MATH-UA-240 / UY-4314 — Combinatorics

Professor Charles Stine

Name: **Homework Assignment #2**
NetID: **Due Date:** September 16, 2024, 11:59 PM

- This homework should be submitted via Gradescope by 11:59 PM on the date listed above.
- There are two main ways you might want to write up your work.
  - Write a PDF using a tablet
  - Write your answers on paper, clearly numbering each question and part.
    - You can use an app such as OfficeLens to take pictures of your work with your phone and convert them into a single pdf file. Gradescope will only allow pdf files to be uploaded.
- You must show all work. You may receive zero or reduced points for insufficient work. Your work must be neatly organized and written. You may receive zero or reduced points for incoherent work.
- Please start a fresh page for each numbered problem. You may have parts a), b) and c) on the page for example, but problems 1) and 2) should be on separate pages.
- When uploading to Gradescope, **you must match each question to the page that your answer appears on**. If you do not, we will be unable to grade the unmatched problems.
- When appropriate, please put a box or circle around your final answer.
- The problems on this assignment will be graded on correctness and completeness.

Lecture 3

1. (10 points) Euler Cycles and Hamiltonian circuits by hand.

   (a) For which values of n, a, b does the complete graph K_n and the complete, bipartite graph $K_{a,b}$ have an Euler cycle? Draw the smallest two such graphs and their Euler cycles.

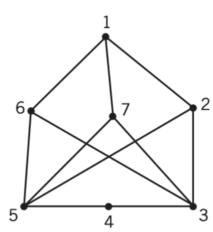
   (b) Hamiltonian circuits by hand: using logic, and the three ideas from class, prove that each of the graphs below does not have a Hamiltonian circuit. We recall the ideas here:

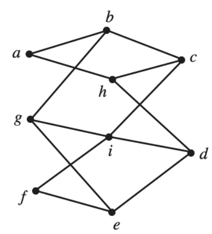
   (1) If a vertex has degree 2, then both edges connected to it must be included in every H.C.

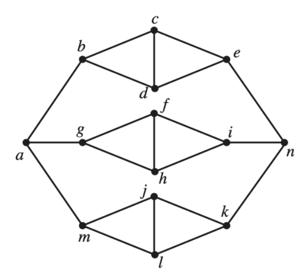
   (2) No H.C. can have a proper sub-circuit.

   (3) No H.C. can use more than two edges connected to the same vertex.

   You may not use Dirac, Chvatal, or Grinberg's theorems. It may help to re-draw the graphs to make them more symmetric. If you choose to do this you must show the new drawing is isomorphic to the original one.







2. (10 points) Chvatal and Grinberg's Theorems.

   (a) Consider the degree sequence $(2, 3, 4, 4, 5, 5, 5)$. Prove that any graph with this degree sequence must have a Hamiltonian circuit, then draw one such graph and find its Hamiltonian circuit.

   (b) There was a famous conjecture made by Peter Guthrie Tait in 1884 that every cubic polyhedral graph had a Hamiltonian circuit. *Cubic* means that every vertex has degree three, and *polyhedral* means that if you remove any two edges from the graph then the graph remains connected. This conjecture stood until 1946 *(62 years, not bad)* before William Thomas Tutte proved it false by drawing a counter-example.

   Use Grinberg's theorem (proven in 1968!) to show that the planar graph in Figure 1, P, is a counter-example to Tait's conjecture.

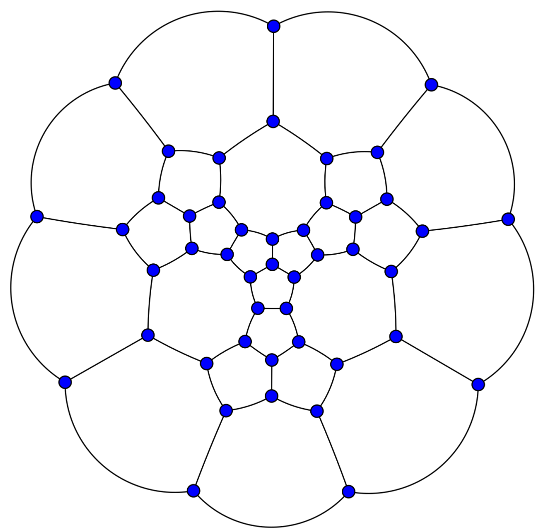


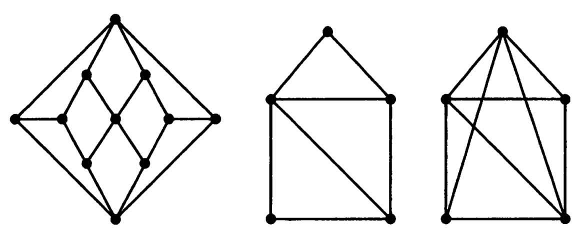
Figure 1: Tutte's counterexample, P, to Tait's conjecture.

Lecture 4

3. (10 points) Chromatic Numbers:

   (a) Prove that the chromatic number, $\chi(G)$, of a disconnected graph is the largest of the chromatic numbers of its connected components.

   (b) Determine the chromatic numbers of the following graphs:

   Graphs A, B, C

   (c) What is the chromatic number of the graph obtained by removing one edge from the graph K_n?

4. (**20** points) Chromatic Polynomials:

   (a) Recall that the Lose-Fuse relationship for the chromatic polynomial, $\chi_G(k)$, says that:

   $$\chi_G(k) = \chi_{G(Lose)}(k) - \chi_{G(Fuse)}(k)$$

   Use the Lose-Fuse relationship to determine the chromatic polynomial of the graph obtained by removing one edge from K_n.

(b) Prove that the chromatic polynomial of a tree, T, is precisely:

$$\chi_T(k) = k(k-1)^{|V_T|-1}$$

(Hint: pick a vertex and orient all the edges of the tree to point away from that vertex. Imagine coloring the graph starting from the chosen vertex)

(c) One consequence of the Lose-Fuse algorithm for computing $\chi_G(k)$ is that there is a unique null graph with the same order as G at the bottom of the Lose-Fuse tree. This is because Fusing always lowers the number of vertices, so the only null graph at the bottom of the tree with the same number of vertices as G is the graph which we have obtained by losing an edge at every step.

(i) Use the observation above to give an interpretation of the largest exponent which appears in $\chi_G(k)$. Going further: what must the coefficient of this term be?

(ii) Use your answer to (i) to determine whether there is a graph with chromatic polynomial $P(k) = k^4 - 4k^3 + 6k^2$.

Hint: you will need to check at most 11 cases.