HABIB UNIVERSITY

Midterm-II Exam
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Probability and Statistics

Max Points: 100
Date: Apr. 02, 2024
Duration: 60 min

Q1 [40 pt]: An electromechanical geared transmission (EGT) system consists of two components – an electric motor (first) and a gearbox (second).

Suppose that the probabilities that the first and second components meet specifications are **0.95** and **0.98**, respectively.

Assume that the components are independent. Let $\boldsymbol{Z}$ be the number of components in EGT that meet specifications.

(1a) Determine the probability mass function of $\boldsymbol{Z}$. **[20 pt]**

(1b) Find the mean and variance of $\boldsymbol{Z}$. **[20 pt]**

Q2 [40 pt]: Consider a discrete uniform random variable $\boldsymbol{X} \in \boldsymbol{\mathcal{U}}[1, 10]$.
That is $\boldsymbol{X}$ may attain any integer value from 1 to 10 with equal probability.

(2a) Find the conditional PMFs: $P(\boldsymbol{X} = k \mid \boldsymbol{X} > 3)$ and $P(\boldsymbol{X} = k \mid \boldsymbol{X} \leq 3)$. **[10 pt]**

(2b) Using the conditional PMFs, find the conditional first-order moments:
. $E[\boldsymbol{X} \mid \boldsymbol{X} > 3]$ and $E[\boldsymbol{X} \mid \boldsymbol{X} \leq 3]$. **[10 pt]**

(2c) Similarly, obtain the conditional second-order moments:
. $E[\boldsymbol{X}^2 \mid \boldsymbol{X} > 3]$ and $E[\boldsymbol{X}^2 \mid \boldsymbol{X} \leq 3]$. **[10 pt]**

(2d) Finally, using the conditional moments (as evaluated above),
. obtain the values of $E[\boldsymbol{X}]$ and $E[\boldsymbol{X}^2]$. **[10 pt]**

Q3 [20 pt]: The number of flaws in rivets used in steel sheets in container manufacturing is assumed to be Poisson-distributed with a mean of 0.25 flawed rivets per square meter.

(3a) Find the probability that there is one flaw in one square meter of sheet. **[05 pt]**

(3b) Find the probability that there are at most two flaws in 10 square meters of sheet. **[05 pt]**

(3c) Find the probability that there are no flaws in 15 square meters of sheet. **[05 pt]**

(3d) Find the probability that there are at least two flaws in 20 square meters of sheet. **[05 pt]**

Midterm II Exam Solution

Q1. Solution

Let p_f = probability that first component meets specs.

p_s = probability that second component meets specs.

$p_f = 0.95$ (given)

$p_s = 0.98$ (given)

(a)

Let Z be the No. of components that meet specs

The possible values of Z are $\{0, 1, 2\}$

$$p_Z(z) = \begin{cases} (1-p_f)(1-p_s), & \text{for } z=0 \\ p_f(1-p_s) + p_s(1-p_f), & \text{for } z=1 \\ p_f\, p_s, & \text{for } z=2. \end{cases}$$

$$= \begin{cases} 0.001 & \text{for } z=0 \\ 0.068 & \text{for } z=1 \\ 0.931 & \text{for } z=2 \end{cases}$$

You may check that $\sum_z p_Z(z) = 1$.

(b) Mean of Z = $E(Z) = \sum_{z=0}^{2} z\, p_Z(z) = 1.93$.

$$E(Z^2) = \sum_{z=0}^{2} z^2 p_Z(z) = 3.792$$

Variance of $Z = \sigma_Z^2 = 3.792 - 1.93^2 = 0.0671$.

Q2. Solution:

$$X \sim \mathcal{U}[1, 10]$$

$$p_X(k) = \frac{1}{10} \qquad \text{for } k = 1, 2, \ldots, 10.$$

$p_X(k)$ is zero, otherwise.

2a)
$$p_X(k \mid X > 3) = P[X = k \mid X > 3]$$

$$= \frac{P[\{X = k\} \cap \{X > 3\}]}{P[X > 3]}$$

$$= \frac{P[X = k]}{P[X > 3]} \qquad \text{for } k > 3$$

$$= \frac{1/10}{7/10} = \frac{1}{7} \quad \text{for } k = 4, 5, \ldots, 10.$$

$$p_X(k \mid X \leq 3) = P[X = k \mid X \leq 3]$$

$$= \frac{P[\{X = k\} \cap \{X \leq 3\}]}{P[X \leq 3]}$$

$$= \frac{P[X = k]}{P[X \leq 3]} \qquad \text{for } k \leq 3$$

$$= \frac{1/10}{3/10} \qquad \text{for } k = 1, 2, 3.$$

$$= \frac{1}{3}.$$

2b) $$E[X|X>3] = \sum_k k\, p_{X|X}(k|X>3)$$

$$= \sum_k k \cdot \frac{1}{7} \quad \text{where } k = 4,5,6,\ldots,10$$

$$= \frac{1}{7}\sum_{k=4}^{10} k$$

$$= \frac{1}{7}(4+5+6+7+8+9+10)$$

$$= \frac{1}{7}\cdot 49 = 7.$$

$$E[X|X\leq 3] = \sum_k k\, p_{X|X}(k|X\leq 3)$$

$$= \sum_k k\cdot\frac{1}{3} \quad \text{where } k = 1,2,\&\ 3.$$

$$= \frac{1}{3}\sum_{k=1}^{3} k = \frac{1}{3}(1+2+3) = \frac{6}{3} = 2.$$

2c) $$E[X^2|X>3] = \sum_{k=4}^{10} k^2\cdot\frac{1}{7} = \frac{1}{7}(4^2+5^2+6^2+7^2+8^2+9^2+10^2)$$

$$= \frac{371}{7} = 53.$$

$$E[X^2|X\leq 3] = \sum_{k=1}^{3} k^2\frac{1}{3} = \frac{1}{3}(1^2+2^2+3^2) = \frac{14}{3}$$

2d) $$E[X] = E[X|X\leq 3]\,P[X\leq 3] + E[X|X>3]\,P[X>3]$$

$$= (2)\cdot\frac{3}{10} + (7)\cdot\frac{7}{10} = \frac{6+49}{10} = \frac{55}{10} = 5.5$$

$$E[X^2] = E[X^2|X\leq 3]\,P[X\leq 3] + E[X^2|X>3]\,P[X>3]$$

$$= (14/3)(3/10) + (53)(7/10) = 38.5$$

Q3 Solution:

$$p_X(k) = \frac{e^{-\lambda} \lambda^k}{k!} \qquad k = 0, 1, 2, \ldots$$

(3a) Prob. that there is one flaw in one sq. meter of sheet.

$\lambda = 0.25$ flawed rivets per sq. meter $= 0.25$ (given)

$$P[X=1] = \frac{e^{-0.25} (0.25)^1}{1!} = 0.1947$$

(3b) Probability that there are at most two flaws in 10 square meter of sheet.

$\Rightarrow$ new $\lambda = 0.25 * 10 = 2.5$

$$P[X \leq 2] = P[X=0] + P[X=1] + P[X=2]$$

$$= p_X(0) + p_X(1) + p_X(2)$$

$$= \frac{e^{-2.5}(2.5)^0}{0!} + \frac{e^{-2.5}(2.5)^1}{1!} + \frac{e^{-2.5}(2.5)^2}{2!}$$

$$= e^{-2.5}\left(1 + 2.5 + \frac{2.5^2}{2}\right)$$

$$= (0.0821)(6.625)$$

$$= 0.544$$

(3c) Probability that there are no flaws in 15 sq. meter of sheet.

New $\lambda = 0.25 \times 15 = 3.75$

$$P[X=0] = p_X(0) = \frac{e^{-3.75}(3.75)^0}{0!} = e^{-3.75}$$

$$= 0.0235$$

(3d) Probability that there are at least two flaws in 20 sq. meter of sheet.

New $\lambda = 0.25 * 20 = 5.$

$$P[X \geq 2] = 1 - P[X=0] - P[X=1]$$

$$= 1 - \frac{e^{-5}(5)^0}{0!} - \frac{e^{-5}(5)^1}{1!}$$

$$= 1 - e^{-5}(1+5) = 1 - 6e^{-5}$$

$$= 0.9596.$$