Homework 6 Solutions

via Gradescope

- Failure to submit homework correctly will result in zeroes.
- Handwritten homework is OK. You do not have to type up your work.
- Problems assigned from the textbook are from the 5^{th} edition.
- No late homework accepted. Lateness due to technical issues will not be excused.

1. (3 points) We can apply mathematical induction on $\mathbb{Z}^+ \times \mathbb{Z}^+$. Suppose that you want to prove

$$P(b,d) \equiv \top \text{ for all } (b,d) \in \mathbb{Z}^+ \times \mathbb{Z}^+.$$

The proof goes as follows:

Proof.
Basis Step: Show that $P(b,1) \equiv \top$ for all $b \geq 1$. Notice that your base case is an induction argument.

Inductive Step: Assume b and k are any positive integers such that

$$P(b,k) \equiv \top.$$

You need to show

$$P(b,k+1) \equiv \top.$$

With that said, use the induction method outlined above to prove

$$\sum_{a=1}^{b}\sum_{c=1}^{d}(a+c) = \frac{bd(b+d+2)}{2} \text{ for all } (b,d) \in \mathbb{Z}^+ \times \mathbb{Z}^+.$$

Solution:

Proof.

(1) *Basis Step:* $\sum_{a=1}^{b}\sum_{c=1}^{1}(a+c) = \sum_{a=1}^{b}(a+1)$. Now we must show that this equals $\frac{b(b+3)}{2}$ using induction.

(a) *Basis Step:* When $b = 1$, we have $\sum_{a=1}^{b}(a+1) = 2$ and $\frac{b(b+3)}{2} = 2$.

(b) *Inductive Step:* Let $k \geq 1$ be any integer. Suppose

$$\sum_{a=1}^{k}(a+1) = \frac{k(k+3)}{2}.$$

Let's add $k+2$ to both sides. Then we have

$$\begin{aligned}\sum_{a=1}^{k+1}(a+1) &= k(k+3)/2 + k + 2 \\ &= \frac{k(k+3)+2(k+2)}{2} \\ &= \frac{k^2+5k+4}{2} \\ &= \frac{(k+1)(k+4)}{2}\end{aligned}$$

(2) *Inductive Step:* Let $k \geq 1$ be any integer. Suppose

$$\sum_{a=1}^{b}\sum_{c=1}^{k}(a+c) = \frac{bk(b+k+2)}{2}$$

for all $b \geq 1$. Then

$$\begin{aligned}\sum_{a=1}^{b}\sum_{c=1}^{k+1}(a+c) &= \sum_{a=1}^{b}\sum_{c=1}^{k}(a+c) + \sum_{a=1}^{b}(a+k+1) \\ &= \frac{bk(b+k+2)}{2} + \sum_{a=1}^{b}(a+k+1) \\ &= \frac{bk(b+k+2)}{2} + \frac{b(b+1)}{2} + (k+1)b \\ &= \frac{bk(b+k+2)}{2} + \frac{b(b+1)}{2} + \frac{2(k+1)b}{2} \\ &= \frac{b\left(k(b+k+2)+b+1+2(k+1)\right)}{2} \\ &= \frac{b\left(kb+k^2+2k+b+1+2(k+1)\right)}{2} \\ &= \frac{b\left(kb+(k+1)^2+b+2(k+1)\right)}{2}\end{aligned}$$

$$= \frac{b\left((k+1)b+(k+1)^2+2(k+1)\right)}{2}$$
$$= \frac{b(k+1)\left(b+k+1+2\right)}{2}$$

□

2. (6 points) Section 5.2 #14, 18.

Solution: #14.

Proof.

Basis Step:

$$\sum_{i=1}^{0+1} i(2^i) = 1(2^1) = 2 = 0+2 = 0(2^{0+2})+2$$

Inductive Step:

Let $k \in \mathbb{Z}$ and $k \geq 0$.

Suppose

$$\sum_{i=1}^{k+1} i(2^i) = k(2^{k+2})+2.$$

Then

$$\begin{aligned}
\sum_{i=1}^{k+2} i(2^i) &= \sum_{i=1}^{k+1} i(2^i) + (k+2)(2^{k+2}) \\
&= k(2^{k+2})+2+(k+2)(2^{k+2}) && \text{(via induction hypothesis)} \\
&= (k+k+2)(2^{k+2})+2 \\
&= (2k+2)(2^{k+2})+2 \\
&= (k+1)(2)(2^{k+2})+2 \\
&= (k+1)(2^{k+1+2})+2.
\end{aligned}$$

□

#18.

Proof.

Basis Step:

$$\prod_{i=2}^{2}\left(1-\frac{1}{i}\right) = 1-\frac{1}{2} = \frac{1}{2}$$

Inductive Step:

Let $k \in \mathbb{Z}$ and $k \geq 2$.

Suppose

$$\prod_{i=2}^{k}\left(1-\frac{1}{i}\right)=\frac{1}{k}.$$

Then

$$\begin{aligned}
\prod_{i=2}^{k+1}\left(1-\frac{1}{i}\right) &= \left(1-\frac{1}{k+1}\right)\prod_{i=2}^{k}\left(1-\frac{1}{i}\right) \\
&= \left(\frac{k+1}{k+1}-\frac{1}{k+1}\right)\left(\frac{1}{k}\right) \qquad \text{(via induction hypothesis)} \\
&= \left(\frac{k+1-1}{k+1}\right)\left(\frac{1}{k}\right) \\
&= \left(\frac{k}{k+1}\right)\left(\frac{1}{k}\right) \\
&= \frac{1}{k+1}.
\end{aligned}$$

□

3. (3 points) Section 5.2 #40.

Solution: #40.

Proof. Let $n \in \mathbb{Z}$ and $p \geq 5$ be prime.

$$\begin{aligned}
\sum_{i=0}^{p-1}(n+i)^2 &= \sum_{i=0}^{p-1} n^2+2ni+i^2 \\
&= pn^2 + 2n\frac{(p-1)(p)}{2} + \frac{(p-1)(p)(2p-1)}{6} \\
&= p\left(n^2+n(p-1)+\frac{(p-1)(2p-1)}{6}\right)
\end{aligned}$$

Note that we've used the formulas

$$\begin{aligned}
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

We need to ensure that the term

$$\frac{(p-1)(2p-1)}{6} \in \mathbb{Z}.$$

$r = p \bmod 6$ is equal to $0, 1, 2, 3, 4$ or 5 via the Quotient-Remainder Theorem. There exists a unique $q \in \mathbb{Z}$ such that $p = 6q + r$.

Case 1: $p \bmod 6 \neq 0$ because, if $p = 6q + 0$, then $6 \mid p$ but $p \geq 5$ is prime.

Case 2: $p \bmod 6 \neq 2$ because, if $p = 6q + 2 = 2(3q + 1)$, then $2 \mid p$ but $p \geq 5$ is prime.

Case 3: $p \bmod 6 \neq 3$ because, if $p = 6q + 3 = 3(2q + 1)$, then $3 \mid p$ but $p \geq 5$ is prime.

Case 4: $p \bmod 6 \neq 4$ because, if $p = 6q + 4 = 2(3q + 2)$, then $2 \mid p$ but $p \geq 5$ is prime.

Case 5: Suppose $p \bmod 6 = 1$. $p = 6q + 1$ so $(p - 1)(2p - 1) = 6q(12q + 1)$ and $6 \mid (p - 1)(2p - 1)$. I.e. $l_1 := q(12q + 1) \in \mathbb{Z}$ such that $(p - 1)(2p - 1) = 6l_1$ and

$$\frac{(p-1)(2p-1)}{6} = l_1.$$

Case 6: Suppose $p \bmod 6 = 5$. $p = 6q + 5$ so

$$(p - 1)(2p - 1) = (6q + 4)(12q + 9) = 6(3q + 2)(4q + 3)$$

and $6 \mid (p - 1)(2p - 1)$. I.e. $l_2 := (3q + 2)(4q + 3) \in \mathbb{Z}$ such that $(p - 1)(2p - 1) = 6l_2$ and

$$\frac{(p-1)(2p-1)}{6} = l_2.$$

Thus

$$\frac{(p-1)(2p-1)}{6} \in \mathbb{Z}.$$

Therefore $p \mid \sum_{i=0}^{p-1} (n + i)^2$. $\square$

4. (12 points) Section 5.3 #3, 12, 21.

Solution: #3. We will leave part (a) to you. Here's part (b).

Proof.
Basis Step:

$$28 = 5(4) + 8(1),$$

i.e. obtain 28 stamps with 4 packets of 5 and 1 packet of 8.

Inductive Step:

Let $k \in \mathbb{Z}$ and $k \geq 28$.

Suppose there exist $x, y \in \mathbb{Z}^+ \cup \{0\}$ such that $k = 5x + 8y$. I.e. suppose we may obtain k stamps with x packets of 5 stamps and y packets of 8 stamps, for some non-negative integers x, y.

$y > 2$ or $y \leq 2$.

Case 1: Suppose $y \leq 2$. $y \in \mathbb{Z}^+ \cup \{0\}$ so $y = 0, 1, 2$. $x \geq 3$ since $k \geq 28$. Define $a := x - 3 \in \mathbb{Z}^+ \cup \{0\}$ and $b := y + 2 \in \mathbb{Z}^+ \cup \{0\}$. I.e. replace 3 packets of 5 with 2 packets of 8.

$$\begin{aligned} 5a + 8b &= 5(x-3) + 8(y+2) \\ &= 5x - 15 + 8y + 16 \\ &= 5x + 8y + 1 \\ &= k + 1 \qquad \text{(via induction hypothesis)} \end{aligned}$$

Case 2: Suppose $y > 2$. Since $y \in \mathbb{Z}^+ \cup \{0\}$, $y \geq 3$. Define $b := y - 3 \in \mathbb{Z}^+ \cup \{0\}$. Since $k \geq 28$, $x \geq 1$. Define $a := x + 5 \in \mathbb{Z}^+$. I.e. replace 3 packets of 8 with 5 packets of 5.

$$\begin{aligned} 5a + 8b &= 5(x+5) + 8(y-3) \\ &= 5x + 25 + 8y - 24 \\ &= 5x + 8y + 1 \\ &= k + 1 \qquad \text{(via induction hypothesis)} \end{aligned}$$

So, if we may obtain k stamps in packets of 5 or 8 stamps, then we may obtain $k + 1$ stamps, for any integer $k \geq 28$. □

#12.

Proof.

Basis Step: $0 \in \mathbb{Z}$ such that $7^0 - 2^0 = 1 - 1 = 0 = 5(0)$, so $5 \mid (7^0 - 2^0)$.

Inductive Step: Let $k \geq 0$ be any integer. Suppose $5 \mid (7^k - 2^k)$. So we have

$$7^k - 2^k = 5m$$

for some $m \in \mathbb{Z}$ and

$$\begin{aligned} 7(7^k) - 7(2^k) &= 7(5m) \\ 7^{k+1} &= 5(7m) + 7(2^k) \end{aligned}$$

Then

$$\begin{aligned} 7^{k+1} - 2^{k+1} &= 5(7m) + 7(2^k) - 2(2^k) \qquad \text{(via induction hypothesis)} \\ &= 5(7m) + (7-2)2^k = 5(7m) + 5(2^k) \\ &= 5(7m + 2^k) \end{aligned}$$

and $7m + 2^k \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under products and sums.
Therefore $5 \mid (7^{k+1} - 2^{k+1})$

□

#21.

Proof.

Basis Step:

$$\begin{aligned}
1 &< 2 \\
\sqrt{1} &< \sqrt{2} && \text{(since } 0<x<y \Rightarrow \sqrt{x}<\sqrt{y}\text{)} \\
1 &< \sqrt{2} \\
2 &< \sqrt{2}+1 \\
\sqrt{2}\sqrt{2} &< \sqrt{2}+1 \\
\sqrt{2} &< \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = \sum_{m=1}^{2} \frac{1}{\sqrt{m}}
\end{aligned}$$

Inductive Step: Let $k \in \mathbb{Z}$ and $k \geq 2$. Suppose

$$\sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} = \sum_{m=1}^{k} \frac{1}{\sqrt{m}}.$$

Then

$$\begin{aligned}
0 < 2 \leq k& \\
k^2 &< k^2 + k = k(k+1) \\
k &< \sqrt{k}\sqrt{k+1} \\
k+1 &< \sqrt{k}\sqrt{k+1}+1 \\
\frac{k+1}{\sqrt{k+1}} &< \sqrt{k} + \frac{1}{\sqrt{k+1}} \\
\sqrt{k+1} &< \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} && \text{(via induction hypothesis)} \\
\sqrt{k+1} &< \sum_{m=1}^{k+1} \frac{1}{\sqrt{m}}.
\end{aligned}$$

□

5. (6 points) Section 5.3 #27, 28.

Remark. There is a typo in problem #28. The denominator should be

$$(2n+1)+(2n+3)+\cdots+(2n+2n-1).$$

Solution: #27.

Proof.

Basis Step:

$$d_1 = 2 = \frac{2}{1} = \frac{2}{1!}$$

Inductive Step:
Let $k \in \mathbb{Z}$ and $k \geq 1$. Suppose

$$d_k = \frac{2}{k!}.$$

Then

$$\begin{aligned} d_{k+1} &= \frac{d_k}{k+1} = \frac{1}{k+1}(d_k) \\ &= \frac{1}{k+1}\left(\frac{2}{k!}\right) \qquad \text{(via induction hypothesis)} \\ &= \frac{2}{(k+1)k!} = \frac{2}{(k+1)!} \end{aligned}$$

□

#28.

Proof.

Basis Step:

$$\frac{1}{3} = \frac{1}{2(1)+1}$$

Inductive Step: Let $k \geq 1$ be an integer. Suppose

$$\frac{1}{3} = \frac{1+3+5+\cdots+(2k-1)}{(2k+1)+(2k+3)+\cdots+(2k+(2k-1))}.$$

From this assumption,

$$\begin{aligned}
\frac{1}{3} &= \frac{\sum_{i=1}^{k}(2i-1)}{\sum_{i=1}^{k}(2k+(2i-1))} \\
\sum_{i=1}^{k}(2k+(2i-1)) &= 3\sum_{i=1}^{k}(2i-1) \\
k(2k)+\sum_{i=1}^{k}(2i-1) &= 3\sum_{i=1}^{k}(2i-1) \\
2k^2 &= 2\sum_{i=1}^{k}(2i-1) \\
k^2 &= \sum_{i=1}^{k}(2i-1) \qquad \text{(via induction hypothesis)}.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1+3+5+\cdots+(2k-1)+(2k+1)}{(2k+3)+(2k+5)+\cdots+(2(k+1)+(2(k+1)-1))} \\
&= \frac{\sum_{i=1}^{k}(2i-1)+(2k+1)}{\sum_{i=1}^{k+1}(2k+(2i+1))} = \frac{k^2+(2k+1)}{2k(k+1)-1+1+\sum_{i=2}^{k+2}(2i-1)} \\
&= \frac{k^2+2k+1}{2k^2+2k-1+(2k+1)+(2k+3)+\sum_{i=1}^{k}(2i-1)} \\
&= \frac{k^2+2k+1}{2k^2+6k+3+k^2} = \frac{k^2+2k+1}{3k^2+6k+3} = \frac{(k+1)^2}{3(k+1)^2} = \frac{1}{3}.
\end{aligned}$$

$\square$

6. (9 points) Section 5.3 #33, 45, 46.

Solution:

#33. We will leave the drawing to you. Otherwise please visit your instructor's scheduled office hours for a solution.

#45. Tricky one indeed! We know $k \geq 1$ so that $k+1 \geq 2$. The constructions of sets B and C imply that $k+1 \geq 3$. For us to legally construct sets B and C we would need $k \geq 2$, hence a second base case. Of course a second base case is not possible.

#46. Nothing is wrong with the induction step. However there is no base case. In fact, the base case is not possible. Hence, one could never iterate the induction step over and over to conclude $3^n - 2$ is even.

7. (3 points) Section 5.3 #36.

   Solution:

   #36. There hint is a rather good hint. In the inductions step, the team we remove (denoted by T') can either win right away, win somewhere in the middle, or at the end. I hope that makes sense. Ok, now on to the proof.

   Proof.

   Base Case: Can you write the base case for two teams?

   Induction Step: Assume $k \geq 2$ is any integer such that $P(k)$ is true. Now consider $k+1$ teams. Let's remove a team. We know we can label the k remaining teams by $T_1, T_2, \ldots T_k$ such that T_i defeats T_{i+1} for $i = 1, 2, 3, \ldots k-1$.

   Case 1: The team we removed, T', defeated T_1. Then we can label the T' by T_1 and the remaining k teams by $T_2, \ldots T_{k+1}$.

   Case 2: The team we removed, T', lost to all teams. Then we can label T' by T_{k+1}.

   Case 3: The team we removed, T', losts to teams $T_1, T_2, \ldots T_m$ and beats T_{m+1} where $1 \leq m \leq k-1$. Then label T' by T_{m+1} and the remaining teams by $T_{m+2}, T_{m+3}, \ldots T_{k+1}$.

   □