LINEAR ALGEBRA

LECTURE 16

MATH 205

RESULT: (P. 233 8TH ED. P. 244 7TH ED.)

IF $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ IS A BASIS FOR A VECTOR SPACE $\boxed{V}$, THEN EVERY VECTOR $\underline{v}$ IN $\boxed{V}$ CAN BE EXPRESSED IN THE FORM

$$\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_n\underline{v}_n$$

IN EXACTLY $\boxed{\text{ONE}}$ WAY.

PROOF:

LET

$$\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_n\underline{v}_n \quad \text{AND}$$

$$\underline{v} = k_1\underline{v}_1 + k_2\underline{v}_2 + \cdots + k_n\underline{v}_n$$

SUBTRACTING THE SECOND EQUATION FROM THE FIRST GIVES

$$\underline{0} = (c_1 - k_1)\underline{v}_1 + (c_2 - k_2)\underline{v}_2 + \cdots + (c_n - k_n)\underline{v}_n$$

THE LINEAR INDEPENDENCE OF VECTORS IN $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ IMPLIES THAT

$$c_1 - k_1 = 0,\ c_2 - k_2 = 0,\ \ldots,\ c_n - k_n = 0$$

$$\Rightarrow \boxed{c_1 = k_1,\ c_2 = k_2,\ \ldots,\ c_n = k_n}$$

WHICH COMPLETES THE PROOF.

PREVIOUS RESULTS:

(1) $\{(1,0), (0,1)\}$ IS STANDARD BASIS FOR R^2.

(2) $\{(-3,7), (5,5)\}$ IS A BASIS BUT NOT A STANDARD BASIS FOR R^2

RECALL: $\boxed{(x,y) = x(1,0) + y(0,1)}$

AND $(x,y) = \frac{(y-x)}{10}(-3,7) + \frac{(7x+3y)}{50}(5,5)$

$$\Rightarrow \boxed{(x,y) = \left(\frac{y-x}{10}\right)(-3,7) + \frac{(7x+3y)}{50}(5,5)}$$

HOMOGENEOUS LINEAR SYSTEM

P.16 8TH ED.
P.17 7TH ED.

FOR $AX = 0$, EXACTLY ONE OF THE FOLLOWING IS TRUE:

(1) THE SYSTEM HAS ONLY THE TRIVIAL SOLUTION (ZERO SOLUTION) IF A IS INVERTIBLE

(2) SYSTEM HAS INFINITELY MANY SOLUTIONS (NONTRIVIAL) IN ADDITION TO THE TRIVIAL SOLUTION. i.e. A IS SINGULAR OR UNKNOWNS ARE MORE THAN EQUATIONS.

3) IF $S = \{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}\}$ IS A NON-EMPTY SET OF VECTORS, THEN IN VECTOR EQUATION

$$k_1\underline{v_1} + k_2\underline{v_2} + \cdots + k_r\underline{v_r} = \underline{0}$$

IF ANY $\boxed{\text{ONE}}$ OF THE SCALARS $\boxed{k_i \neq 0}$, $1 \leq i \leq r$, THEN $\boxed{S}$ IS LINEARLY DEPENDENT AND ALL THE VECTORS $\underline{v_i}$, $1 \leq i \leq r$, ARE LINEARLY DEPENDENT VECTORS. (P.222 6th ED.)
(P.232 7th ED.)

EXAMPLE: CHECK WHETHER $\{(2,2),(1,1)\}$ IS INDEPENDENT OR DEPENDENT?

SOLUTION: LET

$$k_1(2,2) + k_2(1,1) = (0,0) \quad \text{— } 1$$

$$\Rightarrow \quad 2k_1 + k_2 = 0$$
$$2k_1 + k_2 = 0$$

$$\Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

4 $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\because \det\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = 0$

$\therefore$ NONTRIVIAL (NONZERO) SOLUTIONS EXIST, $\therefore$ GIVEN VECTORS ARE LINEARLY DEPENDENT.

NOTE: IN $2k_1 + k_2 = 0$

LET $\boxed{k_1 = t}$, $\boxed{k_2 = -2t}$,

IF $\boxed{t=1}$, $k_1 = 1$, $k_2 = -2$

$\therefore$ WE GET $(2,2) - 2(1,1) = (0,0)$

$\Rightarrow \boxed{(2,2) = 2(1,1)}$ FROM ①

NOTE: RECALL THAT IF $A\underline{X} = \underline{0}$ REPRESENTS A HOMOGENEOUS SYSTEM OF EQUATIONS THEN INFINITELY MANY NONTRIVIAL (NONZERO) SOLUTIONS EXIST IF $\boxed{A}$ IS SINGULAR i.e. $\boxed{\det(A) = 0}$ OR $\boxed{A^{-1}}$ DOESN'T EXIST.

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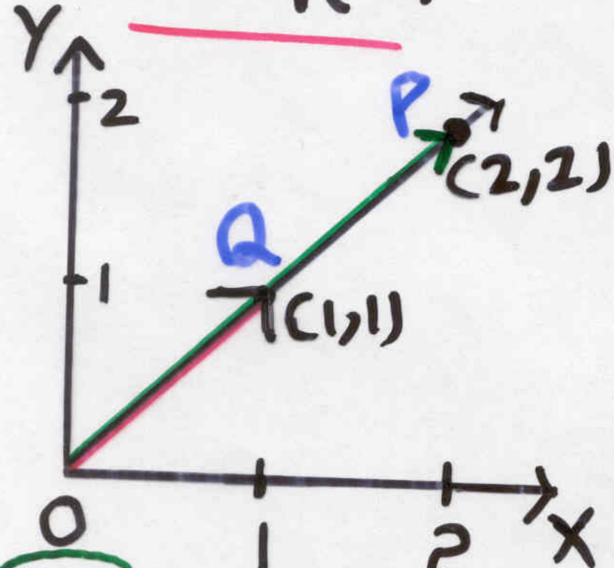
RESULTS:

① $\{(1,1),(2,2)\}$ IS LINEARLY DEPENDENT AND

② $\{(5,5),(-3,7)\}$ IS LINEARLY INDEPENDENT (PROVED LAST TIME)

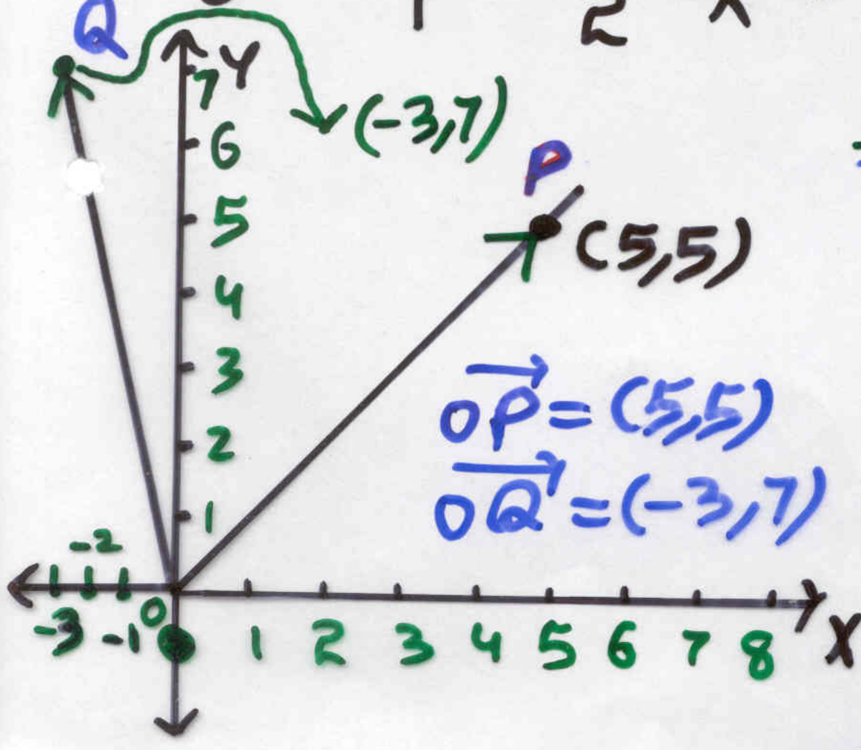
GEOMETRIC INTERPRETATION IN R^2:



IN R^2 TWO VECTORS ARE LINEARLY DEPENDENT IF THEY LIE ON THE SAME LINE.

$\overrightarrow{OQ} = (1,1)$

$\overrightarrow{OP} = (2,2)$



IN R^2 TWO VECTORS ARE LINEARLY INDEPENDENT IF THEY DO NOT LIE ON THE SAME LINE.

NOTE: ALSO FOR ANY VECTOR SPACE V, THE SET $\{\underline{v_1}, \underline{v_2}\}$ IS DEPENDENT IF $\underline{v_1}, \underline{v_2}$ ARE SCALAR MULTIPLES OF EACHOTHER AND $\{\underline{v_1}, \underline{v_2}\}$ IS INDEPENDENT IF AND ONLY IF NEITHER VECTOR IS A SCALAR MULTIPLE OF EACH OTHER.

AS WE SAW THAT $\{(1,1), (2,2)\}$ IS DEPENDENT AND $(2,2) = 2(1,1)$ i.e. $(1,1), (2,2)$ ARE SCALAR MULTIPLES OF EACHOTHER.

BUT

$\{(-3,7), (5,5)\}$ IS INDEPENDENT AND NONE OF $(-3,7), (5,5)$ IS A SCALAR MULTIPLE OF THE OTHER.

HERE $\{\underline{v_1}, \underline{v_2}\} \longrightarrow$ A SET CONTAINING EXACTLY TWO VECTORS

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EXAMPLE: P.222 (6th ED.)
P.232 (7th ED.)

LET $\underline{v_1} = (2,-1,0,3)$, $\underline{v_2} = (1,2,5,-1)$ AND $\underline{v_3} = (7,-1,5,8)$ THEN

(a) CHECK WHETHER $\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ IS LINEARLY DEPENDENT OR INDEPENDENT.

SOLUTION: LET $k_1\underline{v_1} + k_2\underline{v_2} + k_3\underline{v_3} = \underline{0}$, $\Rightarrow k_1(2,-1,0,3) + k_2(1,2,5,-1) + k_3(7,-1,5,8) = (0,0,0,0)$ — ①

COMPARING BOTH SIDES

$$2k_1 + k_2 + 7k_3 = 0 \rightarrow ②$$
$$-k_1 + 2k_2 - k_3 = 0 \rightarrow ③$$
$$k_2 + k_3 = 0 \rightarrow ④$$
$$3k_1 - k_2 + 8k_3 = 0 \rightarrow ⑤$$

WHICH IS A HOMOGENEOUS SYSTEM OF ④ EQUATIONS WITH THREE UNKNOWNS. SO THERE ARE TWO POSSIBILITIES

8) TO SOLVE THIS SYSTEM, SOLVE ②, ③, ④ AND CHECK IF THE SOLUTION SATISFIES ⑤ OR ONLY SOLVE ②, ③, ④ $\because$ ⑤ CAN BE OBTAINED BY SUBTRACTING ③ FROM ②

$\because$

$$\left.\begin{array}{l} 2k_1 + k_2 + 7k_3 = 0 \rightarrow ② \\ \underset{+}{-}k_1 \underset{-}{+} 2k_2 \underset{+}{-} k_3 = 0 \rightarrow ③ \end{array}\right\} \text{SUBTRACTING}$$

$$3k_1 - k_2 + 8k_3 = 0 \rightarrow ⑤$$

NOW IN MATRIX NOTATION, WE HAVE

$$\begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - ⑥$$

BUT $\det \begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix} = 0$ (CHECK)

$\therefore$ NONTRIVIAL SOLUTIONS EXIST $\Rightarrow$ THE GIVEN VECTORS ARE LINEARLY $\boxed{\text{DEPENDENT}}$ $\because$ ②, ③, ④ ARE SATISFIED BY $\boxed{\text{NONZERO}}$ VALUES OF $\boxed{k_1, k_2}$ AND $\boxed{k_3}$.

(9) SINCE THE SYSTEM IS HOMOGENEOUS AND THE COEFFICIENT MATRIX IS $\boxed{\text{NOT}}$ INVERTIBLE i-e. $\boxed{A^{-1}}$ DOESN'T EXIST THEREFORE INFINITE NONTRIVIAL (NONZERO) SOLUTIONS EXIST FOR

$A\underline{X} = \underline{0}$ WHERE

$$A = \begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \underline{X} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

(b) PROVE THAT $\underline{V_3}$ CAN BE WRITTEN AS THE LINEAR COMBINATION OF $\underline{V_2}$ AND $\underline{V_1}$.

SOLUTION: HINT: FOR THIS WE HAVE TO SOLVE THE ABOVE SYSTEM i-e. $A\underline{X} = \underline{0}$ FOR $\boxed{k_1, k_2 \text{ AND } k_3}$, i.e.

10) SOLUTION OF $\begin{cases} 2k_1 + k_2 + 7k_3 = 0 & \text{— (1)} \\ -k_1 + 2k_2 - k_3 = 0 & \text{— (2)} \\ k_2 + k_3 = 0 & \text{— (3)} \end{cases}$

TO FIND THE

LET $\boxed{k_2 = t}$ (3) $\Rightarrow$ $\boxed{k_3 = -k_2 = -t}$

(2) $\Rightarrow$ $k_1 = 2k_2 - k_3 = 2t - (-t) = \boxed{3t = k_1}$

$$\therefore \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 3t \\ t \\ -t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

ALSO SATISFIES (1)

FOR $\boxed{t = 1,\ k_1 = 3,\ k_2 = 1,\ k_3 = -1}$

$k_1\underline{v_1} + k_2\underline{v_2} + k_3\underline{v_3}$

$= 3(2,-1,0,3) + (1,2,5,-1)$

$-(7,-1,5,8) = (6,-3,0,9) +$

$(-6,3,0,-9) = (0,0,0,0)$

$\Rightarrow\ 3\underline{v_1} + \underline{v_2} - \underline{v_3} = \underline{0}$

OR $\boxed{\underline{v_3} = \underline{v_2} + 3\underline{v_1}}$

RESULT: (P.224 8th ED.)(P.234 7th ED.)

A SET S WITH TWO OR MORE VECTORS IS

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(a) LINEARLY DEPENDENT IF AND ONLY IF AT LEAST $\boxed{\text{ONE}}$ OF THE VECTORS IN $\boxed{S}$ IS EXPRESSIBLE AS A LINEAR COMBINATION OF THE OTHER VECTORS IN $\boxed{S}$.

E.G. IN THE LAST EXAMPLE $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ IS DEPENDENT AND $\underline{v}_3 = \underline{v}_2 + 3\underline{v}_1$ i.e. $\underline{v}_3$ IS A LINEAR COMBINATION OF $\underline{v}_1$ AND $\underline{v}_2$.

(b) LINEARLY INDEPENDENT IF AND ONLY IF $\boxed{\text{NO}}$ VECTOR IN $\boxed{S}$ IS EXPRESSIBLE AS A LINEAR COMBINATION OF THE OTHER VECTORS IN $\boxed{S}$.

E.G. $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ IS LINEARLY INDEPENDENT $\because$ NONE OF $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ IS A LINEAR COMBINATION OF THE OTHER TWO.

LET $\underline{e}_3 = k_1\underline{e}_1 + k_2\underline{e}_2$

$$\Rightarrow \underline{e}_3 = k_1 \underline{e}_1 + k_2 \underline{e}_2$$

$$\Rightarrow (0,0,1) = k_1(1,0,0) + k_2(0,1,0)$$

$$\Rightarrow (0,0,1) = (k_1, k_2, 0)$$

WHICH IS $\boxed{\text{NOT}}$ POSSIBLE

$\therefore$ $\underline{e}_3$ IS NOT EXPRESSIBLE AS A LINEAR COMBINATION OF $\underline{e}_1$ AND $\underline{e}_2$.