2D Euler Fourier Transform

Consider 2D Euler:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p, \qquad \nabla \cdot u = 0$$

where $\mathbf{u}(x,t)$ is a 2D vector that is periodic in both dimensions and p is pressure.

To find an expression for p we take the divergence of both sides:

$$\begin{aligned}\nabla \cdot (-\nabla p) &= \nabla \cdot (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) \\ &= \nabla \cdot \mathbf{u}_t + \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})\end{aligned}$$

Since $\nabla \cdot u = 0$, we have:

$$\begin{aligned}\nabla \cdot \begin{bmatrix} -\frac{\partial p}{\partial x} \\ -\frac{\partial p}{\partial y} \end{bmatrix} &= \nabla \cdot \begin{bmatrix} u^x \frac{\partial u^x}{\partial x} + u^y \frac{\partial u^x}{\partial y} \\ u^x \frac{\partial u^y}{\partial x} + u^y \frac{\partial u^y}{\partial y} \end{bmatrix} \\ -\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) &= \frac{\partial u^x}{\partial x}\frac{\partial u^x}{\partial x} + u^x \frac{\partial^2 u^x}{\partial x^2} + \frac{\partial u^y}{\partial x}\frac{\partial u^x}{\partial y} + u^y \frac{\partial^2 u^x}{\partial x \partial y} \\ &\quad + \frac{\partial u^x}{\partial y}\frac{\partial u^y}{\partial x} + u^x \frac{\partial^2 u^y}{\partial x \partial y} + \frac{\partial u^y}{\partial y}\frac{\partial u^y}{\partial y} + u^y \frac{\partial^2 u^y}{\partial y^2}\end{aligned}$$

Notice that

$$u^x \frac{\partial^2 u^x}{\partial x^2} + u^x \frac{\partial^2 u^y}{\partial x \partial y} = u^x \frac{\partial}{\partial x}(\nabla \cdot u) = 0$$

and

$$u^y \frac{\partial^2 u^x}{\partial x \partial y} + u^y \frac{\partial^2 u^y}{\partial y^2} = u^y \frac{\partial}{\partial y}(\nabla \cdot u) = 0.$$

So,

$$-\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) = \frac{\partial u^x}{\partial x}\frac{\partial u^x}{\partial x} + \frac{\partial u^y}{\partial x}\frac{\partial u^x}{\partial y} + \frac{\partial u^x}{\partial y}\frac{\partial u^y}{\partial x} + \frac{\partial u^y}{\partial y}\frac{\partial u^y}{\partial y}.$$

Since $\mathbf{u}$ is periodic, we can represent it as a Fourier series

$$\mathbf{u}(x,t) = \sum_{\mathbf{k}} \mathbf{u}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}$$

where $\mathbf{k}$ is a 2D wavevector and we sum over all possible integer-valued wavevectors. We can do the same for p:

$$p = \sum_{\mathbf{k}} p_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}$$

Now we can write the left-hand side as

$$
\begin{aligned}
-\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right) &= -\left(\sum_{\mathbf{k}} -k_x^2 p_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + \sum_{\mathbf{k}} -k_y^2 p_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\right) \\
&= \sum_{\mathbf{k}} (k_x^2+k_y^2) p_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= \sum_{\mathbf{k}} |\mathbf{k}|^2 p_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}
\end{aligned}
$$

and the right-hand side as

$$
\begin{aligned}
&\left(\sum_{\mathbf{p}} ip_x u^x_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}\right)\left(\sum_{\mathbf{q}} iq_x u^x_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}\right) + \left(\sum_{\mathbf{p}} ip_x u^y_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}\right)\left(\sum_{\mathbf{q}} iq_y u^x_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}\right) \\
&+ \left(\sum_{\mathbf{p}} ip_y u^x_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}\right)\left(\sum_{\mathbf{q}} iq_x u^y_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}\right) + \left(\sum_{\mathbf{p}} ip_y u^y_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}\right)\left(\sum_{\mathbf{q}} iq_y u^y_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}\right) \\
&= -\sum_{\mathbf{k}} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} (p_x q_x u^x_{\mathbf{p}} u^x_{\mathbf{q}} + p_x q_y u^y_{\mathbf{p}} u^x_{\mathbf{q}} + p_y q_x u^x_{\mathbf{p}} u^y_{\mathbf{q}} + p_y q_y u^y_{\mathbf{p}} u^y_{\mathbf{q}}) e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= -\sum_{\mathbf{k}} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} (\mathbf{p}\cdot u_{\mathbf{q}})(\mathbf{q}\cdot u_{\mathbf{p}}) e^{i\mathbf{k}\cdot\mathbf{x}}
\end{aligned}
$$

Since $\nabla\cdot\mathbf{u}=0$,

$$
\begin{aligned}
0 &= \sum_{\mathbf{k}} (ik_x u^x_{\mathbf{k}} + ik_y u^y_{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= i\sum_{\mathbf{k}} (\mathbf{k}\cdot u_{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{x}}
\end{aligned}
$$

We multiply by $e^{-i\mathbf{j}\cdot\mathbf{x}}$ and integrate:

$$
\begin{aligned}
0 &= \int_0^{2\pi} i\sum_{\mathbf{k}} (\mathbf{k}\cdot u_{\mathbf{k}}) e^{i(\mathbf{k}-\mathbf{j})\cdot\mathbf{x}} d\mathbf{x} \\
&= i(\mathbf{j}\cdot u_{\mathbf{j}})
\end{aligned}
$$

Since $i(\mathbf{q}\cdot u_{\mathbf{q}})=0$ and $i\neq 0$, $\mathbf{q}\cdot u_{\mathbf{q}}=0$. Therefore, $\mathbf{p}\cdot u_{\mathbf{q}} = \mathbf{p}\cdot u_{\mathbf{q}} + \mathbf{q}\cdot u_{\mathbf{q}} = \mathbf{k}\cdot u_{\mathbf{q}}$ and, similarly, $\mathbf{q}\cdot u_{\mathbf{p}} = \mathbf{k}\cdot u_{\mathbf{p}}$. We can now rewrite our equation.

$$
\sum_{\mathbf{k}} |\mathbf{k}|^2 p_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} = -\sum_{\mathbf{k}} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} (\mathbf{k}\cdot u_{\mathbf{q}})(\mathbf{k}\cdot u_{\mathbf{p}}) e^{i\mathbf{k}\cdot\mathbf{x}}
$$

Again, we multiply by $e^{-i\mathbf{j}\cdot\mathbf{x}}$ and integrate:

$$
\begin{aligned}
\int_0^{2\pi} \sum_{\mathbf{k}} |\mathbf{k}|^2 p_{\mathbf{k}} e^{i(\mathbf{k}-\mathbf{j})\cdot\mathbf{x}} d\mathbf{x} &= -\int_0^{2\pi} \sum_{\mathbf{k}} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} (\mathbf{k}\cdot u_{\mathbf{q}})(\mathbf{k}\cdot u_{\mathbf{p}}) e^{i(\mathbf{k}-\mathbf{j})\cdot\mathbf{x}} d\mathbf{x} \\
|\mathbf{j}|^2 p_{\mathbf{j}} &= -\sum_{\mathbf{p}+\mathbf{q}=\mathbf{j}} (\mathbf{j}\cdot u_{\mathbf{q}})(\mathbf{j}\cdot u_{\mathbf{p}})
\end{aligned}
$$

So we can express p as

$$p_{\mathbf{k}} = -\frac{1}{|\mathbf{k}|^2} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} (\mathbf{k} \cdot u_{\mathbf{q}})(\mathbf{k} \cdot u_{\mathbf{p}})$$

If we return to the original equation, we can find a Fourier representation for each term.

$$\mathbf{u}_t = \begin{bmatrix} \frac{\partial u^x}{\partial t} \\ \frac{\partial u^y}{\partial t} \end{bmatrix} = \sum_{\mathbf{k}} \frac{d\mathbf{u}_{\mathbf{k}}}{dt} e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\begin{aligned} \mathbf{u} \cdot \nabla \mathbf{u} = \begin{bmatrix} u^x \frac{\partial u^x}{\partial x} + u^y \frac{\partial u^x}{\partial y} \\ u^x \frac{\partial u^y}{\partial x} + u^y \frac{\partial u^y}{\partial y} \end{bmatrix} &= \sum_{\mathbf{k}} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \begin{bmatrix} iq_x u^x_{\mathbf{p}} u^x_{\mathbf{q}} + iq_y u^y_{\mathbf{p}} u^x_{\mathbf{q}} \\ iq_x u^x_{\mathbf{p}} u^y_{\mathbf{q}} + iq_y u^y_{\mathbf{p}} u^y_{\mathbf{q}} \end{bmatrix} e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= i \sum_{\mathbf{k}} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} (\mathbf{q} \cdot u_{\mathbf{p}}) u_{\mathbf{q}} e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= i \sum_{\mathbf{k}} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} (\mathbf{k} \cdot u_{\mathbf{p}}) u_{\mathbf{q}} e^{i\mathbf{k}\cdot\mathbf{x}} \end{aligned}$$

$$\begin{aligned} -\nabla p = -\begin{bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \end{bmatrix} &= -\sum_{\mathbf{k}} \begin{bmatrix} ik_x p_{\mathbf{k}} \\ ik_y p_{\mathbf{k}} \end{bmatrix} e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= -i \sum_{\mathbf{k}} \mathbf{k} p_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= i \sum_{\mathbf{k}} \frac{\mathbf{k}}{|\mathbf{k}|^2} \left(\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} (\mathbf{k} \cdot u_{\mathbf{q}})(\mathbf{k} \cdot u_{\mathbf{p}}) \right) e^{i\mathbf{k}\cdot\mathbf{x}} \end{aligned}$$

We plug these in to the original equation, and then we multiply by $e^{-i\mathbf{j}\cdot\mathbf{x}}$ and integrate.

$$\begin{aligned} &\int_0^{2\pi} \sum_{\mathbf{k}} \frac{d\mathbf{u}_{\mathbf{k}}}{dt} e^{i(\mathbf{k}-\mathbf{j})\cdot\mathbf{x}} d\mathbf{x} \\ &+ \int_0^{2\pi} i \sum_{\mathbf{k}} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} (\mathbf{k} \cdot u_{\mathbf{p}}) u_{\mathbf{q}} e^{i(\mathbf{k}-\mathbf{j})\cdot\mathbf{x}} d\mathbf{x} \quad = \int_0^{2\pi} i \sum_{\mathbf{k}} \frac{\mathbf{k}}{|\mathbf{k}|^2} \left(\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} (\mathbf{k} \cdot u_{\mathbf{q}})(\mathbf{k} \cdot u_{\mathbf{p}}) \right) e^{i(\mathbf{k}-\mathbf{j})\cdot\mathbf{x}} d\mathbf{x} \end{aligned}$$

$$\frac{d\mathbf{u}_{\mathbf{j}}}{dt} + i \sum_{\mathbf{p}+\mathbf{q}=\mathbf{j}} (\mathbf{j} \cdot u_{\mathbf{p}}) u_{\mathbf{q}} = i \frac{\mathbf{j}}{|\mathbf{j}|^2} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{j}} (\mathbf{j} \cdot u_{\mathbf{q}})(\mathbf{j} \cdot u_{\mathbf{p}})$$

So we have our final Fourier transform

$$\frac{d\mathbf{u}_{\mathbf{k}}}{dt} = -i \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \left((\mathbf{k} \cdot u_{\mathbf{p}}) u_{\mathbf{q}} - \frac{\mathbf{k}}{|\mathbf{k}|^2} (\mathbf{k} \cdot u_{\mathbf{q}})(\mathbf{k} \cdot u_{\mathbf{p}}) \right)$$

which we can represent more clearly using a matrix A as

$$\frac{d\mathbf{u}_{\mathbf{k}}}{dt} = -i \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \mathbf{k} \cdot u_{\mathbf{p}} A_{\mathbf{k}} u_{\mathbf{q}}$$

where $A_{\mathbf{k}} = I - \frac{\mathbf{k}\mathbf{k}^T}{|\mathbf{k}|^2}$.