Homework 7 Solutions

via Gradescope

- Failure to submit homework correctly will result in zeroes.
- Handwritten homework is OK. You do not have to type up your work.
- Problems assigned from the textbook are from the 5th edition.
- No late homework accepted. Lateness due to technical issues will not be excused.

1. (9 points) Section 5.4 #6, 13, 20.

   Solution:

   #6.

   Proof.
   Basis Step:

   $f_0 = 5 = 3 + 2 = 3(1) + 2(1) = 3(2^0) + 2(5^0)$

   $f_1 = 16 = 6 + 10 = 3(2) + 2(5) = 3(2^1) + 2(5^1)$

   Inductive Step: Let $k \in \mathbb{Z}$ and $k \geq 1$.

   Suppose $f_i = 3\left(2^i\right) + 2\left(5^i\right)$ for all $i \in \{0, \ldots, k\}$.

$$\begin{aligned} f_{k+1} &= 7f_{k-1} - 10f_{k-2} \\ &= 7\left(3\left(2^k\right) + 2\left(5^k\right)\right) - 10\left(3\left(2^{k-1}\right) + 2\left(5^{k-1}\right)\right) \quad \text{(via induction hypothesis)} \\ &= (21)(2^k) + (14)(5^k) - 5(3)(2^k) - (2^2)(5^k) \\ &= (21 - 15)(2^k) + (14 - 4)(5^k) \\ &= 6(2^k) + 10(5^k) = 3(2^{k+1}) + 2(5^{k+1}) \end{aligned}$$

   □

   #13.

   Proof.
   Basis Step:

   2 is prime so, by generalization, 2 is prime or a product of primes.

   Inductive Step:

   Let $k \in \mathbb{Z}$ and $k \geq 2$. Suppose i is prime or a product of primes for all $i \in \{2, \ldots, k\}$.

Since $k \geq 2$, $k+1 \geq 3$ is prime or not prime.

Case 1: Suppose $k+1$ is prime. $k+1$ is prime or a product of primes via generalization.

Case 2: Suppose $k+1$ is not prime, so $k+1$ is composite. There exist $r, s \in \mathbb{Z}^+$ such that $k+1 = rs$ and $1 < r < k+1$ and $1 < s < k+1$. Since $r, s \in \mathbb{Z}$, $2 \leq r \leq k$ and $2 \leq s \leq k$. So r, s are prime or products of primes via induction hypothesis. Since $k+1 = rs$, $k+1$ is a product of primes. Thus $k+1$ is prime or a product of primes via generalization. □

Remark. Notice that for all integers $k \geq 2$, the least prime $k+1$ is 3. The least composite $k+1$ is $4 = 2(2)$ and $1 < 2 < 4$ so $2 \leq 2 \leq 3 = k$, compatible with the induction hypothesis.

#20.

Theorem. *Let $\{b_k\}_{k=1}^{\infty}$ such that $b_1 = 0$, $b_2 = 3$, and, for all integers $k \geq 3$, $b_k = 5b_{\lfloor k/2 \rfloor} + 6$. Then $3 \mid b_n$ for any $n \in \mathbb{Z}^+$.*

Proof.
Basis Step:

$0 \in \mathbb{Z}$ such that $b_1 = 0 = 3(0)$, so $3 \mid b_1$.

$1 \in \mathbb{Z}$ such that $b_2 = 3 = 3(1)$, so $3 \mid b_2$.

Inductive Step:

Let $k \in \mathbb{Z}$ and $k \geq 2$. Suppose $3 \mid b_i$ for all $i \in \{1, \ldots, k\}$.

$k+1 \in 2\mathbb{Z}$ or $k+1 \notin 2\mathbb{Z}$.

Case 1: Suppose $k+1 \in 2\mathbb{Z}$. There exists $l_1 \in \mathbb{Z}$ such that $k+1 = 2l_1$, so $\frac{k+1}{2} = l_1$.

$$\left\lfloor \frac{k+1}{2} \right\rfloor = \lfloor l_1 \rfloor = l_1$$

since $l_1 \leq l_1 = \frac{k+1}{2} < l_1 + 1$. Since $k \geq 2$,

$$\begin{aligned} k &\geq 2 \\ 2k = k + k > k + 1 &\geq 3 \\ k > \frac{k+1}{2} = l_1 &\geq \frac{3}{2} > 1 \end{aligned}$$

so

$$\begin{aligned} b_{k+1} &= 5b_{\lfloor (k+1)/2 \rfloor} + 6 = 5b_{l_1} + 6 \\ &= 5(3m_1) + 6 = 3(5m_1 + 2) \qquad \text{(via induction hypothesis)} \end{aligned}$$

for some $m_1 \in \mathbb{Z}$ and $5m_1 + 2 \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under products and sums. Hence $3 \mid b_{k+1}$.

Case 2: Suppose $k+1 \notin 2\mathbb{Z}$. There exists $l_2 \in \mathbb{Z}$ such that $k+1 = 2l_2+1$, so $\frac{k+1}{2} = l_2 + \frac{1}{2}$.

$$\left\lfloor \frac{k+1}{2} \right\rfloor = \left\lfloor l_2 + \frac{1}{2} \right\rfloor = l_2$$

since $l_2 \leq l_2 + \frac{1}{2} = \frac{k+1}{2} < l_2 + 1$. Since $k \geq 2$,

$$\begin{aligned} k+1 &= 2l_2 + 1 \\ k &= 2l_2 \geq 2 \\ k \geq \frac{k}{2} &= l_2 \geq 1 \end{aligned}$$

so

$$\begin{aligned} b_{k+1} &= 5b_{\lfloor (k+1)/2 \rfloor} + 6 = 5b_{l_2} + 6 \\ &= 5(3m_2) + 6 = 3(5m_2 + 2) \qquad \text{(via induction hypothesis)} \end{aligned}$$

for some $m_2 \in \mathbb{Z}$ and $5m_2 + 2 \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under products and sums. Hence $3 \mid b_{k+1}$. □

2. (3 points) Let $\{f_k\}_{k=0}^{\infty}$ be the Fibonacci sequence $f_0 = f_1 = 1$ and $f_k = f_{k-1} + f_{k-2}$ for all integers $k \geq 2$. Use the Principle of Strong Mathematical Induction to prove that, for all $n \in \mathbb{Z}^+ \cup \{0\}$,

$$f_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right).$$

Solution:

Proof.
Basis Step:

$$\begin{aligned} f_0 = 1 &= \frac{1}{\sqrt{5}} \left(\sqrt{5} \right) = \frac{1}{\sqrt{5}} \left(\frac{1}{2}\sqrt{5} + \frac{1}{2}\sqrt{5} + \frac{1}{2} - \frac{1}{2} \right) \\ &= \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right) - \left(\frac{1-\sqrt{5}}{2} \right) \right) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{0+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{0+1} \right) \end{aligned}$$

$$\begin{aligned} f_1 = 1 &= \frac{1}{\sqrt{5}} \left(\sqrt{5} \right) = \frac{1}{\sqrt{5}} \left(\frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2} + \frac{1}{4} - \frac{1}{4} + \frac{5}{4} - \frac{5}{4} \right) \\ &= \frac{1}{\sqrt{5}} \left(\frac{1}{4} + \frac{5}{4} + \frac{\sqrt{5}}{2} - \left(\frac{1}{4} + \frac{5}{4} - \frac{\sqrt{5}}{2} \right) \right) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^2 - \left(\frac{1-\sqrt{5}}{2} \right)^2 \right) \\ &= \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{1+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{1+1} \right) \end{aligned}$$

Inductive Step:

Let $k \in \mathbb{Z}^+$. Suppose

$$f_i = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{i+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{i+1}\right)$$

for all $i \in \{0, \ldots, k\}$. Then

$$\begin{aligned}
f_{k+1} &= f_k + f_{k-1} \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right) + \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right) \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k}\right) \\
&+ \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right) \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1}{2}+\frac{\sqrt{5}}{2}+1\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1}{2}-\frac{\sqrt{5}}{2}+1\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k}\right) \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k}\right) \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1}{4}+\frac{5}{4}+\frac{\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1}{4}+\frac{5}{4}-\frac{\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k}\right) \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{2}\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{2}\left(\frac{1-\sqrt{5}}{2}\right)^{k}\right) \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right).
\end{aligned}$$

$\square$

3. (6 points) Section 5.4 #25, 32.

Solution: #25. With only one base case and $k \geq 0$, if $k = 0$, then $k - 1 = -1$.

#32. No. $P(4)$ and $P(5)$ are not necessarily true. For any $k \in \mathbb{Z}$, $3k \neq 4$ and $3k \neq 5$.

4. (12 points) Suppose you wish to show that $P(n)$ is true for all integers $n \geq a$. You begin by defining the set

$$S = \{\, n \geq a : n \in \mathbb{Z} \wedge P(n) \equiv \bot \,\}.$$

Your goal is to show that $S = \varnothing$. You have trouble showing $S = \varnothing$ so you try contradiction.

Proof. Suppose that $S \neq \varnothing$.

(a) Explain why S has a smallest element in your contradiction proof.

Solution: Since S is a non-empty set of integers that is bounded from below (namely by a), it follows from the Well-Ordering Principle.

(b) If you know that $P(a)$ is $\top$, then explain why the smallest element of S, let's denote it by x, satisfies $x > a$ in your contradiction proof.

Solution: Every element in $y \in S$ satisfies $a \leq y$. Since $a \notin S$ it follows that the smallest element in S is greater than a.

(c) Explain why $P(x)$ is $\bot$ and $P(x-1)$ is $\top$ in your contradiction proof.

Solution: Since $x > a$, then $x - 1 \geq a$. Since $x - 1 \notin S$ (otherwise x is not the least element) we must have $P(x-1)$ is $\top$.

(d) Suppose you don't know that $P(a)$ is $\top$. Explain why you cannot say $P(x-1)$ is $\top$ in your contradiction proof.

Solution: That is because x could be a. This means that $x - 1$ is $a - 1$ and P is defined for $n \geq a$.

5. (6 points) Section 5.4 #26, 27.

Solution: #26.

Proof. Define

$$S := \{\, n \in \mathbb{Z}^+ - \{1\} : p \nmid n \text{ for any prime } p \,\}.$$

(1) $S \subseteq \mathbb{Z}$ is a set of integers by definition.

(2) S is bounded below, since $1 \in \mathbb{Z}$ such that $1 \leq x$ for any $x \in S$.

(3) Suppose $S \neq \varnothing$. S has a least element, denoted $x \in S$, via Well-Ordering Principle. Since $x > 1$, x is prime or not prime.

Case 1: Suppose x is prime. $1, x \in \mathbb{Z}$ such that $x = x(1)$ so $x \mid x$ but this contradicts that $x \in S$.

Case 2: Suppose x is not prime, so x is composite. $x = mn$ and $1 < m < x$ and $1 < n < x$ for some $m, n \in \mathbb{Z}^+$. $n \notin S$ since $x \in S$ is the least element of S, i.e. $x \leq k$ for all $k \in S$ and $2 \leq n \leq x-1 < x$. So there exists a prime $p > 1$ such that $p \mid n$. There exists $l \in \mathbb{Z}$ such that $n = pl$ since $p \mid n$. $x = mn = m(pl) = (lm)p$ and $lm \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under products. Hence there is a prime p such that $p \mid x$ but this contradicts that $x \in S$.

Therefore $S = \varnothing$.

Thus, for any integer $n > 1$, there exists a prime $p > 1$ such that $p \mid n$. $\square$

#27.

Proof. Define

$$S := \{\, n \in \mathbb{Z}^{+} - \{1\} : n \text{ has no prime factorization} \,\}.$$

(1) $S \subseteq \mathbb{Z}$ is a set of integers by definition.

(2) S is bounded below, since $1 \in \mathbb{Z}$ such that $1 \leq x$ for any $x \in S$.

(3) Suppose $S \neq \varnothing$. S has a least element, denoted $l \in S$, via Well-Ordering Principle. Since $l > 1$, l is prime or not prime.

  Case 1: Suppose l is prime. Then l has a prime factorization, $l = l^1$ but this contradicts that $l \in S$.

  Case 2: Suppose l is not prime, so l is composite. $l = mn$ and $1 < m < l$ and $1 < n < l$ for some $m, n \in \mathbb{Z}^{+}$. $m, n \notin S$ since $l \in S$ is the least element of S, i.e. $l \leq x$ for all $x \in S$ and $m < l$ and $n < l$. So $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{k_1}^{\alpha_{k_1}}$ and $n = q_1^{\beta_1} q_2^{\beta_2} \cdots q_{k_2}^{\beta_{k_2}}$ for some primes $p_1, \ldots, p_{k_1}, q_1, \ldots, q_{k_2}$; $\alpha_1, \ldots, \alpha_{k_1}, \beta_1, \ldots, \beta_{k_2} \in \mathbb{Z}^{+}$; and $k_1, k_2 \in \mathbb{Z}^{+}$. However

$$l = mn = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{k_1}^{\alpha_{k_1}} q_1^{\beta_1} q_2^{\beta_2} \cdots q_{k_2}^{\beta_{k_2}}$$

  is a prime factorization of l but this contradicts that $l \in S$.

  Thus $S = \varnothing$.

Therefore any integer $n > 1$ has a prime factorization. $\square$