Homework 2 Solutions

via Gradescope

- Failure to submit homework correctly will result in zeroes.
- Handwritten homework is OK. You do not have to type up your work.
- Problems assigned from the textbook are from the 5^{th} edition.
- No late homework accepted. Lateness due to technical issues will not be excused.

1. (9 points) Section 3.3 $\#43, 44, 45$.

   Solution: #43.

   $$\exists\varepsilon > 0, \forall\delta > 0, \exists x \in \mathbb{R}, (a-\delta < x < a+\delta) \wedge (x \neq a) \wedge ((L-\varepsilon \geq f(x)) \vee (f(x) \geq L+\varepsilon))$$
   $$\equiv \exists\varepsilon \in \mathbb{R}^{+}\forall\delta \in \mathbb{R}^{+}\exists x \in \mathbb{R}\,((0 < |x-a| < \delta) \wedge (|f(x)-L| \geq \varepsilon))$$

   Remark. We used the purple parenthesis simply because I thought it would be easier to read.

   #44.

   (a) This is true.

   Proof. Choose $x = 1 \in \mathbb{R}$. Let $y \in \mathbb{R}$. $xy = 1y = y = y1 = yx$.
   Suppose there exists $z \in \mathbb{R}$ such that, for any $y \in \mathbb{R}$, $zy = yz = y$. Then

   $$z = z1 = 1.$$

   So $x = 1 \in \mathbb{R}$ is the unique real number such that, for any $y \in \mathbb{R}$, $1y = y1 = y$. □

   (b) This is false.

   Disproof. ß Choose $1, -1 \in \mathbb{Z}$ such that $\frac{1}{1} = 1 \in \mathbb{Z}$ and $\frac{1}{-1} = -1 \in \mathbb{Z}$ but $1 \neq -1$. □

   (c) This is true.

   Proof. Let $x \in \mathbb{R}$. Choose $y := -x \in \mathbb{R}$ such that

   $$x + y = y + x = 0.$$

Suppose there exists $z \in \mathbb{R}$ such that

$$x + z = z + x = 0.$$

Then

$$-x = -x + 0 = -x + x + z = 0 + z = z.$$

So, for any $x \in \mathbb{R}$, $-x \in \mathbb{R}$ is the unique real number such that

$$x + (-x) = (-x) + x = 0.$$

$\square$

#45. There is one and only one value in D for which P is true.

2. (6 points) Section 3.3 #56, 57.

Solution: #56. These are not logically equivalent.

Consider the following counterexample:

Choose $D = \{2, 9\}$ and

$P(x) : x$ is prime

$Q(x) : x$ is odd.

$$\begin{aligned}
\exists x \in D(P(x) \wedge Q(x)) &\equiv (P(2) \wedge Q(2)) \vee (P(9) \wedge Q(9)) \\
&\equiv (\top \wedge \bot) \vee (\bot \wedge \top) \\
&\equiv \bot \vee \bot \equiv \bot \\
&\not\equiv \top \equiv \top \wedge \top \\
&\equiv (\top \vee \bot) \wedge (\bot \vee \top) \\
&\equiv (P(2) \vee P(9)) \wedge (Q(2) \vee Q(9)) \\
&\equiv \exists x \in D(P(x)) \wedge \exists x \in D(Q(x)).
\end{aligned}$$

#57. These are not logically equivalent, according to the same counterexample in #56:

$$\begin{aligned}
\forall x \in D(P(x) \vee Q(x)) &\equiv (P(2) \vee Q(2)) \wedge (P(9) \vee Q(9)) \\
&\equiv (\top \vee \bot) \wedge (\bot \vee \top) \\
&\equiv \top \wedge \top \equiv \top \\
&\not\equiv \bot \equiv \bot \vee \bot \\
&\equiv (\top \wedge \bot) \vee (\bot \wedge \top) \\
&\equiv (P(2) \wedge P(9)) \vee (Q(2) \wedge Q(9)) \\
&\equiv \forall x \in D(P(x)) \vee \forall x \in D(Q(x)).
\end{aligned}$$

3. (9 points) Section 3.4 #30.

   Solution: #30.

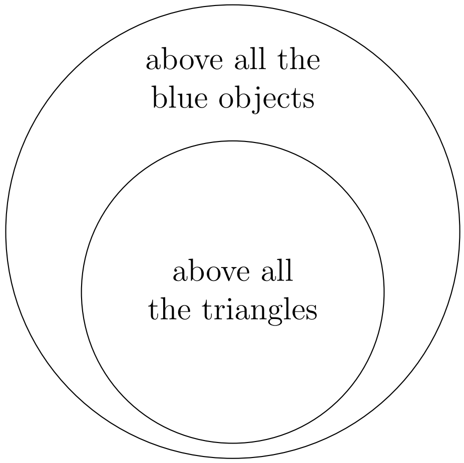
   1. If an object is above all the triangles, then it is above all the blue objects.
   2. If an object is not above all the gray objects, then it is not a square.
   3. Every black object is a square.
   4. Every object that is above all the gray objects is above all the triangles.

   $\therefore$ If an object is black, then it is above all the blue objects.

   1. $\forall$ object x, if x is above all the triangles, then x is above all the blue objects.
   2. $\forall$ object x, if x is a square, then x is above all the gray objects.
   3. $\forall$ object x, if x is black, then x is a square.
   4. $\forall$ object x, if x is above all the gray objects, then x is above all the triangles.

   $\therefore$ $\forall$ object x, if x is black, then x is above all the blue objects.

1.



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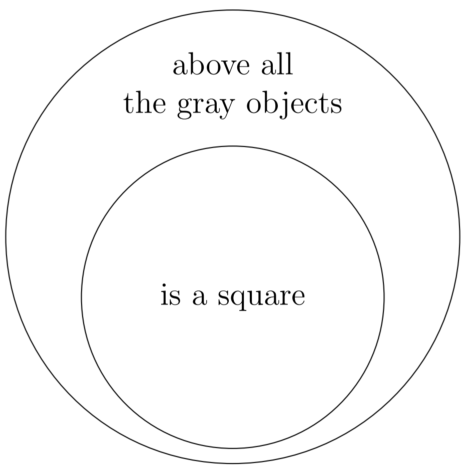

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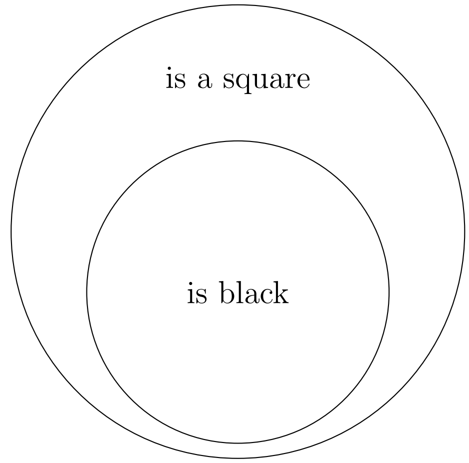

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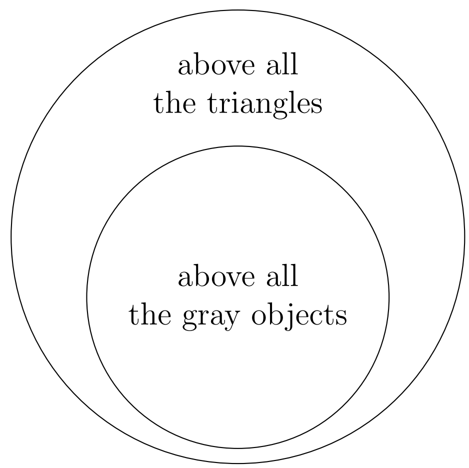


3. $\forall$ object x, if x is black, then x is a square.

2. $\forall$ object x, if x is a square, then x is above all the gray objects.

4. $\forall$ object x, if x is above all the gray objects, then x is above all the triangles.

1. $\forall$ object x, if x is above all the triangles, then x is above all the blue objects.

$\therefore$ $\forall$ object x, if x is black, then x is above all the blue objects.

4. (18 points) Section 3.4 #32, 34.

Solution: #32.

1. When I work a logic example without grumbling, you may be sure it is one I understand.

2. The arguments in these examples are not arranged in regular order like the ones I am used to.

3. No easy examples make my head ache.

4. I can't understand examples if the arguments are not arranged in regular order like the ones I am used to.

5. I never grumble at an example unless it gives me a headache.

$\therefore$ These examples are not easy.

1. If I don't grumble then I can understand. So if I can't understand a logic example then I grumble at it.

2. The arguments in these examples are not arranged in regular order like the ones I am used to.

3. No easy examples make my head ache. If it is an easy example then it does not make my head ache. So if the example makes my head ache then it is not an easy example.

4. If the arguments are not arranged in regular order like the ones I am used to then I can't understand it.

5. I never grumble at an example unless it gives me a headache. I never grumble at an example if it does not give me a headache. I grumble at an example only if it gives me a headache. So if I grumble at an example it gives me a headache.

$\therefore$ These examples are not easy.

1. If I can't understand a logic example then I grumble at it.

2. The arguments in these examples are not arranged in regular order like the ones I am used to.

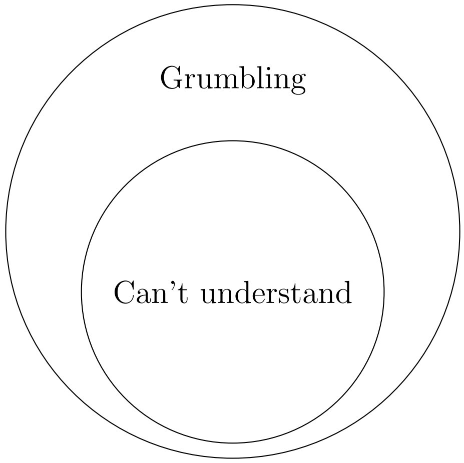
3. If the example makes my head ache then it is not an easy example.

4. If the arguments are not arranged in regular order like the ones I am used to then I can't understand it.

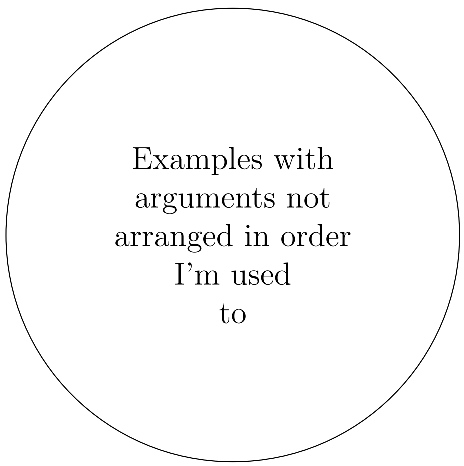
5. If I grumble at a logic example then it gives me a headache.

$\therefore$ These examples are not easy.

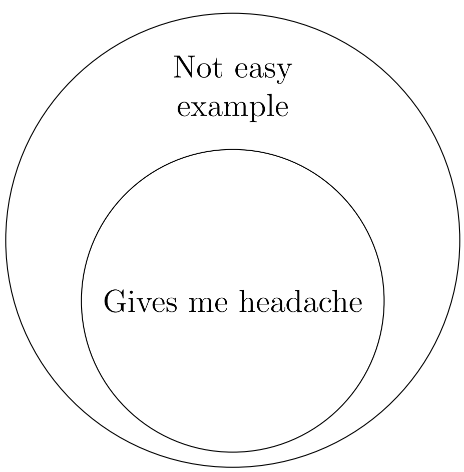
1.


Grumbling
Can’t understand

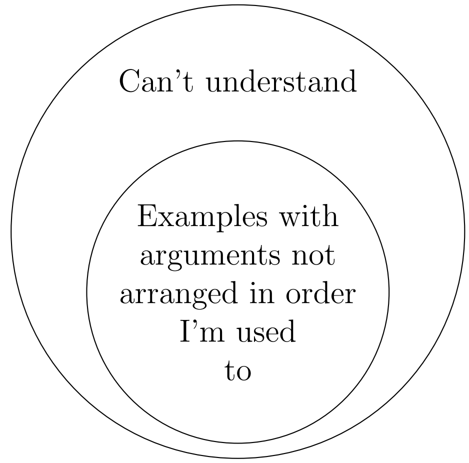
2.


Examples with arguments not arranged in order I’m used to

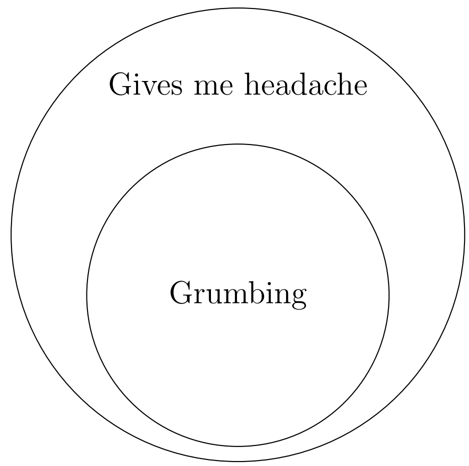
3.


Not easy example
Gives me headache

4.


Can’t understand
Examples with arguments not arranged in order I’m used to

5.


Gives me headache
Grumbing

2. The arguments in these examples are not arranged in regular order like the ones I am used to.

4. If the arguments are not arranged in regular order like the ones I am used to then I can't understand it.

1. If I can't understand a logic example then I grumble at it.

5. If I grumble at a logic example then it gives me a headache.

3. If the example makes my head ache then it is not an easy example.

$\therefore$ These examples are not easy.

For Problem #34, a single conclusion follows when all the given premises are taken into consideration, but it is difficult to see because the premises are jumbled up. reorder the premises to make it clear that a conclusion follows logically, and state the valid conclusion that can be drawn (It may be helpful to rewrite some of the statements in if-then form and to replace some statements by their contrapositives.)

#34.

1. All writers who understand human nature are clever.
2. No one is a true poet unless he can stir the human heart.
3. Shakespeare wrote Hamlet.
4. No writer who does not understand human nature can stir the human heart.
5. None but a true poet could have written Hamlet.

1. All writers who understand human nature are clever. $\forall x$, if x understands human nature then x is clever.

2. No one is a true poet unless he can stir the human heart. $\forall x$, if x can not stir the human heart then x is not a true poet. $\forall x$, if x is a true poet then x can stir the human heart.

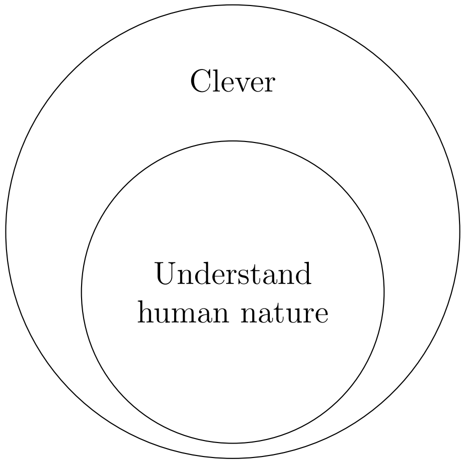
3. Shakespeare wrote Hamlet.

4. No writer who does not understand human nature can stir the human heart. Only writers who understand human nature can stir the human heart. A write can stir human heart only if he understands human nature. $\forall x$, if x can stir the human heart then x understands human nature.

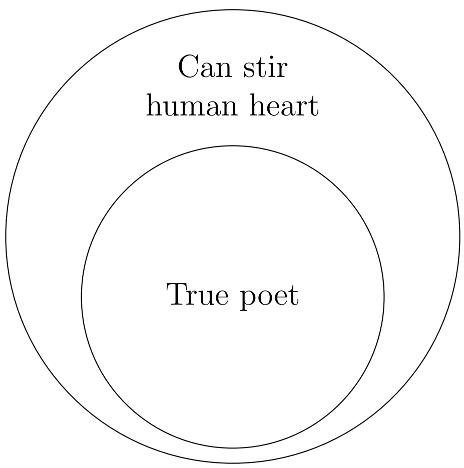
5. None but a true poet could have written Hamlet. $\forall x$, if x can write Hamlet then x is a true poet.

1. $\forall x$, if x understands human nature then x is clever.
2. $\forall x$, if x is a true poet then x can stir the human heart.
3. Shakespeare wrote Hamlet.
4. $\forall x$, if x can stir the human heart then x understands human nature.
5. $\forall x$, if x can write Hamlet then x is a true poet.

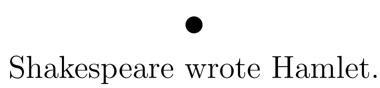
1.



2.



3.



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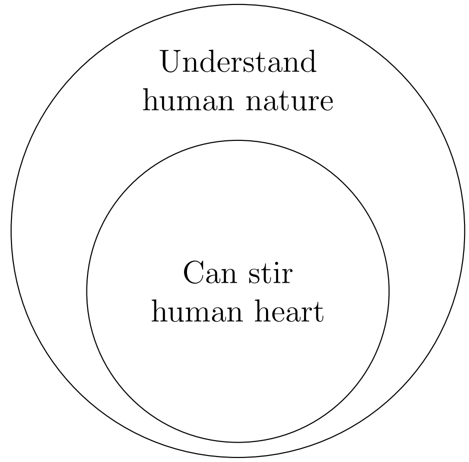
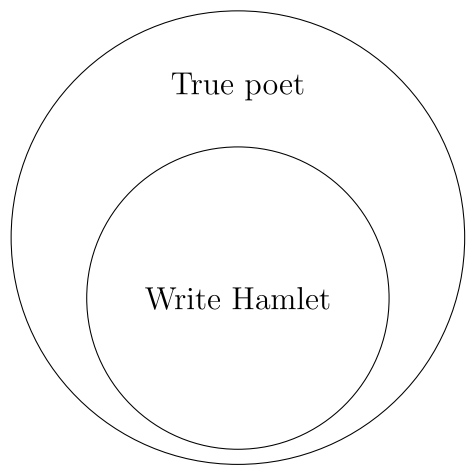


5.



3. Shakespeare wrote Hamlet.

5. $\forall x$, if x can write Hamlet then x is a true poet.

2. $\forall x$, if x is a true poet then x can stir the human heart.

4. $\forall x$, if x can stir the human heart then x understands human nature.

1. $\forall x$, if x understands human nature then x is clever.

$\therefore$ Shakespeare is clever.

5. (9 points) Section 4.1 $\#7, 11, 16$.

   Solution: #7.

   Proof. Choose $a = b = 0 \in \mathbb{R}$ such that $\sqrt{0+0} = \sqrt{0} = 0 = 0 + 0 = \sqrt{0} + \sqrt{0}$. $\square$

   #11.

   Proof. Choose $n = 0 \in \mathbb{Z}$ such that $2n^2 - 5n + 2 = 2$ and 2 is prime. $\square$

   #16.

   Disproof. Choose $n = 8 \in 2\mathbb{Z}$ such that $n^2 + 1 = 8^2 + 1 = 64 + 1 = 65$ and $65 = 5(13)$ is not prime. $\square$

6. (15 points) Section 4.1 $\#22, 29$.

   Solution: #22.

   Proof. Let $p(n) = n^2 - n + 11$.
   $p(1) = 11$ and 11 is prime.
   $p(2) = 13$ and 13 is prime.
   $p(3) = 17$ and 17 is prime.
   $p(4) = 23$ and 23 is prime.
   $p(5) = 31$ and 31 is prime.
   $p(6) = 41$ and 41 is prime.
   $p(7) = 53$ and 53 is prime.
   $p(8) = 67$ and 67 is prime.
   $p(9) = 83$ and 83 is prime.
   $p(10) = 101$ and 101 is prime. $\square$

   # 29.

   (a) By definition of an even integer.
   (b) Substitution.
   (c) The integers are closed under addition and multiplication.
   (d) by definition of an even integer.

7. (9 points) Section 4.2 $\#8, 9, 14$.

   Solution: #8.

   Proof. Let m be any even integer and let n be any odd integer. By the definition of even, $m = 2k$ for some integer k. By the definition of odd, $n = 2l + 1$ for some integer l.

$$\begin{aligned} 5m + 3n &= 5(2k) + 3(2l + 1) \quad \text{(Substition)} \\ &= 10k + 6l + 3 \quad \text{(Distribution)} \\ &= 10k + 6l + 2 + 1 \\ &= 2(5k + 3l + 1) + 1 \quad \text{(Factor)} \end{aligned}$$

   Set $t = 5k + 3l + 1$. Then t is an integer since integers are closed under addition and multiplication. Hence $5m + 2n = 2t + 1$, and so it is odd by the definition of an odd integer.h □

   Remark. Our proofs will not be identical. But we should all have something roughly the same. I get it, the annotation is a hard to figure out. For now, do the best you can and try to follow the spirit of Section 4.1.

   #9.

   Proof. Let $n \in \mathbb{Z}$ such that $n > 4$ and n is a perfect square. Since n is a perfect square, there exists $k \in \mathbb{Z}$ such that $n = k^2$.

$$n - 1 = k^2 - 1 = (k - 1)(k + 1)$$

   Since $n > 4$,

$$\begin{aligned} k^2 - 1 &> 3 \\ k^2 - 4 &> 0 \\ (k - 2)(k + 2) &> 0 \end{aligned}$$

   and $k > 2$, since we do not consider integers less than or equal to 1 prime or composite. So

$$n - 1 = (k - 1)(k + 1) > 1(k + 1) = k + 1 > k - 1 > 1.$$

   Therefore $n - 1$ is composite. □

#14.

Proof. Let $k \in \mathbb{Z}$ and $k \geq 4$. Then $2k^2 - 5k + 2 = (2k-1)(k-2)$ where $2k - 1 \geq 7$ and $k - 2 \geq 2$ since $k \geq 4$. So

$$2k^2 - 5k + 2 = (2k-1)(k-2) \geq (2k-1)(2) > 2k - 1 \geq 7 > 1$$

and

$$2k^2 - 5k + 2 = (2k-1)(k-2) \geq 7(k-2) > k - 2 \geq 2 > 1.$$

Hence $2k^2 - 5k + 2$ is composite. □

8. (6 points) Section 4.2 #18, 19.

   Solution: #18. There is an incorrect jump to conclusion in the line $mn = 2p(2q+1) = 2r$.

   #19. Cannot use the same integer k for both m and n.

9. (9 points) Section 4.2 #26, 30, 31.

   Solution: #26. This is false.

Disproof. Choose $a = 0 \in \mathbb{Z}$ such that $b := a + 1 = 1 \in \mathbb{Z}$ and $c := a + 2 = 2 \in \mathbb{Z}$, so $a, b, c \in \mathbb{Z}$ are consecutive, but $a + b + c = 3a + 3 = 3(a+1) = 3(0+1) = 3 \notin 2\mathbb{Z}$. □

#30. This is false.

Disproof. Choose $m = 3 \in \mathbb{Z}$ such that $m = 3 > 2$ but $m^2 - 4 = 9 - 4 = 5$ is not composite, i.e. $m^2 - 4 = 5$ is prime. □

#31. This is false.

Disproof. Choose $n = 11 \in \mathbb{Z}$ such that $n^2 - n + 11 = 11^2 - 11 + 11 = 11^2 = 121$ is not prime. □

Remark. Based on #22. from Section 4.1, I simply started looking for counterexamples larger than 10.

10. (9 points) Section 4.3 #14, 18.

Solution: #14.

(a) $\forall x \in \mathbb{R}$, if $x \in \mathbb{Q}$, then $x^3 \in \mathbb{Q}$.

(b) *Proof.* Let $q \in \mathbb{Q}$. There exist $m \in \mathbb{Z}$ and $n \in \mathbb{Z} - \{0\}$ such that $q = \frac{m}{n}$.

$$q^3 = \left(\frac{m}{n}\right)^3 = \frac{m^3}{n^3}$$

$m^3, n^3 \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under products. $n \neq 0$ since $n \in \mathbb{Z} - \{0\}$ so $n^3 \neq 0$ via Zero Product Property. Therefore $q^3 \in \mathbb{Q}$. $\square$

#18.

Proof. Let $r, s \in \mathbb{Q}$. There exist $m_1, m_2 \in \mathbb{Z}$ and $n_1, n_2 \in \mathbb{Z} - \{0\}$ such that $r = \frac{m_1}{n_1}$ and $s = \frac{m_2}{n_2}$.

$$\frac{r+s}{2} = \frac{1}{2}(r+s) = \frac{1}{2}\left(\frac{m_1}{n_1} + \frac{m_2}{n_2}\right) = \frac{1}{2}\left(\frac{m_1 n_2 + m_2 n_1}{n_1 n_2}\right) = \frac{m_1 n_2 + m_2 n_1}{2 n_1 n_2}$$

$m_1 n_2 + m_2 n_1 \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under products and sums. $2n_1 n_2 \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under products. $n_1 \neq 0$, $n_2 \neq 0$, since $n_1, n_2 \in \mathbb{Z} - \{0\}$, and $2 \neq 0$, so $2n_1 n_2 \neq 0$ via Zero Product Property. Therefore $\frac{r+s}{2} \in \mathbb{Q}$. $\square$