Homework 3 (Solution)

Introduction to Probability and Statistics
EE 354 / CE 361 / MATH 310

Spring 2024

Note: All questions are worth 10 points each.

Question 1

If the height of Habib University students is modeled as a Normal random variable with mean of 65 inches and standard deviation of 3 inches, what is the probability that a randomly chosen HU student has a height of 6ft or more?

Solution:

Let H be the random variable representing the height of HU students.

$$H \sim N(65, 9)$$

$$P(H \geq 72) = P\left(\frac{H-65}{3} \geq \frac{72-65}{3}\right)$$

$$= P(Y \geq 2.33) \quad \text{where } Y \sim N(0,1)$$

$$= 1 - P(Y \leq 2.33)$$

$$= 1 - 0.9901$$

$$P(H \geq 72) = 0.0099 \quad \text{— Ans}$$

Question 2

The time until the next flight departs from an airport follows the following distribution:

$$f(x) = \frac{1}{20} \quad for\ 25 \le x \le 45$$

a) What is E[X]?
b) What is standard deviation of X?
c) Find the probability that the time until the departure of next flight is at most 30 minutes?
d) Find the probability that the time until the departure of next flight is between 30 and 40 minutes?

Solution:

(a) As this is a uniform continuous random variable so $E[X] = \frac{a+b}{2} = \frac{25+45}{2} = 35$

(b) Standard Deviation of x $= \frac{(b-a)}{\sqrt{12}} = \frac{45-25}{\sqrt{12}} = 5.8$

(c) $\int_{25}^{30} \frac{1}{20} dx = \frac{1}{20}(30 - 25) = \frac{5}{20} = 0.25$

(d) $\int_{30}^{40} \frac{1}{20} dx = \frac{1}{20}(40 - 30) = \frac{10}{20} = 0.5$

Question 3

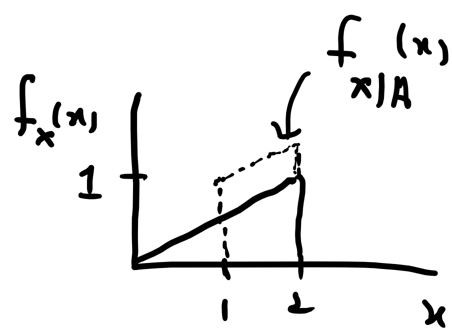
Let X be a random variable with the PDF:

$$f_X(x) = \begin{cases} \frac{x}{2} & if\ 0 \le x \le 2 \\ 0, & otherwise \end{cases}$$

Let A be the event $\{X \ge 1\}$. Let $Y = X^2$. Find the following:

i. $E[X]$ **[2]**
ii. $E[X|A]$ **[3]**
iii. $Var\ [X]$? **[2]**
iv. $Var\ [Y]$? **[3]**

Solution:



(i)

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_0^2 x \cdot \frac{x}{2} \cdot dx = \int_0^2 \frac{x^2}{2} \cdot dx = \frac{x^3}{6}\Big|_0^2$$

$$= \frac{8}{6} = 1.33 \text{ — Ans}$$

(ii)

$$P(A) = P(\{X \geq 1\}) = \int_1^{\infty} f_X(x) \cdot dx = \int_1^{2} \frac{x}{2} \cdot dx$$

$$= \frac{x^2}{4}\Big|_1^2 = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$$

$$f_{X|A} = \begin{cases} \dfrac{f_X(x)}{P(A)} & x \in A \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{x}{1.5} & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X|A] = \int_{-\infty}^{\infty} x \cdot f_{X|A}(x) = \int_1^2 x \cdot \frac{x}{1.5} \cdot dx = \int_1^2 \frac{x^2}{1.5} dx$$

$$= \frac{x^3}{4.5}\Big|_1^2 = \frac{8}{4.5} - \frac{1}{4.5}$$

$$E[X|A] = \frac{7}{4.5} = 1.56 \text{ — Ans}$$

(iii)

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

By Expected Value Rule,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) \cdot dx$$

$$= \int_0^2 x^2 \cdot \frac{x}{2} = \int_0^2 \frac{x^3}{2} \cdot dx = \frac{x^4}{8}\Big|_0^2$$

$$= \frac{16}{8} - 0 = 2$$

$$Var(X) = 2 - (1.33)^2 = 0.23 \quad \text{— Ans}$$

(iv)

$$Y = X^2$$

$$Var(Y) = E[Y^2] - (E[Y])^2 \quad \text{— (A)}$$

$$E[Y] = E[X^2] = 2$$

$$E[Y^2] = E[X^4] = \int_{-\infty}^{\infty} x^4 \cdot f_X(x)\, dx$$

$$= \int_0^2 x^4 \cdot \frac{x}{2} \cdot dx = \int_0^2 \frac{x^5}{2} \cdot dx = \frac{x^6}{12}\Big|_0^2$$

$$= \frac{2^6}{12} = \frac{64}{12}$$

$$E[Y^2] = 5.33$$

From (A) $Var(Y) = 5.33 - 2^2$

$$Var(Y) = 1.33 \text{ — Ans}$$

Question 4

Suppose that the number of kilometers you drive before you get into a car accident is modeled as an exponential random variable with a mean of 15,000 kms. You are planning to drive in your car form Karachi to Khunjerab Pass and back. What is the probability that you will make it back to Karachi without getting into an accident?
(Hint: According to Google Maps, the distance from Karachi to Khunjerab Pass is 2,100 km.)

Solution:

X = Number of kms you drive before getting into an accident

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\text{Mean} = \frac{1}{\lambda} = 15000$$

$$\Rightarrow \lambda = \frac{1}{15000}$$

For an Exponential RV:

$$P(X \geq a) = e^{-\lambda a}$$

$$P(X \geq 4200) = e^{-\left(\frac{1}{15000}\right)(4200)}$$

$$= e^{-0.28}$$

$$= 0.756 \text{ — Ans}$$

Question 5

a) A team-based javelin throw competition involves three athletes from each team. The score of a team is considered to be the maximum of throws achieved by its athletes. Team Pakistan consists of three athletes: Arshad, Akram, and Amjad. All the three athletes can throw between 85m to 95m with all values being equally likely. Throws by each athlete are independent of the other athletes. Let random variable X be the score of Team Pakistan in this competition. What is the PDF of X? **[5]**

b) X and Y are independent continuous random variables that are uniformly distributed with $X \sim U[0,1]$ and $Y \sim U[0,2]$. Find the following:

i. $f_{X,Y}(1.5,1.5)$ **[1]**

ii. $F_{X,Y}(1.5,1.5)$ **[2]**

iii. $E[\,Y|X < 0.5\,]$ **[2]**

(a)

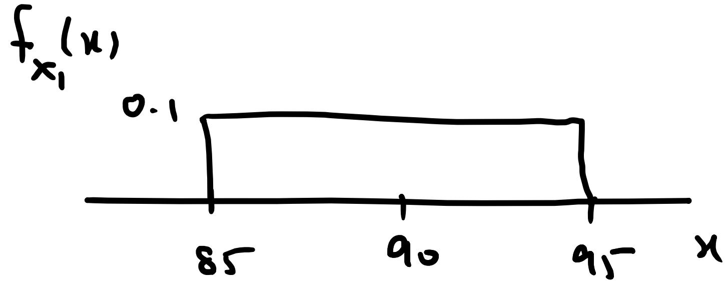
The following strategy can be used:

A. Calculate CDF of X.

B. CDF $\longrightarrow$ PDF

Let X_1, X_2, X_3 be the throws by Arshad, Akram, and Amjad respectively.

$$f_{X_1}(x) = f_{X_2}(x) = f_{X_3}(x) = \begin{cases} 0.1 & 85 \le x \le 95 \\ 0 & \text{otherwise} \end{cases}$$



$$F_X(x) = P(X \le x)$$

$$= P(X_1 \le x, X_2 \le x, X_3 \le x)$$

Due to Independence:

$$= P(X_1 \le x) \cdot P(X_2 \le x) \cdot P(X_3 \le x)$$

$$F_X(x) = \begin{cases} 0 & x \le 85 \\ [0.1(x-85)][0.1(x-85)][0.1(x-85)] & 85 \le x \le 95 \\ 1 & x \ge 95 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x \le 85 \\ (0.1)^3 (x-85)^3 & 85 < x \le 95 \\ 1 & x > 95 \end{cases}$$

Now, for $85 \le x \le 95$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$$= \frac{d}{dx}\left[(0.1)^3 (x-85)^3\right]$$

$$= (0.1)^3 \frac{d}{dx}\left[(x-85)^3\right]$$

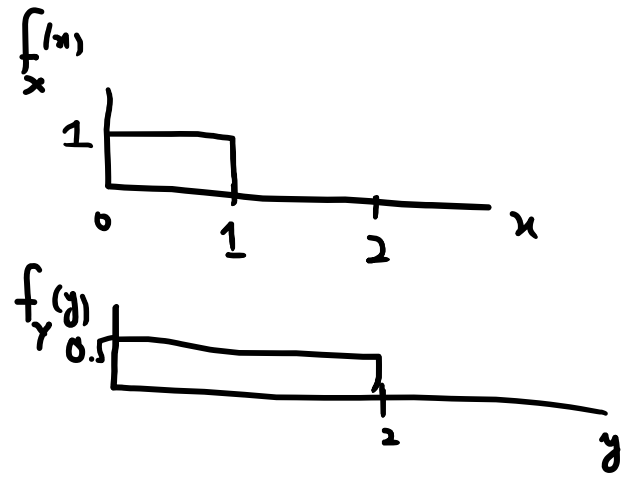
$$= (0.1)^3\, 3 \cdot (x-85)^2 \frac{d}{dx}(x-85)$$

$$f_X(x) = 3(0.1)^3(x-85)^2$$

$$\Rightarrow \quad f_X(x) = \begin{cases} 0 & x \leq 85 \\ 3(0.1)^3 (x-85)^2 & 85 \leq x \leq 95 \\ 0 & x \geq 95 \end{cases} \quad \text{— Ans}$$

(b)

(i)



Due to independence

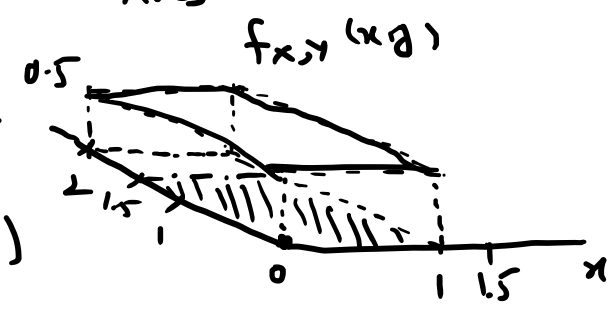
$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

$$f_{X,Y}(x,y) = \begin{cases} 0.5 & 0 \leq x \leq 1,\ 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X,Y}(1.5, 1.5) = 0 \quad \text{— Ans}$$

(ii)

$$F_{X,Y}(1.5, 1.5) = P(X \leq 1.5, Y \leq 1.5)$$



$= $ Volume under the $f_{X,Y}(x,y)$ over the region $\{X \le 1.5, Y \le 1.5\}$

$$= (1.0)(1.5)(0.5)$$

$$= 0.75 \quad \text{— Ans}$$

(iii)

Let A be the event $X < 0.5$

Due to independence of X and Y :-

$$f_{Y|A}(y) = f_Y(y)$$

Now,

$$E[Y|A] = \int_{-\infty}^{\infty} y \, f_{Y|A}(y) \, dy$$

$$= \int_{-\infty}^{\infty} y \, f_Y(y) \, dy = E[Y] = 1 \quad \text{Ans}$$

Question 6

(a)

X and Y are random variables with a joint PDF given by:

$$f_{X,Y}(x,y) = \frac{xy^2}{81}$$

for $0 \le x \le 3$ and $0 \le y \le 3$. The joint PDF is zero for all other values of x and y. Are X and Y independent? Justify your answer.

(b)

X and Y are random variables with a joint PDF given by:

$$f_{X,Y}(x,y) = x + cy^3$$

for $0 \le x \le 1$ and $0 \le y \le 1$. The joint PDF is zero for all other values of x and y.

i. What is the value of constant c?

ii. Are X and Y independent? Justify your answer.

Solution:

(a)

$$f_{X,Y}(x,y) = \begin{cases} \frac{xy^2}{81} & 0 \le x \le 3,\ 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases} \quad \text{—(A)}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,.dy = \int_0^3 \frac{xy^2}{81}\,dy$$

$$= \frac{x}{81}\int_0^3 y^2\,dy = \frac{x}{81}\cdot\left.\frac{y^3}{3}\right|_0^3$$

$$= \frac{x}{81}\left[\frac{3^3}{3} - \frac{0^3}{3}\right] = \frac{x}{81}\cdot 3^2$$

$$f_X(x) = \frac{x}{9} \qquad 0 \le x \le 3 \quad \text{—(1)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,.dx = \int_0^3 \frac{xy^2}{81}\,dx$$

3

$$= \frac{y^2}{81} \cdot \int_0^3 x \, dx \qquad = \frac{y^2}{81} \cdot \left. \frac{x^2}{2} \right|_0^3$$

$$= \frac{y^2}{81} \left(\frac{3^2}{2} - \frac{0^2}{2} \right) = \frac{y^2}{81} \cdot \left(\frac{9}{2} \right)$$

$$f_Y(y) = \frac{y^2}{18} . \qquad 0 \leq y \leq 3 \qquad \text{--- (2)}$$

From (1) and (2) :-

$$f_X(x) . f_Y(y) = \frac{x}{9} \cdot \frac{y^2}{18}$$

$$f_X(x) f_Y(y) = \begin{cases} \frac{xy^2}{162} & 0 \leq x \leq 3 , 0 \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases} \qquad \text{--- (B)}$$

From (A) and (B) :

$$f_{X,Y}(x,y) \neq f_X(x) . f_Y(y)$$

$\Rightarrow$ X and Y are not independent. —— Ans

(b)

$$f_{X,Y}(x,y) = \begin{cases} x + cy^3 & 0 \leq x \leq 1 , 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{--- (A)}$$

i) $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx \, dy = 1$$

$$\int_0^1\int_0^1 (x + cy^3)\, dx\, dy = 1 \quad \text{— (1)}$$

$$\int_0^1 (x + cy^3)\, dx = \frac{x^2}{2} + cy^3 x \Big|_0^1$$

$$= \frac{1}{2} + cy^3$$

From (1)

$$\int_0^1 \left(\frac{1}{2} + cy^3\right) dy = 1$$

$$\frac{1}{2} y + c\frac{y^4}{4} \Big|_0^1 = 1$$

$$\frac{1}{2} + \frac{c}{4} = 1$$

$$\frac{c}{4} = \frac{1}{2}$$

$$c = 2 \quad \text{— Ans}$$

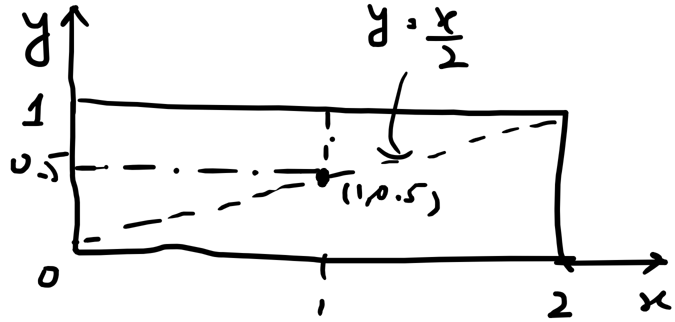
Question 7

X and Y are random variables with a joint PDF given by:

$$f_{X,Y}(x,y)\begin{cases}\frac{1}{3} & 0 \le x < 1, 0 \le y \le 1\\ \frac{2}{3} & 1 \le x \le 2, 0 \le y \le 1\\ 0 & otherwise\end{cases}$$

What is $P(Y \ge \frac{X}{2})$?

Solution:



$$P(Y \geqslant \tfrac{X}{2}) = \int_0^{0.5}\int_0^{2y} f_{X,Y}(x,y)\,dx\,dy + \int_{0.5}^{1}\int_0^{1} f_{X,Y}(x,y)\,dx\,dy + \int_{0.5}^{1}\int_1^{2y} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^{0.5}\int_0^{2y} \frac{1}{3}\,dx\,dy + \int_{0.5}^{1}\int_0^{1} \frac{1}{3}\,dx\,dy + \int_{0.5}^{1}\int_1^{2y} \frac{2}{3}\,dx\,dy$$

$$= \int_0^{0.5} \frac{x}{3}\Big|_0^{2y} dy + \int_{0.5}^{1} \frac{x}{3}\Big|_0^{1} dy + \int_{0.5}^{1} \frac{2x}{3}\Big|_1^{2y} dy$$

$$= \int_0^{0.5} \frac{2y}{3}\,dy + \int_{0.5}^{1} \frac{1}{3}\,dy + \int_{0.5}^{1} \left(\frac{4y}{3} - \frac{2}{3}\right) dy$$

$$= \frac{y^2}{3}\Big|_0^{0.5} + \frac{y}{3}\Big|_{0.5}^{1} + \int_{0.5}^{1} \frac{4y}{3}\,dy - \int_{0.5}^{1} \frac{2}{3}\,dy$$

$$= \frac{0.25}{3} + \frac{0.5}{3} + \frac{2y^2}{3}\Big|_{0.5}^{1} - \frac{2y}{3}\Big|_{0.5}^{1}$$

$$= \frac{0.25}{3} + \frac{0.5}{3} + \left(\frac{2}{3} - \frac{0.5}{3}\right) - \frac{1}{3}$$

$$= \frac{0.25}{3} + \frac{0.5}{3} + \frac{1.5}{3} - \frac{1}{3}$$

$$= \frac{0.25}{3} + \frac{0.5}{3} + \frac{0.5}{3}$$

$$P\left(Y \geq \frac{X}{2}\right) = \frac{1.25}{3} \quad \text{— Ans}$$

Question 8

X and Y are independent continuous random variables that are uniformly distributed with $X \sim U[0,2]$ and $Y \sim U[0,2]$. Find the following:

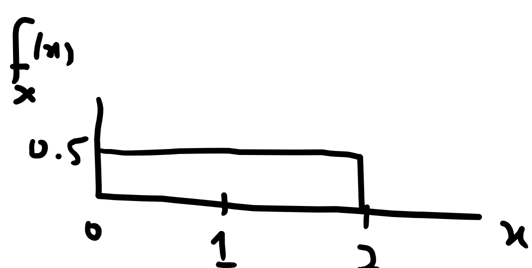
a) $f_{X,Y}(x,y)$

b) $F_{X,Y}(1.5,1.5)$

c) $f_{Y|X}(y|0.5)$

d) $f_{X|Y}(0.5|1)$

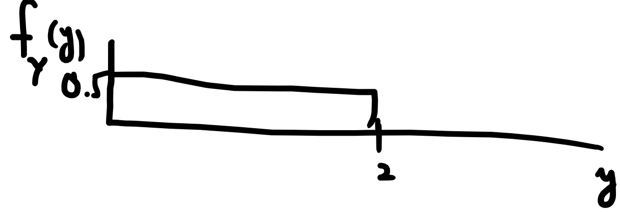
e) $E[X|Y > 1.5]$



(a)

Due to independence,

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$



$$f_{X,Y}(x,y) = \begin{cases} 0.25 & 0 \leq x \leq 2, 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases} \quad \text{— Ans}$$

(b)

$F_{X,Y}(1.5, 1.5) = P(X \le 1.5, Y \le 1.5)$

$=$ Volume under the $f_{X,Y}(x,y)$ over the region $\{X \le 1.5, Y \le 1.5\}$

$= (1.5)(1.5)(0.25)$

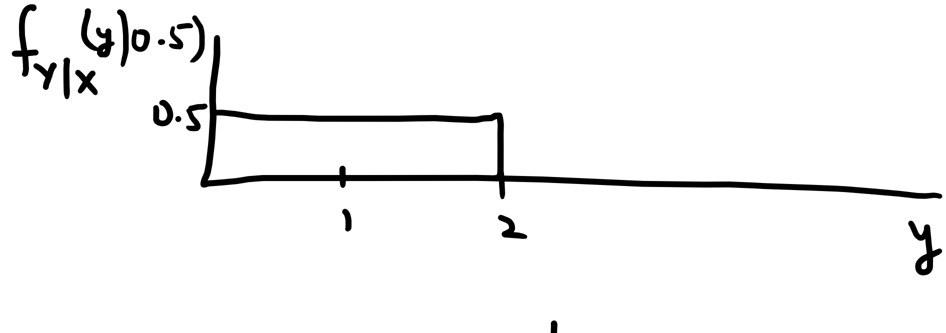
$= 0.5625$ — Ans

(c)

Due to Independence:

$$f_{Y|X}(y|x) = f_Y(y)$$

$$\Rightarrow \quad f_{Y|X}(y|0.5) = f_Y(y)$$



(d)

Due to independence:

$$f_{X|Y}(x|y) = f_X(x)$$

$$\Rightarrow \quad f_{X|Y}(0.5|1) = f_X(0.5)$$

$$= 0.5 \quad \text{— Ans}$$

(e)

Let A be the event $Y > 1.5$.

Due to independence of X and Y :-

$$f_{X|A}(x) = f_X(x)$$

Now,

$$E[X|A] = \int_{-\infty}^{\infty} x \, f_{X|A}(x) \, dx$$

$$= \int_{-\infty}^{\infty} x \, f_X(x) \, dx = E[X] = 1 \quad \text{Ans}$$

Question 9

a) Binary signal S is transmitted across a communication channel. The received signal is $Y = N + S$, where N is normal noise with zero mean and unit variance, independent of S. Let A be the event that the noise is between -0.25 and 0.25 (i.e. $N\epsilon[-0.25, 0.25]$). What is $E[N|A]$? Justify your answer. **[5]**

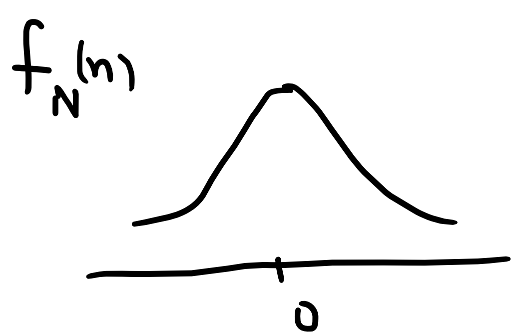
b) Suppose you were told that the average odometer reading (number of kilometers a car has traveled in its lifetime so far) of all the red cars in Karachi was 55,000 km. Express this information in terms of two random variables X and Y. Define the random variables X and Y appropriately. (You can use any of the concepts applicable to two random variables such as joint PDF, conditional PDF, etc.) **[5]**

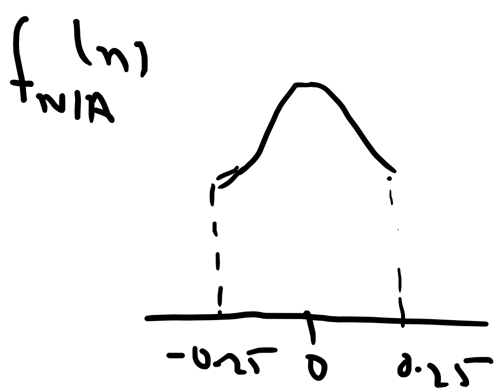
(a)

$N \sim N(0,1)$

$A = N \in [-0.25, 0.25]$

$$f_{N|A}(n) = \begin{cases} \dfrac{f_N(n)}{P(N \in [-0.25, 0.25])} & -0.25 \le n \le 0.25 \\ 0 & \text{otherwise} \end{cases}$$





By Symmetry

$\Rightarrow$ $E[N|A] = 0$ — Ans

(b)

X = The odometer reading of a randomly selected car in Karachi

Y = The color of a randomly selected car in Karachi

$$E[X \mid Y = \text{Red}] = 55{,}000 \text{ km}$$

—— Ans

Question 10

Ahsan frequently drives between Lahore and Islamabad. If he takes the Motorway, his drive time between Lahore and Islamabad is 4 to 5 hours with all times being equally likely. If he does not take the Motorway, his drive time between Lahore and Islamabad is 4.5 to 6.5 hours with all times being equally likely. Assume that Ahsan makes decision of taking the Motorway or not by flipping a fair coin. Find the PDF of Ahsan's drive time between Lahore and Islamabad.

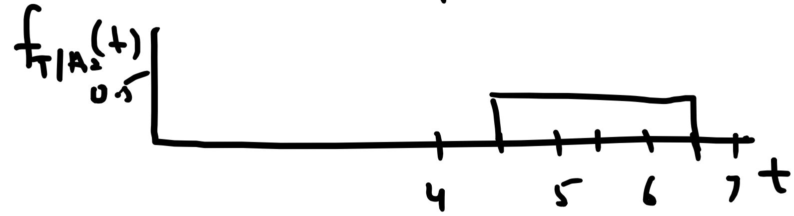
Solution:

Let A_1 be the event that Ahsan takes the Motorway and A_2 be the event that Ahsan does not take the Motorway. $P(A_1) = 0.5$ $\quad P(A_2) = 0.5$

Let T be the random variable representing Ahsan's drive time between Lahore and Islamabad.

$f_{T|A_1}(t) = U(4,5)$

$f_{T|A_2}(t) = U(4.5, 6.5)$

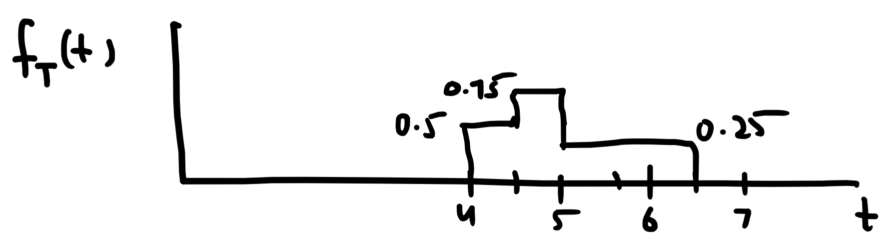


Now,

$$f_T(t) = P(A_1)\, f_{T|A_1}(t) + P(A_2)\, f_{T|A_2}(t)$$

$$f_T(t) = 0.5\left[U(4,5)\right] + 0.5\left[U(4.5, 6.5)\right]$$

Therefore:



— Ans

Question 11

A city has two internet service providers, Charter and Xfinity, with the market share of 30% and 70% respectively. At any given time, the internet speed experienced by a customer of Charter is between 7Mbps and 15Mbps with all speeds being equally likely. At any given time, the internet speed experienced by a customer of Xfinity is between 10Mbps and 20Mbps with all speeds being equally likely. If an internet customer in the city is currently experiencing an internet speed of 12Mbps, what is the probability that the customer is served by Charter?

Solution:

Let I be the random variable that takes on following values:

I = 0 when ISP is Charter

I = 1 when ISP is Xfinity

$$p_I(i) = \begin{cases} 0.3 & i = 0 \\ 0.7 & i = 1 \end{cases}$$

Let S be the speed experienced by an internet customer.

$$f_{S|I}(s|0) = \begin{cases} 1/8 & 7 \le s \le 15 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{S|I}(s|1) = \begin{cases} 1/10 & 10 \le s \le 20 \\ 0 & \text{otherwise} \end{cases}$$

Req. Prob. $= p_{I|S}(0|12) = ?$

By Total Prob. Theorem:

$$f_S(s) = \sum_i p_I(i) f_{S|I}(s|i)$$

$$f_S(s) = p_I(0) f_{S|I}(s|0) + p_I(1) f_{S|I}(s|1) \quad \text{—(1)}$$

Using Bayes' Rule:

$$p_{I|S}(i|s) = \frac{p_I(i) f_{S|I}(s|i)}{f_S(s)}$$

$$\Rightarrow \quad p_{I|S}(0|12) = \frac{p_I(0) f_{S|I}(12|0)}{f_S(12)} \quad \text{—(2)}$$

From (1)

$$f_S(12) = p_I(0) f_{S|I}(12|0) + p_I(1) f_{S|I}(12|1)$$

$$= 0.3\left(\frac{1}{8}\right) + 0.7\left(\frac{1}{10}\right)$$

$$f_S(12) = 0.1075$$

From (2) :-

$$p_{I|S}(0|12) = \frac{0.3\left(\frac{1}{8}\right)}{0.1075}$$

$$= \frac{0.0375}{0.1075}$$

$$= 0.349$$ — Ans

Question 12

A binary signal S is transmitted and we are given that $P(S=0)=0.4$ and $P(S=1)=0.6$. The received signal is $Y = N + S$, where N is normal noise with zero mean and unit variance, independent of S. What is the probability that $S = 1$ if the observed value of received signal Y is 0.7.

Solution:

$$p_S(s) = \begin{cases} 0.4 & \text{when } s=0 \\ 0.6 & \text{when } s=1 \end{cases}$$

$$f_{Y|S}(y|1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}}$$

$$f_{Y|S}(y|0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

By Total Prob. Theorem:-

$$f_Y(y) = \sum_s p_S(s) f_{Y|S}(y|s)$$

$$= p_S(0) f_{Y|S}(y|0) + p_S(1) f_{Y|S}(y|1)$$

$$f_Y(y) = \frac{0.4}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} + \frac{0.6}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}}$$

$$p_{S|Y}(1|0.7) = ?$$

Using Bayes' Rule:-

$$p_{S|Y}(s|y) = \frac{p_S(s)\, f_{Y|S}(y|s)}{f_Y(y)}$$

$$p_{S|Y}(1|0.7) = \frac{p_S(1)\, f_{Y|S}(0.7|1)}{f_Y(0.7)}$$

$$= \frac{0.6\left(\frac{1}{\sqrt{2\pi}}\, e^{-\frac{(0.7-1)^2}{2}}\right)}{\frac{0.4}{\sqrt{2\pi}}\, e^{-\frac{0.7^2}{2}} + \frac{0.6}{\sqrt{2\pi}}\, e^{-\frac{(0.7-1)^2}{2}}}$$

$$= \frac{0.6\, e^{-0.045}}{0.4\, e^{-0.245} + 0.6\, e^{-0.045}}$$

$$= \frac{0.6\,(0.956)}{0.4\,(0.7827) + 0.6(0.956)}$$

$$= \frac{0.5736}{0.3131 + 0.5736}$$

$$p_{S|Y}(1|0.7) = 0.647 \text{ — Ans}$$

Question 13

On a sunny day, it takes a student between 30 to 45 minutes to get to Habib University campus with all times being equally likely. On a rainy day, it takes the same student between 40 to 60 minutes to get to campus with all times being equally likely. Assume a day is sunny with probability 0.75 and rainy with probability 0.25.

a) Find the PDF of the time T that it takes the student to get to campus.
b) On a given day, it took the student 42 minutes to get to campus. What is the probability that this particular day was rainy?

Solution:

(a)

Let X be the random variable that takes the value 0 when its a sunny day and value 1 when its a rainy day.

$$p_X(x) = \begin{cases} 0.75 & \text{when } x = 0 \\ 0.25 & \text{when } x = 1 \end{cases}$$

Now,

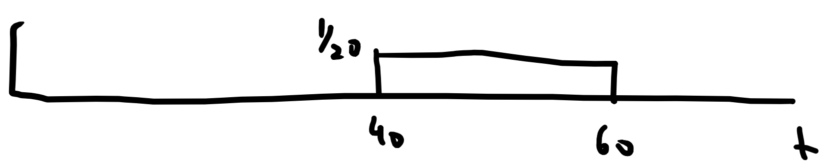
$$f_T(t) = \sum_x p_X(x) f_{T|X}(t|x)$$

$$= p_X(0) f_{T|X}(t|0) + p_X(1) f_{T|X}(t|1)$$

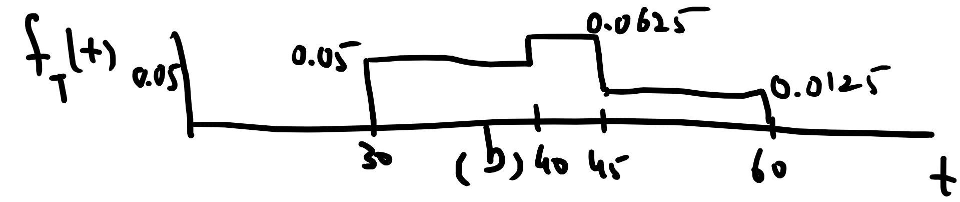
$$f_T(t) = 0.75 \left[U(30,45) \right] + 0.25 \left[U(40,60) \right] \quad \text{—(1)}$$

$f_{T|X}(t|0)$

$f_{T|X}(t|1)$



From 1,



$$p_{X|T}(1|42) = ?$$

Using Bayes' Rule :- (Discrete X, Continuous Y)

$$p_{X|Y}(x|y) = \frac{p_X(x)\, f_{Y|X}(y|x)}{f_Y(y)}$$

$$\Rightarrow \quad p_{X|T}(x|t) = \frac{p_X(x)\, f_{T|X}(t|x)}{f_T(t)}$$

$$p_{X|T}(1|42) = \frac{p_X(1)\, f_{T|X}(42|1)}{f_T(42)} \quad \text{--- (2)}$$

$$p_X(1) = 0.25$$

$$f_{T|X}(42|1) = \frac{1}{20} = 0.05$$

$$f_T(42) = 0.0625$$

From (2)

$$p_{X|T}(1|42) = \frac{0.25 \times 0.05}{0.0625}$$

$$p_{X|T}(1|42) = 0.2 \quad \text{—— Ans}$$

Question 14

Consider a Bernoulli random variable X with parameter $p = 0.1$. We observe X in the presence of additive noise $N \sim U[-2, 2]$. As a result, our observation is random variable Y = X + N. If our observation is 0.8, what is the "most likely" value of X?

(Note: Provide a justification of your answer based on the various theorems/rules/laws/results discussed in this course. Only intuitive argument is not acceptable for this question and will be given no partial credit)

We can approach this using General Version of Bayes' Rule:

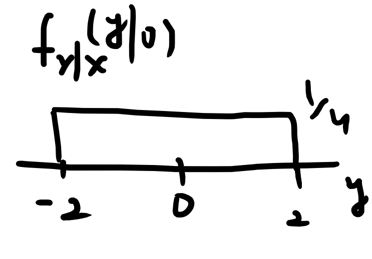
$$X \longrightarrow \boxed{\text{Noisy Measurement}} \longrightarrow Y$$

Prior Belief:

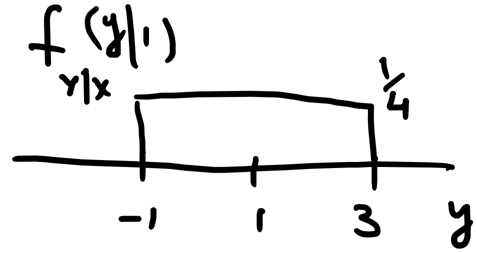
$$p_X(x) = \begin{cases} 0.1 & x = 1 \\ 0.9 & x = 0 \end{cases}$$

Quality of Measurement:-

$$f_{Y|X}(y|0) = \begin{cases} \frac{1}{4} & -2 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$



$$f_{Y|X}(y|1) = \begin{cases} \frac{1}{4} & -1 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$



Posterior Belief:-

$$p_{X|Y}(x|y) = \frac{p_X(x)\, f_{Y|X}(y|x)}{f_Y(y)}$$

$$p_{X|Y}(x|y) = \frac{p_X(x)\, f_{Y|X}(y|x)}{p_X(0) f_{Y|X}(y|0) + p_X(1) f_{Y|X}(y|1)} \quad \text{— (A)}$$

From (A),

$$p_{X|Y}(0|0.8) = \frac{p_X(0)\, f_{Y|X}(0.8|0)}{p_X(0) f_{Y|X}(0.8|0) + p_X(1) f_{Y|X}(0.8|1)}$$

$$= \frac{(0.9)(0.25)}{(0.9)(0.25) + (0.1)(0.25)}$$

$$p_{X|Y}(0|0.8) = 0.9 \quad \text{— (B)}$$

From (A):-

$$p_{X|Y}(1|0.8) = \frac{p_X(1)\, f_{Y|X}(0.8|1)}{p_X(0) f_{Y|X}(0.8|0) + p_X(1) f_{Y|X}(0.8|1)}$$

$$= \frac{(0.1)(0.25)}{(0.9)(0.25) + (0.1)(0.25)}$$

$$p_{X|Y}(1|0.8) = 0.1 \quad \text{———} \quad (C)$$

Comparing (B) and (C), if the observation is 0.8, the "most likely" value of X is 0.