Homework 4 Solutions

via Gradescope

- Failure to submit homework correctly will result in zeroes.
- Handwritten homework is OK. You do not have to type up your work.
- Problems assigned from the textbook are from the 5^{th} edition.
- No late homework accepted. Lateness due to technical issues will not be excused.

1. (6 points) Section 4.6 $\#13, 14$.

   Solution: #13.

   Theorem. *For any $n \in 2\mathbb{Z}$, $\lfloor n/2 \rfloor = n/2$.*

   Proof. Let $n \in 2\mathbb{Z}$. $n = 2k$ for some $k \in \mathbb{Z}$.

$$\lfloor n/2 \rfloor = \left\lfloor \frac{2k}{2} \right\rfloor = \lfloor k \rfloor = k,$$

   since $k \in \mathbb{Z}$ and $k \leq k = n/2 < k + 1$, so

$$\lfloor n/2 \rfloor = k = n/2.$$

   □

   #14.

   Conjecture. $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$ *for any $x, y \in \mathbb{R}$.*

   Disproof. Choose $x = 0 \in \mathbb{R}$ and $y = -\frac{1}{2} \in \mathbb{R}$ such that $\left\lfloor 0 - \left(-\frac{1}{2}\right) \right\rfloor = \lfloor \frac{1}{2} \rfloor = 0$ since $0 \in \mathbb{Z}$ and $0 \leq \frac{1}{2} < 1$. However $-1 \in \mathbb{Z}$ such that $\lfloor 0 \rfloor - \left\lfloor -\frac{1}{2} \right\rfloor = 0 - (-1) = 1$ since $0 \leq 0 < 1$ and $-1 \leq -\frac{1}{2} < 0$. Therefore

$$\lfloor x - y \rfloor = \left\lfloor 0 - \left(-\frac{1}{2}\right) \right\rfloor = 0 \neq 1 = \lfloor 0 \rfloor - \left\lfloor -\frac{1}{2} \right\rfloor = \lfloor x \rfloor - \lfloor y \rfloor.$$

   □

2. (9 points) Section 4.6 #15, 18, 23.

Solution: #15.

Conjecture. $\lfloor x-1 \rfloor = \lfloor x \rfloor - 1$ *for any* $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$. $\lfloor x-1 \rfloor = n \in \mathbb{Z}$ if and only if $n \le x-1 < n+1$.

$$\begin{aligned} n &\le x-1 < n+1 \\ n+1 &\le x-1+1 < n+1+1 \\ n+1 &\le x < n+2 \end{aligned}$$

so $\lfloor x \rfloor = n+1 \in \mathbb{Z}$. Thus

$$\begin{aligned} \lfloor x \rfloor - 1 &= n+1-1 = n+0 \\ &= n = \lfloor x-1 \rfloor. \end{aligned}$$

□

#18.

Conjecture. $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$ *for any* $x, y \in \mathbb{R}$.

Disproof. Choose $x = y = \frac{1}{2}$ such that $\left\lceil \frac{1}{2} + \frac{1}{2} \right\rceil = \lceil 1 \rceil = 1$ since $0 < 1 \le 1$ but $\left\lceil \frac{1}{2} \right\rceil = 1$ since $0 < \frac{1}{2} \le 1$ so

$$\lceil x+y \rceil = 1 \ne 2 = 1+1 = \lceil x \rceil + \lceil y \rceil.$$

□

#23.

Theorem. $\lfloor x \rfloor + \lfloor -x \rfloor = -1$ *for any* $x \in \mathbb{R} - \mathbb{Z}$.

Proof. Let $x \in \mathbb{R} - \mathbb{Z}$.

$\lfloor x \rfloor = n$ if and only if $n < x < n+1$ since $x \notin \mathbb{Z}$.

$-n > -x > -n-1$ so $\lfloor -x \rfloor = -n-1$ since $-n-1 \in \mathbb{Z}$.

Hence

$$\lfloor x \rfloor + \lfloor -x \rfloor = n + (-n-1) = n - n - 1 = 1.$$

□

3. (3 points) Use contradiction to prove the following statement

$$\forall x \in \mathbb{R},\ \text{If } |x| < \varepsilon \text{ for any } \varepsilon > 0,\ \text{then } x = 0.$$

Solution: Written formally: $\forall\, x \in \mathbb{R}\,(\,(\forall\, \varepsilon > 0\,(\,|x| < \varepsilon\,)\,) \to x = 0\,)$.

Proof. Suppose there exists a real number x such that $|x| < \varepsilon$ for any $\varepsilon > 0$ and $x \neq 0$. Written formally: Suppose $\exists\, x \in \mathbb{R}\,(\,(\forall\, \varepsilon > 0\,(\,|x| < \varepsilon\,)\,) \wedge x \neq 0\,)$. Let $\varepsilon = \frac{|x|}{2}$. Since $x \neq 0$, $|x| \neq 0$. Then $|x| < \frac{|x|}{2}$ so $2 < 1$ but $2 \geq 1$ is a contradiction. □

4. (9 points) Section 4.7 #9, 18.

Solution: #9(a). If you negate the beginning of the student's proof, you have

There exists an irrational number and there exists a irrational number such that their difference is irrational

However, this is not the statement you wish to prove.

Remark. Let's recall that

$$\neg p \to \mathbf{c}$$
$$\therefore \text{ p}$$

is a valid argument. If p is the statement **The difference of any irrational number and any rational number is irrational**, then student's proof should begin

Proof. Assume there exists a rational number and there exists an irrational number such that their difference is rational. □

#9(b).

Proof. Suppose there exists $r \in \mathbb{R} - \mathbb{Q}$ and $q \in \mathbb{Q}$ such that $r - q \in \mathbb{Q}$. There exists $m_1, m_2 \in \mathbb{Z}$ and $n_1, n_2 \in \mathbb{Z} - \{0\}$ such that $q = \frac{m_1}{n_1}$ and $r - q = \frac{m_2}{n_2}$.

$$\begin{aligned} \frac{m_2}{n_2} &= r - q = r - \frac{m_1}{n_1} \\ \frac{m_2}{n_2} + \frac{m_1}{n_1} &= r \\ \frac{m_2 n_1 + m_1 n_2}{n_1 n_2} &= r \end{aligned}$$

$m_2 n_1 + m_1 n_2 \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under products and sums. $n_1 n_2 \in \mathbb{Z} - \{0\}$ via Zero Product Property, since $n_1 \neq 0$, $n_2 \neq 0$, and $\mathbb{Z}$ is closed under products. Thus $r \in \mathbb{Q}$ but $r \in \mathbb{R} - \mathbb{Q}$ is a contradiction. Therefore $r - q \notin \mathbb{Q}$. □

#18.

Proof. Suppose $a \in \mathbb{Q}$, $b \in \mathbb{Q} - \{0\}$, and $r \in \mathbb{R} - \mathbb{Q}$. Suppose $a + br \in \mathbb{Q}$. Then there exist $m_1, m_3 \in \mathbb{Z}$ and $m_2, n_1, n_2, n_3 \in \mathbb{Z} - \{0\}$ such that $a = \frac{m_1}{n_1}$, $b = \frac{m_2}{n_2}$, and $a + br = \frac{m_3}{n_3}$. So

$$\begin{aligned}
\frac{m_3}{n_3} = a + br &= \frac{m_1}{n_1} + \frac{m_2}{n_2} r \\
\frac{m_3}{n_3} - \frac{m_1}{n_1} &= \frac{m_2}{n_2} r \\
\frac{n_2}{m_2}\left(\frac{m_3}{n_3} - \frac{m_1}{n_1}\right) &= r \\
\frac{m_3 n_1 n_2 - m_1 n_2 n_3}{m_2 n_1 n_3} &= r
\end{aligned}$$

where $m_3 n_1 n_2 - m_1 n_2 n_3 \in \mathbb{Z}$ and $m_2 n_1 n_3 \in \mathbb{Z} - \{0\}$ via Zero Product Property since $m_2 \neq 0$, $n_1 \neq 0$, and $n_3 \neq 0$. Thus $r \in \mathbb{Q}$ but $r \in \mathbb{R} - \mathbb{Q}$ is a contradiction. Therefore $a + br \notin \mathbb{Q}$. □

5. (9 points) Section 4.7 #22, 24.

Solution: #22(a).

Proof. Suppose there exists a real number r such that r^2 is irrational but r is rational. We NTS (need to show) that this yields a contradiction. For instance, we may show that r^2 is rational to obtain the contradiction. □

#22(b).

Proof. Suppose that r is rational. We need to show that r^2 is rational. □

#24.

Proof by Contradiction. Suppose there exists $x \in \mathbb{R} - \mathbb{Q}$ such that $\frac{1}{x} \in \mathbb{Q}$. There exist $m, n \in \mathbb{Z} - \{0\}$ such that $0 \neq \frac{1}{x} = \frac{m}{n}$. So $x = \frac{n}{m} \in \mathbb{Q}$ but $x \in \mathbb{R} - \mathbb{Q}$ is a contradiction. Therefore $\frac{1}{x} \in \mathbb{R} - \mathbb{Q}$ for any $x \in \mathbb{R} - \mathbb{Q}$. □

Let's now do contrapositive. Note that you will have to write the given statement in if-then form. In doing so, we obtain

If x is irrational, then $\frac{1}{x}$ is irrational.

$$\forall x \in \mathbb{R} - \{0\} \left(x \notin \mathbb{Q} \rightarrow \frac{1}{x} \notin \mathbb{Q}\right)$$

It follows that we want to prove

If $\frac{1}{x}$ is rational, then x is rational.

$$\forall x \in \mathbb{R} - \{0\} \left(\tfrac{1}{x} \in \mathbb{Q} \to x \in \mathbb{Q}\right)$$

Proof by Contraposition. Let $x \in \mathbb{R} - \{0\}$. Suppose $\frac{1}{x} \in \mathbb{Q}$. There exist $m, n \in \mathbb{Z} - \{0\}$ such that $0 \neq \frac{1}{x} = \frac{m}{n}$ so $x = \frac{n}{m} \in \mathbb{Q}$. Therefore $x \notin \mathbb{Q}$ implies $\frac{1}{x} \notin \mathbb{Q}$ for any $x \in \mathbb{R} - \{0\}$. □

6. (3 points) Section 4.7 #28. Prove this using both contradiction and the contrapositive.

   Solution: #28.

   Proof by Contradiction. Assume there exists $a, b, c \in \mathbb{Z}$ such that $a \mid b$, $a \nmid c$, and $a \mid (b + c)$. Since $a \mid b$, $b = ka$ for some $k \in \mathbb{Z}$. Since $a \mid (b + c)$, $b + c = la$ for some $l \in \mathbb{Z}$. So $ka + c = la$ and $c = (l - k)a$. Thus $a \mid c$ but $a \nmid c$ is a contradiction. Therefore $a \mid b$ and $a \nmid c$ implies $a \nmid (b + c)$ for any $a, b, c \in \mathbb{Z}$. □

   Proof by Contraposition. Assume that $a \mid (b+c)$. Therefore $b+c = ka$ for some $k \in \mathbb{Z}$. $a \nmid b$ or $a \mid b$ via the law of the excluded middle.

   Case 1: Suppose $a \nmid b$. Then we're done: $a \nmid b$ or $a \mid c$ via generalization. (Note that the negation of the hypothesis is $a \nmid b \vee a \mid c$.)

   Case 2: Suppose $a \mid b$. Therefore $b = la$ for some $l \in \mathbb{Z}$. The equation $b + c = ak$ now reads $la + c = ka$ so $c = ka - la$ and $c = a(k - l)$, i.e. $a \mid c$. $a \nmid b$ or $a \mid c$ via generalization.

   Therefore $a \mid b$ and $a \nmid c$ implies $a \nmid (b + c)$ for any $a, b, c \in \mathbb{Z}$. □

7. (9 points) Section 4.7 #31.

   Solution: #31.

   (a) *Proof.* Assume $n, r, s \in \mathbb{Z}$ such that $r > \sqrt{n}$ and $s > \sqrt{n}$. We multiply $r > \sqrt{n}$ by s and obtain $rs > s\sqrt{n}$. Note that $s\sqrt{n} > \sqrt{n}\sqrt{n} = n$. Therefore $rs > n$. □

   (b) *Proof.* Let $n > 1$ be any integer that is not prime. Then $n = rs$ where $1 < s < n$ and $1 < r < n$. By part (a) we know that $r \leq \sqrt{n}$ or $s \leq \sqrt{n}$.

   Case 1: Suppose $r \leq \sqrt{n}$. By Theorem 4.4.4, we know there exists a prime p_1 such that $p_1 \mid r$. So, via Theorem 4.4.1, $p_1 \leq r \leq \sqrt{n}$. Since $p_1 \mid r$ and $r \mid n$, via the transitivity of divides, i.e. Theorem 4.4.3, $p_1 \mid n$.

   Case 2: Suppose $s \leq \sqrt{n}$. Now repeat the argument as in **Case 1**. By Theorem 4.4.4, we know there exists a prime p_2 such that $p_2 \mid s$. So, via Theorem 4.4.1, $p_2 \leq s \leq \sqrt{n}$. Since $p_2 \mid s$ and $s \mid n$, via the transitivity of divides, i.e. Theorem 4.4.3, $p_2 \mid n$. □

   (c) For each integer $n > 1$, if for any prime number p, $p > \sqrt{n}$ or $p \nmid n$ then n is prime.