Adversarial Kelly Betting

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Abstract

Bla bla bla

1 Introduction

2 Kelly Betting

3 Algorithm

Define $\theta_{t-1} = \sum_{i=1}^{t} g_i$ where $|g_t| \leq 1$. Predict with $w_t = \beta_t \left(\epsilon + \sum_{i=1}^{t-1} w_i g_i\right)$ where β_t will be defined in the following. Define $L_t = \epsilon + \sum_{i=1}^{t} w_i g_i$, so that $L_0 = \epsilon$. We have $L_t = L_{t-1} + w_t g_t = L_{t-1} + \beta_t\, g_t\, L_{t-1} = L_{t-1}(1 + \beta_t g_t)$. Note that to be a proper betting strategy, $|\beta_t|$ must be strictly less than 1, otherwise the algorithm could lose all the money in one round. For the L_0 we know, by definition that $L_0 = \epsilon$.

We will prove by induction a lower bound on L_T.

3.1 Data independent bound

We will make use of the following lower bound

Lemma 1. *There exist a, b such that $\log(1 + x) \geq x - bx^2$, for $-2/a \leq x \leq 2/a$, $a > 2$, $a \geq 4b$.*

Proof. For example, we can set $a > 3.44$ and $b = a/4$ and verify the inequality numerically. □

Note that the inequality above is equivalent to $1 + x \geq \exp(x - bx^2)$.

Assume that $L_{t-1} \geq \epsilon \exp(\frac{\theta_{t-1}^2}{a(t-1)} - \sum_{i=1}^{t-1} \frac{1}{ai})$. We have to prove that $L_t \geq \epsilon \exp(\frac{\theta_t^2}{at} - \sum_{i=1}^{t} \frac{1}{ai})$.

$$
\begin{aligned}
L_t &= L_{t-1}(1+\beta_t g_t) && (1)\\
&\geq (1+\beta_t g_t)\epsilon \exp(\frac{\theta_{t-1}^2}{a(t-1)} - \sum_{i=1}^{t-1} \frac{1}{ai}) && (2)\\
&= \epsilon \exp(\frac{\theta_{t-1}^2}{a(t-1)} + \log(1+\beta_t g_t) - \sum_{i=1}^{t-1} \frac{1}{ai}). && (3)
\end{aligned}
$$

Hence, we lower bound the quantity $\frac{\theta_{t-1}^2}{a(t-1)} + \log(1+\beta_t g_t) - \frac{\theta_t^2}{at} - \sum_{i=1}^{t-1} \frac{1}{ai}$.

$$
\begin{aligned}
&\frac{\theta_{t-1}^2}{a(t-1)} + \log(1+\beta_t g_t) - \frac{\theta_{t-1}^2 + 2\theta_{t-1} g_t + g_t^2}{at} - \sum_{i=1}^{t-1} \frac{1}{ai} && (4)\\
&= \frac{\theta_{t-1}^2}{a}(\frac{1}{t-1} - \frac{1}{t}) + \log(1+\beta_t g_t) - \frac{2\theta_{t-1} g_t + g_t^2}{at} - \sum_{i=1}^{t-1} \frac{1}{ai} && (5)\\
&\geq \frac{\theta_{t-1}^2}{a\,t\,(t-1)} + \log(1+\beta_t g_t) - \frac{2\theta_{t-1} g_t}{at} - \sum_{i=1}^{t} \frac{1}{ai} && (6)\\
&\geq \frac{\theta_{t-1}^2}{a\,t\,(t-1)} + \beta_t g_t - b\,(\beta_t g_t)^2 - \frac{\theta_{t-1} g_t}{at} - \sum_{i=1}^{t} \frac{1}{ai}. && (7)
\end{aligned}
$$

If we set $\beta_t = \frac{2\theta_{t-1}}{at}$ we have

$$
\begin{aligned}
&\frac{\theta_{t-1}^2}{a(t-1)} + \log(1+\beta_t g_t) - \frac{\theta_{t-1}^2 + 2\theta_{t-1} g_t + g_t^2}{at} - \sum_{i=1}^{t-1} \frac{1}{ai} && (8)\\
&\geq \frac{\theta_{t-1}^2}{a\,t\,(t-1)} - b\frac{4\theta_{t-1}^2}{a^2 t^2} - \sum_{i=1}^{t} \frac{1}{ai} && (9)\\
&\geq -\sum_{i=1}^{t} \frac{1}{ai} && (10)
\end{aligned}
$$

where we used the fact that $a > 4b$ that is $1/a > 4b/a^2$. Hence, by induction, we have

$$
\begin{aligned}
L_T &\geq \epsilon \exp(\frac{\theta_T^2}{aT} - \sum_{i=1}^{T} \frac{1}{ai}) && (11)\\
&\geq \epsilon \exp(\frac{\theta_T^2}{aT} - \frac{1}{a}\log(T) - \frac{1}{a}) && (12)\\
&= \frac{\epsilon \exp(-\frac{1}{a})}{T^{\frac{1}{a}}} \exp(\frac{\theta_T^2}{aT}) && (13)
\end{aligned}
$$

3.2 Data dependent bound

Assume that $|g_t| \leq G_t$ and that you receive at each round t the quantity G_t before making the bet.

Define $a_t = 2\max_{i\leq t} G_i$, $S_t = \sum_{i=1}^t |g_i| + \delta$ and $\theta_t = \sum_{i=1}^t g_i$.

Assume that $L_{t-1} \geq \epsilon \exp(\frac{\theta_{t-1}^2}{a_{t-1}S_{t-1}} - \sum_{i=1}^{t-1} \frac{|g_i|G_i}{a_i S_{i-1}+a_i G_i})$. We have to prove that $L_t \geq \epsilon \exp(\frac{\theta_t^2}{a_t S_t} - \sum_{i=1}^{t} \frac{|g_i|G_i}{a_i S_{i-1}+a_i G_i}))$.

$$L_t = L_{t-1}(1+\beta_t g_t) \tag{14}$$

$$\geq (1+\beta_t g_t)\epsilon \exp(\frac{\theta_{t-1}^2}{a_{t-1}S_{t-1}} - \sum_{i=1}^{t-1} \frac{|g_i|G_i}{a_i S_{i-1}+a_i G_i}) \tag{15}$$

$$= \epsilon \exp(\frac{\theta_{t-1}^2}{a_{t-1}S_{t-1}} + \log(1+\beta_t g_t) - \sum_{i=1}^{t-1} \frac{|g_i|G_i}{a_i S_{i-1}+a_i G_i}) \tag{16}$$

$$\geq \epsilon \exp(\frac{\theta_{t-1}^2}{a_{t}S_{t-1}} + \log(1+\beta_t g_t) - \sum_{i=1}^{t-1} \frac{|g_i|G_i}{a_i S_{i-1}+a_i G_i}). \tag{17}$$

Consider the function

$$\phi(x) = -\log(1+\beta_t x) + \frac{(\theta_{t-1}+x)^2}{a_t S_{t-1} + a_t|x|}.$$

We have that $\phi(x)$ is piece-wise convex on $[-\infty, 0]$ and $[0, \infty]$. Hence, we have that

$$\phi(x) \leq \phi(0) + \frac{x}{G_t}(\phi(G_t) - \phi(0)), \forall 0 \leq x \leq G_t \tag{18}$$

$$\phi(x) \leq \phi(0) + \frac{x}{G_t}(\phi(0) - \phi(-G_t)), \forall -G_t \leq x \leq 0. \tag{19}$$

We now use set β_t such that $\phi(G_t) = \phi(-G_t)$, that is

$$\beta_t = \frac{1}{G_t}\frac{A_{t-1}-1}{A_{t-1}+1} = \frac{1}{G_t}\left(2\,sigmoid\left(\frac{4\theta_{t-1}G_t}{a_t S_{t-1} + a_t G_t}\right) - 1\right)$$

where $A_{t-1} = \exp\left(\frac{4\theta_{t-1}G_t}{a_t S_{t-1}+a_t G_t}\right)$ and $sigmoid(x) = \frac{1}{1+\exp(-x)}$. Hence we have

$$\phi(x) \leq \phi(0) + \frac{|x|}{G_t}(\phi(G_t) - \phi(0)), \forall -G_t \leq x \leq G_t$$

that is

$$\frac{\theta_{t-1}^2}{a_t S_{t-1}} - \frac{(\theta_{t-1}+x)^2}{a_t S_{t-1} + a_t|x|} + \log(1+\beta_t g_t) = \phi(0) - \phi(x) \geq \frac{|x|}{G_t}(\phi(0) - \phi(G_t)) \tag{20}$$

$$= \frac{|x|}{G_t}(\frac{\theta_{t-1}^2}{a_t S_{t-1}} - \frac{(\theta_{t-1}+G_t)^2}{a_t S_{t-1} + a_t G_t} + \log(1+\beta_t G_t)), \forall -G_t \leq x \leq G_t. \tag{21}$$

Using this relation we have that

$$
\begin{aligned}
&-\frac{(\theta_{t-1}+g_t)^2}{a_t S_{t-1}+a_t|g_t|}+\frac{\theta_{t-1}^2}{a_t S_{t-1}}+\log(1+\beta_t g_t)-\sum_{i=1}^{t-1}\frac{|g_i|G_i}{a_i S_{i-1}+a_i G_i} && (22)\\
&\geq \frac{|g_t|}{G_t}\Big(\frac{\theta_{t-1}^2}{a_t S_{t-1}}-\frac{(\theta_{t-1}+G_t)^2}{a_t S_{t-1}+a_t G_t}+\log(1+\beta_t G_t)\Big)-\sum_{i=1}^{t-1}\frac{|g_i|G_i}{a_i S_{i-1}+a_i G_i} && (23)\\
&= \frac{|g_t|}{G_t}\Big(\frac{a_t G_t \theta_{t-1}^2-2G_t\theta_{t-1}S_{t-1}}{a_t S_{t-1}(a_t S_{t-1}+a_t G_t)}+\log(1+\beta_t G_t)\Big)-\sum_{i=1}^{t}\frac{|g_i|G_i}{a_i S_{i-1}+a_i G_i} && (24)\\
&\geq \frac{|g_t|}{G_t}\Big(\frac{a_t G_t \theta_{t-1}^2}{a_t S_{t-1}(a_t S_{t-1}+a_t G_t)}-\frac{2G_t\theta_{t-1}}{a_t S_{t-1}+a_t G_t}+\log(1+\beta_t G_t)\Big)-\sum_{i=1}^{t}\frac{|g_i|G_i}{a_i S_{i-1}+a_i G_i}. && (25)
\end{aligned}
$$

We now use the Taylor expansion, to obtain

$$
\log\left(1+\frac{\exp(x)-1}{\exp(x)+1}\right) \geq \frac{x}{2}-\frac{x^2}{8} \qquad \forall x \in \mathbb{R}
$$

and, using the expression of β_t, have

$$
\log\left(1+\beta_t G_t\right)=\log\left(1+\frac{\exp\left(\frac{4\theta_{t-1}G_t}{a_t S_{t-1}+a_t G_t}\right)-1}{\exp\left(\frac{4\theta_{t-1}G_t}{a_t S_{t-1}+a_t G_t}\right)+1}\right) \geq \frac{2\theta_{t-1}G_t}{a_t S_{t-1}+a_t G_t}-\frac{2\theta_{t-1}^2 G_t^2}{(a_t S_{t-1}+a_t G_t)^2}.
$$

Hence the expression

$$
\frac{a_t G_t \theta_{t-1}^2}{a_t S_{t-1}(a_t S_{t-1}+a_t G_t)}-\frac{2G_t\theta_{t-1}}{a_t S_{t-1}+a_t G_t}+\log(1+\beta_t G_t))
$$

is greater than zero if $a_t \geq 2G_t$, that is true by definition of a_t.

By induction, the final lower bound is

$$
L_T \geq \epsilon \exp\left(\frac{\theta_T^2}{a_T S_{T-1}+a_T G_T}-\sum_{i=1}^{T}\frac{|g_i|G_i}{a_i S_{i-1}+a_i G_i}\right)
$$

3.3 Data dependent bound, version 2

Assume that $|g_t| \leq G_t$ and that you receive at each round t the quantity G_t before making the bet.

Define $S_t=\sum_{i=1}^t |g_i|G_i+\delta$ and $\theta_t=\sum_{i=1}^t g_i$.

Assume that $L_{t-1} \geq \epsilon \exp(\frac{\theta_{t-1}^2}{aS_{t-1}}-\sum_{i=1}^{t-1}\frac{|g_i|G_i}{aS_{i-1}+aG_i^2})$. We have to prove that

$L_t \geq \epsilon \exp(\frac{\theta_t^2}{aS_t} - \sum_{i=1}^{t} \frac{|g_i|G_i}{aS_{i-1}+aG_i^2}))$.

$$L_t = L_{t-1}(1+\beta_t g_t) \tag{26}$$
$$\geq (1+\beta_t g_t)\epsilon \exp(\frac{\theta_{t-1}^2}{aS_{t-1}} - \sum_{i=1}^{t-1} \frac{|g_i|G_i}{aS_{i-1}+aG_i^2}) \tag{27}$$
$$= \epsilon \exp(\frac{\theta_{t-1}^2}{aS_{t-1}} + \log(1+\beta_t g_t) - \sum_{i=1}^{t-1} \frac{|g_i|G_i}{aS_{i-1}+aG_i^2}). \tag{28}$$

Consider the function

$$\phi(x) = -\log(1+\beta_t x) + \frac{(\theta_{t-1}+x)^2}{aS_{t-1}+a|x|G_t}.$$

We have that $\phi(x)$ is piece-wise convex on $[-\infty, 0]$ and $[0, \infty]$. Hence, we have that

$$\phi(x) \leq \phi(0) + \frac{x}{G_t}(\phi(G_t) - \phi(0)), \forall 0 \leq x \leq G_t \tag{29}$$
$$\phi(x) \leq \phi(0) + \frac{x}{G_t}(\phi(0) - \phi(-G_t)), \forall -G_t \leq x \leq 0. \tag{30}$$

We now use set β_t such that $\phi(G_t) = \phi(-G_t)$, that is

$$\beta_t = \frac{1}{G_t}\frac{A_{t-1}-1}{A_{t-1}+1} = \frac{1}{G_t}\left(2\, sigmoid\left(\frac{4\theta_{t-1}G_t}{aS_{t-1}+aG_t^2}\right) - 1\right)$$

where $A_{t-1} = \exp\left(\frac{4\theta_{t-1}G_t}{a_t S_{t-1}+a_t G_t}\right)$ and $sigmoid(x) = \frac{1}{1+\exp(-x)}$. Hence we have

$$\phi(x) \leq \phi(0) + \frac{|x|}{G_t}(\phi(G_t) - \phi(0)), \forall -G_t \leq x \leq G_t$$

that is

$$\frac{\theta_{t-1}^2}{aS_{t-1}} - \frac{(\theta_{t-1}+x)^2}{aS_{t-1}+a|x|G_t} + \log(1+\beta_t g_t) = \phi(0) - \phi(x) \geq \frac{|x|}{G_t}(\phi(0) - \phi(G_t)) \tag{31}$$
$$= \frac{|x|}{G_t}(\frac{\theta_{t-1}^2}{aS_{t-1}} - \frac{(\theta_{t-1}+G_t)^2}{aS_{t-1}+aG_t^2} + \log(1+\beta_t G_t)), \forall -G_t \leq x \leq G_t. \tag{32}$$

Using this relation we have that

$$-\frac{(\theta_{t-1}+g_t)^2}{aS_{t-1}+a|g_t|G_t}+\frac{\theta_{t-1}^2}{aS_{t-1}}+\log(1+\beta_t g_t)-\sum_{i=1}^{t-1}\frac{|g_i|G_i}{aS_{i-1}+aG_i^2} \tag{33}$$

$$\geq \frac{|g_t|}{G_t}(\frac{\theta_{t-1}^2}{aS_{t-1}}-\frac{(\theta_{t-1}+G_t)^2}{aS_{t-1}+aG_t^2}+\log(1+\beta_t G_t))-\sum_{i=1}^{t-1}\frac{|g_i|G_i}{a_iS_{i-1}+aG_i^2} \tag{34}$$

$$=\frac{|g_t|}{G_t}(\frac{aG_t^2\theta_{t-1}^2-2G_t\theta_{t-1}S_{t-1}}{aS_{t-1}(aS_{t-1}+aG_t^2)}+\log(1+\beta_t G_t))-\sum_{i=1}^{t}\frac{|g_i|G_i}{aS_{i-1}+aG_i^2} \tag{35}$$

$$\geq \frac{|g_t|}{G_t}(\frac{aG_t^2\theta_{t-1}^2}{aS_{t-1}(aS_{t-1}+aG_t^2)}-\frac{2G_t\theta_{t-1}}{aS_{t-1}+aG_t^2}+\log(1+\beta_t G_t))-\sum_{i=1}^{t}\frac{|g_i|G_i}{aS_{i-1}+aG_i^2}. \tag{36}$$

We now use the Taylor expansion, to obtain

$$\log\left(1+\frac{\exp(x)-1}{\exp(x)+1}\right)\geq\frac{x}{2}-\frac{x^2}{8} \qquad \forall x\in\mathbb{R}$$

and, using the expression of β_t, have

$$\log(1+\beta_t G_t)=\log\left(1+\frac{\exp\left(\frac{4\theta_{t-1}G_t}{aS_{t-1}+aG_t^2}\right)-1}{\exp\left(\frac{4\theta_{t-1}G_t}{aS_{t-1}+aG_t^2}\right)+1}\right)\geq\frac{2\theta_{t-1}G_t}{aS_{t-1}+aG_t^2}-\frac{2\theta_{t-1}^2G_t^2}{(aS_{t-1}+aG_t^2)^2}.$$

Hence the expression

$$\frac{aG_t^2\theta_{t-1}^2}{aS_{t-1}(aS_{t-1}+aG_t^2)}-\frac{2G_t\theta_{t-1}}{aS_{t-1}+aG_t^2}+\log(1+\beta_t G_t))$$

is greater than zero if $a_t\geq 2$, that is true by definition of a_t.

By induction, the final lower bound is

$$L_T\geq\epsilon\exp\left(\frac{\theta_T^2}{aS_T}-\sum_{i=1}^{T}\frac{|g_i|G_i}{aS_{i-1}+aG_i^2}\right)$$

3.4 Data dependent bound, version 3

Assume that $|g_t|\leq G_t$ and that you receive at each round t the quantity G_t before making the bet.

Define $S_t=\sum_{i=1}^t|g_i|G_i$ and $\theta_t=\sum_{i=1}^t g_i$.

Assume that $L_{t-1}\geq\epsilon\exp(\frac{\theta_{t-1}^2}{a(S_{t-1}+\delta_{t-1})}-\frac{1}{a}\sum_{i=1}^{t-1}\frac{|g_i|G_i}{S_{i-1}+G_i^2+\delta_i})$. We have to prove that $L_t \geq \epsilon \exp(\frac{\theta_t^2}{a(S_t+\delta_t)} - \frac{1}{a}\sum_{i=1}^t \frac{|g_i|G_i}{S_{i-1}+G_i^2+\delta_i}))$.

$$
\begin{aligned}
L_t &= L_{t-1}(1+\beta_t g_t) && (37)\\
&\geq (1+\beta_t g_t)\epsilon \exp(\frac{\theta_{t-1}^2}{a(S_{t-1}+\delta_{t-1})} - \frac{1}{a}\sum_{i=1}^{t-1}\frac{|g_i|G_i}{S_{i-1}+G_i^2+\delta_i}) && (38)\\
&= \epsilon \exp(\frac{\theta_{t-1}^2}{a(S_{t-1}+\delta_{t-1})} + \log(1+\beta_t g_t) - \frac{1}{a}\sum_{i=1}^{t-1}\frac{|g_i|G_i}{S_{i-1}+G_i^2+\delta_i}). && (39)
\end{aligned}
$$

Consider the function

$$\phi(x) = -\log(1+\beta_t x) + \frac{(\theta_{t-1}+x)^2}{aS_{t-1}+a|x|G_t+a\delta_t}.$$

We have that $\phi(x)$ is piece-wise convex on $[-\infty, 0]$ and $[0, \infty]$. Hence, we have that

$$
\begin{aligned}
\phi(x) &\leq \phi(0) + \frac{x}{G_t}(\phi(G_t)-\phi(0)), \forall 0 \leq x \leq G_t && (40)\\
\phi(x) &\leq \phi(0) + \frac{x}{G_t}(\phi(0)-\phi(-G_t)), \forall -G_t \leq x \leq 0. && (41)
\end{aligned}
$$

We now use set β_t such that $\phi(G_t) = \phi(-G_t)$, that is

$$\beta_t = \frac{1}{G_t}\frac{A_{t-1}-1}{A_{t-1}+1} = \frac{1}{G_t}\left(2\, sigmoid\left(\frac{4\theta_{t-1}G_t}{aS_{t-1}+aG_t^2+a\delta_t}\right)-1\right)$$

where $A_{t-1} = \exp\left(\frac{4\theta_{t-1}G_t}{a_t S_{t-1}+aG_t+a\delta_t}\right)$ and $sigmoid(x) = \frac{1}{1+\exp(-x)}$. Hence we have

$$\phi(x) \leq \phi(0) + \frac{|x|}{G_t}(\phi(G_t)-\phi(0)), \forall -G_t \leq x \leq G_t$$

that is

$$
\begin{aligned}
&\frac{\theta_{t-1}^2}{a(S_{t-1}+\delta_{t-1})} - \frac{(\theta_{t-1}+x)^2}{aS_{t-1}+a|x|G_t+a\delta_t} + \log(1+\beta_t g_t) && (42)\\
&\geq \frac{\theta_{t-1}^2}{a(S_{t-1}+\delta_t)} - \frac{(\theta_{t-1}+x)^2}{aS_{t-1}+a|x|G_t+a\delta_t} + \log(1+\beta_t g_t) && (43)\\
&= \phi(0) - \phi(x) \geq \frac{|x|}{G_t}(\phi(0)-\phi(G_t)) && (44)\\
&= \frac{|x|}{G_t}(\frac{\theta_{t-1}^2}{a(S_{t-1}+\delta_t)} - \frac{(\theta_{t-1}+G_t)^2}{aS_{t-1}+aG_t^2+a\delta_t} + \log(1+\beta_t G_t)), \forall -G_t \leq x \leq G_t. && (45)
\end{aligned}
$$

Using this relation we have that

$$
\begin{aligned}
&- \frac{(\theta_{t-1}+g_t)^2}{a(S_{t-1}+|g_t|G_t+\delta_t)} + \frac{\theta_{t-1}^2}{a(S_{t-1}+\delta_{t-1})} + \log(1+\beta_t g_t) - \frac{1}{a}\sum_{i=1}^{t-1} \frac{|g_i|G_i}{S_{i-1}+G_i^2+\delta_i} && (46)\\
&\geq \frac{|g_t|}{G_t}\left(\frac{\theta_{t-1}^2}{a(S_{t-1}+\delta_t)} - \frac{(\theta_{t-1}+G_t)^2}{a(S_{t-1}+G_t^2+\delta_t)} + \log(1+\beta_t G_t)\right) - \frac{1}{a}\sum_{i=1}^{t-1} \frac{|g_i|G_i}{S_{i-1}+G_i^2+\delta_i} && (47)\\
&= \frac{|g_t|}{G_t}\left(\frac{G_t^2\theta_{t-1}^2 - 2G_t\theta_{t-1}(S_{t-1}+\delta_t)}{a(S_{t-1}+\delta_t)(S_{t-1}+G_t^2+\delta_t)} + \log(1+\beta_t G_t)\right) - \frac{1}{a}\sum_{i=1}^{t} \frac{|g_i|G_i}{S_{i-1}+G_i^2+\delta_i} && (48)\\
&= \frac{|g_t|}{G_t}\left(\frac{G_t^2\theta_{t-1}^2}{a(S_{t-1}+\delta_t)(S_{t-1}+G_t^2+\delta_t)} - \frac{2G_t\theta_{t-1}}{a(S_{t-1}+G_t^2+\delta_t)} + \log(1+\beta_t G_t)\right) - \frac{1}{a}\sum_{i=1}^{t} \frac{|g_i|G_i}{S_{i-1}+G_i^2+\delta_i}. && (49)
\end{aligned}
$$

We now use the Taylor expansion, to obtain

$$\log\left(1+\frac{\exp(x)-1}{\exp(x)+1}\right) \geq \frac{x}{2} - \frac{x^2}{8} \qquad \forall x \in \mathbb{R}$$

and, using the expression of β_t, have

$$\log(1+\beta_t G_t) = \log\left(1+\frac{\exp\left(\frac{4\theta_{t-1}G_t}{a(S_{t-1}+G_t^2+\delta_t)}\right)-1}{\exp\left(\frac{4\theta_{t-1}G_t}{a(S_{t-1}+G_t^2+\delta_t)}\right)+1}\right) \geq \frac{2\theta_{t-1}G_t}{a(S_{t-1}+G_t^2+\delta_t)} - \frac{2\theta_{t-1}^2G_t^2}{a^2(S_{t-1}+G_t^2+\delta_t)^2}.$$

Hence the expression

$$\frac{G_t^2\theta_{t-1}^2}{a(S_{t-1}+\delta_t)(S_{t-1}+G_t^2+\delta_t)} - \frac{2G_t\theta_{t-1}}{a(S_{t-1}+G_t^2+\delta_t)} + \log(1+\beta_t G_t))$$

is greater than zero if $a \geq 2$, that is true by definition of a.

By induction, the final lower bound is

$$L_T \geq \epsilon \exp\left(\frac{\theta_T^2}{a(S_T+\delta_T)} - \frac{1}{a}\sum_{i=1}^{T}\frac{|g_i|G_i}{S_{i-1}+G_i^2+\delta_i}\right)$$

We can set δ_t such that $\frac{|g_i|G_i}{S_{i-1}+G_i^2+\delta_i} \leq \frac{1}{i}$ and δ_i are increasing.

3.5 Data dependent bound, no δ

Assume that $|g_t| \leq G_t$ and that you receive at each round t the quantity G_t before making the bet.

Define $a_t = 2\max_{i\leq t} G_i^2$, $S_t = \sum_{i=1}^{t} \frac{|g_i|}{G_t} + 1$ and $\theta_t = \sum_{i=1}^{t} g_i$.

Assume that $L_{t-1} \geq \epsilon \exp(\frac{\theta_{t-1}^2}{a_{t-1}S_{t-1}} - \sum_{i=1}^{t-1} \frac{|g_i|G_i}{a_i S_{i-1}+a_i})$. We have to prove that $L_t \geq \epsilon \exp(\frac{\theta_t^2}{a_t S_t} - \sum_{i=1}^{t} \frac{|g_i|G_i}{a_i S_{i-1}+a_i}))$.

$$\begin{aligned}
L_t &= L_{t-1}(1+\beta_t g_t) && (50)\\
&\geq (1+\beta_t g_t)\epsilon \exp(\frac{\theta_{t-1}^2}{a_{t-1}S_{t-1}} - \sum_{i=1}^{t-1} \frac{|g_i|G_i}{a_i S_{i-1}+a_i}) && (51)\\
&= \epsilon \exp(\frac{\theta_{t-1}^2}{a_{t-1}S_{t-1}} + \log(1+\beta_t g_t) - \sum_{i=1}^{t-1} \frac{|g_i|G_i}{a_i S_{i-1}+a_i}) && (52)\\
&\geq \epsilon \exp(\frac{\theta_{t-1}^2}{a_t S_{t-1}} + \log(1+\beta_t g_t) - \sum_{i=1}^{t-1} \frac{|g_i|G_i}{a_i S_{i-1}+a_i}). && (53)
\end{aligned}$$

Consider the function

$$\phi(x) = -\log(1+\beta_t x) + \frac{(\theta_{t-1}+x)^2}{a_t S_{t-1} + a_t \frac{|x|}{G_t}}.$$

We have that $\phi(x)$ is piece-wise convex on $[-\infty, 0]$ and $[0, \infty]$. Hence, we have that

$$\phi(x) \leq \phi(0) + \frac{x}{G_t}(\phi(G_t) - \phi(0)), \forall 0 \leq x \leq G_t \quad (54)$$

$$\phi(x) \leq \phi(0) + \frac{x}{G_t}(\phi(0) - \phi(-G_t)), \forall -G_t \leq x \leq 0. \quad (55)$$

We now use set β_t such that $\phi(G_t) = \phi(-G_t)$, that is

$$\beta_t = \frac{1}{G_t}\frac{A_{t-1}-1}{A_{t-1}+1} = \frac{1}{G_t}\left(2\, sigmoid\left(\frac{4\theta_{t-1}G_t}{a_t S_{t-1}+a_t}\right) - 1\right)$$

where $A_{t-1} = \exp\left(\frac{4\theta_{t-1}G_t}{a_t S_{t-1}+a_t}\right)$ and $sigmoid(x) = \frac{1}{1+\exp(-x)}$. Hence we have

$$\phi(x) \leq \phi(0) + \frac{|x|}{G_t}(\phi(G_t) - \phi(0)), \forall -G_t \leq x \leq G_t$$

that is

$$\frac{\theta_{t-1}^2}{a_t S_{t-1}} - \frac{(\theta_{t-1}+x)^2}{a_t S_{t-1} + a_t \frac{|x|}{G_t}} + \log(1+\beta_t g_t) = \phi(0) - \phi(x) \geq \frac{|x|}{G_t}(\phi(0) - \phi(G_t)) \quad (56)$$

$$= \frac{|x|}{G_t}(\frac{\theta_{t-1}^2}{a_t S_{t-1}} - \frac{(\theta_{t-1}+G_t)^2}{a_t S_{t-1}+a_t} + \log(1+\beta_t G_t)), \forall -G_t \leq x \leq G_t. \quad (57)$$

Using this relation we have that

$$
\begin{aligned}
& -\frac{(\theta_{t-1}+g_t)^2}{a_t S_{t-1} + a_t \frac{|g_t|}{G_t}} + \frac{\theta_{t-1}^2}{a_t S_{t-1}} + \log(1+\beta_t g_t) - \sum_{i=1}^{t-1} \frac{|g_i| G_i}{a_i S_{i-1} + a_i} && (58)\\
& \geq \frac{|g_t|}{G_t}\left(\frac{\theta_{t-1}^2}{a_t S_{t-1}} - \frac{(\theta_{t-1}+G_t)^2}{a_t S_{t-1} + a_t} + \log(1+\beta_t G_t)\right) - \sum_{i=1}^{t-1} \frac{|g_i| G_i}{a_i S_{i-1} + a_i} && (59)\\
& = \frac{|g_t|}{G_t}\left(\frac{\theta_{t-1}^2 - 2G_t\theta_{t-1}S_{t-1}}{a_t S_{t-1}(S_{t-1}+1)} + \log(1+\beta_t G_t)\right) - \sum_{i=1}^{t} \frac{|g_i| G_i}{a_i S_{i-1} + a_i} && (60)\\
& = \frac{|g_t|}{G_t}\left(\frac{\theta_{t-1}^2}{a_t S_{t-1}(S_{t-1}+1)} - \frac{2G_t\theta_{t-1}}{a_t S_{t-1} + a_t} + \log(1+\beta_t G_t)\right) - \sum_{i=1}^{t} \frac{|g_i| G_i}{a_i S_{i-1} + a_i}. && (61)
\end{aligned}
$$

We now use the Taylor expansion, to obtain

$$\log\left(1 + \frac{\exp(x)-1}{\exp(x)+1}\right) \geq \frac{x}{2} - \frac{x^2}{8} \qquad \forall x \in \mathbb{R}$$

and, using the expression of β_t, have

$$\log\left(1+\beta_t G_t\right) = \log\left(1 + \frac{\exp\left(\frac{4\theta_{t-1}G_t}{a_t S_{t-1}+a_t}\right) - 1}{\exp\left(\frac{4\theta_{t-1}G_t}{a_t S_{t-1}+a_t}\right) + 1}\right) \geq \frac{2\theta_{t-1}G_t}{a_t S_{t-1}+a_t} - \frac{2\theta_{t-1}^2 G_t^2}{(a_t S_{t-1}+a_t)^2}.$$

Hence the expression

$$\frac{\theta_{t-1}^2}{a_t S_{t-1}(S_{t-1}+1)} - \frac{2G_t\theta_{t-1}}{a_t S_{t-1} + a_t} + \log(1+\beta_t G_t))$$

is greater than zero if $a_t \geq 2G_t^2$, that is true by definition of a_t.

By induction, the final lower bound is

$$
\begin{aligned}
L_T &\geq \epsilon \exp\left(\frac{\theta_T^2}{a_T S_T} - \sum_{i=1}^{T} \frac{|g_i| G_i}{a_i S_{i-1} + a_i}\right) && (62)\\
&\geq \epsilon \exp\left(\frac{\theta_T^2}{2G_T^2 S_T} - \frac{1}{2}\sum_{i=1}^{T} \frac{\frac{|g_i|}{G_i}}{S_i}\right) && (63)\\
&\geq \epsilon \exp\left(\frac{\theta_T^2}{2G_T^2 S_T} - \frac{1}{2}\log\left(1 + \sum_{i=1}^{T} \frac{|g_i|}{G_i}\right)\right) && (64)
\end{aligned}
$$

It is interesting to note that β_t is *not* between -1 and 1, but between $-1/G_t$ and $1/G_t$.

The ideal case of $\exp\left(\frac{\theta_T^2}{\sum_{t=1}^T g_t^2}\right)$ is equal to

$$\exp\left(\frac{\theta_T^2}{T\frac{\sum_{t=1}^T g_t^2}{T}}\right) \approx \exp\left(\frac{\theta_T^2}{\left(\sum_{t=1}^T \frac{|g_t|}{G_t}\right)\frac{\sum_{t=1}^T g_t^2}{T}}\right) \approx \exp\left(\frac{\theta_T^2}{\left(\sum_{t=1}^T \frac{|g_t|}{G_t}\right)\max_{t\leq T} g_t^2}\right)$$

4 Upper Bounds on the Oracle Betting Strategies

We will now upper bound the maximum reward achievable by any betting strategy. First we will analyze the class of betting strategy that bets a fixed amount of the current reward for the entire game. The following theorem is well-known, and it has been shown, for example, in .

Theorem 1. *Define the reward at time t as* $L_t = \epsilon + \sum_{i=1}^t w_t z_t$*, where* $z_t \in \{-1, 1\}$*, and the algorithm bets a fixed fraction of his current reward, i.e.* $w_t = \beta L_{t-1}$*. Then, for any sequence* $z_1, \dots, z_T$*, we have*

$$\sum_{t=1}^T w_t z_t \le \epsilon \exp\left(T\, KL\left(\frac{\sum_{t=1}^T z_t}{T}, \frac{1}{2}\right)\right) \le \epsilon \exp\left(\frac{(\sum_{t=1}^T z_t)^2}{2T} + \frac{(\sum_{t=1}^T z_t)^4}{5T^3}\right).$$

Proof. From the betting strategy we have

$$L_t = L_{t-1} + w_t\, g_t = L_{t-1} + \beta\, L_{t-1}\, z_t = L_{t-1}(1 + \beta\, z_t)$$

Hence

$$L_T = \epsilon \prod_{t=1}^T (1 + \beta z_t) = \epsilon (1+\beta)^{\frac{T+Z}{2}} (1-\beta)^{\frac{T-Z}{2}},$$

where $Z = \sum_{t=1}^T z_t$. It is easy to show that the maximum value of L_T w.r.t. β is in $\beta = \frac{Z}{T}$. Hence, we have

$$L_T = \epsilon(1 + \frac{Z}{T})^{\frac{T+Z}{2}} (1 - \frac{Z}{T})^{\frac{T-Z}{2}} \quad = \epsilon\left[(1 + \frac{Z}{T})^{\frac{1+\frac{Z}{T}}{2}} (1 - \frac{Z}{T})^{\frac{1-\frac{Z}{T}}{2}}\right]^T \le \epsilon \exp\left(\frac{Z^2}{2T} + \frac{Z^4}{5T^3}\right) \tag{65}$$

where we used the simple inequalities

$$\frac{x^2}{2} + \frac{x^4}{12} \le \frac{1+x}{2}\log(1+x) + \frac{1-x}{2}\log(1-x) \le \frac{x^2}{2} + \frac{x^4}{5}.$$

□

The second class of betting strategy we consider can bet any amount of money at each round, but we still require to never have a negative total reward, that is to bet at each round a fraction of the total reward. Theorem 1 tells us that the upper bound is at least $\epsilon \exp\left(\frac{(\sum_{t=1}^T z_t)^2}{2T}\right)$. Of course, this class of betting strategies is bigger than the previous one, so we expect a bigger upper bound. So, it is natural to investigate the possibility to obtain a reward of the form $\epsilon \exp\left(\frac{(\sum_{t=1}^T z_t)^2}{\alpha T}\right)$, for a small α. Precisely, the following theorem shows that α cannot be too small.

Theorem 2. *For any sequence of betting* w_t*, there exist a sequence* z_t *in* $\{-1, 1\}, t = 1, \dots, T$ *that does not depend on the bettings* w_t *such that*

$$\sum_{t=1}^T w_t z_t \le \epsilon \exp\left(\frac{(\sum_{t=1}^T z_t)^2}{\alpha T}\right) - \epsilon,$$

where $\alpha \ge 1.64101792\cdots$.

Proof. We will consider the following minimization problem

$$\min_{z_t} \sum_{t=1}^{T} w_t z_t - \epsilon \exp\left(\frac{(\sum_{t=1}^{T} z_t)^2}{\alpha T}\right).$$

Our aim is to find a condition on α to have a negative upper bound independent from T.

The minimization over $z_1, \ldots, z_t$ can be upper bound with IID variable coming from any stochastic distribution on $\{-1, 1\}$. In particular, we choose z_t to be 1 with probability 0.5 and -1 otherwise. In this way we have that the expectation of $w_t z_t$ is 0 regardless of the choice of w_t. Also, $Z := \sum_{t=1}^{T} z_t$ is a distribuited according to Binomial of parameters $(T, 0.5)$. Hence, we have

$$\begin{aligned}
\min_{z_t} \sum_{t=1}^{T} w_t z_t - \epsilon \exp\left(\frac{(\sum_{t=1}^{T} z_t)^2}{\alpha T}\right) &\leq \mathbb{E}_{z_t} \sum_{t=1}^{T} w_t z_t - \epsilon \left[\exp\left(\frac{(\sum_{t=1}^{T} z_t)^2}{\alpha T}\right)\right] \\
&= -\epsilon \mathbb{E}_{Z \sim B(T,0.5)} \left[\exp\left(\frac{(2Z - T)^2}{\alpha T}\right)\right] \\
&= -\epsilon \mathbb{E}_{Z \sim B(T,0.5)} \left[\exp\left(\frac{T^2 + 4Z^2 - 4ZT}{\alpha T}\right)\right] \\
&= -\epsilon \exp\left(\frac{T}{\alpha}\right) \mathbb{E}_{Z \sim B(T,0.5)} \left[\exp\left(\frac{4Z^2 - 4ZT}{\alpha T}\right)\right] \\
&\leq -\epsilon \exp\left(\frac{T}{\alpha}\right) \mathbb{E}_{Z \sim B(T,0.5)} \left[\exp\left(\frac{-4ZT}{\alpha T}\right)\right] \\
&= -\epsilon \exp\left(\frac{T}{\alpha}\right) 2^{-T} \left(1 + \exp\left(-\frac{4}{\alpha}\right)\right)^{T},
\end{aligned}$$

where in the last equality we used the closed form expression of the moment generating function of the binomial distribution. This upper bound holds for any $\alpha > 0$. However, we know that the betting strategy w_t never bets more than the actual reward, hence the reward at any time cannot be negative. So, any upper bound that is negative for T big enough is trivial. Hence, we select a to have the last expression upper bounded by $-\epsilon$, independent of T. Hence, taking the logarithm of negative sign of last expression we have

$$-T \log 2 + \frac{T}{\alpha} + T \log\left(1 + \exp\left(-\frac{4}{\alpha}\right)\right), \tag{66}$$

that numerically has a negative coefficient multiplying T iff $\alpha \geq 1.64101792 \cdots$. $\square$

5 Applications

5.1 Online Convex Optimization in 1d

We use the duality between Regret and Reward in McMahan and Orabona [2014]. In particular, define

$$Regret(u) := \sum_{t=1}^{T} \langle g_t, w_t - u \rangle \ .$$

and

$$Reward := \sum_{t=1}^{T} \langle -g_t, w_t \rangle \ .$$

We have the following Theorem

Theorem 3. *Let $\Psi : \mathcal{H} \to (-\inf, +\inf]$ be a lower semicontinuos and convex function, with $dom f \neq \emptyset$. An algorithm for the player guarantees*

$$Reward \geq \Psi(-g_{1:T}) - \epsilon \qquad \text{for any } g_1, \ldots, g_T$$

for a constant $\epsilon \in \mathbb{R}$ if and only if it guarantees

$$Regret(u) \leq \Psi^*(u) + \epsilon \qquad \text{for all } u \in \mathcal{H} \ . \tag{67}$$

5.2 Online Convex Optimization in Hilber Spaces

6 Proofs

Lemma 2. *The Lambert function satisfies*

$$W(x) \geq 0.6321 \log(x+1).$$

Proof. The Lambert function that satisfies

$$x = W(x) \exp\left(W(x)\right).$$

Hence,

$$W(x) = \log\left(\frac{x}{W(x)}\right) \tag{68}$$

$$= \log\left(\frac{x}{\log(x/W(x))}\right). \tag{69}$$

From the first equality, for any $a > 0$, we get

$$W(x) \leq \frac{1}{a\,e}\left(\frac{x}{W(x)}\right)^a$$

that is

$$W(x) \leq \left(\frac{1}{a\,e}\right)^{\frac{1}{1+a}} x^{\frac{a}{1+a}}. \tag{70}$$

Using (70) in (68), we have

$$W(x) \geq \log\left(\frac{x}{\left(\frac{1}{a\,e}\right)^{\frac{1}{1+a}} x^{\frac{a}{1+a}}}\right) \tag{71}$$

$$= \frac{1}{1+a}\log\left(a\,e\,x\right). \tag{72}$$

Consider now the function $g(x) = \frac{x}{x+1} - \frac{b}{\log(1+b)(b+1)}\log(x+1), x \geq b$. This function has a maximum in $x^* = (1+\frac{1}{b})\log(1+b) - 1$, the derivative is positive in $[0, x^*]$ and negative in $[x^*, b]$. Hence the minimum is in $x = 0$ and in $x = b$, where it is equal to 0. Using the property just proved on g, we have that for $x \leq b$, setting $a = \frac{1}{x}$, we have

$$W(x) \geq \frac{x}{x+1} \geq \frac{b}{\log(1+b)(b+1)}\log(x+1). \tag{73}$$

For $x > b$, setting $a = \frac{x+1}{ex}$, we have

$$W(x) \geq \frac{e\,x}{(e+1)x+1}\log(x+1) \geq \frac{e\,b}{(e+1)b+1}\log(x+1) \tag{74}$$

Hence, we set b such that

$$\frac{e\,b}{(e+1)b+1} = \frac{b}{\log(1+b)(b+1)}$$

Numberically, $b = 1.71825...$, so

$$W(x) \geq 0.6321\log(x+1) \tag{75}$$

$\square$

Lemma 3. *Define* $f(\theta) = \beta\exp\frac{x^2}{2\alpha}$, *for* $\alpha, \beta > 0$, $x \geq 0$. *Then*

$$f^*(y) = y\sqrt{\alpha W\left(\frac{\alpha y^2}{\beta^2}\right)} - \beta\exp\left(\frac{W\left(\frac{\alpha y^2}{\beta^2}\right)}{2}\right).$$

Moreover

$$f^*(y) \leq y\sqrt{\alpha\log\left(\frac{\alpha y^2}{\beta^2}+1\right)} - \beta.$$

Proof. From the definition of Fenchel dual, we have

$$f^*(y) = \max_x\ x\,y - f(x) = \max_x\ x\,y - \beta\exp\frac{x^2}{2\alpha} \leq x^*\,y - \beta$$

where $x^* = \arg\max_x x\, y - f(x)$. We now use the fact that x^* satisfies $y = f'(x^*)$, to have

$$x^* = \sqrt{\alpha W\left(\frac{\alpha y^2}{\beta^2}\right)},$$

where the function $W : \mathbb{R}_+ \to \mathbb{R}$ is the Lambert function that satisfies

$$x = W(x) \exp\left(W(x)\right).$$

Hence, to obtain an upper bound we need an upper bound to the Lambert function. We use Theorem 2.3 in Hoorfar and Hassani [2008], that says that

$$W(x) \leq \log \frac{x + C}{1 + \log(C)}, \quad \forall x > -\frac{1}{e},\ C > \frac{1}{e}.$$

Setting $C = 1$, we obtain the stated bound. $\square$

Lemma 4 ([Bauschke and Combettes, 2011, Example 13.7]). *Let $\phi : \mathbb{R} \to (-\infty, +\infty]$ be even. Then $(\phi * \|\cdot\|)^* = \phi^* * \|\cdot\|$.*

Corollary 1. *Define $f(\boldsymbol{\theta}) = \beta \exp \frac{\|\boldsymbol{\theta}\|^2}{2\alpha}$, for $\alpha, \beta > 0$. Then*

$$f^*(y) \leq \|\boldsymbol{\theta}\| \sqrt{\alpha \log\left(\frac{\alpha \|\boldsymbol{\theta}\|^2}{\beta^2} + 1\right)} - \beta.$$

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