MATH-UA-240 — Combinatorics

Professor Charles Stine

Name: **Homework Assignment #9**
NetID: **Due Date:** December 8, 2024, 11:59 PM

- This homework should be submitted via Gradescope by 11:59 PM on the date listed above.
- There are two main ways you might want to write up your work.
  - Write a PDF using a tablet
  - Write your answers on paper, clearly numbering each question and part.
    - You can use an app such as OfficeLens to take pictures of your work with your phone and convert them into a single pdf file. Gradescope will only allow pdf files to be uploaded.
- You must show all work. You may receive zero or reduced points for insufficient work. Your work must be neatly organized and written. You may receive zero or reduced points for incoherent work.
- Please start a fresh page for each numbered problem. You may have parts a), b) and c) on the page for example, but problems 1) and 2) should be on separate pages.
- When uploading to Gradescope, **you must match each question to the page that your answer appears on**. If you do not, we will be unable to grade the unmatched problems.
- When appropriate, please put a box or circle around your final answer.
- The problems on this assignment will be graded on correctness and completeness.

Due Sunday night.

Lecture 18

1. (12 points) Writing down (but not solving) recurrence relations.

   (a) Write down a recurrence relation for the the number of binary sequences of length n with no sub-sequences of '111'.

   (b) Find a recurrence relation for the for the number of regions created by n mutually intersecting circles in generic position in the plane.

   (c) Everyday you give away either \$1, \$2, or \$3 to charity. Find a recurrence relation for how many different patterns of donations could you make in n days, if you want to give \$1 an even number of times.

   (d) Find a recurrence relation for the number of binary trees with n internal vertices, considered up to isomorphism of graphs.

   Recall that a binary tree is a graph with a unique vertex of degree 2 (the root) and every other vertex must have either degree 3 (internal vertex) or degree 1 (leaf).

2. (10 points) Homogeneous Linear Recurrence: find a closed formula for a_n.

   (a) $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ with $a_0 = a_1 = 1, a_2 = 2$.

   (b) Find the number of ways to arrange flags on an n-foot flagpole using three types of flags: red flags 2 feet high, yellow flags 1 foot high, and blue flags 1 foot high.

Lecture 19

3. (18 points) Non-homogeneous Recurrence: find and solve a recurrence relation for each of the following.

   (a) The number of ways to draw a square sub-board on an $n \times n$ chessboard. Note that the whole board is a sub-board of itself.

   (b) The number of different regions formed when n mutually intersecting planes are drawn in 3-dimensional space in generic position.

   (c) Find the general formula for a_n satisfying the recurrence relation:

   $$a_n = 4a_{n-1} - 4a_{n-2} + 2^n \quad \text{with} \quad a_0 = 0,\ a_1 = 2$$

   Hint: you will need to find and then expand the generating function for the sequence. It may help to recall that,

   $$\frac{1}{(1-x)^p} = \sum_{n=0}^{\infty} \binom{n+(p-1)}{(p-1)} x^n$$