1 | $X = (x_1, \ldots, x_N)$ – нез $\sim U[0, \theta]$

a) Max. likelihood

$$L(\theta) = p(X|\theta) \underset{\uparrow}{=} \prod_{i=1}^{N} p(x_i|\theta)$$

независимость

$$\theta_{ML} = \underset{\theta}{\arg\max}\, L(\theta) = \underset{\theta}{\arg\max} \prod_{i=1}^{N} p(x_i|\theta) =$$

$$= \underset{\theta}{\arg\max} \sum_{i=1}^{N} \log p(x_i|\theta) \ominus$$

NB: $p(x_i|\theta) = \frac{1}{\theta}$

$$= \underset{\theta}{\arg\max} (-N \log \theta)$$

$$\frac{\partial(-N\log\theta)}{\partial\theta} = -\frac{N}{\theta}$$

Мы знаем, что $\theta \geq x_i$ $i = 1, \ldots, N$, т.е.

что $\theta \geq \max(X) = X_{(N)}$

при этом у нас убывающая ф-ия $\Rightarrow$

надо выбрать наименьшее возможное θ, т.е.

$$\theta_{ML} = x_{(N)}$$

в) conj $p(\theta)$

$$p(X|\theta) \sim U[0, \theta]$$

$$p(\theta|\alpha, \beta) = \frac{\alpha \beta^{\alpha}}{\theta^{\alpha+1}} [\beta \leq \theta], \quad \beta > 0, \quad m = \max(x_{(N)}, \beta)$$

$$p(\theta|X) = \frac{p(X|\theta)\, p(\theta|\alpha, \beta)}{\int_m^{+\infty} \ldots d\theta} = \frac{\theta^{-N-\alpha-1} \cancel{\alpha \beta^{\alpha}} [m \leq \theta]}{\cancel{\alpha \beta^{\alpha}} \int_m^{+\infty} \theta^{-N-\alpha-1} d\theta} =$$

$$= \frac{\theta^{-N-\alpha-1} [m \leq \theta]}{\frac{1}{N+\alpha} m^{-N-\alpha}} = \frac{(N+\alpha)\, m^{(N+\alpha)}}{\theta^{N+\alpha+1}} [m \leq \theta] \circledequal$$

$$\tilde{\alpha} = N + \alpha$$

$$\tilde{\beta} = m$$

$$\circledequal \frac{\tilde{\alpha} \tilde{\beta}^{\tilde{\alpha}}}{\theta^{\tilde{\alpha}+1}} [\tilde{\beta} \leq \theta] \text{ — Pareto}$$

c) статистики

$$p(X|\theta) = \theta^{-N}[x_{(N)} \le \theta]$$

$$p(\theta|\alpha,\beta) = \alpha\beta^{\alpha}\theta^{-\alpha-1}[\beta \le \theta] \quad , \exists\, m = \max(\beta, x_{(N)})$$

$$p(\theta|X) = \frac{p(X|\theta)\,p(\theta|\alpha,\beta)}{\int_0^{+\infty} p(X|\theta)\,p(\theta|\alpha,\beta)\,d\theta}$$

$$\ast \int_0^{+\infty} p(X|\theta)\,p(\theta|\alpha,\beta)\,d\theta = \int_m^{+\infty} \theta^{-N}\alpha\beta^{\alpha}\theta^{-\alpha-1}\,d\theta =$$

$$= \alpha\beta^{\alpha}\int_m^{+\infty}\theta^{-N-\alpha-1}\,d\theta = \frac{\alpha\beta^{\alpha}}{(-N-\alpha)}\,\theta^{-N-\alpha}\Big|_m^{+\infty} \ominus$$

Если $-N-\alpha-1 \ge -1 \Rightarrow -N-\alpha \ge 0$, то $\int$-л расходится

т.е. $N+\alpha < 0$

$$\ominus \frac{\alpha\beta^{\alpha}}{N+\alpha}\,m^{-N-\alpha} = \frac{\alpha\beta^{\alpha}}{(N+\alpha)m^{N+\alpha}}$$

$$p(\theta|X) = (N+\alpha)m^{N+\alpha}\cdot\theta^{-N-\alpha-1}[m \le \theta]$$

$$\mathbb{E}\theta = \int_0^{+\infty} \theta\, p(\theta|x)\,d\theta = (N+\alpha)\, m^{N+\alpha} \int_m^{+\infty} \theta^{-N-\alpha}\,d\theta =$$

$$= \left[\mathcal{I} = N+\alpha\right] = \mathcal{I}\, m^{\mathcal{I}} \int_m^{+\infty} \theta^{-\mathcal{I}}\,d\theta = \mathcal{I}\, m^{\mathcal{I}} \frac{(-1)}{\mathcal{I}-1}\, \theta^{-\mathcal{I}+1}\Big|_m^{+\infty} =$$

$$= \mathcal{I}\, m^{\mathcal{I}} \frac{1}{\mathcal{I}-1}\, m^{-\mathcal{I}+1} = \frac{\mathcal{I}}{\mathcal{I}-1}\, m = \mu$$

$$* \; P(c \le \theta < +\infty) = (N+\alpha)\, m^{N+\alpha} \int_c^{+\infty} \theta^{-N-\alpha-1}\,d\theta =$$

$c > m$

$$= (N+\alpha)\, m^{N+\alpha} \frac{(-1)}{N+\alpha}\, \theta^{-N-\alpha}\Big|_c^{+\infty} = m^{N+\alpha}\, c^{-N-\alpha} =$$

$$= \left(\frac{m}{c}\right)^{N+\alpha} = \frac{1}{2} \quad / \cdot \ln(\cdot)$$

$$(N+\alpha)(\ln m - \ln c) = -\ln 2$$

$$(N+\alpha)\ln c = (N+\alpha)\ln m + \ln 2$$

$$\ln c = \ln m + \frac{\ln 2}{N+\alpha}$$

$$c = e^{\ln m + \frac{\ln 2}{N+\alpha}} = m\left(e^{\ln 2}\right)^{\frac{1}{N+\alpha}} = m\, 2^{\frac{1}{N+\alpha}}$$

мода: $\underset{\theta}{\operatorname{argmax}}\, p(\theta|x) = \underset{\theta}{\operatorname{argmax}}\, \theta^{-N-\alpha-1} [m \le \theta] \Rightarrow$

убывающая ф-ия от θ $\Rightarrow$ берем наименьшее доступное θ.

$\Rightarrow m$ — мода, где $m = \max(\beta, X_{(N)})$

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