Uvindu Salgado

N19550514

Exercise 1

1. $3x_1 + 9x_2 = 15$ —①

$x_2 = -3x_1 + 5$

rewriting the 2nd equation we get $3x_1 + x_2 = 5$ —②

① − ②

$3x_1 + 9x_2 - 3x_1 - x_2 = 15 - 5$

$8x_2 = 10 \rightarrow x_2 = \frac{5}{4}$

$x_1 = \frac{5 - x_2}{3}$

$x_1 = \frac{5 - \frac{5}{4}}{3} = \frac{5}{4}$

The solution to the system is $x_1 = \frac{5}{4}$ and $x_2 = \frac{5}{4}$

∴ since there is a solution, the statement is FALSE and the lines are not parallel.

2. This statement is FALSE since the lines ~~are~~ could be non parallel but the lines could both be perpendicular to $\vec{u}$.

3. If $\vec{u}$ is perpendicular to $\vec{v}$ and $\vec{w}$ then

$u \cdot v = 0$ and $u \cdot w = 0$

$\vec{s}$ is a linear combination of $\vec{v}$ and $\vec{w}$ so $\vec{s} = (a \cdot v) + (b \cdot w)$

Now lets calculate $\vec{u} \cdot \vec{s}$ to see if they are perpendicular

$\vec{u} \cdot \vec{s} = u \cdot (av + bw)$

rewrite it as $\vec{u} \cdot \vec{s} = a(u \cdot v) + b(u \cdot w)$

$= 0 + 0$

$= 0$

∴ The statement is TRUE since $\vec{u}$ is perpendicular to $\vec{s}$ because the dot product is 0.

Uvindu Salgado

W19590514

Excercise 1 Continued

4. $\vec{u}$ and $\vec{v}$ are perpendicular so $u \cdot v = 0$

$\vec{u}$ and $\vec{v}$ are unit vectors so $\|u\| = 1$ and $\|v\| = 1$

$\|u + 3v\| = \sqrt{(u + 3v)^2}$

$\|u + 3v\| = \sqrt{u^2 + 6uv + 9v^2}$

$\|u + 3v\| = \sqrt{1 + 9 + (6 \times 0)}$

$\|u + 3v\| = \sqrt{10}$

$\therefore$ the statement is TRUE and the magnitude $\|u + 3v\|$ is $\sqrt{10}$

5. The given statement is FALSE because the length of a unit vector is equal to 1 regardless of ~~the~~ the dimension that it's in. So even if it's in n dimensions, the length of a unit vector will still be 1.

Uvindu Salgado
N19550514

Exercise 2:

1. For the pairs to be orthogonal, the dot product must be 0.

$\vec{v_1} \cdot \vec{v_2} = (1 \cdot 4) + (2 \cdot 0) + (-2 \cdot 4) + (1 \cdot 0) = -4$

$\vec{v_1} \cdot \vec{v_3} = (1 \cdot 1) + (2 \cdot -1) + (-2 \cdot -1) + (1 \cdot -1) = 0$ [orthogonal]

$\vec{v_1} \cdot \vec{v_4} = (1 \cdot 1) + (2 \cdot 1) + (-2 \cdot 1) + (1 \cdot 1) = 2$

$\vec{v_2} \cdot \vec{v_3} = (4 \cdot 1) + (0 \cdot -1) + (4 \cdot -1) + (0 \cdot -1) = 0$ [orthogonal]

$\vec{v_2} \cdot \vec{v_4} = (4 \cdot 1) + (0 \cdot 1) + (4 \cdot 1) + (0 \cdot 1) = 8$

$\vec{v_3} \cdot \vec{v_4} = (1 \cdot 1) + (-1 \cdot 1) + (-1 \cdot 1) + (-1 \cdot 1) = -2$

$\therefore$ The pairs $\vec{v_1}$ and $\vec{v_3}$ along with $\vec{v_2}$ and $\vec{v_3}$ are orthogonal.

Umindu Salgado

N19550574

<u>Exercise 2 continued:</u>

2. The pairs that are not orthogonal are $\vec{v_1}$ and $\vec{v_2}$, $\vec{v_1}$ and $\vec{v_4}$, $\vec{v_2}$ and $\vec{v_4}$, $\vec{v_3}$ and $\vec{v_4}$.

To find the angle between the pairs we use $\theta = \cos^{-1}\left(\frac{a \cdot b}{|a||b|}\right)$

Angle between $\vec{v_1}$ and $\vec{v_2}$:

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

$$|\vec{v_1}| = \sqrt{1^2+2^2+(-2)^2+1^2} = \sqrt{10}$$

$$|\vec{v_2}| = \sqrt{4^2+0^2+4^2+0^2} = \sqrt{32}$$

$$\vec{v_1} \cdot \vec{v_2} = (1 \cdot 4) + (2 \cdot 0) + (-2 \cdot 4) + (1 \cdot 0) = -4$$

$$\theta = \cos^{-1}\left(\frac{-4}{\sqrt{10} \times \sqrt{32}}\right) = 102.9°$$

Angle between $\vec{v_1}$ and $\vec{v_4}$:

$$|\vec{v_1}| = \sqrt{1^2+2^2+(-2)^2+1^2} = \sqrt{10}$$

$$|\vec{v_4}| = \sqrt{1^2+1^2+1^2+1^2} = 2$$

$$\vec{v_1} \cdot \vec{v_4} = (1 \cdot 1) + (2 \cdot 1) + (-2 \cdot 1) + (1 \cdot 1) = 2$$

$$\theta = \cos^{-1}\left(\frac{2}{2\sqrt{10}}\right) = 71.6°$$

Angle between $\vec{v_2}$ and $\vec{v_4}$:

$$|\vec{v_2}| = \sqrt{4^2+0^2+4^2+0^2} = \sqrt{32}$$

$$|\vec{v_4}| = \sqrt{1^2+1^2+1^2+1^2} = 2$$

$$\vec{v_2} \cdot \vec{v_4} = (4 \cdot 1) + (0 \cdot 1) + (4 \cdot 1) + (0 \cdot 1) = 8$$

$$\theta = \cos^{-1}\left(\frac{8}{2\sqrt{32}}\right) = 45°$$

Angle between $\vec{v_3}$ and $\vec{v_4}$:

$$|\vec{v_3}| = \sqrt{1^2+(-1)^2+(-1)^2+(-1)^2} = 2$$

$$|\vec{v_4}| = \sqrt{1^2+1^2+1^2+1^2} = 2$$

$$\vec{v_3} \cdot \vec{v_4} = (1 \cdot 1) + (-1 \cdot 1) + (-1 \cdot 1) + (-1 \cdot 1) = -2$$

$$\theta = \cos^{-1}\left(\frac{-2}{4}\right) = 120°$$

All the angles have been calculated above

Uvindu Salgado
N19550514

Exercise 2 continued:

3. To find a unit vector, implement the formula $\frac{v}{\|v\|}$

Unit vector for $\vec{v_1}$:

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\|\vec{v_1}\| = \sqrt{1^2+2^2+(-2)^2+1^2} = \sqrt{10}$$

$$\frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}$$

Let $\vec{u_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|}$ so $\vec{u_1} = \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ -2/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$

Unit vector for $\vec{v_2}$:

$$\vec{v_2} = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

$$\|\vec{v_2}\| = \sqrt{4^2+0^2+4^2+0^2} = \sqrt{32}$$

$$\frac{\vec{v_2}}{\|\vec{v_2}\|} = \frac{1}{\sqrt{32}} \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

Let $\vec{u_2} = \frac{\vec{v_2}}{\|\vec{v_2}\|}$ so $\vec{u_2} = \begin{bmatrix} 4/\sqrt{32} \\ 0 \\ 4/\sqrt{32} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

Unit vector for $\vec{v_3}$:

$$\vec{v_3} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

$$\|\vec{v_3}\| = \sqrt{1^2+(-1)^2+(-1)^2+(-1)^2} = 2$$

$$\frac{\vec{v_3}}{\|\vec{v_3}\|} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

Let $\vec{u_3} = \frac{\vec{v_3}}{\|\vec{v_3}\|}$ so $\vec{u_3} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$

$\therefore$ the set of 3 unit vectors out of $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$ are $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$

Uvindu Salgado
N19550514

Exercise 3:

1. $$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$$

where $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$

2. $$\begin{bmatrix} 2 & 5 & 1 \\ 4 & 11 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

where $A = \begin{bmatrix} 2 & 5 & 1 \\ 4 & 11 & 1 \\ 0 & 1 & -1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

3. $$\begin{bmatrix} 3 & 5 \\ -2 & 0 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}$$

where $A = \begin{bmatrix} 3 & 5 \\ -2 & 0 \\ 8 & 9 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}$

Uvindu Salgado
W1955051 4

Exercise 4 :

We are given the matrix below

$$\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}$$

To know whether the coloumn vectors are linearly independant or dependant we must use an approach :

$$c_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1 + 5c_3 = 0 \quad - (1)$$
$$-2c_1 + c_2 - 6c_3 = 0 \quad - (2)$$
$$2c_2 + 8c_3 = 0 \quad - (3)$$

Solve these equations :

$(2) + (1) \times 2 = 2'$

$$-\cancel{2c_1} + c_2 - 6c_3 + \cancel{2c_1} + 10c_3 = 0 \quad - (2')$$
$$c_2 + 4c_3 = 0 \quad - (2')$$

$(3) - 2 \times 2'$

$$2c_2 + 8c_3 - 2c_2 - 8c_3 = 0$$
$$0 = 0$$

This system of equations has only one solution which is $c_1 = c_2 = c_3 = 0$

$\therefore$ since all the coefficients are zero the coloumn vectors of the matrix are linearly independant.

Uvindu Salgado

N19550514

Exercise 5:

Let the set be s, $s = \{v_1, v_2, v_3\}$

s is a linearly dependent set of vectors in $\mathbb{R}^n$

$\therefore$ there are scalars c_1, c_2 and c_3 and not all of them are zero

we can say that $c_1v_1 + c_2v_2 + c_3v_3 = 0$ because the set of vectors is linearly dependant. Now we have v_4.

$$c_1v_1 + c_2v_2 + c_3v_3 + 0 \cdot v_4 = 0$$

If we let $c_4 = 0$ and not all of c_1, c_2, c_3 and c_4 are zero

then $\{v_1, v_2, v_3, v_4\}$ is also a linearly dependant set.