MATH-UA-240 — Combinatorics

Professor Charles Stine

Name: **Homework Assignment #8**
NetID: **Due Date:** November 17, 2024, 11:59 PM

- This homework should be submitted via Gradescope by 11:59 PM on the date listed above.
- There are two main ways you might want to write up your work.
  - Write a PDF using a tablet
  - Write your answers on paper, clearly numbering each question and part.
    * You can use an app such as OfficeLens to take pictures of your work with your phone and convert them into a single pdf file. Gradescope will only allow pdf files to be uploaded.
- You must show all work. You may receive zero or reduced points for insufficient work. Your work must be neatly organized and written. You may receive zero or reduced points for incoherent work.
- Please start a fresh page for each numbered problem. You may have parts a), b) and c) on the page for example, but problems 1) and 2) should be on separate pages.
- When uploading to Gradescope, **you must match each question to the page that your answer appears on**. If you do not, we will be unable to grade the unmatched problems.
- When appropriate, please put a box or circle around your final answer.
- The problems on this assignment will be graded on correctness and completeness.

Lecture 16

1. (10 points) Generating function models: build, but do not simplify, a generating function for a_r the number of distributions of r identical objects into

   (a) Five different boxes with at most three objects in each box.

   (b) Three different boxes with between three and six objects in each box.

   (c) Six different boxes with at least one object in each box.

   (d) Three different boxes with at most five objects in the first box.

   (e) Find a generating function for the number of integers between 0 and 999,999 whose digits sum to the number r.

2. (10 points) Computing coefficients of generating functions:

   (a) How many ways are there to place an order for 12 ice cream sundaes if there are five types of sundaes, and at most four sundaes of one type are allowed?

   (b) If a coin is flipped 25 times with eight tails occurring, what is the probability that no run of six or more consecutive heads occurs?

   (c) Show that $(1 - x - x^2 - x^3 - x^4 - x^5 - x^6)^{-1}$ is the generating function for the number of ways a sum of r can occur if a die is rolled in order any number of times.

   Clarification: the number of rolls is not fixed! For example, consider $r = 5$, there is 1 way to get 5 from rolling the die once, there are 4 ways to get it from rolling the die twice ($1 + 4$, $4 + 1$, $2 + 3$, and $3 + 2$), there are 6 ways to get it from rolling thrice ($1 + 1 + 3$, $1 + 3 + 1$, $3 + 1 + 1$, $1 + 2 + 2$, $2 + 1 + 2$, and $2 + 2 + 1$), 4 ways from rolling four times and 1 way from rolling five times. There is no way to make 5 from six or more rolls so the numbers of ways add up to $1 + 4 + 6 + 4 + 1 = 16$ ways to obtain 5 from rolling <u>any</u> number of times so $a_5 = 16$.

Lecture 17

3. (10 points) Partitions:

   (a) Show with generating functions that every positive integer has a unique decimal representation.

   (Recall that a decimal representation of a number X is a sequence of numbers $(a_0, a_1, \ldots, a_n)$ such that $X = \sum_{k=0}^{n} a_k \cdot 10^k$)

   (b) Show that the number of partitions of the number $2r + k$ into $r + k$ parts depends on r but NOT on k.

(Recall that a partition of a number X is a list of non-negative numbers which sum to X. Repeats are allowed, and we do not care about the order of the list.)

(c) 🌶🌶 Show that,

$$A(x) = \frac{1}{2(1-x)^3}\left(\frac{1}{(1-x)^3} + \frac{1}{(1+x)^3}\right)$$

is the generating function for the number of ways to toss r identical dice and obtain an even sum.

(It may help to think about the values of the dice: 1,...,6 as buckets containing infinitely many copies of each number. Rolling r identical dice is the same as picking r numbers out of the six buckets.)

4. (10 points) Exponential generating functions:

(a) Consider the polynomial $P(x) = 3 + 4x + 17x^2 + 24x^3 + 5x^4$. What is the value of a_3 if $P(x)$ is an ordinary generating function for the sequence $\{a_0, a_1, a_2, a_3, \ldots\}$? What if $P(x)$ is an exponential generating function?

(b) How many 10-letter words *(a word may contain any number of any letter of the alphabet)* are there in which each of the letters e, n, r, s occurs at most once? At least once?

(c) Show that $e^x \cdot e^y = e^{x+y}$ by formally multiplying the expansions of e^x and e^y together.

(d) Show that

$$E(x) = \frac{e^x}{(1-x)^n}$$

is the exponential generating function for the number of ways to choose some subset (possibly empty) of r distinct objects and distribute them into n different boxes while remembering the order in each box.