SOLUTIONS

MATH102-CALCULUS 2

TEST 2

SPRING 2022

NAME:______________________________

STUDENT ID:____________________________

SECTION:______________________________

TOTAL MARKS: 25

DATE: 12.03.2022

1. a. Show that every vertical line in the xy –plane has a polar equation of the form $r = a\sec\theta$.
   b. Find the analogous polar equation for horizontal lines in the xy –plane. [3 marks]

a) Every vertical line has equation $x = a$ for some a real number

$\Rightarrow r\cos\theta = a$

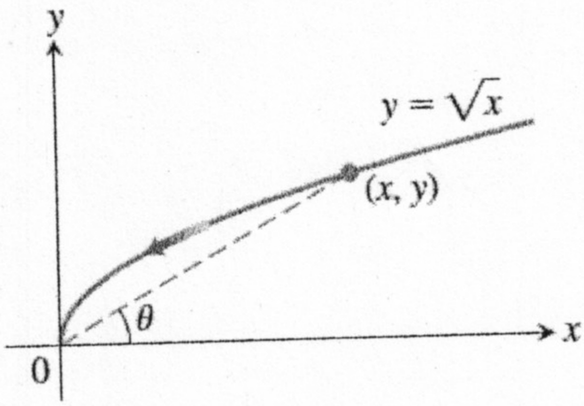
$\Rightarrow \boxed{r = a\sec\theta}$ is the polar equation for vertical line

b) Every horizontal line has equation $y = a$ for some $a \in \mathbb{R}$

$\Rightarrow r\sin\theta = a$

$\boxed{r = a\operatorname{cosec}\theta}$ represents any horizontal line in polar coordinates.

2. Find a parametrization for the curve $y = \sqrt{x}$ ending at the point $(0, 0)$ using the angle θ (see figure) as the parameter.



[4 marks]

In polar coordinates, $x = r\cos\theta$
$y = r\sin\theta$

$y = \sqrt{x}$ in polar becomes,

$$r\sin\theta = \sqrt{r\cos\theta}$$

$$\Rightarrow r^2\sin^2\theta = r\cos\theta$$

$$\Rightarrow r = \frac{\cos\theta}{\sin^2\theta}$$

$$\Rightarrow r = \cot\theta \operatorname{cosec}\theta, \quad 0 < \theta \leq \frac{\pi}{2}$$

$\Rightarrow$ Parametrization in terms of θ

$$x = \frac{\cos\theta}{\sin^2\theta} \cdot \cos\theta = \cot^2\theta$$

$$y = \frac{\cos\theta}{\sin^2\theta} \cdot \sin\theta = \frac{1}{\tan\theta}, \quad 0 < \theta \leq \frac{\pi}{2}$$

the curve $:= \langle \cot^2\theta, \frac{1}{\tan\theta} \rangle, \quad 0 < \theta \leq \frac{\pi}{2}$

3. Find the equation of the plane passing through the points $P_1 = (1, 2, 1), P_2 = (6, 5, 2)$ and $P_3 = (10, 6, 4)$. Explain your result. [5 marks]

Two vectors in the plane with tails of P_1

$P_{12} = \langle 6, 5, 2 \rangle - \langle 1, 2, 1 \rangle$
$= \langle 5, 3, 1 \rangle$

$P_{13} = \langle 10, 6, 4 \rangle - \langle 1, 2, 1 \rangle$
$= \langle 9, 4, 3 \rangle$

A normal to the plane $:= P_{12} \times P_{13}$

$= \begin{vmatrix} i & j & k \\ 5 & 3 & 1 \\ 9 & 4 & 3 \end{vmatrix}$

$= 5\hat{i} - 6\hat{j} - 7\hat{k}$

An equation of plane passing through P_1, P_2, P_3

$5(x-1) - 6(y-2) - 7(z-1) = 0$

$5x - 6y - 7z - 5 + 12 + 7 = 0$

$\boxed{5x - 6y - 7 = -14}$

4. Let $\vec{u} = (u_1, u_2), \vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ be three non-zero vectors in $\mathbb{R}^2$ such that $\vec{u} \cdot \vec{v} = 0$ and $\vec{u} \cdot \vec{w} = 0$. Show that $\vec{v} = k\vec{w}$ for some scalar k. [3 marks]

We have,

$$u_1 v_1 + u_2 v_2 = 0 \quad -(1)$$
$$u_1 w_1 + u_2 w_2 = 0 \quad -(2)$$

We want,

$v_1 = k w_1$, $v_2 = k w_2$, for some scalar 'k'.

1) $\Rightarrow$ $u_1 v_1 = -u_2 v_2$

2) $\Rightarrow$ $u_1 w_1 = -u_2 w_2$

Dividing both equations:

$$\frac{\cancel{u_1} v_1}{\cancel{u_1} w_1} = \frac{-\cancel{u_2} v_2}{-\cancel{u_2} w_2}$$

$$\frac{v_1}{w_1} = \frac{v_2}{w_2}$$

as desired,

$$\frac{v_1}{w_1} = \frac{v_2}{w_2} = k \quad \text{(same ratio)}$$

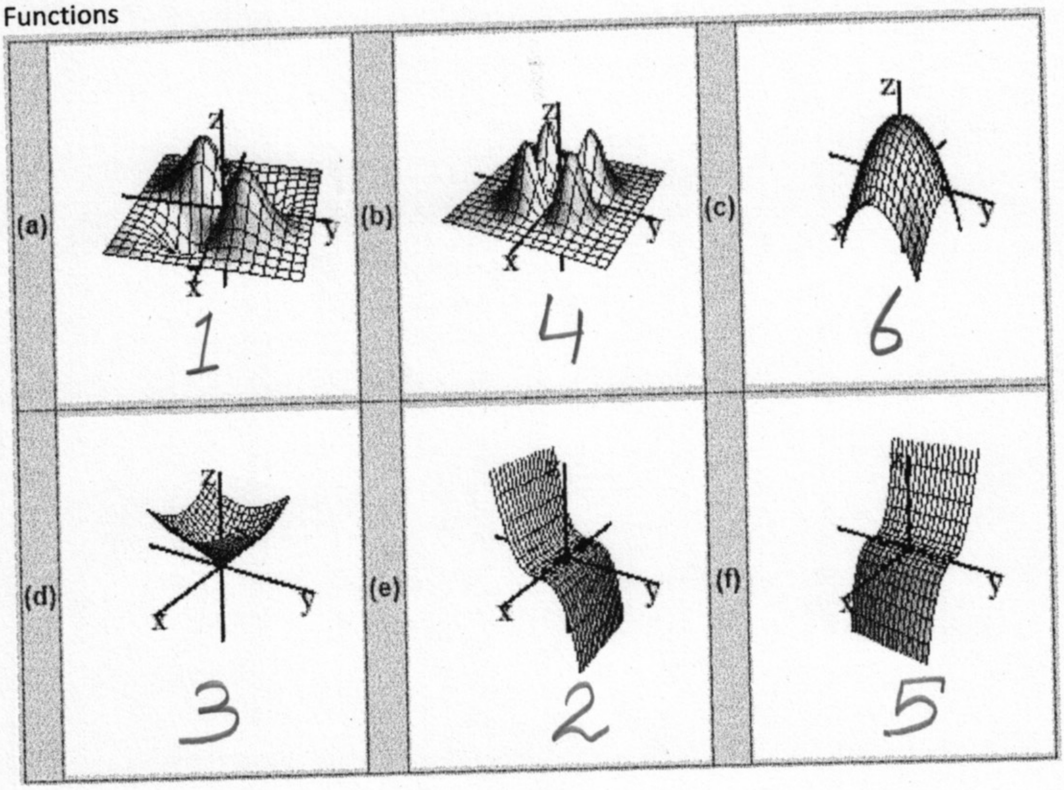
i.e, $v_1 = k w_1$, $v_2 = k w_2$

or, $\vec{v} = (v_1, v_2) = k(w_1, w_2)$

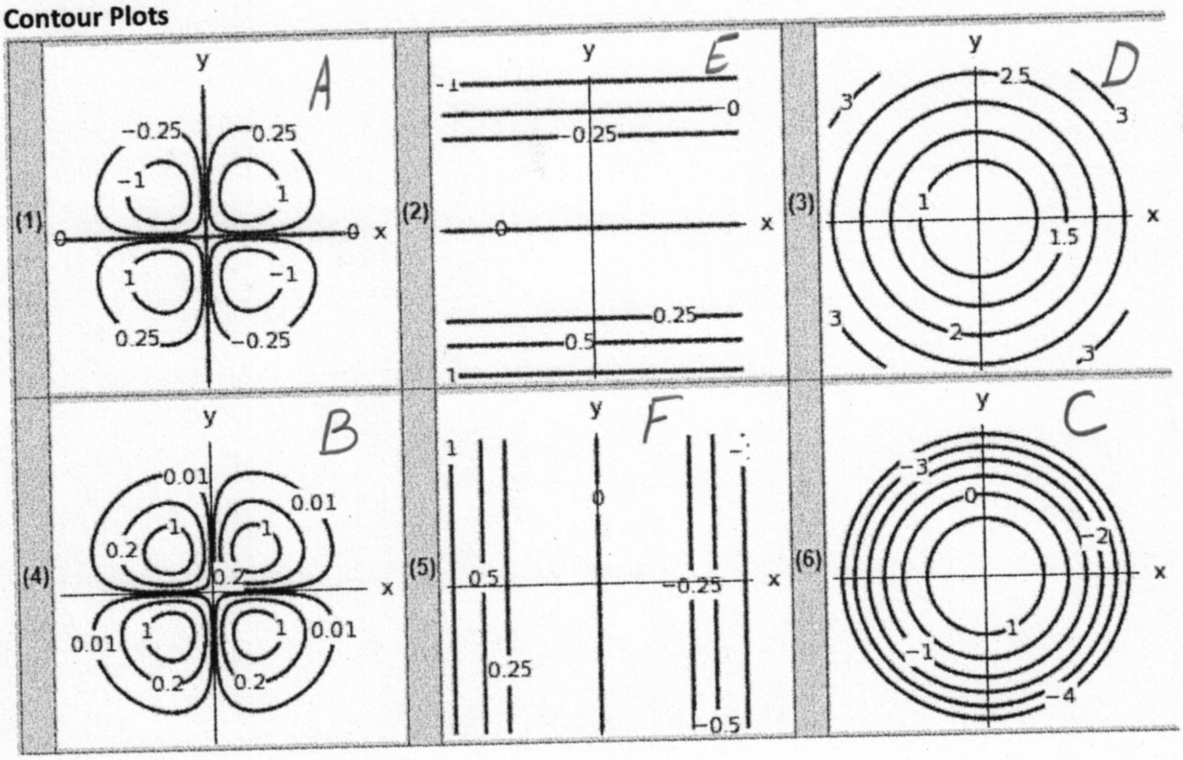
$$\vec{v} = k\vec{w}$$

5. Match the functions with their contour plots. [3 marks]

Functions



Contour Plots



6. Find the equation of the tangent plane for the function $f(x,y) = x^2 + y^2 - 1$ at the point $(1,3)$. [3 marks]

$f_x = 2x \quad , \quad f_y = 2y$

Normal $:= \langle -f_x, -f_y, 1 \rangle$

$:= \langle -2x, -2y, 1 \rangle$

At $(1,3)$

Normal $= \langle -2, -6, 1 \rangle$

$f(1,3) = 9$

Equation of tangent plane

$(z - z_0) - f_x(x - x_0) - f_y(y - y_0) = 0$

$(z - 9) - 2(x - 1) - 6(y - 3) = 0$

$z - 9 - 2x + 2 - 6y + 18 = 0$

$\boxed{z = 2x + 6y - 11}$

7. Find the derivative of $f(x,y) = \frac{x-y}{xy+2}$ at the point $(1,-1)$ in the direction of $\vec{u} = 12\hat{\imath} + 5\hat{\jmath}$

[4 marks]

unit vector $\hat{u} = \frac{1}{\sqrt{12^2+5^2}}(12\hat{i}+5\hat{j}) = \frac{12}{13}\hat{i} + \frac{5}{13}\hat{j}$

$$f_x = \frac{xy+2 - y(x-y)}{(xy+2)^2} = \frac{2+y^2}{(xy+2)^2}$$

$$f_x \Big|_{(1,-1)} = 3$$

$$f_y = \frac{-(xy+2) - x(x-y)}{(xy+2)^2} = \frac{-2-x^2}{(xy+2)^2}$$

$$f_y \Big|_{(1,-1)} = \frac{-3}{1} = -3$$

$$Df(1,-1)\Big|_{\vec{u}} = \langle f_x, f_y \rangle - \langle \tfrac{12}{13}, \tfrac{5}{13} \rangle$$

$$= \frac{36}{13} - \frac{15}{13}$$

$$= \boxed{\frac{21}{13}}$$