Solving 3D frictional contact problems: Formulations and comparisons of numerical methods.

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Contents

Abstract TBW We continue by showing that this problem, central in the simulation of mechanical systems with contact and Coulomb's friction is an particular instance of general problem in Mathematical programming. We focus on the reduced problem, where the gloval velocities have been substituted and the reaction forces is the only unknown.

keywords Multibody systems, nonsmooth Mechanics, unilateral constraints, Coulomb friction, impact, numerical methods

Notation

The following notation is used throughout the article: the 2-norm for a function g is denoted by $\|g\|$ and for a vector $x \in \mathbb{R}^n$ by $\|x\|$. The index $\alpha \in \mathbb{N}$ is used to identify the variable pertaining to a single contact. A multivalued mapping $T\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an operator whose images are sets. The second order cone, also known as Lorentz or ice–cream cone, is defined as $K_\mu \coloneqq \{(x,t) \in \mathbb{R} \times \mathbb{R}_+ \mid \|x\| \leqslant \mu t\}$, $\mu \geqslant 0$. By polarity, the dual convex cone to a convex cone K defined by

$$K^\star = \{x \in \mathbb{R}^n \mid y^\top x \geqslant 0, \quad \text{for all } y \in K\}.$$

The normal cone $N_K\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ to a closed convex set X is the set

$$N_K(x) = \{d \in \mathbb{R}^n \mid d^\top (y - x) \leqslant 0\}$$

The notation $0 \leqslant x \perp y \geqslant 0$ denotes that $x \geqslant 0$, $y \geqslant 0$ and $x^\top y = 0$. A complementarity problem associated with a function $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is to find $x \in \mathbb{R}^n$ such that $0 \leqslant F(x) \perp x \geqslant 0$. The generalized complementarity problem is given by $K^\star \ni F(x) \perp x \in K$, where K is a closed convex cone. Finite-dimentional Variational Inequality (VI) problems subsumes complementarity problems, system of equations. Solving a $\mathrm{VI}(X, F)$ is to find $x \in X$ such that

$$F(x)^\top (y - x) \geqslant 0 \qquad \text{for all } y \in X.$$

It is easy to see this problem is equivalent to solving a *generalized equation*

$$0 \in F(X) + N_X(x).$$

The Euclidean projector on a set X is denoted by P_X.

1 Introduction

More than thirty years after the pioneering work of [Panagiotopoulos, 1975], [Nečas et al., 1980], [Haslinger, 1983, 1984, Haslinger and Panagiotopoulos, 1984], [Del Piero and Maceri, 1983, 1985], [Katona, 1983], [Chaudhary and Bathe, 1986], [Jean and Moreau, 1987], [Mitsopoulou and Doudoumis, 1988] on numerically solving mechanical problems with contact and friction, there are still active research activities on this subject in the computational mechanics and applied mathematics communities. This can be explain by the fact that problems from mechanical systems with unilateral contact and Coulomb friction are difficult to numerically solve and the mathematical results of convergence of the numerical algorithms are rare and most of these require rather strong assumptions. In this article, we want to give some insights of the advantages and weaknesses of standard solvers found in the literature by comparing them on the large sets of examples coming from a wide range of mechanical systems.

1.1 Problem statement

In this section, we formulate an abstract, algebraic finite–dimensional frictional contact problem. We cast this problem as a complementarity problem over cones, and discuss the properties of the latter. We end by presenting some instances with contact and friction phenomenon that fits our problem description.

Abstract problem We want to discuss possible numerical solution procedures for the following three–dimensional finite–dimensional frictional contact problem and some of its variants. Let $n_c \in \mathbb{N}$ be the number of contact points and $n \in \mathbb{N}$ the number of degree of freedom of a discrete mechanical system.

The problem data are: a symmetric positive (semi-) definite matrix $M \in \mathbb{R}^{n\times n}$, a vector $f \in \mathbb{R}^n$, a matrix $H \in \mathbb{R}^{n\times m}$ with $m = 3n_c$, a vector $w \in \mathbb{R}^m$ and a vector of coefficients of friction $\mu \in \mathbb{R}^{n_c}$. The unknowns are two vectors $v \in \mathbb{R}^n$, a velocity–like vector and $r \in \mathbb{R}^m$, a contact reaction or impulse, solution to

$$\begin{cases} Mv = Hr + f \\ K^\star \ni \hat{u} \perp r \in K \end{cases} \qquad \text{with} \qquad \begin{array}{l} u := H^\top v + w \\ \hat{u} := u + g(u), \end{array} \tag{1}$$ {eq:soccp1-in

where the set K is the cartesian product of Coulomb's friction cone at each contact, that is

$$K = \prod_{\alpha=1\ldots n_c} K^\alpha = \prod_{\alpha=1\ldots n_c} \{r^\alpha, \|r^\alpha_{\mathrm{T}}\| \leqslant \mu^\alpha |r^\alpha_{\mathrm{N}}|\}$$

and $K^\star$ is dual. The function $g\colon \mathbb{R}^m \to \mathbb{R}^m$ is a nonsmooth function defined as

$$g(u) = [[\mu^\alpha \|u^\alpha_{\mathrm{T}}\|, 0, 0]^\top, \alpha = 1\ldots n_c]^\top.$$

Note that the variable u and $\hat{u}$ do not appear as unknowns since they can be directly obtained from v.

Redaction note O.H. 1.1.
I think I got the gist of the following paragraph, but I think we should work on it a bit. For me we discuss approximations in the literature that don't solve the true problem. I believe figure to illustrate the effect of neglecting g would be useful (for the optimizer).

I also think that we should be more explicit that we focus on the reduced form, and also explain that in the rigid case, it may not destroy the sparse structure of the problem.

A Second Order Cone Complementarity Problem (SOCCP). From the mathematical programming point of view, the problem appears to be a Second Order Cone Complementarity Problem (SOCCP) [Facchinei and Pang, 2003] which can be generically defined as

$$\begin{cases} y = f(x) \\ K^\star \ni y \perp x \in K, \end{cases}$$

where K is a second order cone. If the nonlinear part of the problem (1) is neglected ($g(u) = 0$), the problem is an associated friction problem with dilatation, and by the way, is a gentle Second Order Cone Linear Complementarity Problem (SOCLCP) with a positive definite matrix $H^\top M^{-1} H$ (possibly semi–definite).The assumption of associated frictional law, i.e, a friction law where the local sliding velocity is normal the friction cone differs dramatically from the standard Coulomb friction since it generates a non–vanishing normal velocity when we slide. In other terms, the sliding motion implies the separation of the bodies. When the non-associated character of the friction is taken into account through $g(u)$, the problem is non monotone and nonsmooth, and therefore is very hard to solve efficiently. For a given numerical algorithm, it is not so difficult to design mechanical example to run the algorithm into troubles. Proof of convergence of the numerical algorithms are rare and most of these required strong assumptions on either the friction coefficients or full rank assumptions of matrices or operators or the dimension of the space in which the problem is formulated. Among these results, we can cite the Czech school where the coefficient of friction is assumed to be bounded and small. This assumption allows us to use fixed point methods on the convex sub–problems of Tresca friction (friction threshold that does depend on the normal reaction and then transform the cone into a semi-cylinder). We can also mention the results from [Pang and Trinkle, 1996, Stewart and Trinkle, 1996, Anitescu and Potra, 1997] where the friction cone is polyhedral (in 2D or by a faceting process). In that case, if $w = 0$ or $w \in \mathrm{im}(H^\top)$, Lemke's algorithm is able to solve the problem. The question of existence of solutions has been treated in [Klarbring and Pang, 1998, Acary et al., 2011] under similar assumptions but with different techniques. The question of uniqueness remains a difficult problem in the general case.

Range of applicability. We clearly choose to simplify a lot the general problems of formulating the contact problems with friction by avoiding including too much side effects that are themselves interesting but render the study too difficult to carry out in a single article. We choose finite dimensional systems where the time dependence does not appear explicitly.

Nevertheless, we believe that there are a strong interest to study this problem since it appears to be relatively generic in numerous simulations of systems with contact and friction. This problem is indeed at the heart of the simulation of mechanical systems with 3D Coulomb's friction and unilateral constraints in the following cases:

- it might be the result of the time–discretization by event–capturing time–stepping methods or event–detecting (event–driven) techniques of dynamical systems with friction; the variables are homogeneous to pairs velocity/impulses or acceleration forces
- it might be the result of space–discretization (by FEM for instance) of the elastic quasi-static problems of frictional contact mechanics; in that case, the variables are homogenous to displacements/forces of displacement rate/forces.
- if the system is a dynamical mechanical system composed of solids, the problem is again obtained by a space and time discretization
- if the material follows a nonlinear mechanical bulk behavior, we can use this model after a standard Newton linearization procedure.

For a description of the derivation of such problems in various practical situations we refer to [Laursen, 2003, Wriggers, 2006, Acary and Brogliato, 2008, Acary and Cadoux, 2013].

1.2 Objectives and outline of the article

In this article, after stating the problem with more details in Section 2, we recall the existence result of [Acary et al., 2011] for the problem in (1) in Section 2.3. In this framework, we briefly present in Section 3 a few alternative formulations of the problem that enable the design of numerical solution procedures : a) finite–dimensional Variational Inequalities(VI) and Quasi-Variational Inequalities(QVI) b) Nonsmooth equations and c) Optimization based formulations.

Right after these formulations, we list some of the most standard algorithms dedicated to one of the previous formulation :

1. the fixed point and projection numerical methods for solving VI are reviewed with a focus on self-adaptive time–step rules (Section 4),
2. the nonsmooth (semi-smooth) Newton methods are described based on the various nonsmooth equations formulations (Section 5),
3. Section 6 is devoted to the presentation of splitting and proximal point techniques,
4. and finally, in Section 7, the Panagiotopoulos alternating optimization technique, the successive approximation technique and the SOCLCP approach are outlined.

Since it is difficult to be exhaustive on the approaches developed in the literature to solve frictional contact problems, we decided to leave out the scope of the article the following approaches:

- the approaches that alter the fundamental assumptions of the 3D Coulomb friction model by faceting the cone as in the pioneering work of [Klarbring, 1986] and followed by [Al-Fahed et al., 1991, Pang and Trinkle, 1996, Stewart and Trinkle, 1996, Anitescu and Potra, 1997, Haslinger et al., 2004], or by convexifying the Coulomb law (associated friction law with normal dilatancy) [Heyn et al., 2013, Tasora and Anitescu, 2013, 2011, Anitescu and Tasora, 2010, Tasora and Anitescu, 2009] or finally by regularizing the friction law [Kikuchi and Oden, 1988]. In the same way, we are discussing recent developments methods for the frictionless case [Morales et al., 2008, Miyamura et al., 2010, Temizer et al., 2014].

- the approaches that are based on domain decomposition and parallel computing. We choose in this article to focus on single domain computation and to skip the discussion about distributed computing mainly for a sake of length of the article. (cite a bit a literature Krause, Koziara, Renouf, Heyn, ...)

Finally, some possible interesting approaches have not been reported. We are thinking mainly to the interior point methods approach [Christensen and Pang, 1998, Miyamura et al., 2010, Kleinert et al., 2014]. some basic implementations of such methods do not give satisfactory results. One of reason is the fact that we were not able to get robustness and efficiency on a large class of problems. As it is reported in [Kleinert et al., 2014, Krabbenhoft et al., 2012], it seems that it is needed to alter the friction Coulomb's law by adding regularization or dilatency in the model. In the same spirit, we skip also the comparison for the possibly very promising methods developed in [Heyn et al., 2013, Heyn, 2013] that are based on Krylov subspace and spectral methods. It could be very interesting to bench also these methods on the actual Coulomb friction model, that is to say, in the nonmonotone case. Finally, our preliminary results on the use of direct general SOCP or SOCLCP solvers were not convincing. Indeed, the structure of contact problems (product of a large number of small second order cones) has to be taken into account to get efficiency and unfortunately, these solvers are difficult to adapt to this structure.

Other comparisons articles have already been published in the literature. One of the first comparison study has been done in [?] and in [Chabrand et al., 1998]. In this work, several formulations are detailed in the bidimensional case (variational inequality, linear complementarity problem (LCP) and augmented Lagrangian formulation) and comparisons of fixed point methods with projection, splitting methods and Lemke's method for solving LCP. In [Christensen et al., 1998], a very interesting comparison in the three-dimensional case has been carried out.

Cite and comment

1. [Christensen et al., 1998]
2. [Mijar and Arora, 2000b,a, 2004a,b]
3. Mylapilli.Jain_JCND2016.pdf
4. [Heyn et al., 2013]
5. [Krabbenhoft et al., 2012]

Redaction note V.A. 1.1.

- *low number of benchmarks*
- *2D*

The comparison are performed on a large set of examples using performance profiles. Let us summarize the main conclusion from Section 8: on one hand, the algorithms based on Newton methods for nonsmooth equations solve quickly the problem when they succeed, but suffer from robustness issues mainly if the matrix H has not full rank. On the other hand, the iterative methods dedicated to solving variational inequalities are quite robust but with an extremely slow rate of convergence. To sum up, as far as we know there is no option that combines time efficiency and robustness. The set of problems used here are from the FCLIB collection[1], which aims at providing many problems to compare algorithms on a fair basis. In this work, this collection is solved with the software Siconos and its component Siconos/Numerics[2][Acary et al., 2015].

[1] http://fclib.gforge.inria.fr

[2] http://siconos.gforge.inria.fr

2 Description of the 3D frictional contact problems

2.1 Signorini's condition and Coulomb's friction.

Let us consider the contact between two bodies $A \subset \mathbb{R}^3$ and $B \subset \mathbb{R}^3$ with sufficiently smooth boundaries, as depicted on Figure 1.

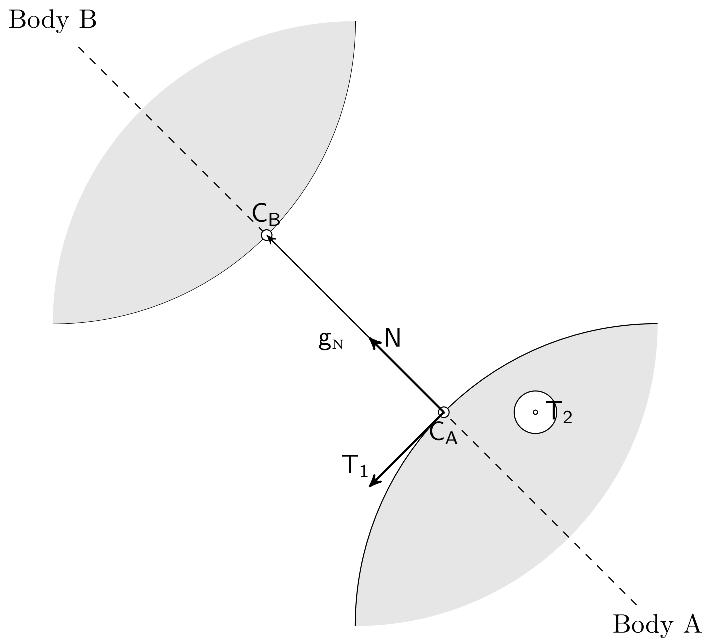

Figure 1: Contact kinematic

From the body A "perspective", the point $C_A \in \partial A$ is called a *master point to contact*. The choice of this master point C_A to write the contact condition is crucial in practice and amounts to consistently discretizing the contact surface. The vector N defines an outward unit normal vector to A at the point C_A. With $\mathsf{T}_1, \mathsf{T}_2$ two vectors in the plane orthogonal to N, we can build an orthornormal frame $(C_A, \mathsf{N}, \mathsf{T}_1, \mathsf{T}_2)$ called the *local frame at contact*. The slave contact point $C_B \in \partial B$ is defined as the projection of the point C_A on ∂B in the direction given by N. Note that we assume that such a point exists. The gap function is defined as the signed distance between C_A and C_B

$$g_{\mathrm{N}} = (C_B - C_A)^\top \mathsf{N}.$$

Consider two strictly convex bodies, which are non penetrating, *i.e.* $A \cap B = \emptyset$, the master and slave contact points can be chosen as the proximal points of each bodies and the normal vector N can be written as

$$\mathsf{N} = \frac{C_B - C_A}{\|C_B - C_A\|}.$$

The contact force exerted by A on B is denoted by $r_G \in \mathbb{R}^3$ and is decomposed as

$$r_G = r_{\mathrm{N}}\mathsf{N} + r_{\mathrm{T}_1}\mathsf{T}_1 + r_{\mathrm{T}_2}\mathsf{T}_2, \qquad \text{with } r_{\mathrm{N}} \in \mathbb{R} \text{ and } r_{\mathrm{T}} := [r_{\mathrm{T}_1}, r_{\mathrm{T}_2}]^\top \in \mathbb{R}^2.$$

In the local frame, the reaction force is given by

$$r := [r_{\mathrm{N}}, r_{\mathrm{T}}] \in \mathbb{R}^3.$$

The *Signorini condition* states that

$$0 \leqslant g_{\mathrm{N}} \perp r_{\mathrm{N}} \geqslant 0, \qquad (2)$$ {eq:signo}

and models the unilateral contact. The condition (2), written at the *position level*, can also be defined at the *velocity level*. To this end, the relative velocity $u_G \in \mathbb{R}^3$ of the point C_B with respect to C_A is also decomposed as

$$u_G = u_{\mathrm{N}}\mathsf{N} + u_{\mathrm{T}_1}\mathsf{T}_1 + u_{\mathrm{T}_2}\mathsf{T}_2 \quad \text{with } u_{\mathrm{N}} \in \mathbb{R} \text{ and } u_{\mathrm{T}} = [u_{\mathrm{T}_1}, u_{\mathrm{T}_2}]^\top \in \mathbb{R}^2.$$

In the local frame, the relative velocity is given by

$$u \coloneqq [u_{\mathrm{N}}, u_{\mathrm{T}}] \in \mathbb{R}^3.$$

At the velocity level, the Signorini condition is written

$$\begin{cases} 0 \leqslant u_{\mathrm{N}} \perp r_{\mathrm{N}} \geqslant 0 & \text{if } g_{\mathrm{N}} \leqslant 0 \\ r_{\mathrm{N}} = 0 & \text{otherwise.} \end{cases} \tag{3}$$ {eq:signo-vel

The Moreau's viability Lemma [Moreau, 1988] ensures that (3) implies (2) if $g_{\mathrm{N}} \geqslant 0$ holds in the initial configuration.

Coulomb's friction models the frictional behavior of the contact force law in the tangent plane spanned by (T_1, T_2). Let us define the Coulomb friction cone K which is chosen as the isotropic second order cone (Lorentz or ice–cream cone)

$$K = \{r \in \mathbb{R}^3 \mid \|r_{\mathrm{T}}\| \leqslant \mu r_{\mathrm{N}}\},$$

where μ is the coefficient of friction. The Coulomb friction states for the sticking case that

$$u_{\mathrm{T}} = 0, \quad r \in K, \tag{4}$$ {eq:Coulom-st

and for the sliding case that

$$u_{\mathrm{T}} \neq 0, \quad r \in \partial K, \quad \text{and} \quad \exists\, \alpha > 0 \text{ such that } r_{\mathrm{T}} = -\alpha u_{\mathrm{T}}. \tag{5}$$ {eq:Coulom-sl

With the Coulomb friction model, there are two relations between u_{T} and r_{T}. The distinction is based on the value of the relative velocity u_{T} between the two bodies. If $u_{\mathrm{T}} = 0$ (sticking case), we have $r_{\mathrm{T}} \leqslant \mu r_{\mathrm{N}}$. On the other hand, we get the sliding case. takeoff case?

Disjunctive formulation of the Signorini-Coulomb model If we consider the velocity-level Signorini condition (3) together with the Coulomb friction (4)–(5) which is naturally expressed in terms of velocity, we obtain a disjunctive formulation of the frictional contact behavior as

$$\begin{cases} r = 0 & \text{if } g_{\mathrm{N}} > 0 & \text{(no contact)} \\ r = 0, u_{\mathrm{N}} \geqslant 0 & \text{if } g_{\mathrm{N}} \leqslant 0 & \text{(take–off)} \\ r \in K, u = 0 & \text{if } g_{\mathrm{N}} \leqslant 0 & \text{(sticking)} \\ r \in \partial K, u_{\mathrm{N}} = 0, \exists\, \alpha > 0, u_{\mathrm{T}} = -\alpha r_{\mathrm{T}} & \text{if } g_{\mathrm{N}} \leqslant 0 & \text{(sliding)} \end{cases} \tag{6}$$ {eq:contact-d

In the computational practice, the disjunctive formulation is not suitable for solving the Coulomb problem as it suggests the use of enumerative solvers, with exponential behavior. In the sequel, alternative formulations of the Signorini-Coulomb model suitable for numerical applications are delineated. The core idea is to translate the cases in (6) into complementarity relations.

Inclusion into normal cones The Signorini condition (2) and (3), in their complementarity forms can be equivalently written as an inclusion into a normal cone to $\mathbb{R}_+$

$$-g_{\mathrm{N}} \in N_{\mathbb{R}_+}(r_{\mathrm{N}}) \qquad \text{and} \qquad -u_{\mathrm{N}} \in N_{\mathbb{R}_+}(r_{\mathrm{N}}), \tag{7}$$ {eq:signo-inc

if $g_{\mathrm{N}} \leqslant 0$ and $r_{\mathrm{N}} = 0$ otherwise. An inclusion form of the Coulomb friction for the tangential part can be also proposed: let $D(\cdot)$ be the Coulomb disk:

$$D(c) \coloneqq \{x \in \mathbb{R}^2 \mid \|x\| \leqslant c\}.$$

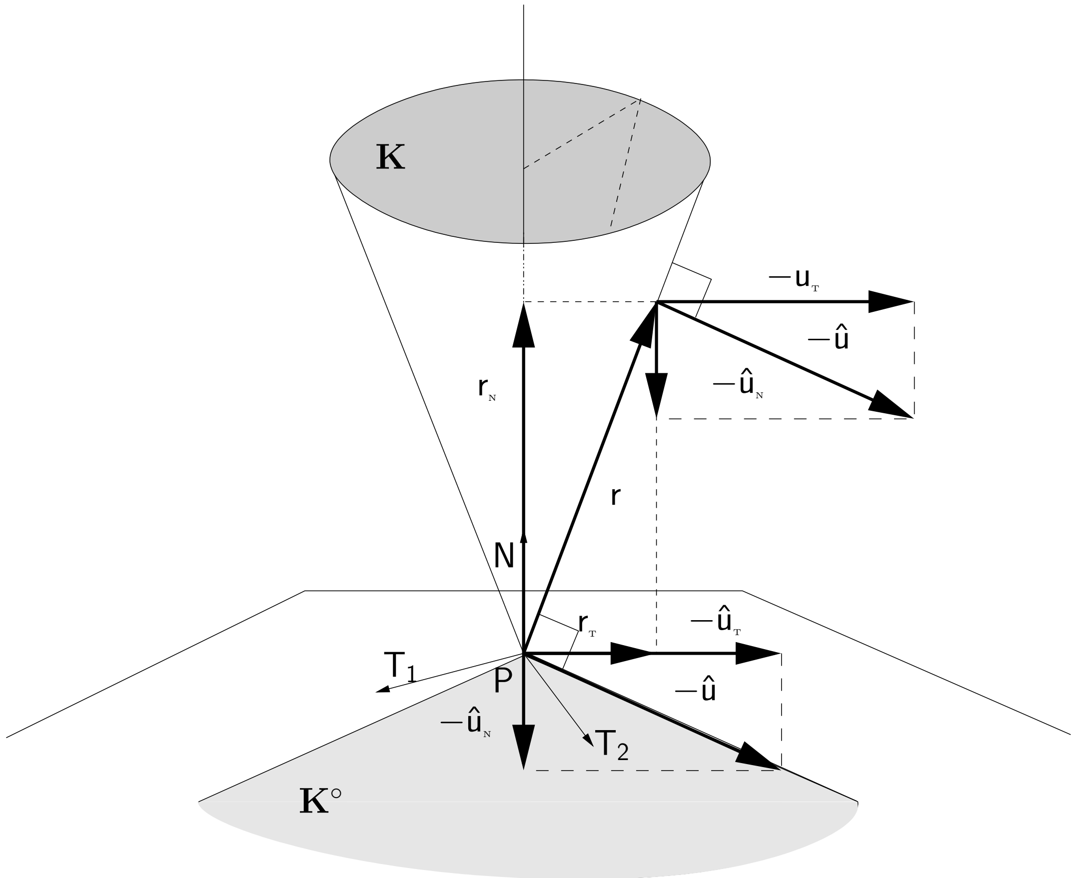

Figure 2: Coulomb's friction law in the sliding case.

For the Coulomb friction, we get

$$-u_T \in N_{D(\mu r_N)}(r_T). \tag{8}$$ {eq:Coulomb-i

Since $D(\mu r_N)$ is not a cone, the inclusion (8) is not a complementarity problem, but a variational inequality. The formulation (8) is often related to Moreau's maximum dissipation principle of the frictional behavior:

$$r_T \in \underset{\|z\| \leqslant \mu r_N}{\arg\max}\ z^\top u_T.$$

This means that the couple (r_T, u_T) maximizes the energy lost through dissipation.

SOCCP formulation of the Signorini-Coulomb model In [Acary and Brogliato, 2008, Acary et al., 2011], another formulation is proposed inspired by the so-called bipotential [De Saxcé, 1992, De Saxcé and Feng, 1991, Saxcé and Feng, 1998]. The goal is to form a complementarity problem out of (7) and (8) To this end, we introduce the modified relative velocity $\hat{u} \in \mathbb{R}^3$ defined by

$$\hat{u} = u + [\mu \|u_T\|, 0, 0]^\top.$$

The entire contact model (6) can be put into a Second-Order Cone Complementarity Problem (SOCCP) as

$$K^\star \ni \hat{u} \perp r \in K \tag{9}$$ {eq:contact-S

if $g_N \leqslant 0$ and $r = 0$ otherwise.

2.2 Frictional contact discrete problems

In this section, we formulate two basic discrete frictional contact problems considering a finite number n of degrees of freedom together with a discrete linear dynamics. The first problem is directly related to (1) and keep the unknown pair (v, r). The second problem will assume that the matrix M is invertible. In that later case, we case eliminate the variable v and reduce the single unknown r.

We assume that a finite set of n_c contact points and their associated local frames has been defined. In general, this task is not straightforward and amounts to correctly discretizing the contact surfaces. For more details, we refer to [Wriggers, 2006, Laursen, 2003].

For each contact $\alpha \in \{1, \ldots, n_c\}$, the local velocity is denoted by $u^\alpha \in \mathbb{R}^3$, the normal velocity by $u^\alpha_{\mathrm{N}} \in \mathbb{R}$ and the tangential velocity by $u^\alpha_{\mathrm{T}} \in \mathbb{R}^2$ with $u^\alpha = [u^\alpha_{\mathrm{N}}, (u^\alpha_{\mathrm{T}})^\top]^\top$. The vectors $u, u_{\mathrm{N}}, u_{\mathrm{T}}$ respectively collect all the local velocity $u = [(u^\alpha)^\top, \alpha = 1 \ldots n_c]^\top$, all the normal velocity $u_{\mathrm{N}} = [u^\alpha_{\mathrm{N}}, \alpha = 1 \ldots n_c]^\top$, and all the tangential velocity $u_{\mathrm{T}} = [(u^\alpha_{\mathrm{T}})^\top, \alpha = 1 \ldots n_c]^\top$. For a contact α, the modified local velocity, denoted by $\hat{u}^\alpha$, is defined by

$$\hat{u}^\alpha = u^\alpha + g^\alpha(u) \qquad \text{where} \qquad g^\alpha(u^\alpha) = [\mu^\alpha \|u^\alpha_{\mathrm{T}}\|, 0, 0]^\top. \tag{10}$$ {eq:modified}

The vector $\hat{u}$ and the function g collect all the modified local velocity at each contact $\hat{u} = [\hat{u}^\alpha, \alpha = 1 \ldots n_c]^\top$ and the function $g(u) = [[\mu^\alpha \|u^\alpha_{\mathrm{T}}\|, 0, 0]^\top, \alpha = 1 \ldots n_c]^\top$.

For each contact α, the reaction vector $r^\alpha \in \mathbb{R}^3$ is also decomposed in its normal part $r^\alpha_{\mathrm{N}} \in \mathbb{R}$ and the tangential part $r^\alpha_{\mathrm{T}} \in \mathbb{R}^2$ as $r^\alpha = [r^\alpha_{\mathrm{N}}, (r^\alpha_{\mathrm{T}})^\top]^\top$ The Coulomb friction cone for a contact α is defined by $K^\alpha = \{r^\alpha \in \mathbb{R}^3 \mid \|r^\alpha_{\mathrm{T}}\| \leqslant \mu^\alpha |r^\alpha_{\mathrm{N}}|\}$ and the set $K^{\alpha,\star}$ is its dual. The set K is the cartesian product of Coulomb's friction cone at each contact, that is

$$K = \prod_{\alpha=1,\ldots,n_c} K^\alpha \quad \text{and} \quad K^\star \text{ is its dual.}$$

In this article, we investigate the case where the problem is given in its *reduced* form. We consider that the discretized dynamics are of the form

$$Mv = Hr + f,$$

with M an symmetric positive-definite matrix. The local velocities at the point of contact are given by

$$u = H^\top v + w. \tag{11}$$ {eq:local_v}

More information on the term w is given later in this section. The (global) velocities v can be substitute in (11) by using a Schur-complement technique. This yields

$$u = H^\top M^{-1} H r + H^\top M^{-1} f + w.$$

Let us define W, often called the *Delassus matrix*, as

$$W := H^\top M^{-1} H$$

and the vector q as

$$q := H^\top M^{-1} f + w.$$

We are now ready to define the mathematical problem that we solve.

Problem FC (Discrete frictional contact problem)**.** *Given*

- *a symmetric positive semi–definite matrix* $W \in \mathbb{R}^{m \times m}$,
- *a vector* $q \in \mathbb{R}^m$,
- *a vector* $\mu \in \mathbb{R}^{n_c}$ *of coefficients of friction,*

find a vector $r \in \mathbb{R}^m$*, such that*

$$\begin{cases} K^\star \ni \hat{u} \perp r \in K \\ u = Wr + q \\ \hat{u} = u + g(u) \end{cases} \tag{12}$$ {eq:soccp2}

with $g(u) = [[\mu^\alpha \|u^\alpha_T\|, 0, 0]^\top, \alpha = 1 \ldots n_c]^\top$. *An instance of the problem is denoted by* $\mathrm{FC}(W, q, \mu)$ □

2.3 Existence of solutions

The question of the existence of solution for the Problem FC have been studied in [Klarbring and Pang, 1998] and [Acary et al., 2011] with different analysis techniques. The key assumption for existence of solutions in both articles is as follows

$$\exists v \in \mathbb{R}^m \; : \; H^\top v + w \in \operatorname{int} K^\star, \tag{13}$$ {ass}

or equivalently

$$w \in \operatorname{im} H + \operatorname{int} K^\star.$$

Under the assumption, the Problem FC have a solution. Therefore, it makes sense to design a procedure to solve the problem. In the sequel, we will compare numerical methods only when this assumption is satisfied.

This assumption is easy verified in numerous applications. For applications in nonsmooth dynamics where the unknown v is a relative contact velocity, the term w vanishes if we have only scleronomic constraints. For $w \in \operatorname{im}(H^\top)$ (and especially $w = 0$), the assumption is trivially satisfied. As it is explained in [Acary and Cadoux, 2013], the term w has several possible sources. If the constraints are written at the velocity level, one is given in dynamics by the impact law and in that case, we can show that of the Newton impact law it holds that $w \in \operatorname{im}(H^\top)$. For other impact law, this is not so clear. Another source for w is given by constraints that depends explicitly on time. In that case, we can have $w \notin \operatorname{im}(H^\top)$ and non existence of solutions. If the constraints are written at the position level, w can be given by initial of terms that comes velocity discretization. In that cases, the existence is not clear.

The assumption is also satisfied whenever $\operatorname{im} H = \mathbb{R}^m$ or in other words if $H^\top$ has full row rank. Unfortunately, in large number of applications $H^\top$ is rank deficient. From the mechanical point of view, the rank deficiency of H and the amount of friction seems to play a fundamental role in the question of the existence (and uniqueness) of solutions. In the numerical comparisons, we will attempt to get a deeper understanding on the role of these assumptions on the convergence of the algorithms. The rank deficiency of H is related to the number of constraints that are imposed to the system with respect to the number of degree of freedom in the system. It is closely related to the concept of hyperstaticity in overconstrained systems. In the most favorable cases, it yields indeterminate Lagrange multipliers but also to unfeasible problems and then to the lost of solutions in the worth cases. The second assumption on the amount of friction is also well–know. The frictionless problem is easy to solve if it is feasible. It is clear that large friction coefficients prevent from sliding and the therefore increase the degree of hyperstaticity of the system.

3 Alternative formulations

In this section, various equivalent formulations of Problem FC are given. Our goal is to show that such problems can be recast into several well-known problems in the mathematical programming and optimization community. These formulations will serve as a basis for numerical solution procedures that we develop in later sections.

3.1 Variational Inequalities (VI) formulations

Let us recall the definition a finite-dimensional VI(X, F): find $z \in X$ such that

$$F^\top(z)(y-z) \geqslant 0 \quad \text{for all } y \in X, \tag{14}$$ {eq:vi}

with X a nonempty subset of $\mathbb{R}^n$ and F a mapping from $\mathbb{R}^n$ into itself. We refer to [Harker and Pang, 1990, Facchinei and Pang, 2003] for the standard theory of finite–dimensional variational inequalities. The easiest way to state equivalent VI formulations of Problem FC is to use the following equivalences:

$$K^\star \ni \hat{u} \perp r \in K \quad \Longleftrightarrow \quad -\hat{u} \in N_K(r) \quad \Longleftrightarrow \quad \hat{u}^\top(s-r) \geqslant 0, \text{ for all } s \in K.$$

For Problem FC, the following equivalent formulation in VI is directly obtained from

$$-(Wr + q + g(Wr+q)) \in N_K(r).$$

The resulting VI is denoted by VI$(F_{\text{vi}}, X_{\text{vi}})$ with

$$F_{\text{vi}}(r) \coloneqq Wr + q + g(Wr+q) \quad \text{and} \quad X_{\text{vi}} \coloneqq K. \tag{15}$$ {eq:vi-II}

Uniqueness properties. In the general case, it is difficult to prove uniqueness of solutions to (15). If the matrix H has full rank and that friction coefficients are "small", a classical argument for the uniqueness of solution of VIs can be satisfied. Note that the full rank hypothesis on H implies that W is positive-definite. Therefore, we have $(x-y)^\top W(x-y) \geqslant C_W\|x-y\|^2$ with $C_W > 0$. Using this relation in (15) yields

$$\begin{array}{rcl}(F_{\text{vi}}(x) - F_{\text{vi}}(y))^\top(x-y) & = & (x-y)^\top W(x-y) \\ & & \quad + \sum_{\alpha=1}^{n_c} \mu^\alpha(x_{\text{N}}^\alpha - y_{\text{N}}^\alpha)[\|[Wx+q]_{\text{T}}^\alpha\| - \|[Wy+q]_{\text{T}}^\alpha\|] \\ & \geqslant & C_W\|x-y\|^2 + \sum_{\alpha=1}^{n_c} \mu^\alpha(x_{\text{N}}^\alpha - y_{\text{N}}^\alpha)[\|[Wx+q]_{\text{T}}^\alpha\| - \|[Wy+q]_{\text{T}}^\alpha\|].\end{array}$$

Note that for small values of the coefficients of friction the first term in the right-hand side dominates the second one. Hence, the mapping F_{vi} is strictly monotone and this ensures that the VI has at most one solution [Facchinei and Pang, 2003, Theorem 2.3.3]. The fact that H is full rank also implies that the Assumption (13) for the existence of solutions is trivially satisfied. Hence, there exists a unique solution to the VI$(F_{\text{vi}}, X_{\text{vi}})$.

Redaction note V.A. 3.1.
Use of homotopy (Haslinger ?)

3.2 Quasi-Variational Inequalities (QVI)

Let us recast Problem FC into the QVI framework. A QVI is a generalization of the VI, where the feasible set is allowed to depend on the solution. Let us define this precisely: let X be a multi-valued mapping $\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and let F be a mapping from $\mathbb{R}^n$ into itself. The quasi-variational inequality problem, denoted by QVI(X, F), is to find a vector $z \in X(z)$ such that

$$F^\top(z)(y-z) \geqslant 0, \forall y \in X(z). \tag{16}$$ {eq:qvi}

The QVI formulation of the frictional contact problems is obtained by considering the inclusions (7) and (8). We get

$$u^\top(s-r) \geqslant 0, \text{ for all } s \in C(\mu, r_{\text{N}}), \tag{17}$$ {eq:qvi-1}

where $C(\mu, r_{\mathrm{N}})$ is the Cartesian product of the semi–cylinders of radius $\mu^\alpha r_{\mathrm{N}}^\alpha$ defined as

$$C(\mu, r_{\mathrm{N}}) \coloneqq \prod_{\alpha=1}^{n_c} \left\{ s \in \mathbb{R}^3 \mid s_{\mathrm{N}} \geqslant 0, \|s_{\mathrm{T}}\| \leqslant \mu^\alpha r_{\mathrm{N}}^\alpha \right\}.$$

Note that the QVI (17) involves only u and not $\hat{u}$: this is the main interest of this formulation. The price to pay is the dependence on r of the set $C(\mu, r_{\mathrm{N}})$. Problem FC can be expressed as a QVI by substituting the expression of u, which yields

$$(Wr + q)^\top (s - r) \geqslant 0, \text{ for all } s \in C(\mu, r_{\mathrm{N}}).$$

This expression is compactly rewritten as $\mathrm{QVI}(F_{\mathrm{qvi}}, X_{\mathrm{qvi}})$, with

$$F_{\mathrm{qvi}}(r) \coloneqq Wr + q \quad \text{and} \quad X_{\mathrm{qvi}}(r) \coloneqq C(\mu, r_{\mathrm{N}}). \tag{18}$$ {eq:qvi-II}

Since W is assumed to be positive semi-definite matrix, F_{qvi} is monotone. Thus we get an affine monotone $\mathrm{QVI}(F_{\mathrm{qvi}}, X_{\mathrm{qvi}})$ for Problem FC.

3.3 Nonsmooth Equations

In this section, we expose a classical approach to solving a VI or a QVI, based on a reformulation of the inclusion as an nonsmooth equation. The term nonsmooth equation highlights that the mapping we consider fails to be differentiable. This is the price to pay to for this reformulation. We can apply fixed-point and Newton-like algorithms to solve the resulting equation. Given the nonsmooth nature of the problem, applying Newton's method appears challenging, but it can still be done for some reformulations.

More precisely for Problem FC, we search for an equation of the type

$$G(r) = 0 \tag{19}$$ {eq:ne-1}

where G is generally only locally Lipschitz continuous. The mapping G is such that the zeroes (r) of (19) are the solutions of (12).

Natural and normal maps for the VI formulations A general-purpose reformulation of VI is obtained by using the normal and natural maps, see [Facchinei and Pang, 2003] for details. The natural map $F^{\mathsf{nat}} \colon \mathbb{R}^n \to \mathbb{R}^n$ associated with the VI (14) is defined by

$$F^{\mathsf{nat}}(z) \coloneqq z - P_X(z - F(z)), \tag{20}$$ {eq:naturalma

where P_X is the Euclidean projector on the set X. A well-known result (see [Facchinei and Pang, 2003]) states that the solutions of a VI are related to the zeroes of the natural map :

$$z \text{ solves } \mathrm{VI}(X, F) \quad \Longleftrightarrow \quad F^{\mathsf{nat}}(z) = 0, \tag{21}$$ {eq:VI-normal

Using (14), it is easy to see that if z solves $\mathrm{VI}(X, F)$, then it is also a solution to $\mathrm{VI}(X, \rho F)$ for any $\rho > 0$. Therefore, we can define a parametric variant of the natural map by

$$F^{\mathsf{nat}}_\rho(z) = z - P_X(z - \rho F(z)).$$

The relations given in (21) continue to hold for the parametric mapping. Using those equivalences, the frictional contact problem can be restated as zeroes of nonsmooth functions. With the natural map, Problem FC under the VI form (15) can be reformulated as

$$F^{\mathsf{nat}}_{\mathrm{vi}}(r) \coloneqq \left[\ r - P_K\left(r - \rho(Wr + q + g(Wr + q))\right)\ \right] = 0. \tag{22}$$ {eq:natural-I

Remark 1. *Following the same lines, the normal map may also be used to derive algorithms. The normal map $F^{\mathsf{nor}} \colon \mathbb{R}^n \to \mathbb{R}^n$ is defined by*

$$F^{\mathsf{nor}}(x) \coloneqq F(P_X(x)) + x - P_X(x),$$

and its parametric variant

$$F^{\mathsf{nor}}_\rho(x) = \rho F(P_X(x)) + x - P_X(x).$$

An equivalent results apply

$$z \text{ solves } \mathrm{VI}(X,F) \quad \Longleftrightarrow \quad z = P_X(x) \text{ for some } x \text{ such that } F^{\mathsf{nor}}(x) = 0.$$

The normal map based formulation of VI are also obtained in the same way.

In the seminal work of [Sibony, 1970], iterative methods for solving monotone VIs are based on the natural map and fixed point iterations. The role of ρ is recognized to be very important for the rate of convergence. To improve the methods, Sibony [1970] proposes to use "skewed" projector based on a non-Euclidean metric. Given a positive definite matrix $R \in R^{n\times n}$, a skewed projector $P_{X,R}$ onto X is defined as follows: $z = P_{X,R}(x)$ is the unique solution of the convex programm

$$\begin{cases} \min \dfrac{1}{2}(y-x)^\top R(y-x), \\ \quad s.t. \quad y \in X. \end{cases}$$

The skew natural map can be also defined and yield the following nonsmooth equation

$$F^{\mathsf{nat}}_R(z) = z - P_{X,R}(z - R^{-1}F(z)).$$

The zeros of $F^{\mathsf{nat}}_R(z)$ are also solution of the $\mathrm{VI}(X,F)$. Considering the skew natural map, we obtain for Problem FC under the VI form (15),

$$F^{\mathsf{nat}}_{\mathrm{vi},R}(r) := \left[\; r - P_{K,R}\left(r - R^{-1}(Wr + q + g(Wr+q))\right) \;\right].$$

The previous case is retrieved by choosing $R = \rho^{-1} I_{n\times n}$.

Redaction note V.A. 3.2.
It can be interesting to use something R as a preconditionner of the problem ? $R = diag(W)$ or incomplete LU. Warning, W is only SPR, so we cannot have $R = W$.

Jean–Moreau's and Alart-Curnier's functions Using the alternative inclusions formulations (7)–(8) with a given set of parameters $\rho_{\mathrm{N}}, \rho_{\mathrm{T}}$ such that

$$\begin{cases} -\rho_{\mathrm{N}} u_{\mathrm{N}} \in N_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N}}), & \rho_{\mathrm{N}} > 0, \\ -\rho_{\mathrm{T}} u_{\mathrm{T}} \in N_{D(\mu,(r_n)_+)}(r_{\mathrm{T}}), & \rho_{\mathrm{T}} > 0, \end{cases}$$

we can replace P_K into $P_{\mathbb{R}^{n_c}_+}$ and $P_{D(\mu,(r_n)_+)}$ where

$$D(\mu,(r_n)_+) = \prod_{\alpha=1\ldots n_c} D(\mu^\alpha (r^\alpha_{\mathrm{N}})_+).$$

defines the Cartesian product of the Coulomb disks for each contact. The notation x_+ stands for $x_+ = \max(0,x)$. Using this procedure, Jean and Moreau [1987], Christensen et al. [1998] propose the following nonsmooth equation formulation of the frictional contact condition

$$\begin{cases} r_{\mathrm{N}} - P_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) = 0, \\ r_{\mathrm{T}} - P_{D(\mu,(r_{\mathrm{N}})_+)}(r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}) = 0. \end{cases} \tag{23}$$ {eq:Moreau-Je

The parameters $\rho_{\mathrm{N}}, \rho_{\mathrm{T}}$ may be also chosen contact by contact. Problem FC is then reformulated as

$$F_{\mathrm{mj}}(r) := \begin{bmatrix} r_{\mathrm{N}} - P_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N}} - \rho_{\mathrm{N}}(Wr+q)_{\mathrm{N}}) \\ r_{\mathrm{T}} - P_{D(\mu,(r_{\mathrm{N}})_+)}(r_{\mathrm{T}} - \rho_{\mathrm{T}}(Wr+q)_{\mathrm{T}}) \end{bmatrix} = 0. \tag{24}$$ {eq:MJ-II}

In the seminal work of Alart & Curnier [Curnier and Alart, 1988, Alart and Curnier, 1991], the augmented Lagrangian approach is invoked (see Remark 3) to obtain a similar formulation motivated by the development of nonsmooth (or generalized) Newton methods (see Section 5.2). To be accurate, the original Alart–Curnier function is given by

$$\begin{cases} r_{\mathrm{N}} - P_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) = 0, \\ r_{\mathrm{T}} - P_{D(\mu,(r_{\mathrm{N}}-\rho u_{\mathrm{N}})_+)}(r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}) = 0. \end{cases} \tag{25}$$ {eq:AC-1}

The difference between (23) and (25) is in the radius of the disk: $D(\mu,(r_{\mathrm{N}} - \rho u_{\mathrm{N}})_+)$ rather than $D(\mu,(r_{\mathrm{N}})_+)$. Problem FC can be also reformulated as in (24) using (25). This yields

$$F_{\mathrm{ac}}(r) \coloneqq \begin{bmatrix} r_{\mathrm{N}} - P_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N}} - \rho_{\mathrm{N}}(Wr+q)_{\mathrm{N}}) \\ r_{\mathrm{T}} - P_{D(\mu,(r_{\mathrm{N}}-\rho_{\mathrm{N}} u_{\mathrm{N}})_+)}(r_{\mathrm{T}} - \rho_{\mathrm{N}}(Wr+q)_{\mathrm{T}}) \end{bmatrix} = 0. \tag{26}$$ {eq:AC-II}

Remark 2. *From the QVI formulation* (17)*, the following nonsmooth equation can also be written*

$$r = P_{C(\mu,r_N)}(r - \rho u)$$

which corresponds to (23)*.*

Remark 3. *In the literature of computational mechanics [Curnier and Alart, 1988, Simo and Laursen, 1992, Alart and Curnier, 1991], very similar expressions are obtained using the concept of augmented Lagrangian functions. This concept introduced in the general framework of Optimization by [Hestenes, 1969] and developed and popularized by [Rockafellar, 1974, 1993] is a strong theoretical tool for analyzing existence and regularity of solutions of constrained optimization problems. Its numerical interest is still a subject of intense debate in the mathematical programming community. In the nonconvex nonsmooth context of problems of frictional contact problems, its invocation is not so clear, but it has enables the design of robust numerical techniques. Nevertheless, it is worth to note that some of these methods appear as variants of the methods developed to solve variational inequalities in other contexts. The method developed by [Simo and Laursen, 1992] is a dedicated version of fixed point with projection for VI (see Section 1) and the method of [Alart and Curnier, 1991] is a tailored version of semi–smooth Newton methods (see Section 5). Nevertheless, the concept of augmented Lagrangian have never been used in the optimization literature for this purpose.*

Redaction note V.A. 3.3.
+ understand the continuity comment of Alart ? + [12, 30 , 31] of [Tzaferopoulos, 1993]. + check 2 + discussion on $(r_N)_+$ in 3.3 (cf. Olivier comment) To be Discussed

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General SOCC-functions More generally, a large family of reformulations of the SOCCP (9) in terms of equations can be obtained by using a so-called Second Order Cone Complementarity (SOCC) function. Let us consider the following SOCCP over a symmetric cone $K^\star = K$. A SOCC-function ϕ is defined by

$$K \ni x \perp y \in K \iff \phi(x,y) = 0. \tag{27}$$ {eq:SOCC-func

The frictional contact problem can be written as a SOCCP over symmetric cones by applying the following transformations

$$x = T_x \hat{u} = \begin{bmatrix} \hat{u}_{\mathrm{N}} \\ \mu \hat{u}_{\mathrm{T}} \end{bmatrix} \text{ and } y = T_y r \begin{bmatrix} \mu r_{\mathrm{N}} \\ r_{\mathrm{T}} \end{bmatrix}.$$

Clearly, the nonsmooth equations of the previous sections provides several examples of SOCC-functions and the natural map offers the simplest one. In [Fukushima et al., 2001], the standard complementarity functions for Nonlinear Complementarity Problems (NCP) such as the celebrated Fischer-Burmeister function are extended to the SOCCP by means of Jordan algebra. Smoothing functions are also given with theirs Jacobians and they studied their properties in view of the application of Newton's method.

For the second order cone, the Jordan algebra can be defined with the following non-associative Jordan product

$$x \cdot y = \begin{bmatrix} x^\top y \\ y_{\mathrm{N}} x_{\mathrm{T}} + x_{\mathrm{N}} y_{\mathrm{T}} \end{bmatrix}$$

and the usual componentwise addition $x+y$. The vector x^2 denotes $x \cdot x$ and there exists a unique vector $x^{1/2} \in K$, the square root of $x \in K$, defined as

$$(x^{1/2})^2 = x^{1/2} \cdot x^{1/2} = x.$$

A direct calculation for the SOC in $\mathbb{R}^3$ yields

$$x^{1/2} = \begin{bmatrix} s \\ \dfrac{x_{\mathrm{T}}}{2s} \end{bmatrix}, \qquad \text{where } s = \sqrt{(x_{\mathrm{N}} + \sqrt{x_{\mathrm{N}}^2 - \|x_{\mathrm{T}}\|^2})/2}.$$

We adopt the convention that $0^{1/2} = 0$. The vector $|x| \in K$ denotes $(x^2)^{1/2}$. Thanks to this algebra and its associated operator, the projection onto K can be written as

$$P_K(x) = \frac{x + |x|}{2}.$$

This formula provides a new expression for the natural map and its associated nonsmooth equations. This is exactly what is done in [Hayashi et al., 2005] where the natural map (20) is used together with an expression of the projection operator based on the Jordan algebra calculus. The resulting SOCCP is then solved with a semi–smooth Newton method, and a smoothing parameter can be added.

Most of the calculus in Jordan algebra are based on the spectral decomposition, a basic concept in Jordan algebra, see [Fukushima et al., 2001] for more details. For $x = (x_{\mathrm{N}}, x_{\mathrm{T}}) \in \mathbb{R} \times \mathbb{R}^2$, the spectral decomposition is defined by

$$x = \lambda_1 u_1 + \lambda_2 u_2,$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ and $u_1, u_2 \in \mathbb{R}^3$ are the spectral values and the spectral vectors of x given by

$$\lambda_i = x_{\mathrm{N}} + (-1)^i \|x_{\mathrm{T}}\|, \quad u_i = \begin{cases} \frac{1}{2} \begin{bmatrix} 1 \\ (-1)^i \dfrac{x_{\mathrm{T}}}{\|x_{\mathrm{T}}\|} \end{bmatrix}, \text{ if } x_{\mathrm{T}} \neq 0 \\ \frac{1}{2} \begin{bmatrix} 1 \\ (-1)^i w \end{bmatrix}, \text{ if } x_{\mathrm{T}} = 0 \end{cases} \qquad i = 1, 2$$

with $w \in \mathbb{R}^2$ any unit vector. Note that the decomposition is unique whenever $x_{\mathrm{T}} \neq 0$. The spectral decomposition enjoys very nice properties that simplifies the computation of basic functions such that

$$\begin{aligned} x^{1/2} &= \sqrt{\lambda_1} u_1 + \sqrt{\lambda_2} u_2, \text{ for any } x \in K, \\ P_K(x) &= \max(0, \lambda_1) u_1 + \max(0, \lambda_2) u_2. \end{aligned}$$

More interestingly, general SOCC-functions can also be extended and smoothed version of this function can be also developed (see [Fukushima et al., 2001]). Let us start with the Fischer-Burmeister function

$$\phi_{\mathsf{FB}}(x, y) = x + y - (x^2 + y^2)^{1/2}.$$

It can show that the zeroes of ϕ_{FB} are solutions of the SOCCP (27) using the Jordan algebra associated with K. Using the spectral decomposition, the Fischer-Burmeister function can be easily computed as

$$\phi_{\mathsf{FB}}(x, y) = x + y - (\sqrt{\bar\lambda_1} \bar u_1 + \sqrt{\bar\lambda_2} \bar u_2)$$

where $\bar\lambda_1, \bar\lambda_2 \in \mathbb{R}$ and $\bar u_1, \bar u_2 \in \mathbb{R}^3$ are the spectral values and the spectral vectors of $x^2 + y^2$ that is

$$\begin{array}{rcl}\bar\lambda_i & = & \|x\|^2 + \|y\|^2 + 2(-1)^i \|x_{\mathrm{N}} x_{\mathrm{T}} + y_{\mathrm{N}} y_{\mathrm{T}}\| \\ \bar u_i & = & \begin{cases} \frac{1}{2}\begin{bmatrix} 1 \\ (-1)^i \dfrac{x_{\mathrm{N}} x_{\mathrm{T}} + y_{\mathrm{N}} y_{\mathrm{T}}}{\|x_{\mathrm{N}} x_{\mathrm{T}} + y_{\mathrm{N}} y_{\mathrm{T}}\|}\end{bmatrix}, \text{ if } x_{\mathrm{N}} x_{\mathrm{T}} + y_{\mathrm{N}} y_{\mathrm{T}} \neq 0 \\ \frac{1}{2}\begin{bmatrix} 1 \\ (-1)^i w \end{bmatrix}, \text{ if } x_{\mathrm{N}} x_{\mathrm{T}} + y_{\mathrm{N}} y_{\mathrm{T}} = 0 \end{cases} \quad , i = 1, 2. \end{array}$$

Redaction note O.H. 3.1.
- All this presentation is nice. However, does it fit into the story of this article?
- The notion of C-function is used, but is not defined -> fixed
VA: I think we can transfer a part in Appendix but we use this approach for the Newton/FB method. It is difficult to define the FB function over cones without spectral decomposition. I will try to rework the section

Fischer-Burmeister function for the frictional contact problem

$$F_{FB}(u, r) = \begin{bmatrix} u - Wr - q \\ \Phi_{FB}\left(\begin{bmatrix}\mu r_{\mathrm{N}} \\ r_{\mathrm{T}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\mu}(u_{\mathrm{N}} + \mu\|u_{\mathrm{T}}\|) \\ u_{\mathrm{T}}\end{bmatrix}\right) \end{bmatrix} = 0. \quad (28)$$ {eq:FB-II}

Redaction note V.A. 3.4.
TBW

- *Define $\Phi_F B$ for all contacts*
- *Finally, note that*
$$K^* \ni x \perp y \in K \Longleftrightarrow K^* \ni x \cdot y \in K$$
Check there is no trouble with $\mu = 0$. This has to be done for all of this section.
- *smoothing and its smoothed version with a regularization parameter $\tau > 0$ as*
$$\phi_{FB,\tau}(x, y) = x + y - (x^2 + y^2 + 2\tau^2 e)^{1/2}$$
where $e = (1, 0, 0)$ is the identity element of the Jordan algebra, that is $e \cdot x = x$. In the same vein, the class of smoothing function of the natural map for NCP developed in [Chen and Mangasarian, 1996] is extended to SOCCP in [Fukushima et al., 2001].
[Fukushima et al., 2001] [Zhang et al., 2009] [Hayashi et al., 2005]

3.4 Alternative complementarity formulations

Redaction note O.H. 3.2.
Are we presenting numerical results using this reformulation? If no, we have to think about the usefulness of this paragraph in the article version. It would be great to have more references here. Where does eq:MCP come from? The assumption that $H = I$ (to get eq:kanno1) are stringent. Do we want to keep it in the article? At the end, we don't say much about the efficiency of the method. Does it work well?
VA: We will remove this section at the end,

A direct complementarity formulation over the positive orthant can also be written by introducing the following slack variable $\kappa^\alpha = \|u^\alpha_{\mathrm{T}}\|$ for each contact α. With the help of this new variable, the contact law for one contact can be written

$$\begin{cases} \kappa = \|u_{\mathrm{T}}\| \\ \kappa r_{\mathrm{T}} + \|r_{\mathrm{T}}\| u_{\mathrm{T}} = 0 \\ 0 \leqslant \kappa \perp \mu r_{\mathrm{N}} - \|r_{\mathrm{T}}\| \geqslant 0 \\ 0 \leqslant u_{\mathrm{N}} \perp r_{\mathrm{N}} \geqslant 0. \end{cases}$$

Together with the linear discretized dynamics, the whole problem can be formulated as an NCP (or MCP) as it is defined in [Dirkse and Ferris, 1995]. This formulation enables therefore the use of MCP solvers as the PATH solver for the frictional contact problem. Unfortunately, the fact that the functions involved in the formulation of the MCP are themselves nonsmooth due to the norm prevents a robust application of such solvers. In [Glocker, 1999], another MCP formulation is provided. The attempt to use it in a numerical framework was a success due the redundancy of constraints in the formulation. We need a reference here

In [Kanno et al., 2006], an alternative second–order cone formulation is proposed by introducing a slack variable $\lambda_{\mathrm{N}} \in \mathbb{R}$ for each contact. The disjunctive formulation of the Coulomb contact law is given by

$$\mathbb{R}_+ \times K_1 \ni \begin{bmatrix} u_{\mathrm{N}} \\ \lambda_{\mathrm{N}} \\ u_{\mathrm{T}} \end{bmatrix} \perp \begin{bmatrix} r_{\mathrm{N}} \\ \mu r_{\mathrm{N}} \\ r_{\mathrm{T}} \end{bmatrix} \in \mathbb{R}_+ \times K_1,$$

where $K_1 = \{x \in \mathbb{R}^3 \mid \|x_{\mathrm{T}}\| \leqslant x_{\mathrm{N}}\}$. In the special case that H is the identity matrix (which implies that $n = m$), the linear relations given in Problem **??** between u and r reduce to $Mv = r + f$ and $u = v + w$. We obtain the following Second Order Cone Linear Complementarity problem (SOCLCP)

$$\begin{cases} \begin{bmatrix} r_{\mathrm{N}} \\ \mu r_{\mathrm{N}} \\ r_{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} M_{\mathrm{NN}} & 0 & M_{\mathrm{NT}} \\ \mu M_{\mathrm{NN}} & 0 & \mu M_{\mathrm{NT}} \\ M_{\mathrm{TN}} & 0 & \mu M_{\mathrm{TT}} \end{bmatrix} \begin{bmatrix} u_{\mathrm{N}} \\ \lambda_{\mathrm{N}} \\ u_{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} q_{\mathrm{N}} \\ \mu q_{\mathrm{N}} \\ q_{\mathrm{T}} \end{bmatrix} \\ \mathbb{R}_+ \times K_1 \ni \begin{bmatrix} u_{\mathrm{N}} \\ \lambda_{\mathrm{N}} \\ u_{\mathrm{T}} \end{bmatrix} \perp \begin{bmatrix} r_{\mathrm{N}} \\ \mu r_{\mathrm{N}} \\ r_{\mathrm{T}} \end{bmatrix} \in \mathbb{R}_+ \times K_1. \end{cases}$$

The fact that the SOCCP is linear is quite attractive but the strong rank deficiency make the design of robust numerical algorithms difficult. In [Zhang et al., 2011], this formulation is used with the help of a normal and tangential penalization of the contact.

3.5 Optimization problems

In this section, several optimization-based formulations are proposed. The quest for an efficient optimization formulation of the frictional problem is a hard task. Since the problem is nonsmooth and nonconvex, the use of an associated optimization problem is interesting from the numerical point of view if we want to improve the robustness and the stability of the numerical methods.

A straightforward optimization problem can be written whose cost function to minimize is the scalar product $r^\top \hat{u}$. Indeed, this product is always positive and vanishes at the solution. Let us consider this first optimization formulation

$$\begin{cases} \min\ r^\top \hat{u} = r^\top u + \sum_{\alpha=1}^{n_c} \mu^\alpha r^\alpha_{\mathrm{N}} \|u^\alpha_{\mathrm{T}}\| \\ \quad s.t. \quad \hat{u} \in K^\star, \\ \qquad\quad\ r \in K, \end{cases}$$

which amounts to minimizing the DeSaxcé's bipotential function [De Saxcé, 1992] over $K^\star \times K$. A first simplification can be made by noting that

$$\hat{u} \in K^\star \Longleftrightarrow u_{\mathrm{N}} \geqslant 0,$$

which leads to

$$\begin{cases} \min\ r^\top u + \sum_{\alpha=1}^{n_c} \mu^\alpha r_{\rm N}^\alpha \|u_{\rm T}^\alpha\| \\ \quad s.t. \quad u_{\rm N} \geqslant 0 \\ \qquad\quad\ r \in K. \end{cases}$$

Starting from Problem FC, a direct substitution of $u = Wr + q$ yields

$$\begin{cases} \min\ r^\top (Wr+q) + \sum_{\alpha=1}^{n_c} \mu^\alpha r_{\rm N}^\alpha \|(Wr+q)_{\rm T}^\alpha\| \\ \quad s.t. \quad (Wr+q)\mathsf{N} \geqslant 0, \\ \qquad\quad\ r \in K. \end{cases}$$

which is a nonlinear optimization problem with a nonsmooth and nonconvex cost function. From the numerical point of view this problem may be very difficult and we have to ensure that the cost function have to be zero at the solution which is not guaranteed of some local minima are reached in the minimization process.

Other optimization-based formulations have been proposed in the literature. They are not direct optimization formulation but they try to identify an optimization sub-problem which is well-posed and for which efficient numerical methods are available. Three approaches can be listed in three categories: a) the *alternating optimization* problems, b) the *successive approximation* method and c) the *convex SOCP* approach.

The Panagiotopoulos alternating optimization approach aims at solving the frictional contact problem by alternatively solving the Signorini condition for a fixed value of the tangential reaction $r_{\rm T}$, and solving the Coulomb friction model for a fixed value of the normal reaction $r_{\rm N}$. Let us split the matrix W and the vector q in the following way:

$$u = Wr + q \quad \Longleftrightarrow \quad \begin{bmatrix} u_{\rm N} \\ u_{\rm T} \end{bmatrix} = \begin{bmatrix} W_{\rm NN} & W_{\rm NT} \\ W_{\rm TN} & W_{\rm TT} \end{bmatrix} \begin{bmatrix} r_{\rm N} \\ r_{\rm T} \end{bmatrix} + \begin{bmatrix} q_{\rm N} \\ q_{\rm T} \end{bmatrix}.$$

Two sub-problems can therefore be identified: the first one is to find $u_{\rm N}$ and $r_{\rm N}$ such that

$$\begin{cases} u_{\rm N} = W_{\rm NN} r_{\rm N} + \tilde q_{\rm N}, \\ 0 \leqslant u_{\rm N} \perp r_{\rm N} \geqslant 0, \end{cases}$$

where $\tilde q_{\rm N} = q_{\rm N} + W_{\rm NT} r_{\rm T}$. The second problem is to find $u_{\rm T}$ and $r_{\rm T}$ such that

$$\begin{cases} u_{\rm T} = W_{\rm TT} r_{\rm T} + \tilde q_{\rm T}, \\ -u_{\rm T} \in N_{D(\mu, \tilde r_{\rm N})}(r_{\rm T}), \end{cases}$$

where $\tilde r_{\rm N}$ is fixed and $\tilde q_{\rm T} = q_{\rm T} + W_{\rm TN} r_{\rm N}$. Since W is a symmetric positive semi–definite matrix, $W_{\rm NN}$ and $W_{\rm TT}$ are also symmetric semi–definite positive matrices. Therefore, two convex optimization problems can be formulation:

$$\begin{cases} \min\ \dfrac{1}{2} r_{\rm N}^\top W_{\rm NN} r_{\rm N} + r_{\rm N}^\top \tilde q_{\rm N} \\ \quad \text{s.t.} \quad r_{\rm N} \geqslant 0, \end{cases}$$

and

$$\begin{cases} \min\ \dfrac{1}{2} r_{\rm T}^\top W_{\rm TT} r_{\rm T} + r_{\rm T}^\top \tilde q_{\rm T} \\ \quad \text{s.t.} \quad r_{\rm T} \in D(\mu, \tilde r_{\rm N}). \end{cases}$$

This approach has been proposed by [Panagiotopoulos, 1975] for two–dimensional applications in soil foundation computing. It has also been used in other finite element applications in [Barbosa and Feijóo, 1985, Tzaferopoulos, 1993] and studied from the mathematical point of view in [Haslinger and Panagiotopoulos, 1984, Haslinger et al., 1996].

Redaction note V.A. 3.5.
remark Olivier : de memoire

What about

$$0 \ni \begin{bmatrix} M & -H_{\scriptscriptstyle T} \\ H_{\scriptscriptstyle T} & 0 \end{bmatrix} \begin{bmatrix} v \\ r_{\scriptscriptstyle T} \end{bmatrix} + \begin{bmatrix} \tilde{f}_{\scriptscriptstyle T} \\ w_{\scriptscriptstyle T} \end{bmatrix} + N_{\mathbb{R}^{n-m} \times D(\mu, \tilde{r}_{\scriptscriptstyle N})}$$

The successive approximation method identifies a single optimization problem by introducing a function that maps the normal reaction to itself (or the friction threshold) such that

$$h(r_{\mathrm{N}}) = r_{\mathrm{N}}.$$

Using this artifact, we can define a new problem from Problem FC such that

$$\begin{cases} \theta = h(r_{\mathrm{N}}) \\ u = Wr + q \\ -u_{\mathrm{N}} \in N_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N}}) \\ -u_{\mathrm{T}} \in N_{D(\mu,\theta)}(r_{\mathrm{T}}). \end{cases}$$

Since W is a symmetric positive semi-definite matrix, the last three lines are equivalent to a convex optimization problem over the product of cylinder $C(\mu, \theta)$, that is

$$\begin{cases} \theta = h(r_{\mathrm{N}}) \\ \min \ \dfrac{1}{2} r^\top W r + r^\top q \\ \ \text{s.t.} \quad r \in C(\mu, \theta) \end{cases}$$

The method of successive approximation has been extensively used for proving existence and uniqueness of solutions to the discrete frictional contact problems. We refer to [Haslinger et al., 1996] which summarizes the seminal work of the Czech school [Nečas et al., 1980, Haslinger, 1983, 1984]. We will see in the sequel that this approach also provides us with very efficient numerical solvers in Section 7.2.

Redaction note O.H. 3.3.
Do we implement this method? The last sentence suggest that we are going to show numerical results.
VA :YES TRESCA family

The convex SOCP approach is in the same vein as the previous one, with the difference that a SOCQP sub-problem is identified. To this aim, we augment the problem by introduction a auxiliary variable s, the image of $g(u)$ introduced in (10). We then obtain

$$\begin{cases} s = g(u) \\ \hat{u} = Wr + q + s \\ K^\star \ni \hat{u} \perp r \in K. \end{cases}$$

Since W is a positive semi-definite matrix, a new convex optimization sub-problem can be defined

$$\begin{cases} s = g(u) \\ \begin{cases} \min & \dfrac{1}{2} r^\top W r + r^\top (q + s) \\ \text{s.t.} & r \in K. \end{cases} \end{cases}$$

This formulation introduced in [Cadoux, 2009] and developed in [Acary and Cadoux, 2013, Acary et al., 2011] has been used to give an existence criteria to the discrete frictional contact problems. Furthermore, this existence criteria can be numerically checked.

Redaction note O.H. 3.4.
Vincent, peut-tu vérifier que les 2 dernières phrases sont encore cohérentes (j'ai supprimé la formulation globale)?

VA yes fixed.

4 Numerical methods for VIs

4.1 Fixed point and projection methods for VI

Starting from the VI formulations (14) or more precisely an associated nonsmooth equation through the natural map,

$$F_R^{\mathsf{nat}}(z) = z - P_{X,R}(z - R^{-1}F(z)).$$

The basic idea of the algorithm is to perform fixed point iterations on the mapping

$$z \mapsto P_{X,R}(z - R^{-1}F(z)),$$

yielding to Algorithm 1 with the specific choice of $R = \rho_k^{-1} I$. The choice of the updating rule of ρ_k is detailed in Section 4.2.

Algorithm 1 Fixed point iterations for the VI (14)

Require: F, X Data of VI (14)
Require: z_0 initial values and $\mathsf{tol} > 0$ a tolerance
Require: ρ_0 initial value for ρ
Ensure: z solution of VI (14)
  $\mathsf{k} \leftarrow 0$
  while error $>$ tol **do**
    Update the value of ρ_k
    $\mathsf{z}_{\mathsf{k}+1} \leftarrow \mathsf{P}_{\mathsf{X}}(\mathsf{z}_{\mathsf{k}} - \rho_{\mathsf{k}}\,\mathsf{F}(\mathsf{z}_{\mathsf{k}}))$
    Evaluate error.
    $\mathsf{k} \leftarrow \mathsf{k}+1$
  end while
  $\mathsf{z} \leftarrow \mathsf{z}_{\mathsf{k}}$

For the formulation (15), the following iterations are performed

$$r_{k+1} \leftarrow P_{K,R}(r_k - R^{-1}(Wr_k + q + g(Wr_k + q))). \tag{29}$$ {eq:FP-vi-II}

In the sequel when a parameter ρ is specified, it is assumed that $R = \rho^{-1} I$.

The convergence of such methods are generally shown for strongly monotone VI. In our case, this assumption is not satisfied, but we will see in the sequel that such methods can converge in practice.

Remark 4. *Algorithm 1 with the iteration rule* (29) *and a fixed value of* ρ_k *has been originally proposed in [De Saxcé and Feng, 1991, 1998]. The algorithm is called Uzawa's algorithm by reference to the algorithm due to Uzawa in computing the optimal values of convex program by primal-dual techniques [Glowinski et al., 1976, Fortin and Glowinski, 1983]. Note that the algorithm in [Simo and Laursen, 1992] is similar to the fixed point algorithm with projection though based on augmented Lagrangian concept (see Remark 3).*

Using the other nonsmooth equations (23) and (25) or the QVI formulation (18), similar algorithms can be derived. For QVI (16), we perform fixed point iterations on the mapping

$$z \mapsto P_{X(z),R}(z - R^{-1}F(z)). \tag{30}$$ {eq:skew-fixe

yielding to Algorithm 2. For Problem FC and the Jean–Moreau function (23), we get the following iteration rule

$$\begin{cases} u_{k+1} \leftarrow Wr_k + q \\ r_{\mathrm{N},k+1} \leftarrow P_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N},k} - \rho_{\mathrm{N}} u_{\mathrm{N},k+1}) \\ r_{\mathrm{T},k+1} \leftarrow P_{D(\mu, r_{\mathrm{N},k})}(r_{\mathrm{T},k} - \rho_{\mathrm{T}} u_{\mathrm{T},k+1}). \end{cases} \tag{31}$$ {eq:FP-MJ-II}

Algorithm 2 Fixed point iterations for the QVI (16)

Require: F, X Data of VI (14)
Require: z_0 initial values and tol > 0 a tolerance
Require: ρ_0 initial value for ρ
Ensure: z solution of VI (14)
  $\mathsf{k} \leftarrow 0$
  while error $>$ tol **do**
    Update the value of ρ_k
    $\mathsf{z}_{\mathsf{k}+1} \leftarrow \mathsf{P}_{\mathsf{X}(\mathsf{z}_\mathsf{k})}(\mathsf{z}_\mathsf{k} - \rho_\mathsf{k}\,\mathsf{F}(\mathsf{z}_\mathsf{k}))$
    Evaluate error.
    $\mathsf{k} \leftarrow \mathsf{k}+1$
  end while
  $\mathsf{z} \leftarrow \mathsf{z}_\mathsf{k}$

With the Alart–Curnier approach (25), we get

$$\begin{cases} u_{k+1} \leftarrow W r_k + q \\ r_{\mathrm{N},k+1} \leftarrow P_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N},k} - \rho_\mathrm{N} u_{\mathrm{N},k+1}) \\ r_{\mathrm{T},k+1} \leftarrow P_{D(\mu, r_{\mathrm{N},k} - \rho_\mathrm{N} u_{\mathrm{N},k+1})}(r_{\mathrm{T},k} - \rho_\mathrm{T} u_{\mathrm{T},k+1}) \end{cases} \tag{32}$$ {eq:FP-AC-II}

We will see in Section 7 that more evolved algorithms can be written exploiting the structure of the convex optimization sub-problems.

Redaction note V.A. 4.1.
More interesting in alternating context because we have two convex sub-problems.

Extragradient methods The extragradient method [Korpelevich, 1976] is also a well-known method for VI which improves the previous projection method. It can be described as

$$\begin{array}{rcl} \bar{z}_k & \leftarrow & P_X(z_k - \rho F(z_k)) \\ z_{k+1} & \leftarrow & P_X(z_k - \rho\, F(\bar{z}_k)) \end{array}$$

and formally defined in Algorithm 3. The convergence of this method is guaranteed under the following

Algorithm 3 Extragradient method for the VI (14)

Require: F, X Data of VI (14)
Require: z_0 initial values and tol > 0 a tolerance
Ensure: z solution of VI (14)
  $\mathsf{k} \leftarrow 0$
  while error $>$ tol **do**
    Update the value of ρ_k
    $\bar{\mathsf{z}}_\mathsf{k} \leftarrow \mathsf{P}_\mathsf{X}(\mathsf{z}_\mathsf{k} - \rho_\mathsf{k}\mathsf{F}(\mathsf{z}_\mathsf{k}))$
    $\mathsf{z}_{\mathsf{k}+1} \leftarrow \mathsf{P}_\mathsf{X}(\mathsf{z}_\mathsf{k} - \rho_\mathsf{k}\,\mathsf{F}(\bar{\mathsf{z}}_\mathsf{k}))$
    Evaluate error.
    $\mathsf{k} \leftarrow \mathsf{k}+1$
  end while
  $\mathsf{z} \leftarrow \mathsf{z}_\mathsf{k}$

assumptions: there exists a solution and the function F is Lipschitz–continuous and pseudo–monotone.

In [He and Liao, 2002], the extragradient method is reinterpreted as a prediction-correction method applied to the proximal point algorithm. Indeed, the proximal point algorithm amounts to solving the following VI for a given z_k

$$((z - z_k) + \alpha_k F(z))^\top (y - z) \geqslant 0, \text{ for all } y \in X \text{ and } \alpha_k > 0$$

or equivalently

$$z = P_X(z - [z - z_k + \alpha_k F(z)]) = P_X(z_k - \alpha_k F(z)). \quad (33)$$ {eq:PPA-VI2}

Searching for a solution z_{k+1} of (33) yields

$$z_{k+1} = P_X(z_k - \alpha_k F(z_{k+1}))$$

which can be viewed as an implicit version of the basic fixed point iteration based on (30). A prediction–correction method based may be written as

$$\begin{array}{lcl} \bar{z}_k & \leftarrow & P_X(z_k - \rho_k F(z_k)) \\ z_{k+1} & \leftarrow & P_X(z_k - \alpha_k\, F(\bar{z}_k)) \end{array}$$

where the scalars α_k and ρ_k have to be updated at each iteration.

Redaction note O.H. 4.1.
I think that the analogy with the proximal point algorithm is premature. We only introduce it in Section 6.2.
I think we are presenting result with the extragradient. Maybe add a teaser at the end of the paragraph?
VA OK we can discuss this point. It is difficult to do a (European) perfect linear presentation.

Projection and contraction methods The extragradient method needs two projections on X per iteration. In order to improve the basic fixed point method but with only one projection per step, Solodov and Tseng [1996] and He [1997] consider the following update:

$$z_{k+1} \leftarrow z_k - \gamma(z_k, \rho) D^{-1} \left[T_\rho(z_k) - T_\rho(P_X(z_k - \rho F(z_k)))\right], \quad (34)$$ {eq:VI-Projec

where $D \in \mathbb{R}^{n\times n}$ is a positive definite matrix and $T_\rho := I - \rho F$. The scalar $\rho \in (0, +\infty)$ must be sufficiently small so that T_ρ is a strongly monotone mapping. Substituting the expression of T_ρ in (34), we get

$$\begin{array}{lcl} \bar{z}_k & \leftarrow & P_X(z_k - \rho F(z_k)) \\ z_{k+1} & \leftarrow & z_k - \gamma(z_k, \rho) D^{-1} \left[z_k - \bar{z}_k - \rho(F(z_k) - F(\bar{z}_k))\right]. \end{array} \quad (35)$$ {eq:VI-Projec

With the analogy with a gradient method, the search direction $d(z, \rho))$ may be introduced

$$d(z, \rho) = z - P_X(z - \rho F(z)) - \rho\left[F(z) - F(P_X(z - \rho F(z)))\right].$$

The method (35) can be written as

$$\begin{array}{lcl} \bar{z}_k & \leftarrow & P_X(z_k - \rho F(z_k)) \\ d_k & \leftarrow & z_k - \bar{z}_k - \rho(F(z_k) - F(\bar{z}_k)) \\ z_{k+1} & \leftarrow & z_k - \gamma(z_k, \rho) D^{-1} d_k, \end{array}$$

where the function $\gamma(z, \rho)$ appears as a step-length in the search direction. In [He, 1997], the matrix D is set to the identity matrix and the step-size is chosen as

$$\gamma(z, \rho) = \frac{\theta[z - P_X(z - \rho F(z))]^\top d(z, \rho)}{\|d(z, \rho)\|^2} \qquad \theta \in (0, 2). \quad (36)$$ {eq:VI-projec

with the parameter ρ such that

$$\rho\|F(z_k) - F(\bar{z}_k)\| \leqslant L\|z_k - \bar{z}_k\|, \quad \text{with } L \in (0, 1). \quad (37)$$ {eq:VI-projec

In [Solodov and Tseng, 1996], the step-size is chosen as

$$\gamma(z,\rho) = \frac{\theta(1-L)\|z - P_X(z-\rho F(z))\|^2}{\|P^{1/2}d(z,\rho)\|^2} \qquad \theta \in (0,2), L \in (0,1), \tag{38}$$ {eq:VI-projec

with the parameter ρ such that

$$\rho(z_k - \bar z_k)(F(z_k) - F(\bar z_k) \leqslant L\|z_k - \bar z_k\|^2, \quad \text{with } L \in (0,1). \tag{39}$$ {eq:VI-projec

The resulting algorithm is described in Algorithm 4. The updating rule for ρ_k is detailed in Section 4.2 in order to satisfy the conditions (37) or (39).

Algorithm 4 Projection and Contraction method for the VI (14)

Require: F, X data of VI (14)
Require: D a symmetric positive definite matrix
Require: z_0 initial values and tol > 0 a tolerance
Ensure: z solution of VI (14)
  $k \leftarrow 0$
  while error $>$ tol **do**
    Update the value of ρ_k satisfying (37) or (39)
    $\bar z_k \leftarrow P_X(z_k - \rho_k F(z_k))$
    $d_k \leftarrow z_k - \bar z_k - \rho_k(F(z_k) - F(\bar z_k))$
    $\gamma_k \leftarrow \gamma(z_k, \rho_k)$ with (36) or (38)
    $z_{k+1} \leftarrow z_k - \gamma_k D^{-1} d_k$
    Evaluate error.
    $k \leftarrow k+1$
  end while
  $z \leftarrow z_k$

Other more evolved projected iterative methods can be found in [He et al., 2012a,b].

Hyperplane projection method This approach was introduced by [Konnov, 1993]. The convergence has been proved under the assumptions that F is a continuous pseudo-monotone mapping. The method is described in Algorithm 5.

4.2 Self-adaptive step-size rules

A key ingredient in this efficiency and the convergence of the numerical methods for VI presented above is the choice of the sequence $\{\rho_k\}$. A sensible work has been done in the literature mainly motivated by some convergence proofs under specific assumption. Besides the relaxation of the assumption for the convergence, we are interesting in improving the numerical efficiency and robustness. We present in this section, the most popular approach for choosing the sequence $\{\rho_k\}$.

In [Khobotov, 1987], a method is proposed to improve the extragradient method of Korpelevich [1976] by adapting ρ_k is the following way. The goal is the find ρ_k that satisfies

$$0 < \rho_k \leqslant \min\left\{\bar\rho, L\frac{\|z_k - \bar z_k\|}{\|F(z_k) - F(\bar z_k)\|}\right\} \text{ with } L \in (0,1) \tag{40}$$ {eq:khobotov1

where $\bar\rho$ is the maximum value of ρ_k which is chosen in the light of the specific problem. The objective is to find a coefficient that is bounded by the local Lipschitz constant. The standard way to do that is to use an Armijo–type procedure by successively trying value of $\rho_k = \bar\rho\nu^m$ with $m \in \mathbb{N}$ and $\nu \in (0,1)$, with a typical value of 2/3. In the original article of [Khobotov, 1987], there is no procedure to size $\bar\rho$ or to update it. In [He and Liao, 2002] and in the context of prediction–correction, the authors propose to use the rule $\rho_k = \rho_{k-1}\nu^m$ and if the criteria (40) is largely satisfied for ρ_k, the value is increased. In [Han and Lo, 2002], a similar procedure is used for the extragradient method by adding an increasing step of ρ_k, which is done after the correction as in [He and Liao, 2002].

this sentence needs to be rewritten. VA: It is better ?

Algorithm 5 Hyperplane projection method [Konnov, 1993]

Require: F, X
Require: $\mathsf{z}_0 \in \mathsf{X}, \tau > 0, \sigma \in (0,1)$
Ensure: z solution of $\mathrm{VI}(\mathsf{X},\mathsf{F})$ with F a continuous pseudo-monotone mapping
 $\mathsf{k} \leftarrow 0$
 while error > tol **do**
  $\mathsf{y_k} \leftarrow \mathsf{P_X}(\mathsf{z_k} - \tau \mathsf{F}(\mathsf{z_k}))$
  (Armijo line–search) Find the smallest integer, $\mathsf{i} \in \mathbb{N}$ such that

$$\mathsf{F}(2^{-\mathsf{i}}\mathsf{y_k} + (1-2^{-\mathsf{i}})\mathsf{z_k})^\top(\mathsf{z_k} - \mathsf{y_k}) \geqslant \frac{\sigma}{\tau}\|\mathsf{z_k} - \mathsf{y_k}\|^2$$

  $\mathsf{i_k} \leftarrow \mathsf{i}$
  $\mathsf{x^k} \leftarrow 2^{-\mathsf{i_k}}\mathsf{y_k} + (1-2^{-\mathsf{i_k}})\mathsf{z_k}$
  $\mathsf{w_k} \leftarrow \mathsf{z_k} - \dfrac{\mathsf{F}(\mathsf{x_k})^\top(\mathsf{z_k}-\mathsf{x_k})}{\|\mathsf{F}(\mathsf{x_k})\|^2}\,\mathsf{F}(\mathsf{x_k})$
  $\mathsf{z_{k+1}} \leftarrow \mathsf{P_X}(\mathsf{w_k})$
  $\mathsf{k} \leftarrow \mathsf{k}+1$
  Evaluate error.
 end while

Redaction note O.H. 4.2.
We have to motivate the ratio before presentation the equations.
VA: It is difficult to motivate more since enter into the detail of the convergence proof. To be discussed.

The criteria (40) is verified by computing the ratio

$$r_k \leftarrow \frac{\rho_k\|F(z_k) - F(\bar{z}_k)\|}{\|z_k - \bar{z}_k\|}. \tag{41}$$ {eq:Khobotov-

In [Solodov and Tseng, 1996], similar Armijo–like technique is used, and the ratio r_k is computed as follows:

$$r_k \leftarrow \frac{\rho_k(z_k - \bar{z}_k)^\top(F(z_k) - F(\bar{z}_k))}{\|z_k - \bar{z}_k\|^2}. \tag{42}$$ {eq:SolodovTs

In [Han and Sun, 2004], the ration r_k is evaluated as

$$r_k \leftarrow \frac{\rho_k\|z_k - \bar{z}_k\|^2}{(z_k - \bar{z}_k)^\top(F(z_k) - F(\bar{z}_k))}.$$

but in practice this choice appears to be the worth one.

Redaction note O.H. 4.3.
We have to back up the last claim with a reference of some data
The next paragraph is hard to follow. Maybe we could put some informations in the algorithm description, like the typical values.

The approach is summarized in Algorithm 6. The parameter L typically chosen around 0.9 is a safety coefficient in the evaluation of ρ_k. The parameter $\mathsf{L}_{\mathsf{min}}$ that triggers an increase of ρ_k is chosen around 0.3. In Algorithm 6, a Boolean option is added to the standard Armijo approach. The approximation $\bar{z}_k$ is updated within the self–adaptive loop. This trick is not justifies by any theoretical argument and is most of the articles this operation is not performed but in practice (see Section 9.1) it appears to improve the convergence speed. The update of the Armijo rule $\rho_k \leftarrow \nu\,\rho_k$ can also be replaced by $\rho_k \leftarrow \nu\,\rho_k \min\{1, 1/r_k\}$ but it appears that this trick does improve the self–adaptive procedure. Other

Name	Algo.	Additional informations	parameters
FP-DS	1	iteration rule (29) and fixed ρ	ρ
FP-QVI-MJ	2	iteration rule (31) and fixed $\rho_{\mathrm{N}}, \rho_{\mathrm{T}}$	$\rho_{\mathrm{N}}, \rho_{\mathrm{T}}$
FP-QVI-AC	2	iteration rule (32) and fixed $\rho_{\mathrm{N}}, \rho_{\mathrm{T}}$	$\rho_{\mathrm{N}}, \rho_{\mathrm{T}}$
FP-VI-UPK	1	iteration rule (29) and Algorithm 6 with (41)	$L, L_{\min}, \nu$ isUpdateInTheLoop
FP-VI-UPTS	1	iteration rule (29) and Algorithm 6 with (42)	$L, L_{\min}, \nu$ isUpdateInTheLoop
EG-VI-UPK	3	formulation (15) and Algorithm 6 with (41)	$L, L_{\min}, \nu$ isUpdateInTheLoop
EG-VI-UPTS	3	formulation (15) and Algorithm 6 with (42)	$L, L_{\min}, \nu$ isUpdateInTheLoop
HPA-VI	5	formulation (15)	σ, τ

Table 1: Naming convention for the algorithms based on VI formulations.

more evolved step–lengths strategies can be found in [Wang et al., 2010] that have been tried in this study.

Algorithm 6 Updating rule for ρ_k

Require: F, X
Require: Search and safety parameters. $\mathsf{L} \in (0,1), 0 < \mathsf{L}_{\min} < \mathsf{L}, \nu \in (0,1)$
Require: isUpdateInTheLoop Boolean option
Require: Initial values $z_k \in X, \rho_{k-1} > 0$
  $\rho_k \leftarrow \rho_{k-1}$
  $\bar{z}_k \leftarrow P_X(z_k - \rho_k F(z_k))$
  Evaluate r_k with (41) (or (42))
  while $r_k > L$ **do**
    $\rho_k \leftarrow \nu\, \rho_k$
    if isUpdateInTheLoop **then**
      $\bar{z}_k \leftarrow P_X(\bar{z}_k - \rho_k F(\bar{z}_k))$
    else
      $\bar{z}_k \leftarrow P_X(z_k - \rho_k F(z_k))$
    end if
    Evaluate r_k with (41) (or (42))
  end while
  Perform the correction step of extragradient or prediction–correction method.
  if $r_k < L_{\min}$ **then**
    $\rho_k = \frac{1}{\nu}\rho_k$
  end if

4.3 Nomenclature

A nomenclature for the algorithms based on the VI formulation is given in Table 1.

5 Newton based methods

Redaction note V.A. 5.1.
Objectives:

- *Alart Curnier [Alart and Curnier, 1991]*
- *Christensen et al.[Christensen et al., 1998] [Christensen and Pang, 1998],*
- *Newton sur De Saxce cf P. Joly [Joli and Feng, 2008]. Equivalence natural map.*
- *and others ? [Stadler, 2004] [Hüeber et al., 2008, Hüeber and Wohlmuth, 2005][Koziara and Bićanić, 2008][Renard, 2013]*
- *Update the section with the substitution of u*

5.1 Principle of the nonsmooth Newton methods

In Section 3.3, several formulations of the frictional contact problem by means of nonsmooth equations have been presented. These nonsmooth equations call for the use of nonsmooth Newton's methods. Remember that the standard Newton method is to solve

$$G(z) = 0 \tag{43}$$ {eq:NSN1}

by performing the following Newton iteration

$$z_{k+1} = z_k - J^{-1}(z_k)G(z_k).$$

If the mapping G is smooth enough, the matrix J is the Jacobian matrix of G with respect to z, that is $J(z) = \nabla_z^\top G(z)$. When the G is nonsmooth but Lipschitz continuous, as it is the case in the equation reformulations of VIs, the Jacobian matrix is replaced by an element of the subdifferential at z denoted by $\Phi(z) \in \partial G(z)$. If $\Phi(z)$ is nonsingular, then an iteration of the nonsmooth Newton method is given by

$$z_{k+1} = z_k - \Phi^{-1}(z_k)(G(z_k)).$$

The resulting nonsmooth Newton method is detailed in Algorithm 7.

Algorithm 7 Nonsmooth Newton method for (43)

Require: G data of Problem (43)
Require: D a symmetric positive definite matrix
Require: z_0 initial values and tol > 0 a tolerance
Ensure: z solution of Problem (43)
  $\mathsf{k} \leftarrow 0$
  while error $>$ tol **do**
    compute (select) $\Phi(\mathsf{z}_\mathsf{k}) \in \partial \mathsf{G}(\mathsf{z}_\mathsf{k})$
    $\mathsf{z}_{\mathsf{k}+1} \leftarrow \mathsf{z}_\mathsf{k} - \Phi^{-1}(\mathsf{z}_\mathsf{k})(\mathsf{G}(\mathsf{z}_\mathsf{k}))$
    Evaluate error.
    $\mathsf{k} \leftarrow \mathsf{k}+1$
  end while
  $\mathsf{z} \leftarrow \mathsf{z}_\mathsf{k}$

The convergence of nonsmooth Newton methods is based on the assumption of semi–smoothness of the nonsmooth function in (43). For this reason that are often called semi–smooth Newton methods (see [Facchinei and Pang, 2003, Section 7.5] and references therein)

5.2 Application to the discrete frictional contact problem

Nonsmooth newton based on the natural map Let us consider the natural map $F^{\mathsf{nat}}_{\mathrm{vi}}$ in (22) that enables to write Problem FC as a nonsmooth equation. Algorithm 7 is applied with

$$\Phi(r) \in \partial F^{\mathsf{nat}}_{\mathrm{vi}}(r).$$

The details of a possible computation of Φ can be found in Appendix B.1. Similar computations can also be found in [Joli and Feng, 2008] where a Newton method based on the formulation (22) is used contact by contact in a Gauss–Seidel loop.

Newton method based on the Jean–Moreau and Alart–Curnier functions Let us consider now the Alart–Curnier function $F_{\mathrm{ac}}(u, r)$ in (26) or the Jean–Moreau function $F_{\mathrm{mj}}(u, r)$ (24) for Problem FC. Algorithm 7 is applied with

$$\Phi(r) \in \partial F_{\mathrm{ac}}(r) \quad \text{or} \quad \Phi(r) \in \partial F_{\mathrm{mj}}(r).$$

The details of a possible computation of Φ can be found in Appendix B.2.

Newton method based on SOCC-function Let us consider now the Fischer-Burmeister function $F_{FB}(u, r)$ in (28) for Problem FC. Algorithm 7 is applied with

$$\Phi(r) \in \partial F_{FB}(r).$$

The details of a possible computation of Φ can be found in Appendix ??.

Redaction note O.H. 5.1.
Do we have ode with smoothing? Do we want to do it?
VA: this smoothing is completely different because is is not included in the dynamics. TBD

5.3 Damped Newton and Line-search procedures

- (GP, Armijo, FBLSA, Non-monotone watch dogs)

Redaction note O.H. 5.2.
I believe we only do GP + Armijo. Do we want to do non-monotone LS? From my very limited experiments, it helps, but usually to convergence faster. I'm not sure it really helps to not get stuff in practice. But I may be wrong.
I'm not sure that FBLSA make sense here? What is AC-FBLSA? In siconos, when I used the term FBLSA, it is for FB + line search algo, based on the algorithm name given in F& Pang. What is AC-FBLSA? Whouldn't it be ACLSA or AC-LSA?

5.4 Nomenclature

A nomenclature for the algorithms based on the nonsmooth Newton methods is given listed in Table 2.

Algorithm	Description	parameters
NSN-NM	Algorithm 7 with the natural map formulation (22)	tol, ρ
NSN-AC	Algorithm 7 with the Alart–Curnier formulation (26)	$tol, \rho_{\mathrm{N}}, \rho_{\mathrm{T}}$
NSN-JM	Algorithm 7 with the Jean–Moreau formulation (24)	$tol, \rho_{\mathrm{N}}, \rho_{\mathrm{T}}$
NSN-FB	Algorithm 7 with the Fischer-Burmeister formulation (28)	tol
NSN-MJ-GP	??	
NSN-AC-GP	Algorithm 7 with the Alart–Curnier formulation (26) and the Goldstein–Price (GP) line search	
NSN-AC-FBLSA	Algorithm 7 with the Alart–Curnier formulation (26) and the FBLSA line search	
NSN-FB-GP	Algorithm 7 with the Fischer-Burmeister formulation (28) and the Goldstein–Price (GP) line search	
NSN-FB-FBLSA	Algorithm 7 with the Fischer-Burmeister formulation (28) and the FBLSA line search	
...	...	

Table 2: Naming convention for the algorithms based on nonsmooth Newton (NSN) method

6 Splitting techniques and proximal point algorithm

Splitting techniques are standard techniques to solve VI(F, X) when the function F is affine, that is $F(z) = Mz + q$. Usually, a block splitting of the matrix M is performed and a Projected Successive Over Relaxation (PSOR) method is used to solve the VI. Since the cone K is a product of second-order cones in $\mathbb{R}^3$, a natural way to split the problem is to form sub-problems by using single contact as a building block. The sub-problems can be solved by any method for the VI that have been presented in the previous sections. In the same way, the proximal point algorithm can also be used which amounts to solving the original VI(F, X) by solving a sequence of VI(F_{c,x_k}, X) problems such that $F_{c,x_k}(z) = z - x_k + cF(z), \quad c > 0$ and $\lim_{k\to+\infty} \|x_k - z\| = 0$.

6.1 Splitting techniques

The particular structure of the cone K as a product of second-order cone in $\mathbb{R}^3$ calls for a splitting of the problem contact by contact. For Problem FC, the relation

$$u = Wr + q \tag{44}$$ {eq:delassus-

is split along each contact as follows

$$u^\alpha = W^{\alpha\alpha} r^\alpha + \sum_{\beta\neq\alpha} W^{\alpha\beta} r^\beta + q^\alpha, \text{ for all } \alpha \in 1 \ldots n_c, \tag{45}$$ {eq:delassus-

where the matrices α and β are used to label the variable for each contact. The matrices $W^{\alpha\beta}$ with $\alpha \in 1, \ldots, n_c$ and $\beta \in 1, \ldots, n_c$ are easily identified from (44). From (45), a projected Gauss–Seidel (PGS) method is obtained by using the following update rule at the k-th iterate:

$$u^\alpha_{k+1} = W^{\alpha\alpha} r^\alpha_{k+1} + \sum_{\beta<\alpha} W^{\alpha\beta} r^\beta_{k+1} + \sum_{\beta>\alpha} W^{\alpha\beta} r^\beta_k + q^\alpha, \text{ for all } \alpha \in 1 \ldots n_c.$$

A Projected Successive Over Relaxation (PSOR) scheme is derived by introducing a relaxation parameter $\omega > 0$ such that

$$u^\alpha_{k+1} = \frac{1}{\omega} W^{\alpha\alpha} r^\alpha_{k+1} - \frac{1}{\omega} W^{\alpha\alpha} r^\alpha_k + \sum_{\beta<\alpha} W^{\alpha\beta} r^\beta_{k+1} + \sum_{\beta\geqslant\alpha} W^{\alpha\beta} r^\beta_k + q^\alpha, \text{ for all } \alpha \in 1 \ldots n_c.$$

At the k-th iteration, the following problem is solved for each contact α:

$$\begin{cases} u^\alpha_{k+1} = \bar{W}^{\alpha\alpha} r^\alpha_{k+1} + \bar{q}^\alpha_{k+1}, \\ \hat{u}^\alpha_{k+1} = u^\alpha_{k+1} + g(u^\alpha_{k+1}), \\ K^{\alpha,\star} \ni \hat{u}^\alpha_{k+1} \perp r^\alpha_{k+1} \in K^\alpha, \end{cases} \tag{46}$$ {eq:psor-3}

where

$$\begin{cases} \bar{W}^{\alpha\alpha} = \frac{1}{\omega} W^{\alpha\alpha} \\ \bar{q}^\alpha_{k+1} = -\frac{1}{\omega} W^{\alpha\alpha} r^\alpha_k + \sum_{\beta<\alpha} W^{\alpha\beta} r^\beta_{k+1} + \sum_{\beta\geqslant\alpha} W^{\alpha\beta} r^\beta_k + q^\alpha \end{cases}, \text{ for all } \alpha \in 1 \ldots n_c.$$

The problem (46) has exactly the same structure as Problem FC, but is of lower size since it is only for one contact. It is solved by a *local solver*, which can be any of the algorithms presented in this article or even an analytical method (enumerating all the possible cases). The PSOR algorithm is summarized in Algorithm 8 and the PGS can be recovered by setting $\omega = 1$.

Applications methods in frictional contact date back to the work of [Mitsopoulou and Doudoumis, 1988, 1987] for two-dimensional friction. In [Jourdan et al., 1998], this method is developed in the Gauss-Seidel configuration ($\omega = 1$) with a local Newton solver based on the Alart–Curnier formulation. If the local solver is only one iteration of the VI solver based on projection, we get a standard splitting technique for VI. In Table 3, the methods based on PSOR used in the comparison are summarized.

Algorithm 8 PSOR algorithm for Problem FC

Require: $\mathsf{W}, \mathsf{q}, \mu$
Require: r_0 initial values and tol > 0 a tolerance
Require: ω a relaxation parameter
Ensure: r, u solution of Problem FC
  while error $>$ tol **do**
    for $\alpha = 1 \ldots \mathsf{n_c}$ **do**
      $\bar{\mathsf{W}}^{\alpha\alpha}_{\mathsf{k}+1} \leftarrow \frac{1}{\omega}\mathsf{W}^{\alpha\alpha}$
      $\bar{\mathsf{q}}^{\alpha}_{\mathsf{k}+1} \leftarrow -\frac{1}{\omega}\mathsf{W}^{\alpha\alpha}\mathsf{r}^{\alpha}_{\mathsf{k}} + \sum_{\beta<\alpha}\mathsf{W}^{\alpha\beta}\mathsf{r}^{\beta}_{\mathsf{k}+1} + \sum_{\beta\geqslant\alpha}\mathsf{W}^{\alpha\beta}\mathsf{r}^{\beta}_{\mathsf{k}} + \mathsf{q}^{\alpha}$
      Solve the single contact problem $\mathrm{FC}(\bar{\mathsf{W}}^{\alpha\alpha}, \bar{\mathsf{q}}^{\alpha}_{\mathsf{k}+1}, \mu)$
    end for
    Evaluate error.
    $\mathsf{k} \leftarrow \mathsf{k}+1$
  end while
  $\mathsf{r} \leftarrow \mathsf{r_k}$
  $\mathsf{u} \leftarrow \mathsf{u_k}$

6.2 Proximal points techniques

Redaction note O.H. 6.1.
The following paragraph is a work in progress.

Proximal points methods, or more generally augmented Lagrangian method, finds their roots in the study of variational inequalities. They rely on rather theoretical tools (monotone operators and resolvents), introduced to study VI. However, they are also useful to device numerical methods as we shall see. The basic idea is to design an iterative algorithm to find a solution to the inclusion

$$0 \in F(x) + N_X(x) \qquad \text{also} \quad 0 \in T(x),$$

where T is a maximal monotone operator (see Definition 1). Remember that the first inclusion is a.

Redaction note V.A. 6.1.

Small introduction on proximal functions of Moreau, algo of Martinet and result of Rockafellar (see [Chen and Teboulle, 1993])

Without entering into theoretical aspects, the key idea of proximal points algorithms it to replace the original $\mathrm{VI}(F, X)$ by a sequence of $\mathrm{VI}(F_{\rho,x_k}, X)$ problems such that

$$F_{\rho,x}(z) = z - x_k + \alpha F(z), \quad \rho > 0 \tag{47}$$ {eq:prox-algo

with the property that $\lim_{k\to+\infty}\|x_k - z\| = 0$. This is usually performed by defining a sequence x_k such that

$$x_{k+1} = (1-\omega)x_k + \omega z_{k+1}$$

where ω is a relaxation parameter and z_{k+1} the solution of $\mathrm{VI}(F_{\rho,x_k}, X)$. The algorithm is described in algorithm 9

For solving the sub-problem $\mathrm{VI}(F_{\alpha,x_k}, X)$, any of the previous presented algorithms can be used. In this sense, the proximal point algorithm defined a general family of algorithm. The main interest of the proximal point algorithm is the regularization it introduces in the definition (47). For instance, let consider an affine VI, that is defined with $F(z) = Mz + q$. The function is the sub-problem is given by

$$F_{\alpha,x}(z) = (I + \alpha M)z - x_k + \rho q, \quad \rho > 0.$$

Algorithm 9 Proximal point algorithm for the VI (14)

Require: F, X Data of VI (14)
Require: ω relaxation parameter
Require: α proximal point parameter
Require: x_0 initial values
Require: $\mathsf{tol} > 0, \mathsf{tol}_{\mathsf{in}}$ tolerances
Ensure: z solution of VI (14)
  $\mathsf{k} \leftarrow 0$
  while error $> \mathsf{tol}$ **do**
    Solve $\mathrm{VI}(F_{\alpha,x_k}, X)$ for $\mathsf{z}_{\mathsf{k}+1}$ with tolerance $\mathsf{tol}_{\mathsf{in}}$
    $\mathsf{x}_{\mathsf{k}+1} \leftarrow (1-\omega)\mathsf{x}_{\mathsf{k}} + \omega \mathsf{z}_{\mathsf{k}+1}$
    Evaluate error.
    $\mathsf{k} \leftarrow \mathsf{k}+1$
  end while
  $\mathsf{z} \leftarrow \mathsf{z}_{\mathsf{k}+1}$

We can easily that for sufficiently small ρ, we get a monotone affine VI and even better a strongly monotone VI. For Problem FC with F_{vi}, the proximal point algorithm yields

$$F_{\mathrm{vi},\rho,x}(r) = (I+\alpha W)r - x_k + \alpha(q + g(Wr+q)), \quad \rho > 0.$$

and it should not be difficult to prove that with a small ρ parameter that we get a monotone VI. Naturally, there is a price to pay, smaller the parameter ρ is, easier the VI problem is, but the resulting solution of z_{k+1} is far from the solution.

The use of proximal point algorithm can be very interesting when the solving of the problem suffers for the lack of regularity of the operator. For instance, the Newton methods are in trouble when the Jacobian is not invertible. Thanks to the proximal point algorithm, we can retrieve invertible Jacobian.

Redaction note V.A. 6.2.

- *Prove it !*
- *Is there an analogy with choosing a small r ?*
- *Techniques of [Han, 2008] for sizing alpha*

6.3 Nomenclature

A nomenclature for the algorithms based on the projection/splitting approach is given in Table 3.

7 Optimization based methods

7.1 Alternating optimization problem

The Panagiotopoulos approach described in Section 3.5 generates a family of solvers by choosing a particular solver for the normal contact problem (3.5) and the tangential contact problem (3.5). More precisely, the following choices may be made

The normal contact problem

$$\begin{cases} \min \dfrac{1}{2} r_{\mathrm{N}}^\top W_{\mathrm{NN}} r_{\mathrm{N}} + r_{\mathrm{N}}^\top \tilde{q}_{\mathrm{N}} \\ \text{s.t.} \quad r_{\mathrm{N}} \geqslant 0 \end{cases} \quad \text{with } \tilde{q}_{\mathrm{N}} = q_{\mathrm{N}} + W_{\mathrm{NT}} r_{\mathrm{T},k},$$

Algorithm	Description	parameters
NSGS-AC	Algorithm 8 with $\omega = 1$ with the local solver NSN-AC with tolerance tol_{local}	$tol, tol_{local}, \rho_{\mathrm{N}}, \rho_{\mathrm{T}}$
NSGS-JM	Algorithm 8 with $\omega = 1$ with the local solver NSN-JM with tolerance tol_{local}	$tol, tol_{local}, \rho_{\mathrm{N}}, \rho_{\mathrm{T}}$
NSGS-AC-GP	Algorithm 8 with $\omega = 1$ with the local solver NSN-AC-GP with tolerance tol_{local}	$tol, tol_{local}, \rho_{\mathrm{N}}, \rho_{\mathrm{T}}$
NSGS-JM-GP	Algorithm 8 with $\omega = 1$ with the local solver NSN-JM=GP with tolerance tol_{local}	$tol, tol_{local}, \rho_{\mathrm{N}}, \rho_{\mathrm{T}}$
NSGS-FP-DS-One	Algorithm 8 with $\omega = 1$ with one iteration of FP-DS for the local solver	tol
NSGS-FP-VI-UPK	Algorithm 8 with $\omega = 1$ with FP-VI-UPK for the local solver with tolerance tol_{local}	$tol, tol_{local},$
NSGS-EXACT	Algorithm 8 with $\omega = 1$ with the exact local solver.	tol
PSOR-AC	Algorithm 8 with the local solver NSN-AC with tolerance tol_{local}	ω, tol, tol_{local}
PPA-NSN-AC	Algorithm 9 with the NSN-AC solver as internal solver.	ω, ρ
PPA-NSGS-AC	Algorithm 9 with the NSGS-AC solver as internal solver.	ω, ρ

Table 3: Naming convention for the algorithms based on splitting and proximal algorithms

is a convex quadratic program with simple bound constraints that can be solved with any dedicated solver. In this work, we solve it with a) an active set strategy, b) a conjugate gradient approach with projection as it is developed in [Moré and Toraldo, 1991] or c) an adaptation of one of the splitting techniques detailed in Section 6. It is clear that we can also use semi-smooth Newton methods but our experience has shown that such methods are not efficient when $\ker(W_{\mathrm{NN}}) \neq \{0\}$. Note that there exists also a bunch of methods in the literature that improves the method of [Moré and Toraldo, 1991] for large–scale systems. Unfortunately, to keep the article in a reasonable size, we leave these improvements for further studies.

The tangential problem,

$$\begin{cases} \min \; \frac{1}{2} r_{\mathrm{T}}^\top W_{\mathrm{TT}} r_{\mathrm{T}} + r_{\mathrm{T}}^\top \tilde q_{\mathrm{T}} \\ \text{s.t.} \quad r_{\mathrm{T}} \in D(\mu, \tilde r_{\mathrm{N}}) \end{cases} \quad \text{with } \tilde q_{\mathrm{T}} = q_{\mathrm{T}} + W_{\mathrm{TN}} r_{\mathrm{N},k+1},$$

is also a convex program but with a more complex structure since the constraints are quadratic one. Although it exists also some dedicated methods to solve this specific problem, we will use an adaptation of one of the splitting techniques detailed in Section 6. The method in [Dostál and Kozubek, 2012] (an extension of [Moré and Toraldo, 1991]) which is an algorithm for solving QP over convex constraints (disk constraint for instance) could be used.

In Table 4, we detailed the algorithms we use in the present study. The algorithm in Algorithm 10 shows that this method may be viewed as a two-block Gauss-Seidel method as it has been pointed out by [Tzaferopoulos, 1993].

Redaction note V.A. 7.1.
cite [11, 13, 14] and [25-29] of [Tzaferopoulos, 1993].

7.2 Successive approximation method

The methods of successive approximation successive approximations is a natural tool for the numerical realization of Problem FC. It is based on the Tresca approximation of the Coulomb cone as it is described in Section 3.5 and the work of the celebrated Czech school which summarizes the seminal work of the Czech school [Nečas et al., 1980, Haslinger, 1983, 1984, Haslinger et al., 1996]. Each iterative step is represented by an auxiliary contact problem with given friction described by quadratic program over a

Algorithm 10 Panagiotopoulos decomposition algorithm for Problem FC

Require: $\mathsf{W}, \mathsf{q}, \mu$
Require: r_0 initial values and $\mathsf{tol} > 0$ a tolerance
Ensure: r, u solution of Problem FC
  $\mathsf{r}_\mathsf{k} \leftarrow \mathsf{r}_0$; $\mathsf{k} \leftarrow 0$
  while error $>$ tol **do**
    $\tilde{\mathsf{q}}_\mathsf{N} \leftarrow \mathsf{q}_\mathsf{N} + \mathsf{W}_\mathsf{NT}\mathsf{r}_{\mathsf{T},\mathsf{k}}$
    solve (7.1) for $r_{\mathsf{N},k+1}$
    $\tilde{\mathsf{q}}_\mathsf{T} \leftarrow \mathsf{q}_\mathsf{T} + \mathsf{W}_\mathsf{TN}\mathsf{r}_{\mathsf{N},\mathsf{k}+1}$
    solve (7.1) for $r_{\mathsf{T},k+1}$
    $\mathsf{k} \leftarrow \mathsf{k}+1$
    evaluate error.
  end while
  $\mathsf{r} \leftarrow \mathsf{r}_\mathsf{k}$
  $\mathsf{u} \leftarrow \mathsf{W}\mathsf{r} + \mathsf{q}$

cylinder (3.5), that we recall there:

$$\begin{cases} \theta = h(r_{\mathrm{N}}) \\ \min \ \frac{1}{2} r^\top W r + r^\top q \\ \text{s.t.} \quad r \in C(\mu, \theta). \end{cases}$$

The radius of the cylinder is then updated in a iterative procedure. The algorithm is described in Algorithm ??

Algorithm 11 Tresca approximation algorithm for Problem FC

Require: $\mathsf{W}, \mathsf{q}, \mu$
Require: r_0 initial values and $\mathsf{tol} > 0$ a tolerance
Ensure: r, u solution of Problem FC
  $\mathsf{r}_\mathsf{k} \leftarrow \mathsf{r}_0$; $\mathsf{k} \leftarrow 0$
  while error $>$ tol **do**
    $\theta \rightarrow h(r_{\mathsf{N},k})$
    solve 7.2 for $r_{\mathsf{N},k+1}, r_{\mathsf{T},k+1}$
    $\mathsf{k} \leftarrow \mathsf{k}+1$
    evaluate error.
  end while
  $\mathsf{r} \leftarrow \mathsf{r}_\mathsf{k}$
  $\mathsf{u} \leftarrow \mathsf{W}\mathsf{r} + \mathsf{q}$

In the literature, the successive approximation technique has been used in the bidimensional case in [Haslinger et al., 2002] & [Dostál et al., 2002] with improved and dedicated QP solvers overs box-constraints. Two strategies are implemented : a) the classical Tresca iteration (called FPMI) and b) the Panagiotopoulos decomposition plus a Fixed point (called FPMII). They use a specific QP solver for box constraint [Dostál, 1997] that is an improvement of Moré–Toraldo method [Moré and Toraldo, 1991]. This technique has been direcltly extended in the three-dimensional case with a faceting of the cone in [Haslinger et al., 2004]. In the latter case, the problem is still a box constrained QP since it contains only polyhedral constraints.

[Haslinger et al., 2012] propose a successive approximation technique in 3D with the special solver of [Kučera, 2007, 2008] which is itself an extension to disk constraints of the Polyak method (conjugate gradient with active set on the bounds constraint) and its improvements [Dostál, 1997, Dostál and Schöberl, 2005].

[Dostál and Kozubek, 2012] An algorithm (extension of MPGP) for solving QP over convex constraint (disk constraint for instance). Application to frictional contact.

Name	Algo.	Additional informations	parameters
PANA-NSGS-NSN-FB	10	NSGS-NSN-FB for both normal and tangent problem	tol, tol_{local}
PANA-QP-NSGS-NSN	10	Active set strategy for normal contact problem and NSGS with NSN for tangential problem	tol, tol_{local}
PANA-VI-EG	10	VI-EG for both normal and tangent problem	tol, tol_{local}
PANA-...	10		tol, tol_{local}
TRESCA-NSGS-FP-VI			tol, tol_{local}
ACLM-NSGS-FP-VI			tol, tol_{local}

Table 4: Naming convention for optimization based algorithms

[Dostál and Kučera, 2010] last improvement of the method in [Dostál and Kozubek, 2012] and [Kučera, 2008]

7.3 ACLM approach

7.4 SOCCP approach

7.5 Nomenclature

A nomenclature for the algorithms based on the optimisation approach is given in Table 4.

8 Comparison framework

8.1 Measuring errors

8.2 Parameters

- size (n, m), sparsity,
- matrix storage
- conditionning M W
- rank of H or W on active contact
- parameter $\nu = m_c/n$
- algorithms parameters (ρ)

8.3 Measure of performance

- CPU time and memory or better flops (papi) for a fixed user tolerance
- Reached accuracy for a given CPU effort

Other studies.

- Convergence rate (error w.r.t. flop within the iteration process)
- Scalable properties () study with a set of large size ?

8.4 Benchmarks presentation

8.5 Sofware & implementation details

9 Comparison of methods by family

9.1 Numerical methods for VI

Evaluation of the influence of the self–adaptive procedure for step length In Figure 3, we study the effect of the self–adaptive procedure in Algorithm 6 on the convergence of the of the fixed point method in Algorithm 1.

Comparison between fixed point, extragradient and projection–contraction method.

9.2 Splitting based algorithms

Redaction note V.A. 9.1.
TODO list

- *Effect and influence of the local solver*
- *Influence of the tolerance of the local solver* tol_{local}
- *Influence of the contacts order*
- *Comparison of PSOR algorithm with respect to the relaxation parameter* ω

Effect and influence of the local solver in NSGS algorithms

Test set	required accuracy	average performance
Capsules	10^{-08}	$1.072 \cdot 10^{-03}s$
KaplasTower	10^{-08}	$1.024 \cdot 10^{-03}s$
LMGC_Bridge_PR	10^{-05}	$1.581 \cdot 10^{-02}s$
LMGC_Cubes_H20	10^{-04}	$6.271 \cdot 10^{-02}s$

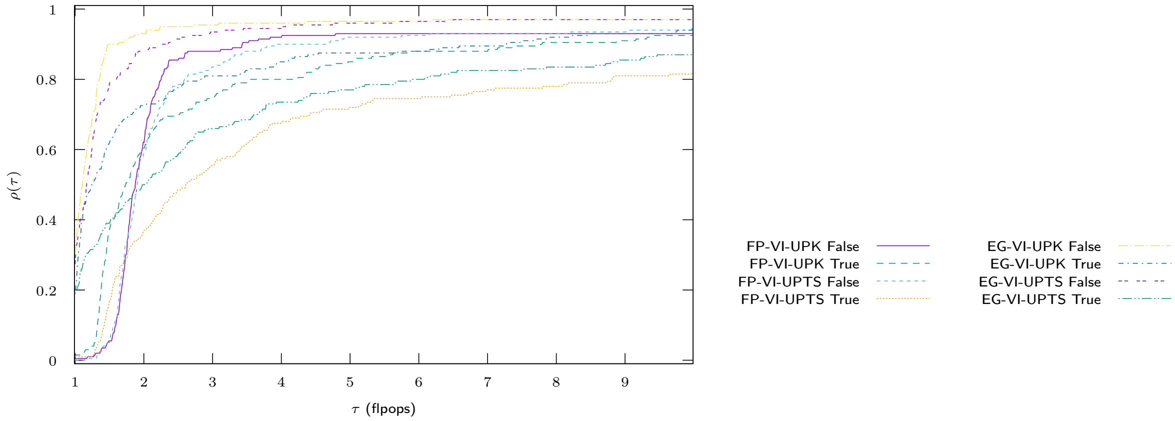
Table 5: The average performance of resolution by contacts for the best solver to reach a given accuracy.

Influence of the tolerance of the local solver tol_{local} in NSGS algorithms In this section, the tolerance of the local solver is varied and its effect on the global convergence of the solver is reported. For the "Capsules" set of examples in figure 5(a), the tolerance of the local solver tol_{local} has almost no effect on the performance of the global solver. Surprisingly, the algorithm is also able to reach the global accuracy of $tol = 10^{-8}$ with a quite low accuracy of the local solver (10^{02} for instance). This mainly due to the fact that at least one iteration of the local solver is always done and the set of examples are not so difficult to solve (see the average performance in Table 5). Let us have a look to more difficult examples in Figures 5(c) and 5(d). Although the required accuracy is lower, the solver with a low local tolerance fails to solve the problems efficiently. In the most difficult test set, it is even required to have a local tolerance tol_{local} at a very low level with respect to the tolerance tol to improve the rate of convergence and even to ensure the success of the solver.

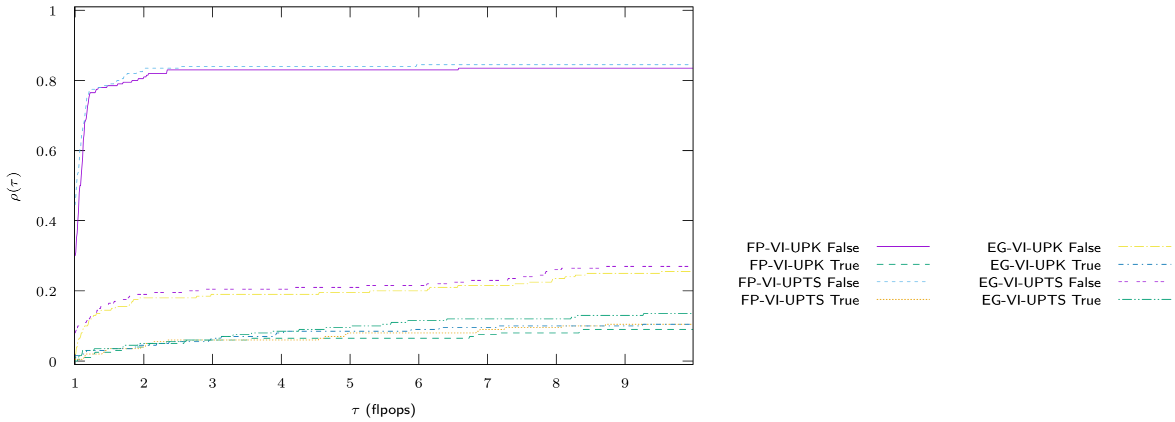
Redaction note V.A. 9.2.

redo this comparison with the flop measure.

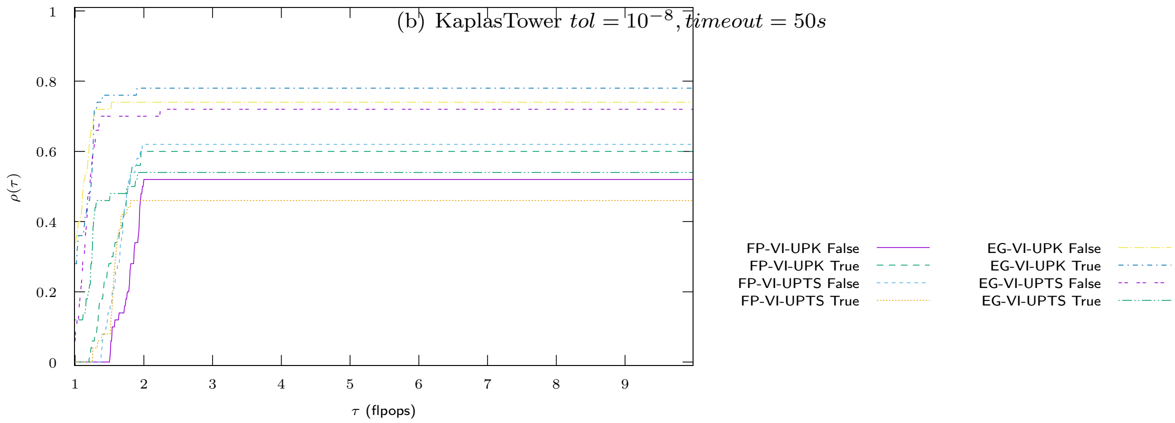
The Capsules test seems very easy to solve



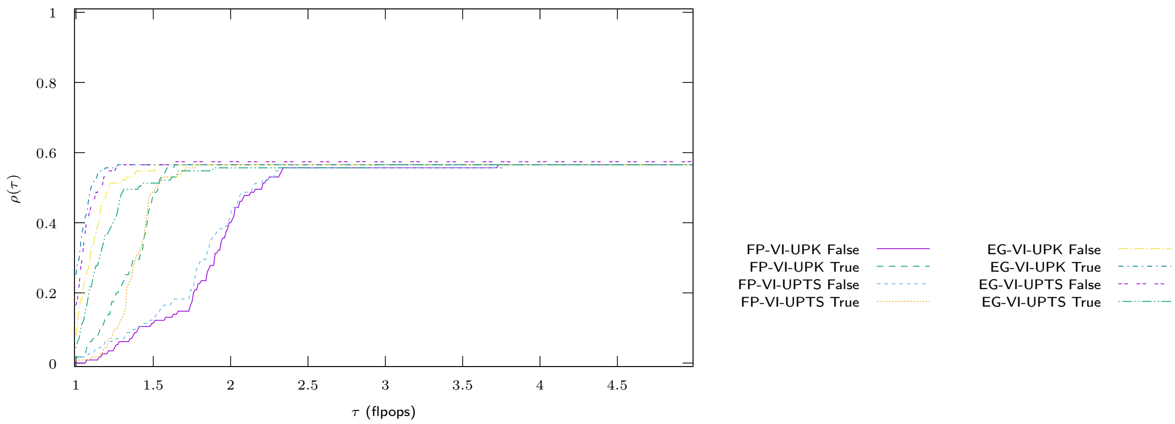
(a) Capsules tests (reduced) $tol = 10^{-8}, timeout = 10s$



(b) KaplasTower $tol = 10^{-8}, timeout = 50s$

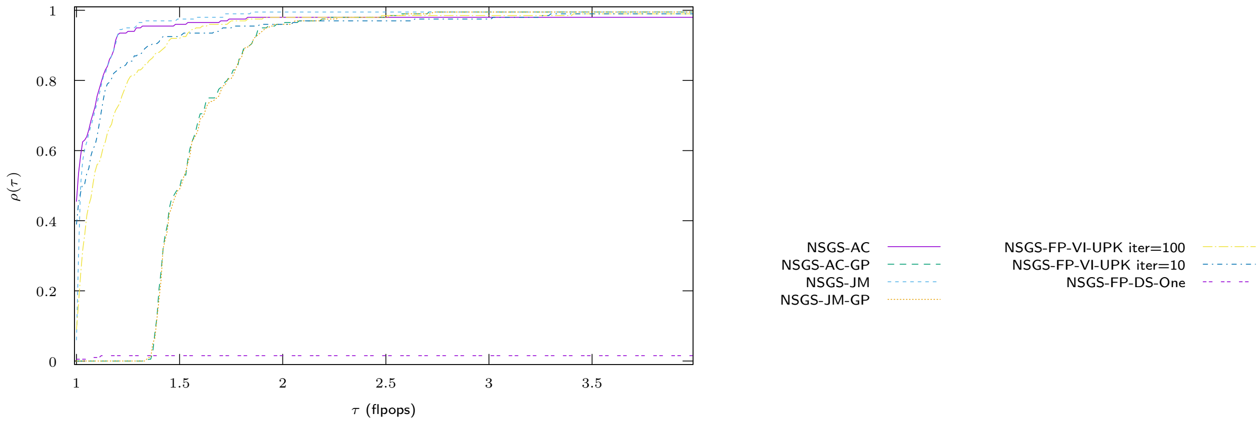


(c) LMGC_Bridge_PR $tol = 10^{-5}, timeout = 50s$

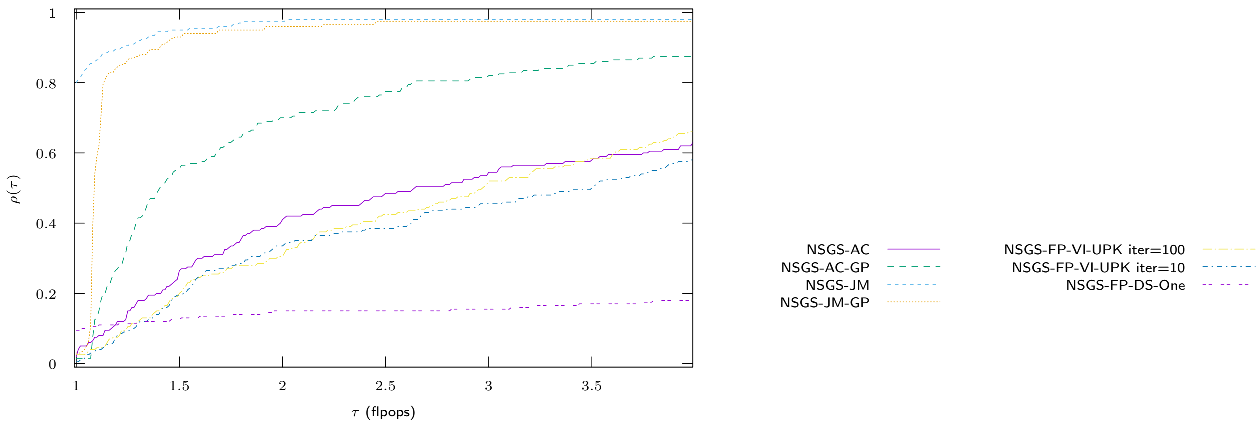


(d) LMGC_Cubes_H8_20 $tol = 10^{-5}, timeout = 50s$

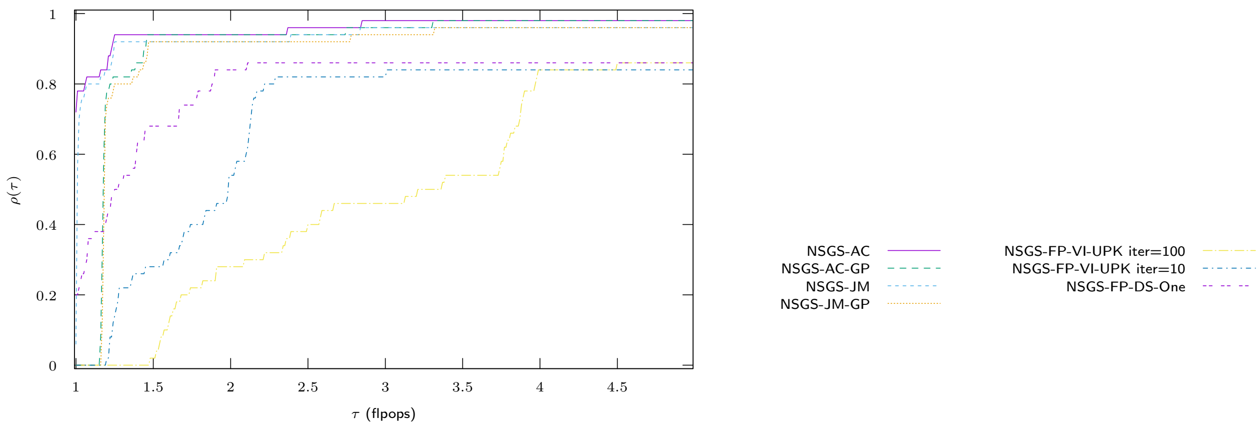
Figure 3: Evaluation of the influence of the self–adaptive procedure for step length.



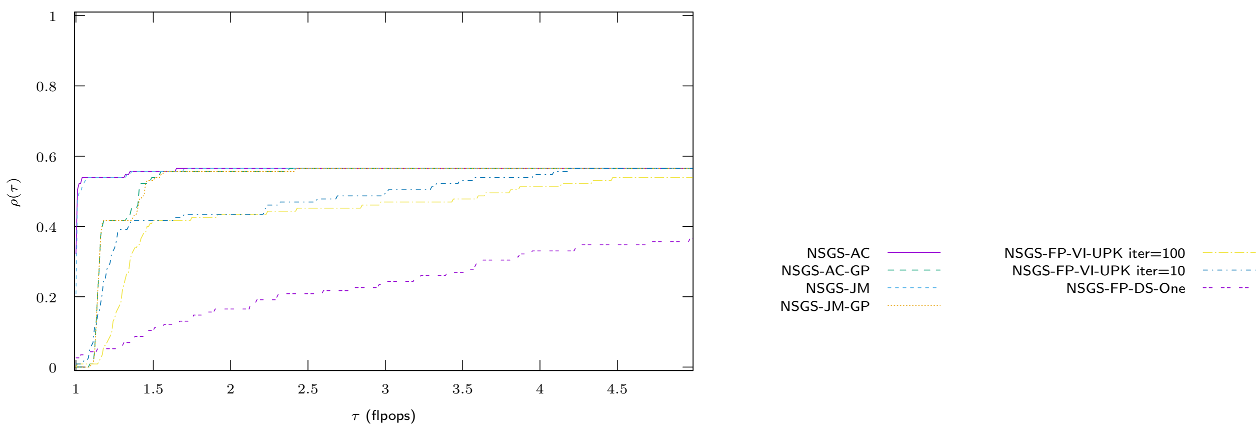
(a) Capsules tests (reduced) $tol = 10^{-8}, timeout = 10s$



(b) KaplasTower $tol = 10^{-8}, timeout = 50s$

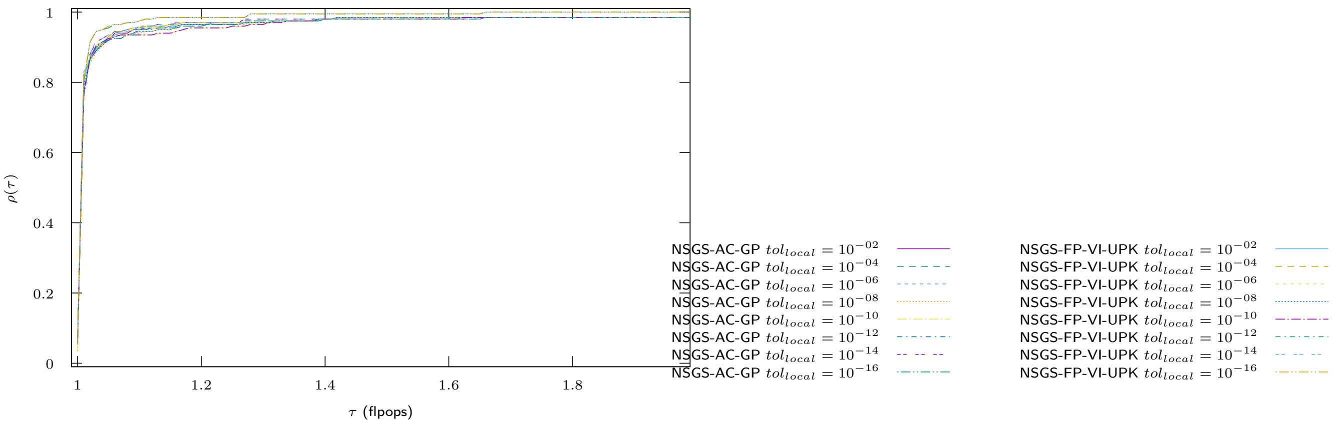


(c) LMGC_Bridge_PR $tol = 10^{-5}, timeout = 50s$

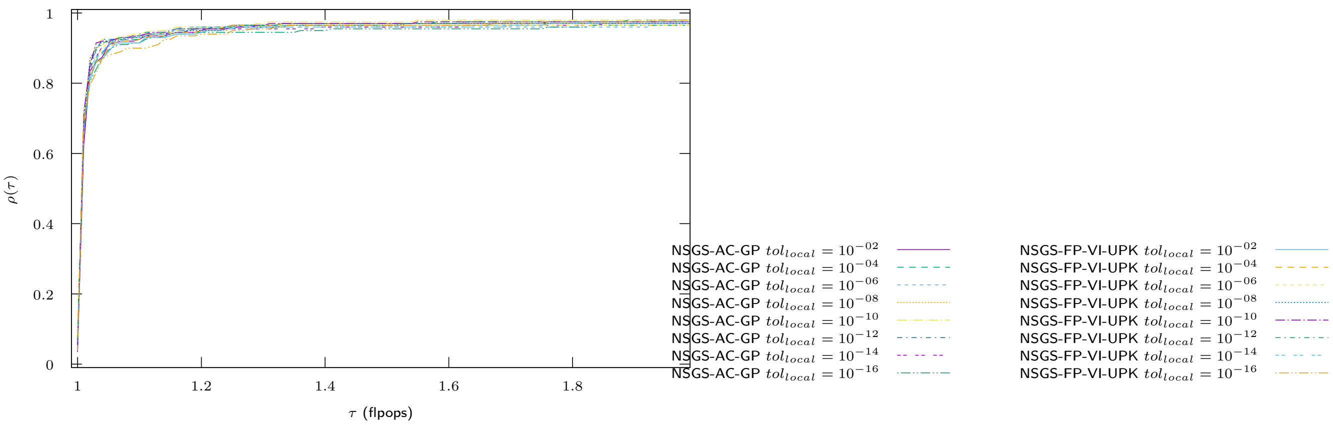


(d) LMGC_Cubes_H8 tests. $tol = 10^{-4}, timeout = 50s$

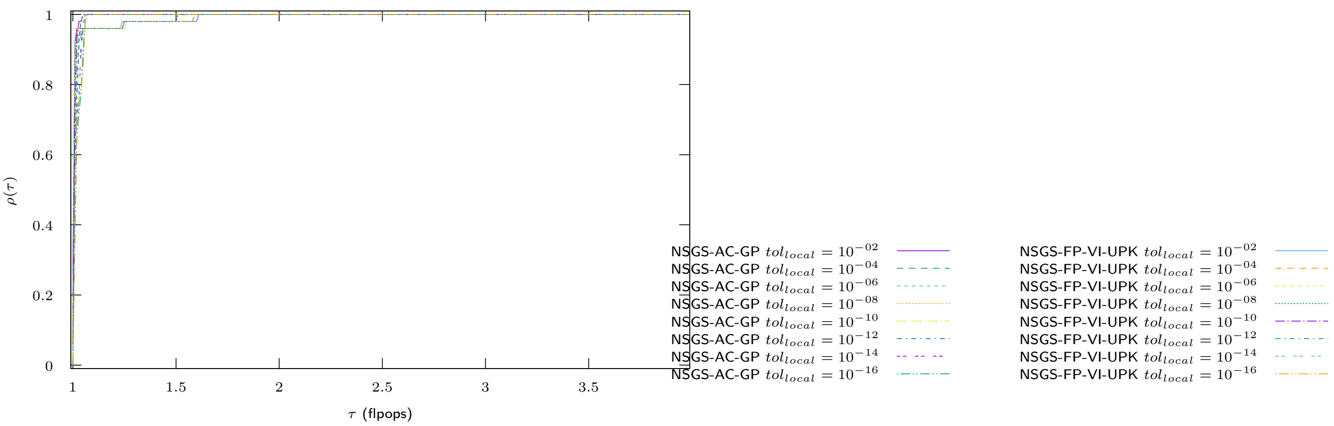
Figure 4: Influence of the local solver in NSGS algorithms.



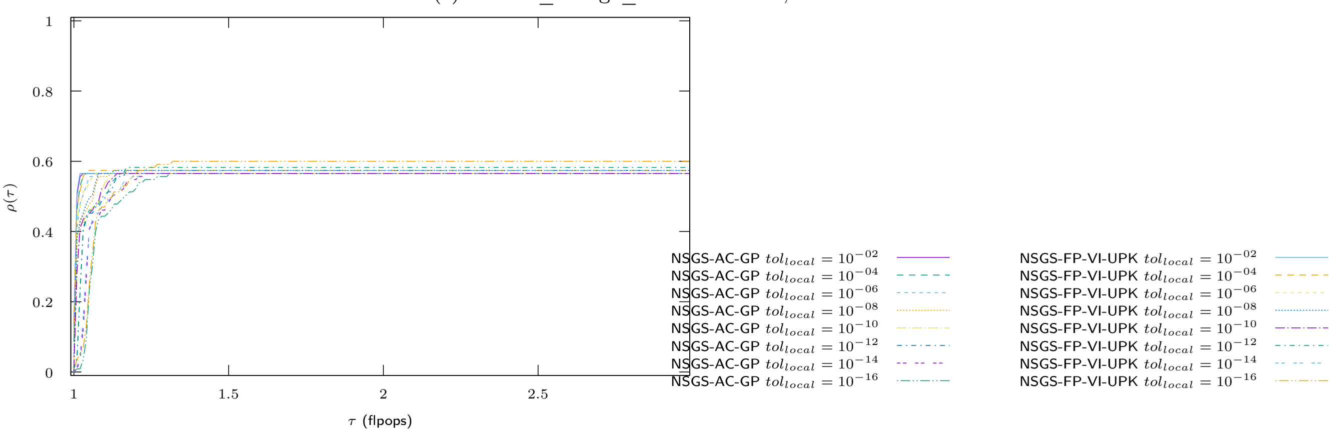
(a) Capsules tests (reduced) $tol = 10^{-8}, timeout = 10s$



(b) KaplasTower $tol = 10^{-8}, timeout = 50s$



(c) LMGC_Bridge_PR $tol = 10^{-5}, timeout = 50s$



(d) LMGC_Cubes_H20 tests. $tol = 10^{-4}, timeout = 50s$

Figure 5: Influence of the tolerance of the local solver tol_{local} in NSGS algorithms.

Influence of the contacts order in NSGS algorithms In this section, we study thein the list of contact that is iterated by the NSGS-AC solver. We reproduce in Figure 6(a) the result of the solvers with the original contact list of the problem (NSGS-AC), with the 10 lists of contacts that are randomly shuffled (NSGS-AC-Shuffles-x) and with a list of contact that is shuffled in each loop ot the solver(NSGS-AC-Shuffled-full). We can observe that the contact order change slightly the behavior of the algorithm and most surprisingly the randomization in each loop of the NSGS algorithm deteriorates its convergence. In Figure 6(b), we restrict our attention to three solvers without noting major difference with the respect of the result of each solvers. In Figure 6(d), the result change drastically with the Cubes_H8_20 examples. For each problem, the randomization in each loop improves the performance by a factor between 20 and 60. Note the scale of the axis. In view of this result, the question of the contact order seems to be an important question for improving the performance of the NSGS solvers. It remains nevertheless an open question since it is difficult to guess a priori what the optimal order.

Redaction note V.A. 9.3.

Not so great interest of this section because it is difficult to conclude something

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Comparison of PSOR algorithm with respect to the relaxation parameter ω In Figure 7, we study the effect of the relaxation parameter ω ranging from $[0.5, 1.5]$ on the computational time. Two conclusions can be drawn a) with increasing values of ω, the PSOR algorithm increases its convergence rate as we van observe for $\tau = 1$ but b) the robustness of the algorithm is weakened. Indeed, $\tau = 15$, we observe that the performance profile is flat and the number of problems solved is higher for valued of ω around 1.

To conclude, it is difficult to advice to use PSOR algorithm with $\omega \neq 1$. If it accelerates drastically the rate of convergence of the algorithm for some problems it deteriorates the convergence for other. Further studies would be needed to design self–adaptive schemes for the choice of ω.

Redaction note V.A. 9.4.

redo this comparison on a set of mixed examples

add some results with $\omega = 1.8$ to show that higher values will destroy the convergence.

is it possible to find sizing rule in the literature ?

9.3 Comparison of PPA-NSN-AC algorithm with respect to the step-size parameter σ, μ

Redaction note V.A. 9.5.

redo this comparison on a set of mixed examples

add some results with $\nu < 1$

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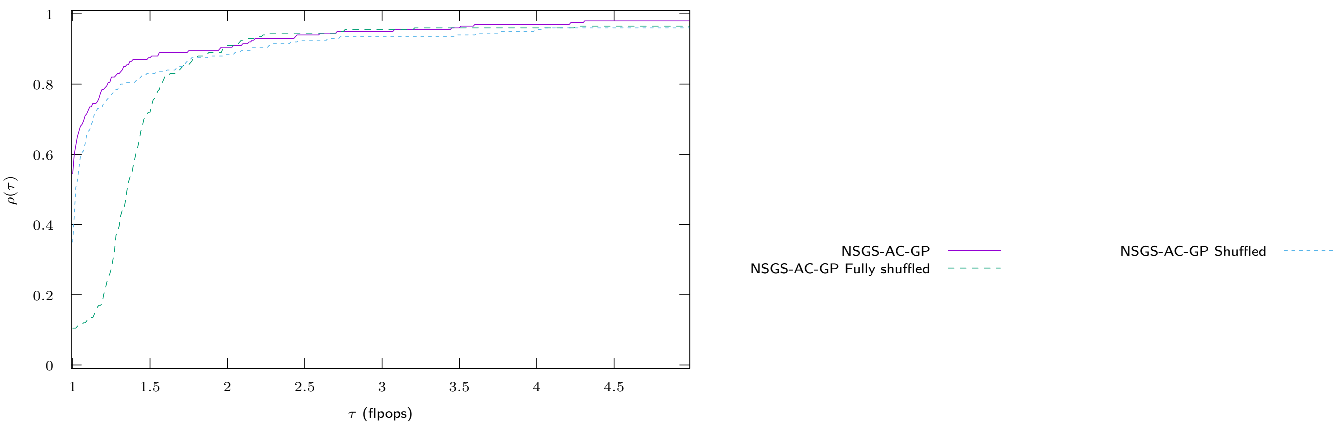
10 Comparison of different families of solvers.

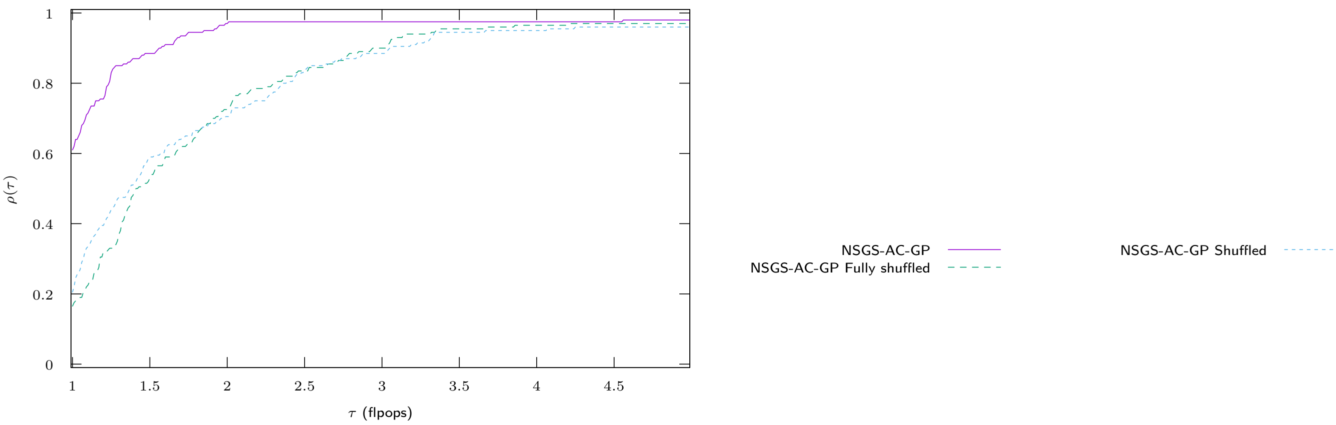
10.1 CPU and memory efforts for a given tolerance

Analyze the quickest one and the more robust one.

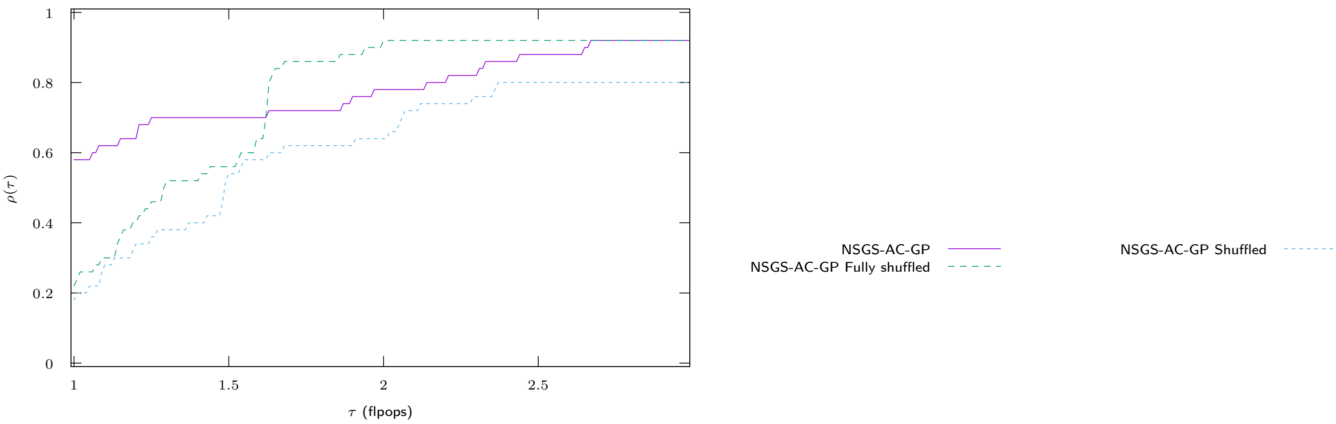
10.2 Analyze reached accuracy for a given time



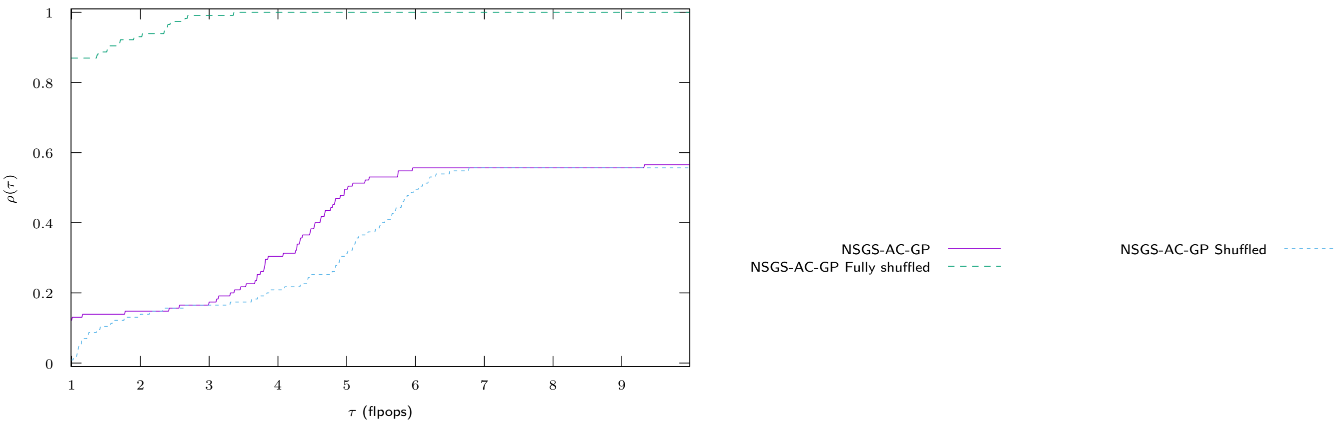
(a) Capsules tests (reduced) $tol = 10^{-8}, timeout = 10s$



(b) KaplasTower $tol = 10^{-8}, timeout = 50s$

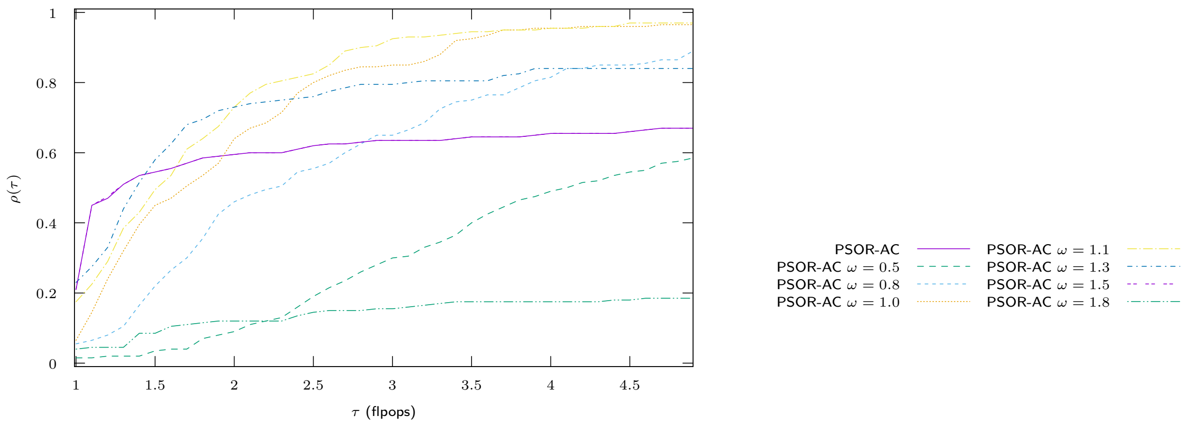


(c) LMGC_Bridge_PR tests. $tol = 10^{-5}, timeout = 50s$

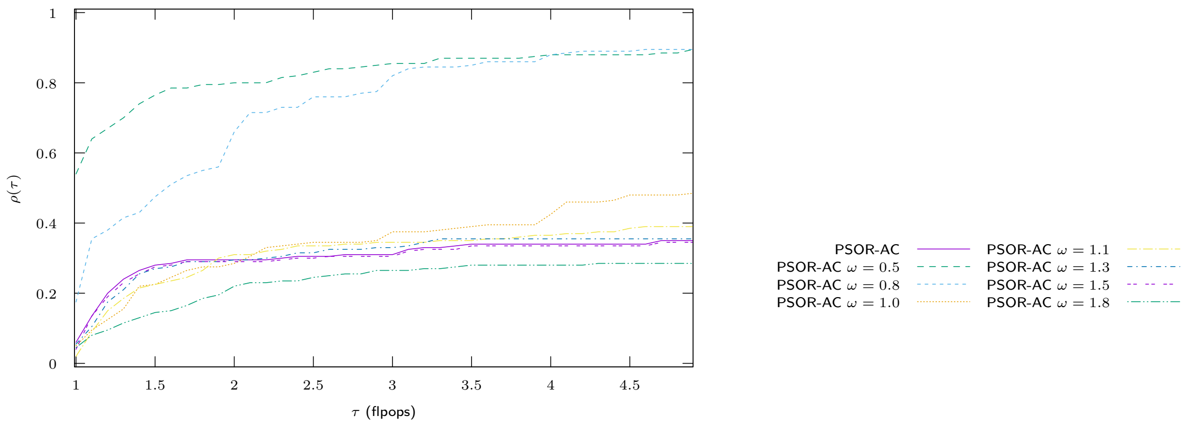


(d) LMGC_Cubes_H20 tests. $tol = 10^{-4}, timeout = 50s$

Figure 6: Influence of the contacts order in NSGS algorithms.

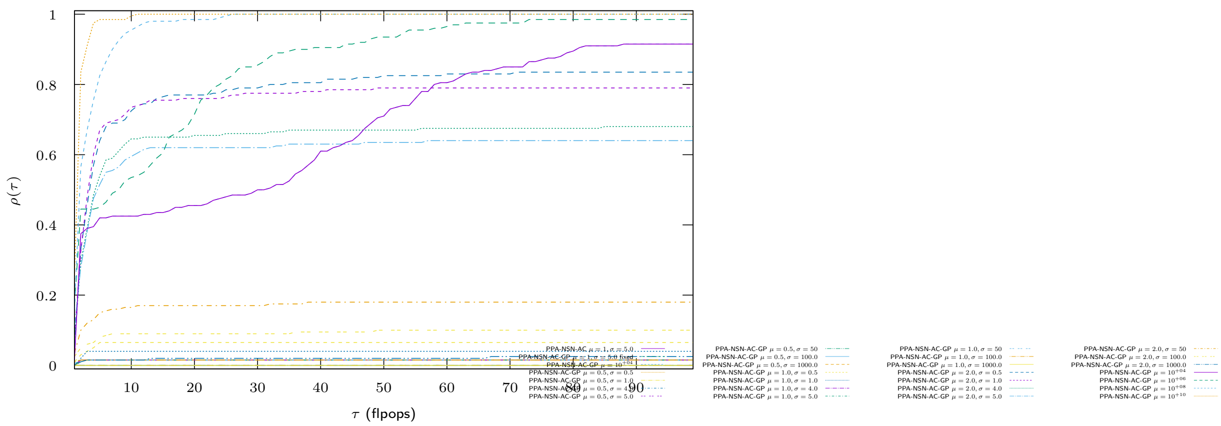


(a) Capsules tests (reduced) $tol = 10^{-8}, timeout = 100s$



(b) KaplasTower $tol = 10^{-8}, timeout = 100s$

Figure 7: Effect of relation coefficient ω in PSOR-AC algorithm($tol_{local} = 10^{-16}$).



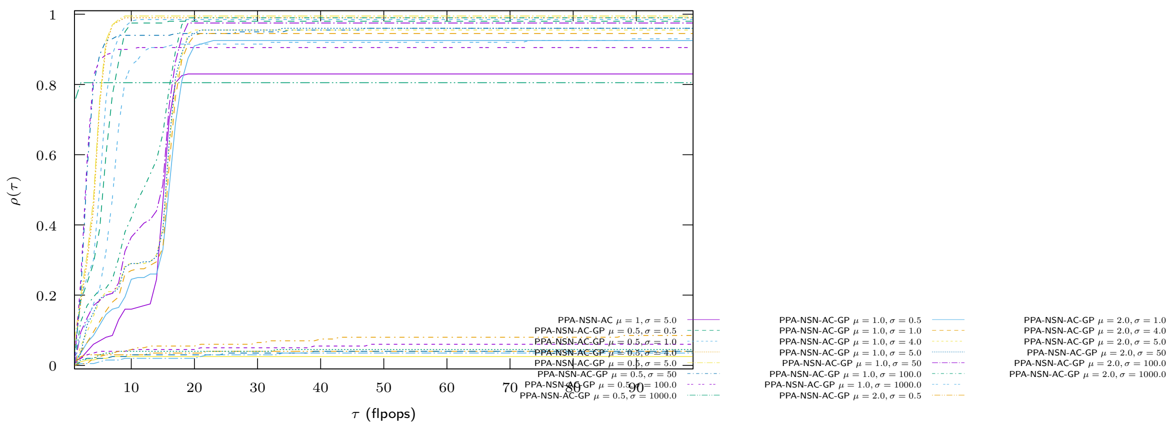


Figure 8: Effect of the step-size parameter σ, μ in PPA-NSN-AC algorithm

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A Basics in Convex Analysis

Definition 1 ([Rockafellar and Wets, 1997]). *Let $X \subseteq \mathbb{R}^n$. A multivalued (or point-to-set) mapping $T\colon X \rightrightarrows X$ is said to be (strictly) monotone if there exists $c(>) \geqslant 0$ such that for all $\hat{x}, \widetilde{x} \in X$*

$$(\hat{v} - \widetilde{v})^\top (\hat{x} - \widetilde{x}) \geqslant c\|\hat{x} - \widetilde{x}\| \qquad \text{with } \hat{v} \in T(\hat{x}), \widetilde{v} \in T(\widetilde{x}).$$

Moreover T is said to be maximal when it is not possible to add a pair (x, v) to the graph of T without destroying the monotonicity.

The Euclidean projector P_X onto a closed convex set X: for a vector $x \in \mathbb{R}^n$, the projected vector $z = P_X(x)$ is the unique solution of the convex quadratic programm

$$\begin{cases} \min \dfrac{1}{2}(y-x)^\top (y-x), \\ \text{s.t.} \quad y \in X. \end{cases}$$

$$\begin{aligned} y = P_K(x) &\iff \begin{array}{ll} \min & \frac{1}{2}(y-x)^\top (y-x) \\ \text{s.t.} & y \in K \end{array} \\ &\iff -(y-x) \in N_K(y) \\ &\iff (x-y)^\top (y-z) \geqslant 0, \forall z \in K \end{aligned} \tag{48}$$

{eq:tutu}

$$\begin{aligned} -F(x) \in N_K(x) &\iff -\rho F(x)^\top (y-x) \geqslant 0, \forall y \in K \\ &\iff (x - (x - \rho F(x))^\top (y-x) \geqslant 0, \forall y \in K \\ &\iff x = P_K(x - \rho F(x)) \text{ thanks to (48)} \end{aligned}$$

$$\partial \|z\|_2 = \begin{cases} \dfrac{z}{\|z\|} \\ \{x, \|x\| = 1\} \end{cases}$$

A.1 Euclidean projection on the disk in $\mathbb{R}^2$.

Let $D = \{x \in \mathbb{R}^3, \|x\| \leqslant 1\}$. We have

$$P_D(z) = \begin{cases} z & \text{if} \quad z \in D \\ \dfrac{z}{\|z\|} & \text{if} \quad z \notin D \end{cases}$$

and

$$\partial P_D(z) = \begin{cases} I & \text{if} \quad z \in D \setminus \partial D \\ I + (s-1)zz^\top, s \in [0,1] & \text{if} \quad z \in \partial D \\ \dfrac{I}{\|z\|} - \dfrac{zz^\top}{\|z\|^3} & \text{if} \quad z \notin D \end{cases}$$

A.2 Euclidean projection on the second order cone of $\mathbb{R}^3$.

Let $K = \{x = [x_{\mathrm{N}} x_{\mathrm{T}}]^T \in \mathbb{R}^3, x_{\mathrm{N}} \in \mathbb{R}, \|x_{\mathrm{T}}\| \leqslant \mu x_{\mathrm{N}}\}$. We have

$$P_K(z) = \begin{cases} z & \text{if} \quad z \in K \\ 0 & \text{if} \quad -z \in K^* \\ \dfrac{1}{1+\mu^2}(z_{\mathrm{N}} + \mu\|z_{\mathrm{T}}\|) \begin{bmatrix} 1 \\ \mu \dfrac{z_{\mathrm{T}}}{\|z_{\mathrm{T}}\|} \end{bmatrix} & \text{if} \quad z \notin K \text{ and } -z \notin K^* \end{cases}$$

Direct computation of an element of the subdifferential The computation of the subdifferential are given as follows

- if $z \in K \setminus \partial K$, $\partial_z P_K(z) = I$,
- if $-z \in K^* \setminus \partial K^*$, $\partial_z P_K(z) = 0$,
- if $z \notin K$ and $-z \notin K^*$ and, $\partial_z P_K(z) = 0$, we get

$$\partial_{z_{\mathrm{N}}} P_K(z) = \frac{1}{1+\mu^2}\begin{bmatrix} 1 \\ \mu z_{\mathrm{T}} \end{bmatrix}$$

and

$$\partial_{z_{\mathrm{T}}}[P_K(z)]_{\mathrm{N}} = \frac{\mu}{1+\mu^2}\frac{z_{\mathrm{T}}}{\|z_{\mathrm{T}}\|}$$

$$\partial_{z_{\mathrm{T}}}[P_K(z)]_{\mathrm{T}} = \frac{\mu}{(1+\mu^2)}\left[\mu\frac{z_{\mathrm{T}}}{\|z_{\mathrm{T}}\|}\frac{z_{\mathrm{T}}^\top}{\|z_{\mathrm{T}}\|} + (z_{\mathrm{N}} + \mu\|z_{\mathrm{T}}\|)(\frac{I_2}{\|z_{\mathrm{T}}\|} - \frac{z_{\mathrm{T}}z_{\mathrm{T}}^\top}{\|z_{\mathrm{T}}\|^3})\right]$$

that is

$$\partial_{z_{\mathrm{T}}}[P_K(z)]_{\mathrm{T}} = \frac{\mu}{(1+\mu^2)\|z_{\mathrm{T}}\|}\left[(z_{\mathrm{N}} + \mu\|z_{\mathrm{T}}\|)\, I_2 + z_{\mathrm{N}}\frac{z_{\mathrm{T}}z_{\mathrm{T}}^\top}{\|z_{\mathrm{T}}\|^2})\right]$$

Redaction note V.A. **A.1.**
to be checked carefully

Computation of the subdifferential using the spectral decomposition In [Hayashi et al., 2005], the computation of the Clarke subdifferential of the projection operator is also done by inspecting the different cases using the spectral decomposition

$$\partial P_K(x) = \begin{cases} I & (\lambda_1 > 0, \lambda_2 > 0) \\ \dfrac{\lambda_2}{\lambda_1+\lambda_2} I + Z & (\lambda_1 < 0, \lambda_2 > 0) \\ 0 & (\lambda_1 < 0, \lambda_2 < 0) \\ \mathrm{co}\{I, I+Z\} & (\lambda_1 = 0, \lambda_2 > 0) \\ \mathrm{co}\{0, Z\} & (\lambda_1 < 0, \lambda_2 = 0) \\ \mathrm{co}\{0 \cup I \cup S\} & (\lambda_1 = 0, \lambda_2 = 0) \end{cases}$$

where

$$Z = \tfrac{1}{2}\begin{bmatrix} -y_{\mathrm{N}} & y_{\mathrm{T}}^\top \\ y_{\mathrm{T}} & -y_{\mathrm{N}} y_{\mathrm{T}} y_{\mathrm{T}}^\top \end{bmatrix},$$
$$S = \left\{ \tfrac{1}{2}(1+\beta)I + \tfrac{1}{2}\begin{bmatrix} -\beta & w^\top \\ w & -\beta w w^\top \end{bmatrix} \mid -1 \leqslant \beta \leqslant 1, \|w\| = 1 \right\}$$

with $y = x/\|x_{\mathrm{T}}\|$. A simple verification shows that the previous computation is an element of the subdifferential.

Redaction note V.A. **A.2.**
to be checked carefully

B Computation of Generalized Jacobians for Nonsmooth Newton methods

B.1 Computation of components of subgradient of $F_{\mathrm{vi}}^{\mathsf{nat}}$

Let us introduce the following notation for an element of the sub–differential

$$\Phi(u,r) = \begin{bmatrix} \rho I & -\rho W \\ \Phi_{ru}(u,r) & \Phi_{rr}(u,r) \end{bmatrix} \in \partial F_{\mathrm{vi}}^{\mathsf{nat}}(u,r)$$

where $\Phi_{xy}(u,r) \in \partial_x [F^{\mathsf{nat}}_{\text{vi}}]_y(u,r)$. Since $\Phi_{uu}(u,r) = I$, a reduction of the system is performed in practise and Algorithm 7 is applied or $z = r$ with

$$\begin{cases} G(z) = [F^{\mathsf{nat}}_{\text{vi}}]_r(Wr+q,r) \\ \Phi(z) = \Phi_{rr}(r,Wr+q) + \Phi_{ru}(r,Wr+q)W \end{cases}$$

Redaction note V.A. B.1.
to be checked carefully. compare with [Hayashi et al., 2005]

Let us introduce the following notation for an element of the sub–differential with a obvious simplification

$$\Phi(v,r) = \left[\begin{array}{ccc} \rho M & -\rho H & \\ -\rho H^\top & \rho I & 0 \\ 0 & \Phi_{ru}(v,u,r) & \Phi_{rr}(v,u,r) \end{array}\right] \in \partial F^{\mathsf{nat}}_{\text{vi}}(u,r)$$

where $\Phi_{xy}(v,u,r) \in \partial_x [F^{\mathsf{nat}}_{\text{vi-1}}]_y(v,u,r)$. A possible computation of $\Phi_{ru}(v,u,r)$ and $\Phi_{rr}(v,u,r)$ is directly given by (50) and (49). In this case, the variable u can be also substituted.

For one contact, a possible computation of the remaining parts in $\Phi(u,r)$ is given by

$$\Phi_{ru}(u,r) = \begin{cases} 0 & \text{if} \quad r - \rho(u + g(u)) \in K \\ I - \partial_r[P_K(r - \rho(u+g(u)))] & \text{if} \quad r - \rho(u+g(u)) \notin K \end{cases} \tag{49}$$ {eq:Phi-natur

$$\Phi_{ru}(u,r) = \begin{cases} \rho\left(I + \left[\begin{array}{ccc} 0 & 0 & 0 \\ \dfrac{u_{\mathrm{T}}}{\|u_{\mathrm{T}}\|} & 0 & 0 \end{array}\right]\right) & \text{if} \quad \begin{cases} r - \rho(u+g(u)) \in K \\ u_{\mathrm{T}} \neq 0 \end{cases} \\ \rho\left(I + \left[\begin{array}{ccc} 0 & 0 & 0 \\ s & 0 & 0 \end{array}\right]\right), s \in \mathbb{R}^2, \|s\| = 1 & \text{if} \quad \begin{cases} r - \rho(u+g(u)) \in K \\ u_{\mathrm{T}} = 0 \end{cases} \\ I + \rho\left(I + \left[\begin{array}{ccc} 0 & 0 & 0 \\ \dfrac{u_{\mathrm{T}}}{\|u_{\mathrm{T}}\|} & 0 & 0 \end{array}\right]\right)\partial_u[P_K(r - \rho(u+g(u)))] & \text{if} \quad r - \rho(u+g(u)) \notin K \end{cases} \tag{50}$$ {eq:Phi-natur

The computation of an element of ∂P_K is given in Appendix A.

B.2 Alart–Curnier function and its variants

For one contact, a possible computation of the remaining parts in $\Phi(u,r)$ is given by

$$\Phi_{r_{\mathrm{N}}u_{\mathrm{N}}}(u,r) = \begin{cases} \rho_{\mathrm{N}} & \text{if } r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}} > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Phi_{r_{\mathrm{N}}r_{\mathrm{N}}}(u,r) = \begin{cases} 0 & \text{if } r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}} > 0 \\ 1 & \text{otherwise} \end{cases}$$

$$\Phi_{r_{\mathrm{T}}u_{\mathrm{N}}}(u,r) = \begin{cases} 0 & \text{if } \|r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}\| \leqslant \mu \max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) \\ 0 & \text{if } \begin{cases} \|r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}\| > \mu \max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) \\ r_{\mathrm{N}} - \rho_{\mathrm{N}} u_n \leqslant 0 \end{cases} \\ \mu\rho_{\mathrm{N}} \dfrac{r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}}{\|r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}\|} & \text{if } \begin{cases} \|r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}\| > \mu \max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) \\ r_{\mathrm{N}} - \rho_{\mathrm{N}} u_n > 0 \end{cases} \end{cases}$$

$$\Phi_{r_{\mathrm{T}}u_{\mathrm{T}}}(u,r) = \begin{cases} \rho_{\mathrm{T}} & \text{if } \|r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}\| \leqslant \mu \max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) \\ \mu\rho_{\mathrm{T}}(r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}})_+ \Gamma(r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}) & \text{if } \begin{cases} \|r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}\| > \mu \max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) \\ r_{\mathrm{N}} - \rho_{\mathrm{N}} u_n > 0 \end{cases} \end{cases}$$

$$\Phi_{r_{\mathrm{T}}r_{\mathrm{N}}}(u,r) = \begin{cases} 0 & \text{if } \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| \leqslant \mu \max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}}) \\ 0 & \text{if } \begin{cases} \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| > \mu \max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}}) \\ r_{\mathrm{N}} - \rho_{\mathrm{N}}u_n \leqslant 0 \end{cases} \\ -\mu \dfrac{r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}}{\|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\|} & \text{if } \begin{cases} \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| > \mu \max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}}) \\ r_{\mathrm{N}} - \rho_{\mathrm{N}}u_n > 0 \end{cases} \end{cases}$$

$$\Phi_{r_{\mathrm{T}}r_{\mathrm{T}}}(u,r) = \begin{cases} 0 & \text{if } \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| \leqslant \mu \max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}}) \\ I_2 - \mu(r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}})_+ \Gamma(r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}) & \text{if } \begin{cases} \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| > \mu \max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}}) \\ r_{\mathrm{N}} - \rho_{\mathrm{N}}u_n > 0 \end{cases} \end{cases}$$

with the function $\Gamma(\cdot)$ defined by

$$\Gamma(x) = \frac{I_{2\times 2}}{\|x\|} - \frac{x\,x^\top}{\|x\|^3}$$

If the variant (23) is chosen, the computation of $\Phi_{r_{\mathrm{T}}\bullet}$ simplify in

$$\Phi_{r_{\mathrm{T}}u_{\mathrm{N}}}(u,r) = 0$$

$$\Phi_{r_{\mathrm{T}}u_{\mathrm{T}}}(u,r) = \begin{cases} \rho_{\mathrm{T}} & \text{if } \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| \leqslant \mu r_{\mathrm{N}} \\ -\mu\rho_{\mathrm{T}} r_{n,+}\Gamma(r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}) & \text{if } \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| > \mu r_{\mathrm{N}} \end{cases}$$

$$\Phi_{r_{\mathrm{T}}r_{\mathrm{N}}}(u,r) = \begin{cases} 0 & \text{if } \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| \leqslant \mu r_{\mathrm{N}} \\ 0 & \text{if } \begin{cases} \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| > \mu r_n \\ r_{\mathrm{N}} \leqslant 0 \end{cases} \\ -\mu \dfrac{r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}}{\|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\|} & \text{if } \begin{cases} \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| > \mu r_n \\ r_{\mathrm{N}} > 0 \end{cases} \end{cases}$$

$$\Phi_{r_{\mathrm{T}}r_{\mathrm{T}}}(u,r) = \begin{cases} 0 & \text{if } \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| \leqslant \mu r_{\mathrm{N}} \\ I_2 - \mu(r_{\mathrm{N}})_+\Gamma(r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}) & \text{if } \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| > \mu r_{\mathrm{N}} \end{cases}$$

REDACTION NOTE V.A. **B.2.**

* *Is there a difference with the computation of Florent in his thesis?*
*

B.3 Fischer-Burmeister function