MATH-UA-240 / UY-4314 — Combinatorics

Professor Charles Stine

Name: **Homework Assignment #3**
NetID: **Due Date:** September 23, 2024, 11:59 PM

- This homework should be submitted via Gradescope by 11:59 PM on the date listed above.
- There are two main ways you might want to write up your work.
  - Write a PDF using a tablet
  - Write your answers on paper, clearly numbering each question and part.
    - You can use an app such as OfficeLens to take pictures of your work with your phone and convert them into a single pdf file. Gradescope will only allow pdf files to be uploaded.
- You must show all work. You may receive zero or reduced points for insufficient work. Your work must be neatly organized and written. You may receive zero or reduced points for incoherent work.
- Please start a fresh page for each numbered problem. You may have parts a), b) and c) on the page for example, but problems 1) and 2) should be on separate pages.
- When uploading to Gradescope, **you must match each question to the page that your answer appears on**. If you do not, we will be unable to grade the unmatched problems.
- When appropriate, please put a box or circle around your final answer.
- The problems on this assignment will be graded on correctness and completeness.

Lecture 5

1. (10 points) Chromatic Numbers and bounds.

   (a) A *tree* is a connected graph which does not contain any circuits. Draw some small trees yourself, then try to work out what their chromatic numbers are. State a conjecture about how to find the chromatic number of any tree, and prove your conjecture is correct.

   (b) Let G be a graph and assume that its complement G^c is bipartite. Show that the size of the largest, complete graph inside of G is equal to the chromatic number of G.

   (c) Give an example of a graph H with the property that its chromatic number is strictly larger than the size of the largest, complete subgraph it contains. Could the chromatic number of H^c ever be 2?

   Hint: try looking for a planar example. The hard part will be showing that it doesn't have a complete subgraph of size equal to H's chromatic number, but you know something about planar diagrams of complete graphs from Homework 1, Problem 4, Part (b).

2. (15 points) *The four color theorem says that every planar graph can be colored with four colors. It was first posed as a conjecture by Francis Guthrie in 1853 (exactly 170 years ago!). It was proved by Kenneth Appell and Wolfgang Haken in 1977 with the help of an early computer which found 4-colorings for 1936 planar graphs! This problem will give you a taste of how the computer found colorings for some of these graphs using Kempe's great idea, shown in class.* Consider the 4-coloring of the planar graph P below:

   (a) Find the mistake in the coloring which has been given above. Fix it by changing the color of only one vertex. Then use that coloring for the remaining questions.

   (b) Create a new graph Q by drawing a vertex next to the graph P, then connect that vertex to each of the four outermost vertices of P. Draw this graph so that none of its edges are crossing.

   The graph Q is planar, so the four color theorem says there must be a way to color it with 4-colors, but we can't color the new vertex right now because it is already adjacent to all of the colors.

   We need to find a way to reorganize the coloring we have of P to get one that allows us to color the additional vertex in Q.

   (c) Recall that a <u>Kempe chain</u> in a colored graph is a path whose vertices alternate between two fixed colors.

   Is there a Red-Green Kempe chain connecting the uppermost vertex of P (not Q!) to the lowermost?

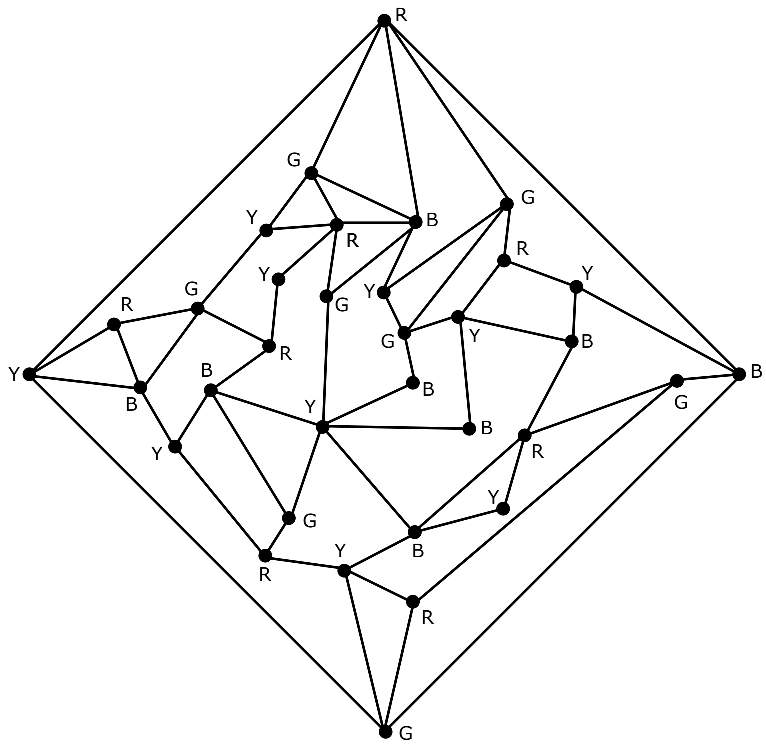


Figure 1: the graph P (above) colored with Red, Green, Blue, and Yellow.

Is there a Blue-Yellow Kempe chain connecting the rightmost vertex of P to the leftmost?

(d) Explain why there could never be both Kempe chains in any coloring of this graph. You should assume the four outermost vertices are the same colors they are now, but the inner vertices could have any legal colorings.

(Hint: remember that every vertex can only be assigned one color in a coloring.)

(e) It follows that at least one of the Red-Green outermost pair of vertices or the Blue-Yellow outermost pair are NOT connected by a Kempe chain. Which pair is it in your coloring of P?

Whichever it is, draw the sub-graph consisting of the vertices with only those colors and the edges which connect them.

(f) Are the two outermost vertices in the same connected component of this two-color sub-graph?

(g) Finally, pick one of the outermost vertices and swap all the colors of the component of the sub-graph from (f) that contains it.

Re-draw the graph Q with this modified coloring on the P. What color can you give to the last vertex of Q?

Lecture 6

3. (10 points) Fundamentals of Trees:

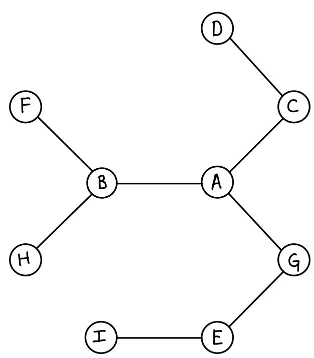
   (a) Show that if a connected graph has fewer edges than vertices, then it must be a tree.

   (b) What is the maximum number of vertices of an m-ary tree of height h?

   (c) Prove that the average height of a leaf in a binary tree is at least the base-two logarithm of the number of leaves.

4. (10 points) Prufer Sequences: these are the sequences of numbers which determine labeled trees. They were first discovered by Heinz Prufer (1896-1934) after whom they are named. In this problem we will work with letters instead of numbers, but same process can be applied. This converts a tree labelled with letters into a word of length $n - 2$ or vice versa.

   (a) Find the Prufer sequence (a word!) for the following graph:



Hint: I didn't know they grow on trees!?

   (b) What is the degree of the vertex labeled D in the letter labeled tree whose Prufer sequence is given by the word $DECIDED$?

   (c) Is there a way to determine the degree of a vertex from the Prufer sequence, without drawing the graph? Prove your answer is correct.