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MATH 205 LINEAR ALGEBRA
MIDTERM PART A
SPRING 2024

TIME: 60 MINUTES
TOTAL MARKS: 50

Question 1: For which values of a will the following system have no solutions? Exactly one solution? Infinitely many solutions?

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

[10 Marks]

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Question 2: If *A* is an *n* x *n* such that $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every *n* x 1 matrix **b**, show that *A* is non-singular. **[10 Marks]**

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Question 3: Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Show that if the matrix $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ satisfies the equation $AX = XB$, then X is a scalar multiple of $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. **[10 Marks]**

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Question 4: Let A be a square matrix.

(a) If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$.

(b) If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.

(Hint: Assume (a) holds and then prove part (b)). **[10 Marks]**

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<u>Question 5:</u> Show that A $= \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$ is not invertible for any values of the entries.

[10 Marks]

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SOLUTION PART A

Q 1

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & (a^2-14) & a+2 \end{array}\right]$$

$$R_2 - 3R_1, R_3 - 4R_1 \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & (a^2-2) & a-14 \end{array}\right]$$

$$R_2 - R_3 \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & (a^2-16) & a-4 \end{array}\right]$$

$$\frac{-1}{7}R_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & (a^2-16) & a-4 \end{array}\right]$$

The Gauss-Jordan process will reduce this system to the equations

$$\begin{aligned} x + 2y - 3z &= 4 \\ y - 2z &= 10/7 \\ \left(a^2 - 16\right) z &= a - 4 \end{aligned}$$

If $a = 4$, then the last equation becomes $0 = 0$, and hence there will be infinitely many solutions-for instance,

$$z = t, y = 2t + \frac{10}{7}, x = -2\left(2t + \frac{10}{7}\right) + 3t + 4$$

. If $a = -4$, then the last equation becomes $0 = -8$, and so the system will have no solutions.

Any other value of a will yield a unique solution for z and hence also for y and x.

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Q 2

$$Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

are consistent. Let x_1, x_2, ..., x_n be solutions of the respective systems, and let us form an $n \times n$ matrix C having these solutions as columns. Thus C has the form

$$C = [x_1 | x_2 | \cdots | x_n]$$

As discussed in Section 1.3, the successive columns of the product AC will be

$$Ax_1, Ax_2, \ldots, Ax_n$$

Thus

$$AC = [Ax_1 | Ax_2 | \cdots | Ax_n] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

By part (*b*) of Theorem 1.6.3, it follows that $C = A^{-1}$. Thus, A is invertible.

Q 3

Since $AX = XB$, this implies that

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Simplifying after multiplication

$$\begin{bmatrix} x & y \\ x + 2x & y + 2t \end{bmatrix} = \begin{bmatrix} 2x - y & -x + 2y \\ 2z - t & -z + 2t \end{bmatrix}$$

Comparing

$$x = y, x = -t, y = -z$$

So, the matrix X will be

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & x \\ -x & -x \end{bmatrix}$$

$$= x\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

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Q 4 (a)

i.e. as long as the matrix is a square matrix, multiplying by another square matrix, B, on either side and getting I is sufficient to know that this is an inverse.

Given: $BA = I$

Using Th. 3.(b):
we need to show that $A\underline{x} = \underline{0}$ has only trivial sol.

Let $\underline{x}_0$ be any sol. of $A\underline{x} = \underline{0}$

$$A\underline{x}_0 = \underline{0}$$
$$(BA)\underline{x}_0 = B\underline{0}$$
$$I\underline{x}_0 = \underline{0}$$
$$\underline{x}_0 = \underline{0} \quad \therefore$$

Th. 3(a) is true i.e. A^{-1} exists

$\vdash A^{-1}$ ✓

$$BAA^{-1} = IA^{-1}$$
$$BI = A^{-1}$$
$$B = A^{-1}$$

(b)

If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.

$\boxed{\text{Assume A is invertible}}$

$$AB = I$$
$$\underbrace{A^{-1}A}B = A^{-1}I$$
$$IB = A^{-1}I$$
$$B = A^{-1}$$ ✓

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Q 5

$$R_1 \leftrightarrows R_2 \left[\begin{array}{ccccc|ccccc} b & 0 & c & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & e & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & f & 0 & g & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_4 \leftrightarrows R_5 \left[\begin{array}{ccccc|ccccc} b & 0 & c & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & e & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & f & 0 & g & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 - \frac{d}{a}R_2 \left[\begin{array}{ccccc|ccccc} b & 0 & c & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & f & 0 & g & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 - \frac{e}{h}R_4 \left[\begin{array}{ccccc|ccccc} b & 0 & c & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & f & 0 & g & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

One can see the third row is zero but the corresponding row of identity matrix is not zero. Hence it is inconsistent for any values of a, b, c, d, e, f, g, and h.

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MATH 205 LINEAR ALGEBRA
MIDTERM PART B
SPRING 2024

TIME: 60 MINUTES
TOTAL MARKS: 50

Question 1: (a) Briefly, give one advantage of using the LU decomposition method. **[2 Marks]**

(b) Determine the LU decomposition of the matrix A = $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$. **[8 Marks]**

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<u>Question 2:</u> (a) Show that if $\boldsymbol{v}$ is a nonzero vector in $\mathbb{R}^n$, then $\frac{v}{||v||}$ has Euclidean norm 1. **[4 Marks]**

(b) Show that if $\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_r}$ are pairwise orthogonal vector in $\mathbb{R}^n$, then

$$\left|\left|v_1 + v_2 + \cdots + v_r\right|\right|^2 = ||v_1||^2 + ||v_2||^2 + \cdots + ||v_r||^2$$ **[6 Marks]**

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Question 3: Let V consist of a single object, denoted by $\mathbf{0}$, and define $\mathbf{0}+\mathbf{0}=\mathbf{0}$ and $k\mathbf{0}=\mathbf{0}$ for all scalars k. Show that V is a vector space. **[10 Marks]**

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Question 4: Consider an n x n system of linear equations $A\,\boldsymbol{x} = \mathbf{B}$. Let matrix D_k be an identity matrix I_n with the k-*th* column replaced by $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Write this matrix D_k, and then use it to prove that $x_k = \frac{det(A_k)}{\det(A)}$ i.e. the Cramer's Rule. **[10 Marks]**

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<u>Question 5:</u> Prove: If $\boldsymbol{u}$ and $\boldsymbol{v}$ are $n \times 1$ matrices and A is $n \times n$ n matrix, then

$$(\boldsymbol{v}^T A^T A\boldsymbol{u})^2 \leq (\boldsymbol{u}^T A^T A\boldsymbol{u})(\boldsymbol{u}^T A^T A\boldsymbol{v})$$ **[10 Marks]**

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SOLUTION B

Q 1 (a) LU decomposition has a particular advantage when the equation system we wish to solve, Ax=b, has more than one right side or when the right sides are not known in advance. (This is because the factors **L** and **U** are obtained explicitly and they can be used for any right sides as they arise without recalculating **L** and **U).**

(b)

We need to find L and U such that A = LU

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 2 & 3 & 1 \end{bmatrix} \quad -2R_1+R_3$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 1 & 3 \end{bmatrix} \quad \tfrac{1}{3}R_2+R_3$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & \frac{13}{3} \end{bmatrix} = U \qquad \frac{4}{3}+\frac{9}{3}=\frac{13}{3}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -\frac{1}{3} & 1 \end{bmatrix}$$

Q 2 (a)

Show that if v is a nonzero vector in $\mathbb{R}^n$, then $\frac{1}{\|v\|}v$ has Euclidean norm 1.

Solution: Let $v=(v_1,v_2,\ldots,v_n)$, so $\frac{1}{\|v\|}v=\frac{1}{\sqrt{v_1^2+v_2^2+\ldots+v_n^2}}(v_1,v_2,\ldots,v_n)$, so the Euclidean norm is $\|\frac{1}{\|v\|}v\|=\sqrt{\left(\frac{v_1}{\sqrt{v_1^2+v_2^2+\ldots+v_n^2}}\right)^2+\left(\frac{v_2}{\sqrt{v_1^2+v_2^2+\ldots+v_n^2}}\right)^2+\ldots+\left(\frac{v_n}{\sqrt{v_1^2+v_2^2+\ldots+v_n^2)}}\right)^2}$

$=\sqrt{\left(\frac{v_1^2}{v_1^2+v_2^2+\ldots+v_n^2}\right)+\left(\frac{v_2^2}{v_1^2+v_2^2+\ldots+v_n^2}\right)+\ldots+\left(\frac{v_n^2}{v_1^2+v_2^2+\ldots+v_n^2)}\right)}=\sqrt{\frac{v_1^2+v^2+\ldots+v_n^2}{v_1^2+v^2+\ldots+v_n^2}}=\sqrt{1}$

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Q 2 (b)

Solution: Proof

$$\begin{aligned}\|v_1 + v_2 + \ldots + v_r\|^2 &= \|v_1 + (v_2 + \ldots + v_r)\|^2 \\ &= \|v_1\|^2 + \|v_2 + v_3 + \ldots + v_r\|^2\end{aligned}$$

We know $\|v_1 + v_2\|^2 = \|v_1\|^2 + \|v_2\|^2$ $\quad\because v_i v_j$ are orthogonal

Apply this procedure r-1 times

$$\vdots$$

$$\vdots$$

$$= \|v_1\|^2 + \|v_2\|^2 + \ldots \|v_r\|^2$$

Q 3

Let $\underline{u} = \underline{0} \in V$
$\underline{v} = \underline{0}$

Q 3 Ax 1 $\underline{0} + \underline{0} = \underline{0} \in V$

Ax 2 $\underline{u} + \underline{v} = \underline{v} + \underline{u} = \underline{0} + \underline{0} = \underline{0}$

Ax 3 $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$

$\Rightarrow \underline{0} + (\underline{0} + \underline{0}) = (\underline{0} + \underline{0}) + \underline{0}$

Ax 4 $\underline{0} + \underline{u} = \underline{u} + \underline{0} = \underline{0} + \underline{0} = \underline{0} = \underline{u}$

Ax 5 Let $-\underline{u} = \underline{0}$

$\underline{u} + (-\underline{u}) = (-\underline{u}) + \underline{u} = \underline{0} + \underline{0} = \underline{0}$

Ax 6 $k\underline{u} = k\underline{0} = \underline{0} \in V$

Ax 7 $k(\underline{u} + \underline{v}) = k(\underline{0} + \underline{0}) = k\underline{0} + k\underline{0} = k\underline{u} + k\underline{v}$

Ax 8 $(k + l)\underline{u} = (k + l)\underline{0} = k\underline{0} + l\underline{0} = k\underline{u} + l\underline{u}$

Ax 9 $k(l\underline{u}) = k(l\underline{0}) = (kl)\underline{0} = (kl)\underline{u}$

Ax 10 $1\underline{u} = 1\underline{0} = \underline{0} = \underline{u}$

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Q 4

Now, for any k, $1 \le k \le n$, take the matrix

$$D_k = \begin{bmatrix} 1 & 0 & \cdots & 0 & x_1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 & x_2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots & & & \vdots \\ \vdots & 0 & 0 & \ddots & \vdots & & & \vdots \\ \vdots & \vdots & & 0 & x_k & 0 & & \vdots \\ \vdots & \vdots & & & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots 0 & x_n & 0 & \cdots 0 & 1 \end{bmatrix}$$

i.e. the identity matrix I_n, with the k^{th} column replaced by $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Recall that $X = A^{-1}B$.

Also, from $(*)$: $e_1 = A^{-1}C_1$, $e_2 = A^{-1}C_2$, $\ldots$, $e_n = A^{-1}C_n$

$$\therefore \quad D_k = [e_1 | e_2 | \ldots | e_{k-1} | X | e_{k+1} | \ldots | e_n]$$

$$D_k = [A^{-1}C_1 | A^{-1}C_2 | \cdots | A^{-1}C_{k-1} | A^{-1}B | A^{-1}C_{k+1} | \ldots | A^{-1}C_n]$$

$$\Rightarrow D_k = A^{-1} [\underline{C_1 | C_2 | \ldots | C_{k-1} | B | C_{k+1} | \ldots | C_n}]$$

Which matrix is this?

This matrix is A_k (in Cramer's Rule)!

$$\therefore \quad D_k = A^{-1} A_k$$

Next, note that $\det(D_k) = x_k$ — ① $\left\{ \text{Just expand along the row containing } x_k. \text{ Everything else is zero on that row} \right.$ & $\det(I) = 1$

However, $\det(D_k) = \det(A^{-1}A_k)$

$$= \det(A^{-1}) \det(A_k)$$

$$= \frac{\det(A_k)}{\det(A)} \quad \text{——②} \quad \because \det(A^{-1}) = \frac{1}{\det(A)}$$

From ① and ②, we have

$$\boxed{x_k = \frac{\det(A_k)}{\det(A)}}$$

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Q 5

Solution: Let $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ be vectors and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in R^n and let A be an $n \times n$ matrix.

So, letting $u = Au$ and $v = Av$ and this multiplication possible. Hence the inner product is $(a, b) = a.b = b^T a$.

$$(u, v) = Au \cdot Av = (Av)^T Au$$

So we have

$$(u, v) = v^T A^T Au$$

In special cases we have

$$(u, u) = u^T A^T Au \text{ and } (v, v) = v^T A^T Av$$

The Cauchy-Schwarz inequality is

$$(u, v)^2 \leqslant (u, u)(v, v)$$

By substituting the previous we get

$$(u^T A^T Au)^2 \leq (u^T A^T Au)(v^T A^T Av)$$