# MATH-UA-240 / UY-4314 — Combinatorics

**Professor Charles Stine**

**Name:** **Homework Assignment #1**

**NetID:** **Due Date:** September 9, 2024, 11:59 PM

- This homework should be submitted via Gradescope by 11:59 PM on the date listed above.
- There are two main ways you might want to write up your work.
  - Write a PDF using a tablet
  - Write your answers on paper, clearly numbering each question and part.
    - You can use an app such as OfficeLens to take pictures of your work with your phone and convert them into a single pdf file. Gradescope will only allow pdf files to be uploaded.
- You must show all work. You may receive zero or reduced points for insufficient work. Your work must be neatly organized and written. You may receive zero or reduced points for incoherent work.
- Please start a fresh page for each numbered problem. You may have parts a), b) and c) on the page for example, but problems 1) and 2) should be on separate pages.
- When uploading to Gradescope, **you must match each question to the page that your answer appears on**. If you do not, we will be unable to grade the unmatched problems.
- When appropriate, please put a box or circle around your final answer.
- The problems on this assignment will be graded on correctness and completeness.

# Lecture 1

1. (10 points) Basics of graphs.

   (a) Four teams: A, B, C, D compete in a tournament. A beats everyone except B, B loses to everyone except A, and D beats C. Draw a directed graph which represents the outcome of every game in this tournament.

   Then determine all the ways to list the four teams in order so that each team in the list has beaten the team to it's right in the list.

   (b) A computer wants to search through the alphabet for a letter. It starts at a null character "$*$" (which is not a letter) and can move from that state to any of the 26 letters of the alphabet, from there it can take one step either to the previous letter or to the next letter, then it stops. Draw a graph which represents all possible ways it can move through the alphabet.

   Now assume that it has the option to return to $*$ any time it is at one of the letters: "c", "k", or "t". What is the longest directed path between any two letters? (a path to a specific letter ends at the first moment you reach that letter)

   (c) Graphs $\subseteq$ Multi-Graphs $\subseteq$ General Graphs. Give an example of each type which is not a member of the previous type.

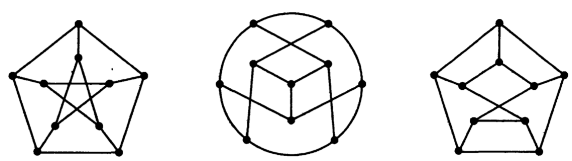
   (d) What is the smallest possible order of a graph with at least one vertex of degree 3 and the degree of all other vertices at least 2? (Recall that *order* means number of vertices). Draw a graph of that order and explain why no smaller graph is possible.

2. (10 points) Isomorphism of Graphs:

   (a) Determine the 11 non-isomorphic graphs of order 4 and give a planar diagram of each.

   (b) Do there exist graphs of order 5 with either of the following degree sequences: $(4, 4, 3, 2, 2)$ or $(4, 4, 4, 2, 2)$? If not, then explain why. If so then draw such a graph. (hint: think about the complements)

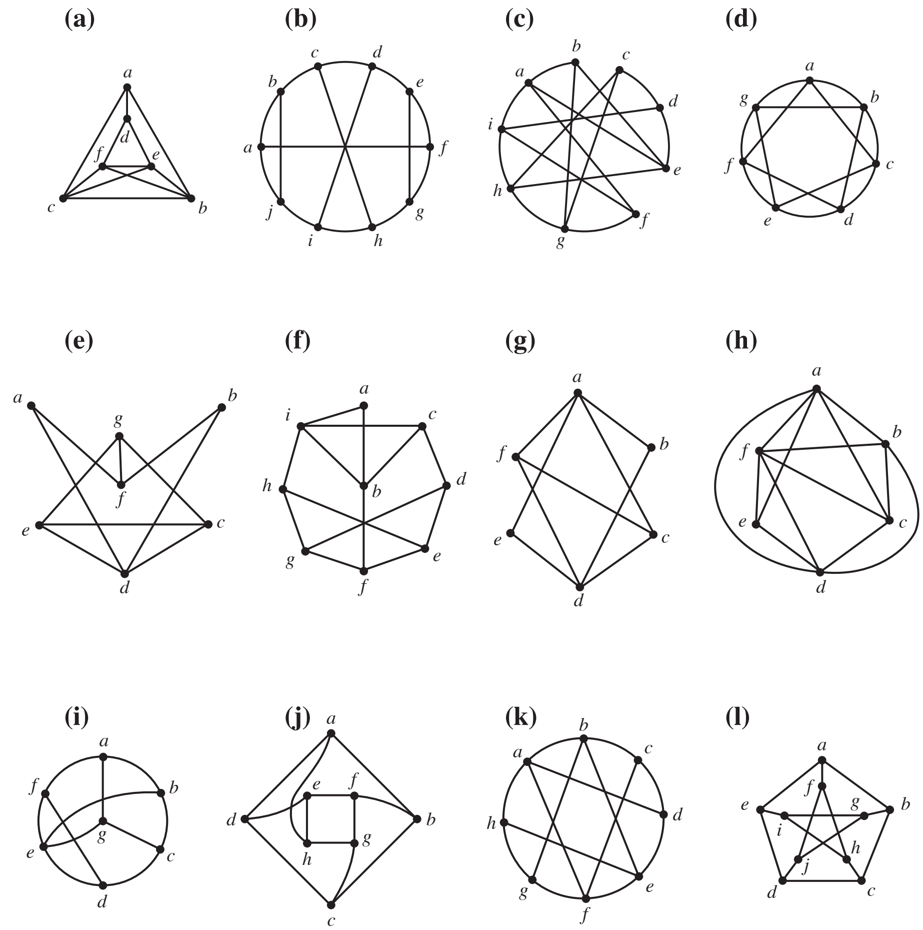
   (c) Determine which pairs of the graphs below are isomorphic, and for those that are isomorphic, find an isomorphism.



Graphs: A, B, C (left to right)

# Lecture 2

3. (10 points) Determine which of the following graphs are planar. For each graph, either draw a planar diagram of it (with vertices labelled), or copy the diagram below and highlight the subset of vertices and edges which form a $K_5$ or $K_{3,3}$ configuration.




4. (10 points) Fun with Euler's Formula:

   (a) If every vertex of a connected, planar graph has degree 4, and the graph has 10 regions, then what is it's order?

   (b) A graph is called *maximally planar* if it is planar, but adding any other edge between two non-adjacent vertices makes it non-planar. Show that every region of a maximally planar graph must be triangular.

   (c) If a connected, maximally planar graph has order $n$, how many regions and edges does it have?