LINEAR ALGEBRA

LECTURE 23

MATH 205

EIGENSPACE: (P. 341 8TH ED.) (P. 359 7TH ED.) THE EIGENVECTORS CORRESPONDING TO $\boxed{\lambda}$ ARE THE NONZERO VECTORS IN THE SOLUTION SPACE OF $\boxed{A\underline{X} = \lambda\underline{X}}$ OR $(A - \lambda I)\underline{X} = \underline{0}$. WE CALL THIS SOLUTION SPACE THE EIGENSPACE OF $\boxed{A}$ CORRESPONDING TO $\boxed{\lambda}$.

QUESTION: FIND THE BASES FOR THE EIGENSPACES OF

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

SOLUTION:

EIGENVALUES OF $\boxed{A}$ ARE $\boxed{1, 2, 3}$ (OBTAINED LAST TIME), THEREFORE THERE ARE THREE EIGENSPACES OF $\boxed{A}$ CORRESPONDING TO $\lambda = \boxed{1}, \boxed{2}$, AND $\boxed{3}$. SO WE PROCEED AS FOLLOWS

2

(1) EIGENSPACE CORRESPONDING TO $\boxed{\lambda=1}$ IS GIVEN BY

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ (DERIVED LAST TIME)

$\therefore$ ITS BASIS = $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(2) EIGENSPACE CORRESPONDING TO $\boxed{\lambda=2}$ IS GIVEN BY

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$ (DERIVED LAST TIME)

(3) AND FINALLY EIGENSPACE CORRESPONDING TO $\boxed{\lambda=3}$ IS GIVEN BY

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$ THEREFORE

$\boxed{\lambda=3}$

BASES FOR THE EIGENSPACES CORRESPONDING TO $\boxed{\lambda=2}$ AND $\boxed{\lambda=2}$ ARE GIVEN BY

$\boxed{3}$

$$\left\{\begin{bmatrix}1\\-2\\-2\end{bmatrix}\right\} \text{ AND } \left\{\begin{bmatrix}1\\-1\\-1\end{bmatrix}\right\}$$

RESPECTIVELY.

NOTE: DIMENSION OF EACH EIGENSPACE OF $\boxed{A}$ DESCRIBED ABOVE $=\boxed{1}$ $\because$ EACH HAS ONLY $\boxed{\text{ONE}}$ BASIS VECTOR.

ASSIGNMENT NO. 6(a)

(ROTATION MATRIX) $\rightarrow$ SPECIAL CASE OF ORTHOGONAL MATRIX

RECALL THAT $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ IS THE ROTATION MATRIX,

(1) PROVE THAT THE TERMINAL POINT OF e_1 IN $\boxed{R^2}$ REACHES $(\cos\theta, \sin\theta)$ AFTER A ROTATION THROU-

4

AN ANGLE θ (COUNTERCLOCKWISE)

(2) FROM (1) IF IT REACHES $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ THEN THROUGH WHICH ANGLE ROTATION HAS TAKEN PLACE. ALSO WRITE DOWN THE ROTATION MATRIX IN THIS CASE.

(3) WRITE DOWN THE ROTATION MATRIX WHEN ROTATION IS ABOUT Z-AXIS IN THREE DIMENSIONS.

(4) CAN WE SAY THAT THE ROTATION MATRIX IS A TRANSITION MATRIX FROM ONE ORTHONORMAL BASIS TO ANOTHER.

(5) IF MATRIX R_1 GIVES ROTATION THROUGH θ (COUNTERCLOCKWISE), R_2 GIVES ROTATION

THROUGH ϕ THEN WHAT IS THE GEOMETRICAL SIGNIFICANCE OF R_1R_2 ?

(6) IF $\boxed{A, B}$ ARE ORTHOGONAL MATRICES OF SAME ORDER THEN

(i) AB IS ORTHOGONAL

(ii) A^T IS ORTHOGONAL AND

(iii) A^{-1} IS ORTHOGONAL

(7) IF $\langle A\underline{u}, A\underline{v}\rangle = A\underline{u} \cdot A\underline{v}$ WHERE $\boxed{A}$ IS AN $n \times n$ MATRIX, THEN FOR WHICH MATRIX IT IS TRUE THAT

$$\langle A\underline{u}, A\underline{v}\rangle = \langle \underline{u}, \underline{v}\rangle$$

FOR EUCLIDEAN INNER PRODUCT

EIGENVALUES/EIGENVECTORS

(8) FIND THE BASES FOR THE EIGENSPACES OF

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$\boxed{6}$

(9) IF $\boxed{k}$ IS A POSITIVE INTEGER, $\boxed{\lambda}$ IS AN EIGENVALUE OF MATRIX $\boxed{A}$, AND $\underline{X}$ IS A CORRESPONDING EIGENVECTOR, THEN PROVE THAT $\boxed{\lambda^k}$ IS AN EIGENVALUE OF $\boxed{A^k}$ AND $\boxed{\underline{X}}$ IS A CORRESPONDING EIGENVECTOR.

<u>HINT</u>: PROVE BY MATHEMATICAL INDUCTION.

(10) PROVE THAT EIGENVECTORS OF $\boxed{A}$ CORRESPONDING TO <u>DISTINCT</u> EIGENVALUES ARE <u>DISTINCT</u>.

$\boxed{7}$

(11) HOW CAN YOU FIND THE EIGENVALUES OF DIAGONAL, UPPER TRIANGULAR AND LOWER TRIANGULAR MATRICES?

(12) IF $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

FIND THE EIGENVALUES OF A^{100}.

(13) IF X IS AN EIGENVECTOR OF $\boxed{A}$ CORRESPONDING TO $\boxed{\lambda}$ THEN PROVE THAT $\boxed{\lambda^3}$ IS THE EIGENVALUE OF $\boxed{A^3}$ CORRESPONDING TO EIGENVECTOR X.