MATH-UA-240 — Combinatorics

Professor Charles Stine

Name: **Homework Assignment #7**
NetID: **Due Date:** November 4, 2023, 11:59 PM

- This homework should be submitted via Gradescope by 11:59 PM on the date listed above.
- There are two main ways you might want to write up your work.
  - Write a PDF using a tablet
  - Write your answers on paper, clearly numbering each question and part.
    * You can use an app such as OfficeLens to take pictures of your work with your phone and convert them into a single pdf file. Gradescope will only allow pdf files to be uploaded.
- You must show all work. You may receive zero or reduced points for insufficient work. Your work must be neatly organized and written. You may receive zero or reduced points for incoherent work.
- Please start a fresh page for each numbered problem. You may have parts a), b) and c) on the page for example, but problems 1) and 2) should be on separate pages.
- When uploading to Gradescope, **you must match each question to the page that your answer appears on**. If you do not, we will be unable to grade the unmatched problems.
- When appropriate, please put a box or circle around your final answer.
- The problems on this assignment will be graded on correctness and completeness.

Lecture 14

It may be helpful to review the examples given in sections 5.1, 5.2, and 5.3 of our textbook: Applied Combinatorics, 6th edition, by Alan Tucker.

1. (10 points) Simple Arrangements and Selections:
   (a) How many arrangements of the letters "JUPITER" are there in which the vowels occur in alphabetical order?
   (b) How many triangles are formed by n non-parallel lines in the plane? (You many assume that only two lines meet at each intersection point.)
   (c) Consider a normal 52-card deck (no Jokers), what are the odds of a 7-card hand having the same number of hearts and spades? (Break it into cases!)

2. (10 points) Arrangement and Selection with Repetitions:
   (a) How many sequences of outcomes are possible if one rolls two identical dice 10 successive times?

   When you roll two dice, and they stop rolling, you don't know which di is "first" or "second" on that roll, but when you roll both again, you know that this roll is the next roll of both.

   (b) How many arrangements are there of the letters of the word "TINKERER" with two but not three successive vowels?
   (c) How many ways are there to place nine different rings on the four fingers of your right hand (excluding the thumb) if the order of the rings on a finger is considered?

   Counting this process is very simple if you split it into the correct pair of sub-processes. If your answer is longer than three lines, you aren't splitting it up correctly!

Lecture 15

1. (10 points) Distributions:
   (a) How many bridge deals are there in which North and South get all the spades? (It may be helpful to Google how the card game 'bridge' is dealt)
   (b) What is the probability that a randomly shuffled deck of cards has exactly k runs of hearts? (a run is a consecutive sequence of length at least 1)

2. (10 points) Finding Combinatorial Arguments: using the guidelines outlined in class, come up with **combinatorial arguments** for each of the following identities. (Note: this does NOT mean to expand each side and check the equality using algebra.)

(a) Give a careful combinatorial proof of the formula below. You may base your argument on the ideas from the Lecture 15 notes, but you must explain each step in your own words. No credit will be given for copying what is in the lecture notes verbatim.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

You are looking for a set which is counted by $\binom{n}{k}$*, and that can be split into two subsets that are disjoint and clearly have sizes given by the two terms on the right. Your task is to clearly say what each set is, why the two on the right are disjoint, and why their union is the whole of the set on the left.*

(b) Write your own combinatorial argument for the following identity:

$$\binom{2n}{2} = \binom{n}{2} + n^2 + \binom{n}{2}$$

Try to find a set counted by the right side formula, and 3 sets counted by the terms on the left, which are disjoint, and whose union is the set on the right.