Homework 8 Solutions

via Gradescope

- Failure to submit homework correctly will result in zeroes.
- Handwritten homework is OK. You do not have to type up your work.
- Problems assigned from the textbook are from the 5^{th} edition.
- No late homework accepted. Lateness due to technical issues will not be excused.

Remark. Let A and B be sets. The set $A \times B$ is called the Cartesian product of A and B and is defined as

$$A \times B = \{ (a, b) : a \in A \text{ and } b \in B \}.$$

Note that the elements of $A \times B$ are ordered pairs. That means that $(a, b) = (c, d)$ if and only if $(a = c) \wedge (b = d)$. See page 12 in the textbook for additional info.

1. (27 points) Section 6.1 $\#6, 15(b)(c), 18$.

   Solution:

   $\#6(a)$. This is false.

   Disproof. Choose $x = 2 \in \mathbb{Z}$.

   (i) $0 \in \mathbb{Z}$ such that $x = 2 = 5(0) + 2 \in A$.

   (ii) Suppose $x \in B$. $x = 10b - 3$ for some $b \in \mathbb{Z}$. $2 = 10b - 3$ so $b = \frac{1}{2} \in \mathbb{Q} - \mathbb{Z}$ but $b \in \mathbb{Z}$ is a contradiction. So $x = 2 \notin B$.

   Therefore $A \not\subseteq B$. $\square$

   $\#6(b)$. This is true.

   Proof. Let $x \in B$. So $x = 10b - 3$ for some $b \in \mathbb{Z}$.

   $$10b - 3 = 5(2b) - 5 + 2 = 5(2b - 1) + 2$$

   and $a := 2b - 1 \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under products and differences. So $x = 5a + 2 \in A$. Therefore $B \subseteq A$. $\square$

   $\#$ 6(c). This is true. Note that we need to show that $B \subseteq C$ and $C \subseteq B$.

Proof. Let $x \in B$. $x = 10b - 3$ for some $b \in \mathbb{Z}$.

$$10b - 3 = 10b - 10 + 7 = 10(b-1) + 7$$

and $c := b - 1 \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under differences. So $x = 10c + 7 \in C$. Thus $B \subseteq C$.
Let $x \in C$. $x = 10c + 7$ for some $c \in \mathbb{Z}$.

$$10c + 7 = 10c + 10 - 3 = 10(c+1) - 3$$

and $b := c + 1 \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under differences. So $x = 10b - 3 \in C$. Thus $C \subseteq B$.
Since $B \subseteq C$ and $C \subseteq B$, $B = C$. $\square$

#15. Visit office hours if you need help with the sketch of the Venn diagrams.

#18(a). No. Nothing belongs to the empty set.

#18(b). No. The right hand side is non-empty.

#18(c). Yes. The $\varnothing$ is an element in $\{\varnothing\}$.

#18(d). No. No element belongs to the empty set.

2. (36 points) Section 6.1 #25, 27.

Remark. There are typos on problems #23, 25. The lower index should be $i = 1$ (not $i = 0$).

Solution:

#25(a). There is a typo in the textbook. The sets R_i are defined for $i \geq 1$. Notice that $R_{i+1} \subseteq R_i$. Therefore the answer here is $[1, 2]$.

#25(b). $[1, 5/4]$.

#25(c). No. $R_1 \cap R_2 = R_2$.

#25(d). Because of the nesting, the answer is $[1, 2]$.

#25(e). $[1, 1 + 1/n]$.

#25(f). $[1, 2]$.

#25(g). $\{1\}$.

#27(a). No. This is because the sets a, d, e and d, f are not disjoint.

#27(b). Yes.

#27(c). No. The sets $\{5, 4\}$ and $\{1, 3, 4\}$ are not disjoint.

#27(d). No. 6 does not belong to any of the given sets in the proposed partition.

#27(e). Yes.

3. (15 points) Section 6.1 #32, 33.

   Solution:

   #32(a) Note that $A \times B = \{(1,u),(1,v)\}$. Therefore

   $$\mathcal{P}(A \times B) = \{\varnothing, \{(1,u)\}, \{(1,v)\}, A \times B\}$$

   #32(b). Here $X \times Y = \{(1,x),(1,y),(2,x),(2,y)\}$. The power set has 16 elements.

   $$\begin{aligned}\mathcal{P}(X \times Y) = \{&\varnothing, \{(1,x)\},\ \{(1,y)\},\ \{(2,x)\},\ \{(2,y)\},\ \{(1,x),(1,y)\},\\ &\{(1,x),(2,y)\},\ \{(2,x),(1,y)\},\ \{(2,x),(2,y)\},\\ &\{(1,x),(2,x)\},\ \{(1,y),(2,y)\}, \{(1,x),(1,y),(2,x)\},\\ &\{(1,x),(2,x),(2,y)\},\ \{(1,y),(2,y),(1,x)\},\\ &\{(1,y),(2,y),(2,x)\},\ X \times Y\}\end{aligned}$$

   #33(a) $\mathcal{P}(\varnothing) = \{\varnothing\}$

   #33(b) $\mathcal{P}(\mathcal{P}(\varnothing)) = \{\varnothing,\ \{\varnothing\}\}$.

   #33(c) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing))) = \{\ \varnothing,\ \{\varnothing\},\ \{\{\varnothing\}\},\ \{\varnothing,\ \{\varnothing\}\}\ \}$

4. (3 points) Write the following union of intervals as a single interval.

   $$\bigcup_{n=2}^{\infty} \left[\sqrt{1+\frac{1}{n}}, \sqrt{2-\frac{1}{n}}\right]$$

   Solution: $\left(1, \sqrt{2}\right)$

5. (9 points) Section 6.2 #6, 11, 15.

   Solution:

   #6(a) $(A \cap B) \cup (A \cap C)$

   #6(b) A

   #6(c) C

   #6(d) $x \in (A \cap B) \cup (A \cap C)$

   #6(a) or

   #6(b) and

   #6(c) $x \in A \cap (B \cup C)$

   #6(d) subset

   #11.

   Remark. This proof is more in the spirit of Epp. You might however approach it using a Theorem 2.1.1. approach.

Proof. Let $x \in A \cap (B - C)$. Therefore $x \in A$ and $x \in B - C$, i.e. $x \in B$ and $x \notin C$. Since $x \in A$ and $x \in B$ it follows that $x \in A \cap B$. Suppose $x \in A \cap C$. $x \in A$ and $x \in C$. So $x \in C$ but $x \notin C$ is a contradiction. Hence $x \notin A \cap C$. Since $x \in A \cap B$ and $x \notin A \cap C$, $x \in (A \cap B) - (A \cap C)$. $\square$

#15.

Proof. Let A be a set.
Let $x \in A \cup \varnothing$. Then $x \in A$ or $x \in \varnothing$. $x \notin \varnothing$ so $x \in A$ by elimination. $A \cup \varnothing \subset A$.
Let $x \in A$. Then $x \in A$ or $x \in \varnothing$ via generalization. So $x \in A \cup \varnothing$. $A \subset A \cup \varnothing$.
$A \cup \varnothing = A$. $\square$

6. (6 points) Section 6.2 #22, 35.

#22.

Proof. Let $\mathbf{v} \in A \times (B \cap C)$. $\mathbf{v} = (x, y)$ such that $x \in A$ and $y \in B \cap C$. $y \in B \cap C$ so $y \in B$ and $y \in C$.
$y \in B$ via specialization. $x \in A$ and $y \in B$ via conjunction, so $\mathbf{v} = (x, y) \in A \times B$.
$y \in C$ via specialization. $x \in A$ and $y \in C$ via conjunction, so $\mathbf{v} = (x, y) \in A \times C$.
$\mathbf{v} \in A \times B$ and $\mathbf{v} \in A \times C$, so $\mathbf{v} \in (A \times B) \cap (A \times C)$.
Thus $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$.
Let $\mathbf{v} \in (A \times B) \cap (A \times C)$. $\mathbf{v} \in (A \times B)$ and $\mathbf{v} \in (A \times C)$.
$\mathbf{v} \in (A \times B)$ so $x \in A$ and $y \in B$ such that $\mathbf{v} = (x, y)$.
$\mathbf{v} \in (A \times C)$ so $x \in A$ and $z \in C$ such that $\mathbf{v} = (x, z)$.
$(x, y) = \mathbf{v} = (x, z)$ so define $w := y = z$ such that $w \in B$ and $w \in C$. $x \in A$ and $w \in B \cap C$ so $\mathbf{v} = (x, w) \in A \times (B \cap C)$.
Thus $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$.
Since $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$ and $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$, $A \times (B \cap C) = (A \times B) \cap (A \times C)$. $\square$

#35. Let's do a contradiction proof.

Proof. Let A, B, C be sets. Suppose $A \subset B$ and $B \cap C = \varnothing$. Suppose $A \cap C \neq \varnothing$. Then there exists $x \in A \cap C$. $x \in A$ and $x \in C$ so $x \in A$ by specialization. Since $A \subset B$ it follows that $x \in B$. $x \in C$ by specialization so $x \in B$ and $x \in C$ by conjunction. Therefore $x \in B \cap C$, so $B \cap C \neq \varnothing$ and this is a contradiction. Thus $A \cap C = \varnothing$. $\square$