Homework 9 Solutions

via Gradescope

- Failure to submit homework correctly will result in zeroes.
- Handwritten homework is OK. You do not have to type up your work.
- Problems assigned from the textbook are from the 5^{th} edition.
- No late homework accepted. Lateness due to technical issues will not be excused.

1. (30 points) Section 6.3 #2, 22, 28.

   Solution: #2. Let $A = \{1, 2\}$, $B = \{2, 3\}$ and let $U = \{1, 2, 3, 4\}$ be the universe. Then $A^c \cup B^c = \{1, 3, 4\}$ while $(A \cup B)^c = \{4\}$.

   #22(a). $\exists$ set S, $\forall$ sets T, $S \cap T \neq \varnothing$.

   #22(b). $\forall$ sets S, $\exists$ set T, $S \cup T \neq \varnothing$.

   #28.

   (a) Set difference laws.
   (b) Set difference laws.
   (c) Commutative law.
   (d) De Morgan's law.
   (e) Double complement law.
   (f) Distribution law.
   (g) Set difference law.

2. (9 points) Section 6.3 #33, 38, 43. Annotate.

   Solution: #33.

   Proof.

$$
\begin{aligned}
(A - B) \cap (A \cap B) &= (A \cap B^c) \cap (A \cap B) \ (Set\ difference\ laws) \\
&= (A \cap B^c) \cap Z \quad (Z := A \cap B) \\
&= A \cap (B^c \cap Z) \quad (Associative) \\
&= A \cap (Z \cap B^c) \quad (Commutative) \\
&= A \cap ((A \cap B) \cap B^c) \quad (Substitution) \\
&= A \cap (A \cap (B \cap B^c)) \quad (Associative) \\
&= A \cap (A \cap \varnothing) \quad (Complement\ law) \\
&= A \cap \varnothing \quad (Identity\ law) \\
&= \varnothing \quad (Identity\ law)
\end{aligned}
$$

$\square$

#38.

Proof.

$$\begin{aligned}(A \cap B)^c \cap A &= (A^c \cup B^c) \cap A \ (De\ Morgan's \quad law)\\ &= A \cap (A^c \cup B^c) \quad (Commutative\ law)\\ &= (A \cap A^c) \cup (A \cap B^c) \quad (Distributive\ law)\\ &= \varnothing \cup (A \cap B^c \quad (Complement\ law)\\ &= (A \cap B^c) \cup \varnothing \quad (Commutative\ law)\\ &= A \cap B^c \quad (Identity\ law)\\ &= A - B \quad (Set\ difference\ law)\end{aligned}$$

$\square$

#43.

$$\begin{aligned}&(A \cap (B \cup C)) \cap (A - B)) \cap (B \cup C^c)\\ &= (A \cap (B \cup C)) \cap (A \cap B^c)) \cap (B \cup C^c) \quad (Set\ difference\ laws)\\ &= (A \cap A \cap (B \cup C) \cap B^c) \cap (B \cup C^c) \quad (Commutative\ law)\\ &= (A \cap (B \cup C) \cap B^c) \cap (B \cup C^c) \quad (Idempotent\ law)\\ &= (A \cap [(B^c \cap B) \cup (B^c \cap C)]) \cap (B \cup C^c) \quad (Distribution\ law)\\ &= (A \cap [\varnothing \cup (B^c \cap C)]) \cap (B \cup C^c) \quad (Complement\ law)\\ &= (A \cap (B^c \cap C)) \cap (B \cup C^c) \quad (Identity\ law)\\ &= A \cap B^c \cap (C \cap (B \cup C^c)) \quad (Associative\ law)\\ &= A \cap B^c \cap ((C \cap B) \cup (C \cap C^c)) \quad (Distributive\ law)\\ &= A \cap B^c \cap ((C \cap B) \cup \varnothing) \quad (Complement\ law)\\ &= A \cap B^c \cap (C \cap B) \quad (Identity\ law)\\ &= A \cap B^c \cap (B \cap C) \quad (Commutative\ law)\\ &= A \cap (B^c \cap B) \cap C \quad (Associative\ law)\\ &= A \cap \varnothing \cap C \quad (Complement\ law)\\ &= \varnothing \quad (Identity\ law)\end{aligned}$$

3. (15 points) Section 6.3 #46, 52. Use Theorem 6.2.2. when doing Problem 52. Annotate as well.

Solution: #46.

(a) $\{1, 2, 5, 6\}$.

(b) $\{3, 4, 7, 8\}$.

(c) $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

(d) $\{1, 2, 7, 8\}$.

#52.

Definition. Given sets A and B, the symmetric difference of A and B, denoted $A \,\Delta\, B$, is

$$A \,\Delta\, B = (A - B) \cup (B - A).$$

Proof.

$$\begin{aligned}
&(A \,\Delta\, B) \,\Delta\, C \\
&= ((A - B) \cup (B - A)) \Delta\, C \\
&= ((A - B) \cup (B - A) - C) \cup (C - ((A - B) \cup (B - A))) \\
&= (((A \cap B^c) \cup (B \cap A^c)) \cap C^c) \cup (C \cap ((A \cap B^c) \cup (B \cap A^c))^c) \\
&= (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap ((A \cap B^c) \cup (B \cap A^c))^c) \\
&= (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap (A \cap B^c)^c \cap (B \cap A^c)^c) \\
&= (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap (A^c \cup B^{cc}) \cap (B^c \cup A^{cc})) \\
&= (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap (A^c \cup B) \cap (B^c \cup A)) \\
&= (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap (((A^c \cup B) \cap B^c) \cup ((A^c \cup B) \cap A))) \\
&= (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap ((A^c \cap B^c) \cup (B \cap B^c) \cup (A^c \cap A) \cup (B \cap A))) \\
&= (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap ((A^c \cap B^c) \cup \varnothing \cup \varnothing \cup (B \cap A))) \\
&= (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap ((A^c \cap B^c) \cup (B \cap A))) \\
&= (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap A^c \cap B^c) \cup (C \cap B \cap A) \\
&= (A \cap B \cap C) \cup (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \\
&= (A \cap B^c \cap C^c) \cup (A \cap C \cap B) \cup (B \cap C^c \cap A^c) \cup (C \cap B^c \cap A^c) \\
&= (A \cap ((B^c \cap C^c) \cup (C \cap B))) \cup (B \cap C^c \cap A^c) \cup (C \cap B^c \cap A^c) \\
&= (A \cap ((B^c \cap C^c) \cup \varnothing \cup \varnothing \cup (C \cap B))) \cup (B \cap C^c \cap A^c) \cup (C \cap B^c \cap A^c) \\
&= (A \cap ((B^c \cap C^c) \cup (C \cap C^c) \cup (B^c \cap B) \cup (C \cap B))) \cup (B \cap C^c \cap A^c) \cup (C \cap B^c \cap A^c) \\
&= (A \cap (((B^c \cup C) \cap C^c) \cup ((B^c \cup C) \cap B))) \cup (B \cap C^c \cap A^c) \cup (C \cap B^c \cap A^c) \\
&= (A \cap (B^c \cup C) \cap (C^c \cup B)) \cup (B \cap C^c \cap A^c) \cup (C \cap B^c \cap A^c) \\
&= (A \cap (B^c \cup C^{cc}) \cap (C^c \cup B^{cc})) \cup (B \cap C^c \cap A^c) \cup (C \cap B^c \cap A^c) \\
&= (A \cap (B \cap C^c)^c \cap (C \cap B^c)^c) \cup (B \cap C^c \cap A^c) \cup (C \cap B^c \cap A^c) \\
&= (A \cap (B \cap C^c)^c \cap (C \cap B^c)^c) \cup (((B \cap C^c) \cup (C \cap B^c)) \cap A^c) \\
&= (A \cap ((B \cap C^c) \cup (C \cap B^c))^c) \cup (((B \cap C^c) \cup (C \cap B^c)) \cap A^c) \\
&= (A - ((B - C) \cup (C - B))) \cup (((B - C) \cup (C - B)) - A) \\
&= A \,\Delta\, ((B - C) \cup (C - B)) \\
&= A \,\Delta\, (B \,\Delta\, C).
\end{aligned}$$

$\square$

4. (12 points) Section 7.1 #4, 14.

Solution: #4(a).

$$\begin{aligned} f_1 &= \{(a,u),(b,u)\} \\ f_2 &= \{(a,u),(b,v)\} \\ f_3 &= \{(a,v),(b,u)\} \\ f_4 &= \{(a,v),(b,v)\} \end{aligned}$$

#4(b). $f = \{(a,u),(b,u),(c,u)\}$.

#4(c).

$$\begin{aligned} f_1 &= \{(a,u),(b,u),(c,u)\} \\ f_2 &= \{(a,v),(b,v),(c,v)\} \\ f_3 &= \{(a,u),(b,u),(c,v)\} \\ f_4 &= \{(a,u),(b,v),(c,u)\} \\ f_5 &= \{(a,v),(b,u),(c,u)\} \\ f_6 &= \{(a,v),(b,v),(c,u)\} \\ f_7 &= \{(a,v),(b,u),(c,v)\} \\ f_8 &= \{(a,u),(b,v),(c,v)\} \end{aligned}$$

#14. No. $H(0) = 1$ while $K(0) = 0$.

5. (15 points) Section 7.1 #22, 25, 27.

Solution: #22.

Proof by Contradiction. Suppose $\log_3(7) \in \mathbb{Q}$. There exist $m, n \in \mathbb{Z}^+$ such that $1 = \log_3(3) < \log_3(7) = \frac{m}{n}$. Otherwise, provided $\log_3(7) = \frac{x}{y}$ for some $x, y \in \mathbb{Z}^-$, define $m := -x \in \mathbb{Z}^+$ and $n := -y \in \mathbb{Z}^+$ such that $\log_3(7) = \frac{x}{y} = \frac{-x}{-y} = \frac{m}{n}$.

$$\log_3(7) = \frac{m}{n} \iff 3^{m/n} = 7$$

so

$$1 = 3^0 < 3^m = 7^n.$$

$3 \neq 7$ are distinct primes, so 3^m and 7^n are distinct factorizations of the same integer, contradicting the uniqueness of integer factorizations. Thus $\log_3(7) \notin \mathbb{Q}$. $\square$

#25(a). $p_1(2, y) = 2$ and $p_2(5, x) = 5$. The range of p_1 is A.

#25(b). $p_1(2, y) = y$ and $p_2(5, x) = x$. The range of p_2 is B.

#27(a). $f(aba) = 0,\ f(bbab) = 2,\ f(b) = 0.$

#27(b). $g(aba) = aba,\ g(bbab) = babb,\ g(b) = b.$

6. (9 points) Section 7.1 $\#32(b), 42, 44, 45$.

   Let X and Y be sets, $A \subset X$, $B \subset X$, $C \subset Y$, $D \subset Y$, and $f\colon X \to Y$.

   Solution: #42. This is true. Let's prove it.

   Proof. The images are defined:

   $f(A) = \{\, y \in Y : \exists\, a \in A\,(\, f(a) = y\,)\,\}$

   $f(B) = \{\, y \in Y : \exists\, b \in B\,(\, f(b) = y\,)\,\}$

   $f(A \cap B) = \{\, y \in Y : \exists\, x \in A \cap B\,(\, y = f(x)\,)\,\}$

   Let $y \in f(A \cap B)$. There exists $x \in A \cap B$ such that $y = f(x)$. $x \in A \cap B$ so $x \in A$ and $x \in B$. $x \in A$ by specialization such that $y = f(x) \in f(A)$. $x \in B$ by specialization such that $y = f(x) \in f(B)$. Thus $y \in f(A) \cap f(B)$. □

   #44. This is false. Here's a counterexample.

   Disproof. Choose $X = \{1, 2\}$, $Y = A = \{1\}$, $B = \{2\}$, and $f = \{(1, 1), (2, 1)\}$. $A - B = \{1\} - \{2\} = \{1\} = A$ so $f(A - B) = f(A) = \{1\}$. Note that $f(B) = f(\{2\}) = \{1\}$ so $f(A - B) = \{1\} \not\subset \varnothing = \{1\} - \{1\} = f(A) - f(B)$. □

   #45. This is true. Let's prove it.

   Proof. The inverse images of subsets are defined:

   $f^{-1}(C) = \{\, x \in X : f(x) \in C\,\}$

   $f^{-1}(D) = \{\, x \in X : f(x) \in D\,\}$

   Suppose $C \subset D$. Let $x \in f^{-1}(C)$. Then $f(x) \in C$. Since $C \subset D$, $f(x) \in D$. Therefore $x \in f^{-1}(D)$. □