Homework 1 Solutions

via Gradescope

- Failure to submit homework correctly will result in zeroes.
- Handwritten homework is OK. You do not have to type up your work.
- Problems assigned from the textbook are from the 5th edition.
- No late homework accepted. Lateness due to technical issues will not be excused.

Theorem *. *Let p and q be statement variables.* $p \to q \equiv \neg p \vee q$.

1. (6 points) Show the logical equivalences using truth tables and say a few words explaining why your truth table shows $\equiv$

   (a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$ (DeMorgan's law).
   Solution: Let's set up the truth table. Note that I will number the columns and rows so that I can refer to them if need be. I encourage you to do the same.

	1	2	3	4	5	6	7
1	p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
2	T	T	T	F	F	F	F
3	T	F	T	F	F	T	F
4	F	T	T	F	T	F	F
5	F	F	F	T	T	T	T

   DeMorgan's law holds since column 4 = column 7.

   Remark. Everyone should have 7 columns. Your first two columns should be the statement variables. The order in which my columns 3-7 are written may not necessarily be order in which your columns are written. That is OK.

   (b) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (Distributive law).
   Note that your truth table will have 8 rows.
   Solution:

	1	2	3	4	5	6	7	8
1	p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
2	T	T	T	T	T	T	T	T
3	T	T	F	F	T	T	T	T
4	T	F	T	F	T	T	T	T
5	T	F	F	F	T	T	T	T
6	F	T	T	T	T	T	T	T
7	F	T	F	F	F	T	F	F
8	F	F	T	F	F	F	T	F
9	F	F	F	F	F	F	F	F

The distributive law holds since column 5 = column 8.

Remark. The order in which you have written columns 4-8 may be different than mine. That is OK.

2. (3 points) Prove that $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$ using Theorem * and Theorem 2.1.1. Annotate your proof. For reference, see example 2.1.14.

Solution:

$$\begin{aligned}
(p \vee q) \to r &\equiv \neg(p \vee q) \vee r \ \ \text{(Theorem *)} && (1)\\
&\equiv (\neg p \wedge \neg q) \vee r \ \ (DeMorgan's) && (2)\\
&\equiv r \vee (\neg p \wedge \neg q) \ \ (Commutative) && (3)\\
&\equiv (r \vee \neg p) \wedge (r \vee \neg q) \ \ (Distributive) && (4)\\
&\equiv (\neg p \vee r) \wedge (\neg q \vee r) \ \ (Commutative) && (5)\\
&\equiv (p \to r) \wedge (q \to r) \ \ \text{(Theorem *)} && (6)
\end{aligned}$$

Remark. I know some of you did not use either of the commutative properties. That is OK. With that said, everyone should have lines (1), (2), (4), (6).

3. (3 points) Find all values of p and q for which $p \to q$ is not equal to $q \to p$. For which values of p, q are the statement forms equal?

Solution: Let's look at the truth table.

	1	2	3	4
1	p	q	$p \to q$	$q \to p$
2	T	T	T	T
3	T	F	F	T
4	F	T	T	F
5	F	F	T	T

It's clear that the statement forms $p \to q$ is equal to $q \to p$ if and only if statement variables p, q have the same truth values.

Remark. The results of problem 3 tell us that the conditional is not logically equivalent to its converse.

4. (3 points) Show that $[(p \to q) \wedge (p \to \neg q)] \to \neg p$ is a tautology using Theorem * and Theorem 2.1.1. Annotate your proof. For reference, see example 2.1.14.

Solution: Let's set $P \equiv p \to q$ and $Q \equiv p \to \neg q$. Let's start.

$$\begin{aligned}
[P \wedge Q] \to \neg p &\equiv \neg[P \wedge Q] \vee \neg p && (7)\\
&\equiv \neg P \vee \neg Q \vee \neg p \ \ (DeMorgan's) && (8)\\
&\equiv \neg P \vee (\neg Q \vee \neg p) \ \ (Associative\ Law) && (9)\\
&\equiv \neg P \vee P && (10)\\
&\equiv \mathbf{t} \ \ (NegationLaws) && (11)
\end{aligned}$$

Of course, I bet you're thinking: How in the world did we get line (11)? Let's now show that $P \equiv \neg Q \vee \neg p$. We have

$$\begin{aligned}
\neg Q \vee \neg p &\equiv \neg(p \to \neg q) \vee \neg p && (12)\\
&\equiv \neg(\neg p \vee \neg q) \vee \neg p \ \ \text{(Theorem *)} && (13)\\
&\equiv (\neg(\neg p) \wedge \neg(\neg q)) \vee \neg p \ \ (DeMorgan's) && (14)\\
&\equiv (p \wedge q) \vee \neg p \ \ (Double\ Negation\ Laws) && (15)\\
&\equiv \neg p \vee (p \wedge q) \ \ (Commutative\ Law) && (16)\\
&\equiv (\neg p \vee p) \wedge (\neg p \vee q) \ \ (Distribution) && (17)\\
&\equiv \mathbf{t} \wedge (\neg p \vee q) \ \ (Negation\ Laws) && (18)\\
&\equiv (\neg p \vee q) \wedge \mathbf{t} \ \ (Commutative\ Law) && (19)\\
&\equiv \neg p \vee q \ \ (Identity\ Laws) && (20)\\
&\equiv (p \to q) \ \ \text{(Theorem *)} && (21)\\
&\equiv P && (22)
\end{aligned}$$

Remark. The argument above is not the only argument. Here's a very clever argument that a student presented in office hours!

$$\begin{aligned}
(p \to q) \wedge (p \to \neg q) &\equiv (\neg p \vee q) \wedge (\neg p \vee \neg(\neg q)) \ \ \text{(Theorem *)} && (23)\\
&\equiv (\neg p \vee q) \wedge (\neg p \vee q) \ \ (Double\ Negative\ Law) && (24)\\
&\equiv \neg p \vee (q \wedge \neg q) \ \ (Distributive\ Law) && (25)\\
&\equiv \neg p \vee \mathbf{c} \ \ (Negation\ Law) && (26)\\
&\equiv \neg p \ \ (Identity\ Law) && (27)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
[(p \to q) \wedge (p \to \neg q)] \to \neg p &\equiv \neg p \to \neg p && (28)\\
&\equiv \neg(\neg p) \vee \neg p \ \ \text{(Theorem *)} && (29)\\
&\equiv p \vee \neg p \ \ (Double\ Negative\ Law) && (30)\\
&\equiv \mathbf{t} \ \ (Negation\ Law) && (31)
\end{aligned}$$

5. (9 points) Section 2.1 #31 (see page 52).

Solution:

(a) This is the set of all strings s of length two where the first entry is a 0 or a 1 and the second must be a 1 or a 2.

Remark. BTW. I am happy with the description given above. If we agree to denote a string of length two as (x, y), then then set of such strings looks like

$$\{(0,1),(0,2),(1,1),(1,2)\}$$

(b) The first entry must be a 2 and the second entry must be a 1 or a 2.

(c) The first entry must be a 1 or a 2. The second entry must be a 1 or a 0.

6. (3 points) Section 2.1 #46(a) (see page 53)

Solution: I will give a solution that will differ from the text and use Theorem 2.1.1.

$$\begin{aligned} p \oplus p &\equiv (p \vee p) \wedge \neg(p \wedge p) \\ &\equiv p \wedge \neg p \ \ (\textit{Idempotent Laws}) \\ &\equiv \mathbf{c} \ \ (\textit{Negation Laws}) \end{aligned}$$

Similarly,

$$\begin{aligned} (p \oplus p) \oplus p &\equiv \mathbf{c} \oplus p && (32) \\ &\equiv (\mathbf{c} \vee p) \wedge \neg(\mathbf{c} \wedge p) && (33) \\ &\equiv (p \vee \mathbf{c}) \wedge \neg(p \wedge \mathbf{c}) \ \ (\textit{Commutative Law}) && (34) \\ &\equiv p \wedge \neg\mathbf{c} \ \ (\textit{Identity Laws and Universal Bound Laws}) && (35) \\ &\equiv p \wedge \mathbf{t} \ \ (\textit{Negation of } \mathbf{c}) && (36) \\ &\equiv p \ \ (\textit{Identity Law}) && (37) \end{aligned}$$

7. (24 points) Section 2.2 #22, 23.

Remark. For both problems, parts (b), (c), (e), (g) only.

Solution: #22.

(b) If tomorrow is not January, then today is not New Years Eve.

(c) If r is irrational, then the decimal expansion is non-terminating.

(e) If x is not positive and x is not zero, then x is negative.

(g) If n is not divisible by 2 or n is not divisible by 3, then n is not divisible by 6.

Solution: #23.

(b) Converse: If tomorrow is January, then today is New Years Eve.
Inverse: If today is not New Years Eve, then tomorrow is not January.

(c) Converse: If r is rational, then the decimal expansion is terminating.
Inverse: If the decimal expansion of r is non-terminating, then r is irrational.

(e) Converse: If x is positive or x is zero, then x is non-negative.
Inverse: If x is negative, then x is not positive and x is not zero.

(g) Converse: If 2 divides n and 3 divides n, then 6 divides n.
Inverse: If n is not divisible by 6, then n is not divisible by 2 or n is not divisible by 3.

8. (9 points) Section 2.2 $\#14, 38, 43$.

   Remark. When doing #14, do not use truth tables. Use Theorem * and Theorem 2.1.1.

   Solution: #14. Only part (a) will be done.

$$\begin{aligned}
p \to q \vee r &\equiv \neg p \vee (q \vee r) \ \ \text{(Theorem *)}\\
&\equiv (\neg p \vee q) \vee r \ \ (Associative)\\
&\equiv \neg(p \wedge \neg q) \vee r \ \ (DeMorgan's)\\
&\equiv p \wedge \neg q \to r \ \ \text{(Theorem *)}\\
&\equiv \neg(p \wedge \neg q) \vee r \ \ \text{(Theorem *)}\\
&\equiv (\neg p \vee q) \vee r \ (DeMorgan's)\\
&\equiv \neg p \vee (q \vee r) \ \ (Associative)\\
&\equiv \neg p \vee (r \vee q) \ \ (Commutative)\\
&\equiv (\neg p \vee r) \vee q \ \ (Associative)\\
&\equiv \neg(p \wedge \neg r) \vee q \ \ (DeMorgan's)\\
&\equiv p \wedge \neg r \to q \ \ \text{(Theorem *)}
\end{aligned}$$

   #38. If it does not rain, then Ann will go.

   #43.

   (a) If Jim passes the course, then Jim does homework regularly.

   (b) If Jim does not do homework regularly, then Jim does not pass the course.

9. (9 points) Section 2.3 $\#9, 12(b), 23$.

   Remark. In addition to the instructions associated to these problems. Complete all truth tables even if argument is invalid. Identify all critical rows even if the argument is invalid.

   Solution: Only solutions to problems #9 and #23 will be provided.

   #9.

	1	2	3	4	5	6	7	8	9	10
	p	q	r	$p \wedge q$	$\neg r$	$\neg q$	$p \wedge q \to \neg r$	$p \vee \neg q$	$\neg q \to p$	$\neg r$
1	T	T	T	T	F	F	F	T	T	F
2	T	T	F	T	T	F	T	T	T	T
3	T	F	T	F	F	T	T	T	T	F
4	T	F	F	F	T	T	T	T	T	T
5	F	T	T	F	F	F	T	F	T	F
6	F	T	F	F	T	F	T	F	T	T
7	F	F	T	F	F	T	T	T	F	F
8	F	F	F	F	T	T	T	T	F	T

The premises are columns 7, 8, and 9. The conclusion is column 10. The critical rows are rows 2, 3, 4. The argument form is invalid due to the conclusion being F in critical row 3.

#23. Here is the associated argument form.

$$\begin{aligned} &p \vee q \\ &p \to r \\ \therefore\ &q \vee \neg r \end{aligned}$$

While I won't construct the truth table here, notice how the premises will be true for $p \equiv r \equiv T$ and $q \equiv F$. Meanwhile the conclusion will be false.

10. (6 points) Section 2.3 $\#29, 38(d)$.

Solution: #29. The associated argument form is

$$\begin{aligned} &p \to q \\ &\neg p \\ \therefore\ &\neg q \end{aligned}$$

The argument form is invalid, an inverse error.

$\#38(d)$.

Solution: Let's assume that U is a knight.

$\therefore$ No one is a knight.

$\therefore$ U is not a knight. (Contradiction. U is a knight.)

Let's assume that V is a knight.

$\therefore$ There are at least three knights.

$\therefore$ $Z \wedge Y$ are knaves.

Since U is also a knave $\to V \wedge X \wedge W$ are the only knights.

$\therefore$ X tells the truth. (contradiction. There are exactly three knights).

Let's assume that X is a knight.

$\therefore$ There are exactly 5 knights.

Since $U \wedge V$ are knaves we have at most 4 knights $(W \wedge X \wedge Y \wedge Z)$. (Contradiction.)

Let's assume Z is a knight.

$\therefore$ Therefore there is exactly one knight.

$\therefore$ Z is a knight.

$\therefore W \wedge Y$ are knaves.

$\therefore W$ lies. (Contradiction. W is telling the truth.)

Notice that both W and Y tell the truth. They are the knights!

11. (6 points) Section 2.3 #42, 44. Annotate.

Solution: Here we are trying to show an argument form is valid using the the chart on page 76.

$$\begin{aligned} & q \to r \\ & \neg r \\ \therefore\ & \neg q \ \ (\textit{Modus Tollens}) \end{aligned}$$

$$\begin{aligned} & p \vee q \\ & \neg q \\ \therefore\ & p \ \ (\textit{Elimination}) \end{aligned}$$

$$\begin{aligned} & \neg q \to u \wedge s \\ & \neg q \\ \therefore\ & u \wedge s \ \ (\textit{Modus Ponens}) \end{aligned}$$

$$\begin{aligned} & u \wedge s \\ \therefore\ & s \ \ (\textit{Specialization}) \end{aligned}$$

$$\begin{aligned} & p \\ & s \\ \therefore\ & p \wedge s \ \ (\textit{Conjunction}) \end{aligned}$$

$$\begin{aligned} & p \wedge s \to t \\ & p \wedge s \\ \therefore\ & t \ \ (\textit{Modus Ponens}) \end{aligned}$$

$$\begin{aligned} & \neg q \vee s \\ & \neg s \\ \therefore\ & \neg q \ \ (\textit{Elimination}) \end{aligned}$$

Remark. Your justification may not be in the same order as the solution provided.

$$\begin{aligned}
& r \vee s \\
& \neg s \\
\therefore\ & r \ \ (\mathit{Elimination})
\end{aligned}$$

$$\begin{aligned}
& \neg s \rightarrow \neg t \\
& \neg s \\
\therefore\ & \neg t \ \ (Modus\ Ponens)
\end{aligned}$$

$$\begin{aligned}
& w \vee t \\
& \neg t \\
\therefore\ & w \ \ (\mathit{Elimination})
\end{aligned}$$

$$\begin{aligned}
& \neg q \vee s \\
& \neg s \\
\therefore\ & \neg q \ \ (\mathit{Elimination})
\end{aligned}$$

$$\begin{aligned}
& p \rightarrow q \\
& \neg q \\
\therefore\ & \neg p \ \ (Modus\ Tollens)
\end{aligned}$$

$$\begin{aligned}
& \neg p \\
& r \\
\therefore\ & \neg p \wedge s \ \ (Conjunction)
\end{aligned}$$

$$\begin{aligned}
& \neg p \wedge s \rightarrow u \\
& \neg p \wedge s \\
\therefore\ & u \ \ (Modus\ Ponens)
\end{aligned}$$

$$\begin{aligned}
& u \\
& w \\
\therefore\ & u \wedge w \ \ (Conjunction)
\end{aligned}$$

Remark. Your justification may not be in the same order as the solution provided.

12. (21 points) Section 3.1 $\#4, 20, 32(b)(d)$.

    Solution: #4.

(a) Notice that $x = -2 < 1 = y$ is true while $x^2 < y^2$ is false.

(b) $x = -1$, $y = 0$ would work. Infinitely many correct values of x and y.

(c) This is because $x = 3 < 8 = y$ and $x^2 < y^2$ are both true.

(d) $x = 8$, $y = 3$ will work. Do you see why?

#20. Every positive real number has a positive square root.

Every real number that has a non-positive square root must be non-positive.

Remark. Non-positive is ≤ 0.

#32. Both parts (b) and (d) are true. They did not ask for a justification. However, you can always see this from the graph of $y = x^2$. Also, about part (d), note that $|x| > 2 \Leftrightarrow x < -2 \vee x > 2$.

13. (9 points) Section 3.2 #15$(b)(d)(e)$.

    Solution: #15.

    (b) True, $x = -8, -14 - 48$ are the only negative values in D and are all even.

    (d) True, $x = 32$ is the only value in D where the ones digit is 2. Note that the tens digit is 3.

    (e) False, $x = 36$ has a ones digit equal to 6 and the tens digit is a 3.

14. (9 points) Section 3.2 #12, 40, 46.

    Solution: #12. The proposed negation is incorrect. I find it useful to write the given statement formally, then negate. Then write the informal negation. The given statement formally reads

$$\forall x \in \mathbb{R} - \mathbb{Q} \forall y \in \mathbb{Q} (xy \in \mathbb{R} - \mathbb{Q}).$$

The formal negation reads

$$\exists x \in \mathbb{R} - \mathbb{Q} \exists y \in \mathbb{Q} (xy \in \mathbb{Q}).$$

The informal negation reads:

There exists an irrational number and a rational number such that their product is rational.

#40. $\forall x \in \mathbb{R}$, if x is divisible by 8, then x is divisible by 4.

#46. The *not necessary* is the negation of a universal necessary. That is,

$$\begin{aligned}\neg\forall x\,(Q(x) \text{ is necessary for } P(x)) &\equiv \neg\forall x\,(P(x) \to Q(x))\\ &\equiv \neg\,(P(x) \Rightarrow Q(x))\end{aligned}$$

This simplifies to

$$\exists x\,(P(x) \wedge \neg Q(x))$$

Here we used the fact that

$$\neg(p \to q) \equiv \neg(\neg p \vee q) \equiv \neg\neg p \wedge \neg q \equiv p \wedge \neg q$$

The final answer reads

Someone is happy but does not have a large income.