MATH 205

# LECTURE 6 LINEAR ALGEBRA

(GAUSS-JORDAN ELIMINATION) (P. 8)

## REDUCED ROW-ECHELON FORM. ↑

A MATRIX IS IN REDUCED ROW-ECHELON FORM IF

(1) IT IS ALREADY IN THE ECHELON FORM

(2) EACH COLUMN THAT CONTAINS A LEADING 1 HAS ZEROS EVERYWHERE ELSE.

EXAMPLES: THE FOLLOWING MATRICES ARE IN REDUCED ROW-ECHELON FORM

(1) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) $$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

EXAMPLE: SOLVE THE FOLLOWING LINEAR SYSTEM BY REDUCING THE AUGMENTED MATRIX TO REDUCED ROW-ECHELON FORM (GAUSS-JORDAN ELIMINATION)

$$3x_1 + 4x_2 + 5x_3 = 12$$

$$x_1 - x_2 + 2x_3 = 2$$

$$2x_1 + x_2 + 3x_3 = 6$$

SOLUTION: HERE THE AUGMENTED MATRIX IS

$$\begin{bmatrix} 3 & 4 & 5 & 12 \\ 1 & -1 & 2 & 2 \\ 2 & 1 & 3 & 6 \end{bmatrix}$$

AND ITS ECHELON FORM IS GIVEN BY

$$\begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(DERIVED LAST TIME)

NOW TO REDUCE IT TO REDUCED ROW-ECHELON FORM WE PROCEED AS FOLLOWS:

3]

$$\begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_1 \to R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 - R_3 \end{matrix}$$

NOW RE-WRITING THE LINEAR SYSTEM AGAIN WE GET

$$\boxed{x_1 = 1, \; x_2 = 1, \; x_3 = 1}$$

→ REQUIRED REDUCED ROW ECHELON FORM. IN THIS CASE ENTRIES IN THE LAST (4th) COLUMN FORM THE SOLUTION SET.

QUESTION:

FOR WHAT VALUES OF 'K' DOES THE FOLLOWING SYSTEM HAVE (a) NO SOLUTION (b) ONLY ONE SOLUTION (c) INFINITELY MANY SOLUTIONS.

$$Kx + Y = 1$$
$$x + KY = 1$$

SOLUTION: THE AUGMENTED MATRIX OF THE GIVEN SYSTEM OF EQUATIONS IS GIVEN BY

$$A = \left[\begin{array}{cc|c} K & 1 & 1 \\ 1 & K & 1 \end{array}\right]$$, LET US TRY TO FIND THE ECHELON FORM OF THIS MATRIX

$$\Rightarrow A \sim \left[\begin{array}{cc|c} 1 & K & 1 \\ K & 1 & 1 \end{array}\right] \quad R_1 \longleftrightarrow R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & K & 1 \\ 0 & 1-K^2 & 1-K \end{array}\right] \quad \begin{array}{l} R_2 \longrightarrow \\ R_2 - KR_1 \end{array} \quad \longrightarrow (1)$$

LET US DISCUSS DIFFERENT CASES:

5)



(I) FOR $\boxed{\text{EXACTLY ONE}}$ SOLUTION ① MUST BE $\boxed{\text{TRANSFORMED}}$ INTO THE $\boxed{\text{ECHELON}}$ FORM BY MAKING THE ENTRY (2,2) $\boxed{\text{ONE}}$ BY PERFORMING $\boxed{R_2 \longrightarrow \frac{R_2}{1-k^2}}$ TO GET

$$\sim \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1 & \frac{1}{1+k} \end{array}\right]$$ PROVIDED $1-k^2 \neq 0 \Rightarrow k^2 \neq 1$

$\Rightarrow \boxed{k \neq \pm 1} \rightarrow$ FOR <u>ONE SOLUTION</u>.

USING $\boxed{k=-1}$ IN ① GIVES

$$\left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1-k^2 & 1-k \end{array}\right] \xrightarrow{k=-1} \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 0 & 2 \end{array}\right],$$ SO WE HAVE $\boxed{\text{NO}}$ SOLUTION FOR $\boxed{k=-1}$ ∵ <u>SECOND ROW</u> GIVES $\boxed{0=2}$ WHICH IS <u>NOT</u> POSSIBLE.

USING $\boxed{k=+1}$ IN ① GIVES

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right],$$ REWRITING THE <u>LINEAR</u> SYSTEM GIVES

$\boxed{x_1 + x_2 = 1}$

HERE NO. OF UNKNOWNS = 2, NO. OF EQUATIONS = 1, WHICH IS LESS THAN NO. OF UNKNOWNS. THIS GIVES INFINITE SOLUTIONS FOR $\boxed{k=+1}$.

NOTE: HERE (INFINITE SOLUTIONS CASE) $\boxed{x_1}$ IS CALLED A LEADING VARIABLE WHICH CORRESPONDS TO THE LEADING 1 IN THE ECHELON FORM OF AUGMENTED MATRIX

i.e. $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$, AND $\boxed{x_2}$ IS CALLED A FREE VARIABLE WHICH CORRESPONDS (DOES NOT) TO THE LEADING 1.

WE WRITE DOWN THE SOLUTIONS BY WRITING THE LEADING VARIABLES IN TERMS OF FREE VARIABLES.

IN LAST EXAMPLE WRITING THE FREE VARIABLE $\boxed{x_2 = t}$ (SAY),

$\boxed{x_1 = 1 - x_2 = 1 - t}$, FOR DIFFERENT VALUES OF $\boxed{t}$ (WHICH IS A REAL NUMBER), WE HAVE INFINITE VALUES OF $x_1, x_2$ $\therefore$ WE HAVE INFINITE SOLUTIONS FOR $\boxed{k=1}$.