exercise 25, show that if $v_1, v_2, \ldots, v_r$ span V, then the coordinate vectors $(v_1)_S, (v_2)_S, \ldots, (v_r)_S$ span R^n, and conversely.

Let $v_1, v_2, \cdots, v_r$ span V and $w \in \mathbb{R}^n$. Since S is a basis of V, we have that there exists $v \in V$, such that $(v)_S = w$. Since $v_1, v_2, \cdots, v_r$ span V, we have $k_1, k_2, \cdots, k_r \in \mathbb{R}$, such that $k_1 v_1 + k_2 v_2 + \cdots + k_r v_r = v$ Hence we have

$$\begin{aligned} & k_1 v_1 + k_2 v_2 + \cdots + k_r v_r = v \\ \Longrightarrow\ & (k_1 v_1 + k_2 v_2 + \cdots + k_r v_r)_S = (v)_S \\ \Longrightarrow\ & k_1 (v_1)_S + k_2 (v_2)_S + \cdots + k_r (v_r)_S = w. \end{aligned}$$

Hence $\{(v_1)_S, (v_2)_S, \cdots, (v_r)_S\}$ spans $\mathbb{R}^n$.

Conversely assume $(v_1)_S, (v_2)_S, \cdots, (v_r)_S$ spans $\mathbb{R}^n$ and $v \in V$. Then $(v)_S = w \in \mathbb{R}^n$. Hence there exis $k_1, k_2, \cdots, k_r \in \mathbb{R}$, such that $k_1 (v_1)_S + k_2 (v_2)_S + \cdots + k_r (v_r)_S = w$. Hence we have

$$\begin{aligned} & k_1 (v_1)_S + k_2 (v_2)_S + \cdots + k_r (v_r)_S = (v)_S \\ \Longrightarrow\ & (k_1 v_1 + k_2 v_2 + \cdots + k_r v_r)_S = (v)_S \\ \Longrightarrow\ & k_1 v_1 + k_2 v_2 + \cdots + k_r v_r = v. \end{aligned}$$

Hence $v_1, v_2, \cdots, v_r$ span V.

13. Prove that the row vectors of an $n \times n$ invertible matrix A form a basis for R^n.

13. Let A be an $n \times n$ invertible matrix. Since A^T is also invertible, it is row equivalent to I_n. It is clear that the column vectors of I_n are linearly independent. Hence, by virtue of Theorem 5.5.5, the column vectors of A^T, which are just the row vectors of A, are also linearly independent. Therefore the rows of A form a set of n linearly independent vectors in R^n, and consequently form a basis for R^n.

(4) CAN WE SAY THAT THE ROTATION MATRIX IS A TRANSITION MATRIX FROM ONE ORTHONORMAL BASIS TO ANOTHER.

Solution: To make notation simpler, let $\mathcal{S}$ and $\mathcal{T}$ be the bases. Form matrices S and T from the respective basis vectors. Since P is an orthogonal matrix, we have

$$PP^T = P^T P = I$$

Since $\mathcal{S}$ is an orthonormal basis, S is an orthogonal matrix and

$$SS^T = S^T S = I$$

Using that P is the transition matrix from S to T,

$$\begin{aligned} T &= PS \\ T^T T &= T^T PS = (PS)^T PS \\ T^T T &= S^T P^T PS = S^T S = I \end{aligned}$$

Similarly,

$$\begin{aligned} T &= PS \\ TT^T &= PST^T = PS(PS)^T \\ TT^T &= PSS^T P^T = P^T P = I. \end{aligned}$$

Since $T^T T = TT^T = I$, we have that T is an orthogonal matrix. Since T was formed from the basis vectors of $\mathcal{T}$, we have that $\mathcal{T}$ is an orthonormal basis.

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① FIND THE EIGENVALUES OF $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ WITHOUT FORMING THE CUBIC EQUATION

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

Solution: Considering linearly dependent equation that holds for eigenvalues in R^3 is

$$k_1 v + k_2 A v + k_3 A^2 v + k^4 A^3 v = 0 \qquad \forall k_i \neq 0$$

Now let $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so $Av = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$, $A^2 v = \begin{bmatrix} 24 \\ 20 \\ 20 \end{bmatrix}$ and $A^3 v = \begin{bmatrix} 176 \\ 168 \\ 168 \end{bmatrix}$. So the matrix would be in the form of $k_1 v + k_2 A v + k_3 A^2 v + k^4 A^3 v = 0 \Rightarrow$

$$\begin{bmatrix} 1 & 4 & 24 & 176 \\ 0 & 2 & 20 & 168 \\ 0 & 2 & 24 & 176 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Echoleon form of above matrix leads to values for k_i.

$$\begin{aligned} k_1 &= 16k_3 + 160k_4 \\ k_2 &= -10k_3 - 84k_4 \\ k_3 &= t \\ k_4 &= s \end{aligned}$$

Lets take $s = 1$ and $t = -12$ for simplication of k_i. Hence $k_1 = 32, k_2 = 36, k_3 = -12$ and $k_4 = 1$.

Now substitute in above equation and solve for the roots of equation, we have 8,2,2.

7 PROVE THAT IF $\boxed{A}$ IS A SYMMETRIC MATRIX THEN THE EIGENVECTORS FROM DIFFERENT EIGENSPACES ARE ORTHOGONAL. 10

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Question 07

If A is a real symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues are orthogonal.

Solution: Proof. Let λ_1 and λ_2 be distinct eigenvalues with associated eigenvectors v_1 and v_2. Then, $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$. Take the inner product of the first equation by v_2 and the inner product of the second equation by v_1 :

$$v_2^T A v_1 = \lambda_1 v_2 v_1, \quad A v_2{}^T v_1 = \lambda_2 v_2 v_1$$

In Equation, $(Av_2)^T v_1 = v_2^T A^T v_1$, so becomes $v_2^T A v_1 = \lambda_1 v_2 v_1$,$v_2^T A^T v_1 = \lambda_2 v_2 v_1$
Since $A^T = A$, in Equation, we have $v_2^T A v_1 = \lambda_1 v_2 v_1$, $\quad v_2^T A v_1 = \lambda_2 v_2 v_1$ and

$$\lambda_1 v_2 v_1 = \lambda_2 v_2 v_1$$

Equation gives

$$(\lambda_1 - \lambda_2) v_2 v_1 = 0.$$

Since $\lambda_1 \neq \lambda_2$, $\langle v_2, v_1 \rangle = 0$, and v_1, v_2 are orthogonal.

25. Let $\{v_1, v_2, v_3\}$ be an orthonormal basis for an inner product space V. Show that if w is a vector in V, then $\|w\|^2 = \langle w, v_1\rangle^2 + \langle w, v_2\rangle^2 + \langle w, v_3\rangle^2$.

$\therefore \|\omega\|^2 = \omega_1^2 + \omega_2^2 + \omega_3^3 \dots + \omega_n^2$

25. By Theorem 6.3.1, we know that

$$\boxed{w = a_1v_1 + a_2v_2 + a_3v_3}$$

where $a_i = \langle w, v_i\rangle$. Thus

$$= a_1^2 (v_1 \cdot v_1) + a_1 \times a_2 (v_1 \cdot v_2) + a_1 \times a_3 (v_1 \cdot v_3)$$

$$\|w\|^2 = \langle w, w\rangle \quad = a_1^2 + a_2^2 + a_3^3$$

25. By Theorem 6.3.1, we know that

$$\boxed{w = a_1v_1 + a_2v_2 + a_3v_3}$$

where $a_i = \langle w, v_i\rangle$. Thus

$$= a_1^2 (v_1 \cdot v_1) + a_1 \times a_2 (v_1 \cdot v_2) + a_1 \times a_3 (v_1 \cdot v_3)$$

$$\|w\|^2 = \langle w, w\rangle \quad = a_1^2 + a_2^2 + a_3^3$$

$$= \sum_{i=1}^{3} a_i^2 \langle v_i, v_i\rangle + \sum_{i \neq 1} a_i a_j \langle v_i, v_j\rangle$$

But $\langle v_i, v_j\rangle = 0$ if $i \neq j$ and $\langle v_i, v_i\rangle = 1$ because the set $\{v_1, v_2, v_3\}$ is orthonormal. Hence

$$\|w\|^2 = a_1^2 + a_2^2 + a_3^2 \qquad \omega \cdot v_1$$

$$= \underbrace{\langle w, v_1\rangle^2}_{a_1} + \underbrace{\langle w, v_2\rangle^2}_{a_2} + \underbrace{\langle w, v_3\rangle^2}_{a_3} \qquad (a_1v_1 + a_2v_2 + a_3v_3)\cdot v_1$$

27. In Step 3 of the proof of Theorem 6.3.6, it was stated that "the linear independence of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ ensures that $\mathbf{v}_3 \neq \mathbf{0}$." Prove this statement.

↓ Base

27. Suppose the contrary; that is, suppose that

scalar

(*) $$\boxed{\mathbf{u}_3 - \langle \mathbf{u}_3, \mathbf{v}_1 \rangle \mathbf{v}_1 - \langle \mathbf{u}_3, \mathbf{v}_2 \rangle \mathbf{v}_2 = \mathbf{0}} \Rightarrow \bar{u}_3 = \overbrace{\langle \bar{u}_3, \bar{v}_1 \rangle}^{\text{scalar}} \bar{v}_1 + \langle \bar{u}_3, \bar{v}_2 \rangle \bar{v}_2$$

Then (*) implies that $\mathbf{u}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. But $\mathbf{v}_1$ is a multiple of $\mathbf{u}_1$ while $\mathbf{v}_2$ is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. Hence, (*) implies that $\mathbf{u}_3$ is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ and therefore that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly dependent, contrary to the hypothesis that $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is linearly independent. Thus, the assumption that (*) holds leads to a contradiction.

$$u_3 = \overbrace{a_1 (b_1}^{\text{scalar}} \bar{u}_1) + a_2 \underbrace{(b_2 \bar{u}_1 + b_3 \bar{u}_2)}_{\text{scalar}}$$

29. **(For Readers Who Have Studied Calculus)**

Let the vector space P_2 have the inner product

15. Prove: If $k \neq 0$, then A and kA have the same rank.

Step 1

1 of 2

Let

$$A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

where r_i for $i = \overline{1, n}$ is the row of the A matrix and

$$kA = \begin{bmatrix} kr_1 \\ kr_2 \\ \vdots \\ kr_n \end{bmatrix}$$

where kr_i for $i = \overline{1, n}$ is the row of the kA matrix. Confirmation is proved by contradiction. Suppose it is rank $(A) = p$ and let $\{(r_1, r_2, \cdots r_p)\}$ are linearly independent. Since k is a scalar then the set $\{(kr_1, kr_2, \cdots kr_p)\}$ are linearly independent. Let kr_i be another row of the matrix A so that it is the set $\{(kr_1, kr_2, \cdots kr_p, kr_i)\}$ are linearly independent for $i \neq \overline{1, p}$. Hence, the set $\{(r_1, r_2, \cdots r_p, r_i)\}$ is also linearly independent which is contrary to the assumption that it is rank$(A) = p$. Then, the maximum independent rows of the kA matrix is $\{(kr_1, kr_2, \cdots kr_p)\}$, hence the matrices A and kA have the same rank.

Result

2 of 2

The matrices A and kA have the same rank.

6. If A is an $m \times n$ matrix, what are the largest possible value for its rank and the smallest possible value for its nullity?

Hint See Exercise 5.

Step 1 1 of 3

The goal of the exercise is to find the largest possible value of the rank of the $m \times n$ matrix

$$[A]_{m \times n}$$

and to find the smallest possible value for its nullity.

Step 2 2 of 3

Now since A is a $m \times n$ matrix therefore the row vectors of the matrix lie in the space $\mathbb{R}^n$ and the column vectors of the matrix lie in space $\mathbb{R}^m$.

We know that the rank of a matrix is the common dimension of its column space and row space. This leads us to conclude that the rank of the matrix A is less than or equal to the minimum of m and n, that is

$$\operatorname{rank}(A) \leq \min\{m, n\}.$$

Thus we get the largest possible value of the rank of the matrix is

$$\min\{m, n\}.$$