Homework 10 Solutions

via Gradescope

- Failure to submit homework correctly will result in zeroes.
- Handwritten homework is OK. You do not have to type up your work.
- Problems assigned from the textbook are from the 5^{th} edition.
- No late homework accepted. Lateness due to technical issues will not be excused.

1. (6 points) Section 7.2 #25.

   Solution: #25. Do you notice the typo in the print edition of the textbook?

   Yes, C is injective. No, C is not surjective.

   Let's first prove C is injective.

   Proof. Let s_1, s_2 be strings in S. Suppose $C(s_1) = C(s_2)$. Then $as_1 = as_2$. Note that strings are ordered tuples of elements. Aside from λ, a string in S of length $n \in \mathbb{Z}^+$ is denoted as an n-tuple $(x_1, x_2, \ldots, x_n)$ where x_i is a or b for all $i \in \{1, \ldots, n\}$. Two n-tuples $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ are equal if and only if $x_i = y_i$ for all $i \in \{1, \ldots, n\}$. So $s_1 = s_2$. Therefore C is injective. $\square$

   Let's disprove that C is surjective.

   Disproof. Recall that

   $$C(S) = \{\, t \in S : \exists\, s \in S\,(\,C(s) = t\,)\,\}.$$

   Choose $t = b \in S$. Suppose $b \in C(S)$. There exists $s \in S$ such that $b = C(s) = as$. However $b \neq as$ since the first characters $b \neq a$ and b is a string of length 1 while as is a string of length 2. So $b \notin C(S)$. $b \in S$ but $b \notin C(S) \subseteq S$, so $C(S) \subsetneq S$ and $C(S) \neq S$. Therefore C is not surjective. $\square$

2. (9 points) Section 7.2 #29, 34.

   Solution: #29. Yes, H is injective and surjective.

Proof of One-to-one. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ such that $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ and $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Suppose $H(\mathbf{a}) = H(\mathbf{b})$, i.e. $H(a_1, a_2) = H(b_1, b_2)$. Then

$$(a_1 + 1, 2 - a_2) = (b_1 + 1, 2 - b_2).$$

So

$$\begin{aligned} a_1 + 1 &= b_1 + 1 \\ a_1 &= b_1 \end{aligned}$$

and

$$\begin{aligned} 2 - a_2 &= 2 - b_2 \\ -a_2 &= -b_2 \\ a_2 &= b_2. \end{aligned}$$

Thus $\mathbf{a} = (a_1, a_2) = (b_1, b_2) = \mathbf{b}$ and H is injective. □

Proof of Onto. Let $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. Choose $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ such that $x_1 = y_1 - 1$ and $x_2 = 2 - y_2$.

$$H(\mathbf{x}) = H(x_1, x_2) = (x_1 + 1, 2 - x_2) = (y_1 - 1 + 1, 2 - (2 - y_2)) = (y_1, y_2) = \mathbf{y}$$

so H is surjective.

□

#34.

Proof. Let $x, y \in (0, \infty)$ and $b \in (0, 1) \cup (1, \infty)$. $\log_b \colon (0, \infty) \to \mathbb{R}$ is well-defined, so

$$\begin{aligned} \log_b(x) &=: z_1 \iff b^{z_1} = x \\ \log_b(y) &=: z_2 \iff b^{z_2} = y \end{aligned}$$

for some $z_1, z_2 \in \mathbb{R}$ and

$$xy = b^{z_1} b^{z_2} = b^{z_1 + z_2} \iff \log_b(xy) = z_1 + z_2 = \log_b(x) + \log_b(y).$$

□

3. (6 points) Section 7.2 #40.

   Solution: #40. Let $F \colon X \to Y$ be injective. For any $x_1, x_2 \in X$, $F(x_1) = F(x_2)$ implies $x_1 = x_2$.

(a) *Proof.* Let $A \subset X$.
$F(A) = \{\, y \in Y : \exists\, a \in A\, (\, F(a) = y\,)\,\}$
$F^{-1}(F(A)) = \{\, x \in X :\, F(x) \in F(A)\,\}$
Case 1: Let $x \in F^{-1}(F(A))$. So $F(x) \in F(A)$. There exists some $a \in A$ such that $F(x) = y = F(a)$. $x = a$ since F is injective. Thus $x \in A$.
Case 2: Let $x \in A$. Since $F\colon X \to Y$ is well-defined and $A \subset X$, there exists a unique $y \in Y$ such that $F(x) = y$. Since $x \in A$, $y \in F(A)$, i.e. $F(x) \in F(A)$. Thus $x \in F^{-1}(F(A))$. □

(b) *Proof.* Let $A_1, A_2 \subset X$.
$F(A_1) = \{\, y \in Y : \exists\, a_1 \in A\, (\, F(a_1) = y\,)\,\}$
$F(A_2) = \{\, y \in Y : \exists\, a_2 \in A\, (\, F(a_2) = y\,)\,\}$
$F(A_1 \cap A_2) = \{\, y \in Y : \exists\, a \in A_1 \cap A_2\, (\, F(a) = y\,)\,\}$
Case 1: Let $y \in F(A_1 \cap A_2)$. $F(a) = y$ for some $a \in A_1 \cap A_2$. $a \in A_1$ and $a \in A_2$. $a \in A_1$ by specialization, so $F(a) \in F(A_1)$. $a \in A_2$ by specialization, so $F(a) \in F(A_2)$. $F(a) \in F(A_1)$ and $F(a) \in F(A_2)$ so $F(a) \in F(A_1) \cap F(A_2)$
Case 2: Let $y \in F(A_1) \cap F(A_2)$. $y \in F(A_1)$ and $y \in F(A_2)$. $F(a_1) = y$ for some $a_1 \in A_1$ and $F(a_2) = y$ for some $a_2 \in A_2$. $F(a_1) = y = F(a_2)$ so $a := a_1 = a_2$ since F is injective. Therefore $F(a) = y$ for some $a \in A_1$ and $a \in A_2$, i.e. there exists $a \in A_1 \cap A_2$ such that $F(a) = y$ and $y \in F(A_1 \cap A_2)$. □

4. (3 points) Section 7.2 #41.

Solution: #41. Let $F\colon X \to Y$ be surjective. For any $y \in Y$, there exists $x \in X$ such that $F(x) = y$.

Proof. Let $B \subset Y$.

$F^{-1}(B) = \{\, x \in X :\, F(x) \in B\,\}$

$F(F^{-1}(B)) = \{\, y \in Y : \exists\, x \in F^{-1}(B)\, (\, F(x) = y\,)\,\}$

Case 1: Let $b \in F(F^{-1}(B))$. There exists $x \in F^{-1}(B)$ such that $F(x) = b$. Since $x \in F^{-1}(B)$ this implies that $F(x) \in B$, i.e. $b \in B$.

Case 2: Let $b \in B$. Since F is surjective, there exists $x \in X$ such that $F(x) = b$. $F(x) \in B$ so $x \in F^{-1}(B)$. $F(x) = b$ for some $x \in F^{-1}(B)$. Thus $b \in F(F^{-1}(B))$. □

5. (15 points) Section 7.3 #8, 10.

Solution: #8.

(a) $T(4) = 1$.
(b) $T(5) = 2$.
(c) $T(3) = 0$.

Solution: #10.

(a) $(F \circ G)(8) = (G \circ F)(8) = 8$ and $(F \circ G)(3) = (G \circ F)(3) = 3$.

(b) No, $F \circ G \neq G \circ F$. Choose $x = 1/2$ such that

$$(G \circ F)(x) = (G \circ F)(1/2) = 1 \neq 0 = (F \circ G)(1/2) = (F \circ G)(x).$$

6. (6 points) Section 7.3 #15.

   Solution: #15. Let $b \in (0,1) \cup (1,\infty)$.

   (a) *Proof.* Let $x \in \mathbb{R}$. Since $\log_b \colon \mathbb{R}^+ \to \mathbb{R}$ is a bijection, $\log_b(b^x) = y$ for some $y \in \mathbb{R}$ if and only if $b^y = b^x$. Since the exponential function $\exp_b \colon \mathbb{R} \to \mathbb{R}^+$ is injective, $\log_b(b^x) = y = x$. $\square$

   (b) *Proof.* Let $x \in \mathbb{R}^+$. Since $\exp_b \colon \mathbb{R} \to \mathbb{R}^+$ is a bijection, $b^{\log_b(x)} = y$ for some $y \in \mathbb{R}$ if and only if $\log_b(y) = \log_b(x)$. Since the logarithmic function $\log_b \colon \mathbb{R}^+ \to \mathbb{R}$ is injective, $b^{\log_b(x)} = y = x$. $\square$

7. (3 points) Section 7.3 #20.

   Solution: #20. Let $f \colon X \to Y$ and $g \colon Y \to Z$ be any functions and $g \circ f$ be surjective. f is not surjective in general:

   Proof. Choose $X := Z := \{1\} \subseteq \{1,2\} =: Y$ and $f := \{(1,1)\}$ and $g := \{(1,1),(2,1)\}$. So $g \circ f = \{(1,1)\}$ is surjective, since $1 \in X$ such that $(g \circ f)(1) = 1$ and $(g \circ f)(X) = (g \circ f)(\{1\}) = \{1\} = Z$. Choose $2 \in Y$ such that, for any $x \in X$, $f(x) \neq 2$, since $X = \{1\}$ and $f(1) = 1 \neq 2$. So f is not surjective. $\square$

8. (6 points) Let f and g be functions such that

   (i) $f \colon X \to Y$ and $g \colon Y \to X$.

   (ii) $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

   Prove that

   (a) f is injective and surjective.

   (b) $f^{-1} = g$.

   Solution:

   Proof. Let f and g be functions such that

   (i) $f \colon X \to Y$ and $g \colon Y \to X$.

   (ii) $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

$$\begin{tikzcd} X \arrow[loop left, "\mathrm{id}_X"] \arrow[rr, bend left, "f"] & & Y \arrow[ll, bend left, "g"] \arrow[loop right, "\mathrm{id}_Y"] \end{tikzcd}$$

(a) Let $x_1, x_2 \in X$. Suppose $f(x_1) = f(x_2)$. Since $g\colon Y \to X$ is well-defined,

$$\begin{aligned} g(f(x_1)) &= g(f(x_2)) \\ (g \circ f)(x_1) &= (g \circ f)(x_2) \\ \mathrm{id}_X(x_1) &= \mathrm{id}_X(x_2) \\ x_1 &= x_2. \end{aligned}$$

So f is injective.
Let $y \in Y$. Since $g\colon Y \to X$ is well-defined, there exists $x \in X$ such that $g(y) = x$.

$$y = \mathrm{id}_Y(y) = (f \circ g)(y) = f(g(y)) = f(x).$$

So f is surjective.
Since f is injective and surjective, f is bijective, so $f^{-1}\colon Y \to X$ is a function.

(b) Let $y \in Y$. Since $f^{-1}\colon Y \to X$ is a function, $f^{-1}(y) = x$ for some $x \in X$ and

$$f^{-1}(y) = x \iff y = f(x).$$

So

$$f^{-1}(y) = x = \mathrm{id}_X(x) = (g \circ f)(x) = g(f(x)) = g(y).$$

Therefore $f^{-1} = g$.

$\square$