On solving frictional contact problems: Formulations and comparisons of numerical methods.

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Abstract: TBW

Key-words: Multibody systems, nonsmooth Mechanics, unilateral constraints, Coulomb friction, impact, numerical methods

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Sur la résolution du problème de frottement tridimensionnel. Formulations and comparaisons des méthodes numériques.

Résumé : TBW

Mots-clés : Systèmes multi–corps, Mécanique non régulière, contraintes unilatérales, frottement de Coulomb, impact, Schémas numériques de résolution

Contents

Notation

The following notation is used throughout the article: the 2-norm for a function g is denoted by $\|g\|$ and for a vector $x \in \mathbb{R}^n$ by $\|x\|$. The index $\alpha \in \mathbb{N}$ is used to identify the variable pertaining to a single contact. A multivalued mapping $T\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an operator whose images are sets. The second order cone, also known as Lorentz or ice–cream cone, is defined as $K_\mu \coloneqq \{(x,t) \in \mathbb{R} \times \mathbb{R}_+ \mid \|x\| \leqslant \mu t\}$, $\mu \geqslant 0$. By polarity, the dual convex cone to a convex cone K defined by

$$K^\star = \{x \in \mathbb{R}^n \mid y^\top x \geqslant 0, \quad \text{for all } y \in K\}. \tag{1}$$

{eq:dual-cone

The normal cone $N_K\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ to a closed convex set X is the set

$$N_K(x) = \{d \in \mathbb{R}^n \mid d^\top (y - x) \leqslant 0\} \tag{2}$$

The notation $0 \leqslant x \perp y \geqslant 0$ denotes that $x \geqslant 0$, $y \geqslant 0$ and $x^\top y = 0$. A complementarity problem associated with a function $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is to find $x \in \mathbb{R}^n$ such that $0 \leqslant F(x) \perp x \geqslant 0$. The generalized complementarity problem is given by $K^\star \ni F(x) \perp x \in K$, where K is a closed convex cone. Finite-dimentional Variational Inequality (VI) problems subsumes complementarity problems, system of equations. Solving a VI(X, F) is to find $x \in X$ such that

$$F(x)^\top (y - x) \geqslant 0 \qquad \text{for all } y \in X. \tag{3}$$

It is easy to see this problem is equivalent to solving a *generalized equation*

$$0 \in F(X) + N_X(x). \tag{4}$$

The Euclidean projector on a set X is denoted by P_X.

1 Introduction

More than thirty years after the pioneering work of [Panagiotopoulos, 1975], [Nečas et al., 1980], [Haslinger, 1983, 1984, Haslinger and Panagiotopoulos, 1984], [Del Piero and Maceri, 1983, 1985], [Katona, 1983], [Chaudhary and Bathe, 1986], [Jean and Moreau, 1987], [Mitsopoulou and Doudoumis, 1988] on numerically solving mechanical problems with contact and friction, there are still active research activities on this subject in the computational mechanics and applied mathematics communities. This can be explain by the fact that problems from mechanical systems with unilateral contact and Coulomb friction are difficult to numerically solve and the mathematical results of convergence of the numerical algorithms are rare and most of these require rather strong assumptions. In this article, we want to give some insights of the advantages and weaknesses of standard solvers found in the literature by comparing them on the large sets of examples coming from the simulation of a wide range of mechanical systems.

1.1 Problem statement

In this section, we formulate an abstract, algebraic finite–dimensional frictional contact problem. We cast this problem as a complementarity problem over cones, and discuss the properties of the latter. We end by presenting some instances with contact and friction phenomenon that fits our problem description.

Abstract problem We want to discuss possible numerical solution procedures for the following three–dimensional finite–dimensional frictional contact problem and some of its variants. Let $n_c \in \mathbb{N}$ be the number of contact points and $n \in \mathbb{N}$ the number of degrees of freedom of a discrete mechanical system.

The problem data are: a symmetric positive (semi-) definite matrix $M \in \mathbb{R}^{n\times n}$, a vector $f \in \mathbb{R}^n$, a matrix $H \in \mathbb{R}^{n\times m}$ with $m = 3n_c$, a vector $w \in \mathbb{R}^m$ and a vector of coefficients of friction $\mu \in \mathbb{R}^{n_c}$. The unknowns are two vectors $v \in \mathbb{R}^n$, a velocity–like vector and $r \in \mathbb{R}^m$, a contact reaction or impulse, solution to

$$\begin{cases} Mv = Hr + f \\ K^\star \ni \hat{u} \perp r \in K \end{cases} \quad \text{with} \quad \begin{aligned} u &\coloneqq H^\top v + w \\ \hat{u} &\coloneqq u + g(u), \end{aligned} \tag{5}$$ {eq:soccp1-in

where the set K is the cartesian product of Coulomb's friction cone at each contact, that is

$$K = \prod_{\alpha=1\ldots n_c} K^\alpha = \prod_{\alpha=1\ldots n_c} \{r^\alpha, \|r^\alpha_{\text{T}}\| \leqslant \mu^\alpha |r^\alpha_{\text{N}}|\} \tag{6}$$ {eq:CC}

and $K^\star$ is dual. The function $g\colon \mathbb{R}^m \to \mathbb{R}^m$ is a nonsmooth function defined as

$$g(u) = [[\mu^\alpha \|u^\alpha_{\text{T}}\|, 0, 0]^\top, \alpha = 1\ldots n_c]^\top. \tag{7}$$ {eq:gg}

Note that the variable u and $\hat{u}$ do not appear as unknowns since they can be directly obtained from v.

Redaction note O.H. 1.1.

I think I got the gist of the following paragraph, but I think we should work on it a bit. For me we discuss approximations in the literature that don't solve the true problem. I believe figure to illustrate the effect of neglecting g would be useful (for the optimizer).

I also think that we should be more explicit that we focus on the reduced form, and also explain that in the rigid case, it may not destroy the sparse stricture of the problem.

A Second Order Cone Complementarity Problem (SOCCP). From the mathematical programming point of view, the problem appears to be a Second Order Cone Complementarity Problem (SOCCP) [Facchinei and Pang, 2003] which can be generically defined as

$$\begin{cases} y = f(x) \\ K^\star \ni y \perp x \in K, \end{cases} \tag{8}$$

where K is a second order cone. If the nonlinear part of the problem (5) is neglected ($g(u) = 0$), the problem is an associated friction problem with dilatation, and by the way, is a gentle Second Order

Cone Linear Complementarity Problem (SOCLCP) with a positive definite matrix $H^\top M^{-1}H$ (possibly semi–definite). The assumption of an associated frictional law, i.e, a friction law where the local sliding velocity is normal the friction cone differs dramatically from the standard Coulomb friction since it generates a non–vanishing normal velocity when we slide. In other terms, the sliding motion implies the separation of the bodies. When the non-associated character of the friction is taken into account through $g(u)$, the problem is non monotone and nonsmooth, and therefore is very hard to solve efficiently. For a given numerical algorithm, it is not so difficult to design mechanical example to run the algorithm into troubles. Proof of convergence of the numerical algorithms are rare and most of these required strong assumptions on either the friction coefficients or full rank assumptions of matrices or operators or the dimension of the space in which the problem is formulated. Among these results, we can cite the Czech school where the coefficient of friction is assumed to be bounded and small. This assumption allows us to use fixed point methods on the convex sub–problems of Tresca friction (friction threshold that does depend on the normal reaction and then transform the cone into a semi-cylinder). We can also mention the results from [Pang and Trinkle, 1996, Stewart and Trinkle, 1996, Anitescu and Potra, 1997] where the friction cone is polyhedral (in 2D or by a faceting process). In that case, if $w = 0$ or $w \in \mathrm{im}(H^\top)$, Lemke's algorithm is able to solve the problem. The question of existence of solutions has been treated in [Klarbring and Pang, 1998, Acary et al., 2011] under similar assumptions but with different techniques. The question of uniqueness remains a difficult problem in the general case.

Range of applicability. We clearly choose to simplify a lot the general problems of formulating the contact problems with friction by avoiding including too much side effects that are themselves interesting but render the study too difficult to carry out in a single article. We choose finite dimensional systems where the time dependency does not appear explicitly.

Nevertheless, we believe that there are a strong interest to study this problem since it appears to be relatively generic in numerous simulations of systems with contact and friction. This problem is indeed at the heart of the simulation of mechanical systems with 3D Coulomb's friction and unilateral constraints in the following cases:

- it might be the result of the time–discretization by event–capturing time–stepping methods or event–detecting (event–driven) techniques of dynamical systems with friction; the variables are homogeneous to pairs velocity/impulses or acceleration forces
- it might be the result of space–discretization (by FEM for instance) of the elastic quasi-static problems of frictional contact mechanics; in that case, the variables are homogenous to displacements/forces of displacement rate/forces.
- if the system is a dynamical mechanical system composed of solids, the problem is again obtained by a space and time discretization

- if the material follows a nonlinear mechanical bulk behavior, we can use this model after a standard Newton linearization procedure.

For a description of the derivation of such problems in various practical situations we refer to [Laursen, 2003, Wriggers, 2006, Acary and Brogliato, 2008, Acary and Cadoux, 2013].

1.2 Objectives and outline of the article

In this article, after stating the problem with more details in Section 2, we recall the existence result of [Acary et al., 2011] for the problem in (5) in Section 2.3. In this framework, we briefly present in Section 3 a few alternative formulations of the problem that enable the design of numerical solution procedures : a) finite–dimensional Variational Inequalities(VI) and Quasi-Variational Inequalities(QVI) b) Nonsmooth equations and c) Optimization based formulations.

Right after these formulations, we list some of the most standard algorithms dedicated to one of the previous formulation :

1. the fixed point and projection numerical methods for solving VI are reviewed with a focus on self-adaptive time–step rules (Section 4),
2. the nonsmooth (semi-smooth) Newton methods are described based on the various nonsmooth equations formulations (Section 5),
3. Section 6 is devoted to the presentation of splitting and proximal point techniques,
4. and finally, in Section 7, the Panagiotopoulos alternating optimization technique, the successive approximation technique and the SOCLCP approach are outlined.

Since it is difficult to be exhaustive on the approaches developed in the literature to solve frictional contact problems, we decided to leave out the scope of the article the following approaches:

- the approaches that alter the fundamental assumptions of the 3D Coulomb friction model by faceting the cone as in the pioneering work of [Klarbring, 1986] and followed by [Al-Fahed et al., 1991, Pang and Trinkle, 1996, Stewart and Trinkle, 1996, Anitescu and Potra, 1997, Haslinger et al., 2004], or by convexifying the Coulomb law (associated friction law with normal dilatancy) [Heyn et al., 2013, Tasora and Anitescu, 2013, 2011, Anitescu and Tasora, 2010, Tasora and Anitescu, 2009, Krabbenhoft et al., 2012] or finally by regularizing the friction law [Kikuchi and Oden, 1988]. In the same way, we are discussing recent developments methods for the frictionless case [Morales et al., 2008, Miyamura et al., 2010, Temizer et al., 2014].
- the approaches that are based on domain decomposition and parallel computing. We choose in this article to focus on single domain computation and to skip the discussion about distributed computing mainly for a sake of length of the article. (cite a bit a literature Krause, Koziara, Renouf, Heyn, ...)

Finally, some possible interesting approaches have not been reported. We are thinking mainly to the interior point methods approach [Christensen and Pang, 1998, Miyamura et al., 2010, Kleinert et al., 2014]. some basic implementations of such methods do not give satisfactory results. One of reasons is the fact that we were not able to get robustness and efficiency on a large class of problems. As it is reported in [Kleinert et al., 2014, Krabbenhoft et al., 2012], it seems that it is needed to alter the friction Coulomb's law by adding regularization or dilatency in the model. In the same spirit, we skip also the comparison for the possibly very promising methods developed in [Heyn et al., 2013, Heyn, 2013] that are based on Krylov subspace and spectral methods. It could be very interesting to bench also these methods on the actual Coulomb friction model, that is to say, in the nonmonotone case. Finally, our preliminary results on the use of direct general SOCP or SOCLCP solvers were not convincing. Indeed, the structure of contact problems (product of a large number of small second order cones) has to be taken into account to get efficiency and unfortunately, these solvers are difficult to adapt to this structure.

Other comparisons articles have already been published in the literature. One of the first comparison study has been done in [Raous et al., 1988] and in [Chabrand et al., 1998]. In this work, several formulations are detailed in the bidimensional case (variational inequality, linear complementarity problem (LCP) and augmented Lagrangian formulation) and comparisons of fixed point methods with projection, splitting methods and Lemke's method for solving LCP. Other comparisons have been done on 2D systems in [Mijar and Arora, 2000b,a, 2004a,b]. In [Christensen et al., 1998], a very interesting comparison in the three-dimensional case has been carried out which shows the superiority of the semi-smooth Newton methods over the interior point methods. Comparisons on simple multi-body systems composed of kinematic chains can be found in [Mylapilli and Jain, 2017].

The comparison are performed on a large set of examples using performance profiles. Let us summarize the main conclusion from Section 8: on one hand, the algorithms based on Newton methods for nonsmooth equations solve quickly the problem when they succeed, but suffer from robustness issues mainly if the matrix H has not full rank. On the other hand, the iterative methods dedicated to solving variational inequalities are quite robust but with an extremely slow rate of convergence. To sum up, as far as we know there is no option that combines time efficiency and robustness. The set of problems used here are from the FCLIB collection[1]. In this work, this collection is solved with the software Siconos and its component Siconos/Numerics[2][Acary et al., 2015].

[1] `https://frictionalcontactlibrary.github.io/index.html`, which aims at providing many problems to compare algorithms on a fair basis

[2] `http://siconos.gforge.inria.fr`

2 Description of the 3D frictional contact problems

2.1 Signorini's condition and Coulomb's friction.

Let us consider the contact between two bodies $A \subset \mathbb{R}^3$ and $B \subset \mathbb{R}^3$ with sufficiently smooth boundaries, as depicted on Figure 1.

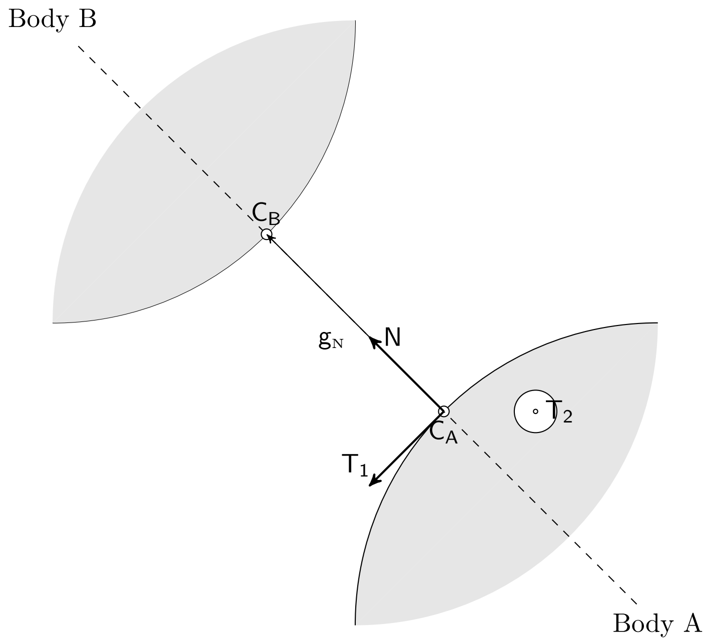

Figure 1: Contact kinematic

From the body A "perspective", the point $C_A \in \partial A$ is called a *master point to contact*. The choice of this master point C_A to write the contact condition is crucial in practice and amounts to consistently discretizing the contact surface. The vector N defines an outward unit normal vector to A at the point C_A. With $\mathsf{T}_1, \mathsf{T}_2$ two vectors in the plane orthogonal to N, we can build an orthornormal frame $(C_A, \mathsf{N}, \mathsf{T}_1, \mathsf{T}_2)$ called the *local frame at contact*. The slave contact point $C_B \in \partial B$ is defined as the projection of the point C_A on ∂B in the direction given by N. Note that we assume that such a point exists. The gap function is defined as the signed distance between C_A and C_B

$$g_{\mathrm{N}} = (C_B - C_A)^\top \mathsf{N}. \tag{9}$$ {eq:gap}

Consider two strictly convex bodies, which are non penetrating, *i.e.* $A \cap B = \emptyset$, the master and slave contact points can be chosen as the proximal points of each bodies and the normal vector N can be written as

$$\mathsf{N} = \frac{C_B - C_A}{\|C_B - C_A\|}. \tag{10}$$ {eq:normal}

The contact force exerted by A on B is denoted by $r \in \mathbb{R}^3$ and is decomposed in the local frame as

$$r \coloneqq r_{\mathrm{N}}\mathsf{N} + r_{\mathrm{T}_1}\mathsf{T}_1 + r_{\mathrm{T}_2}\mathsf{T}_2, \quad \text{with } r_{\mathrm{N}} \in \mathbb{R} \text{ and } r_{\mathrm{T}} \coloneqq [r_{\mathrm{T}_1}, r_{\mathrm{T}_2}]^\top \in \mathbb{R}^2. \tag{11}$$ {eq:reaction}

The *Signorini condition* states that

$$0 \leqslant g_{\mathrm{N}} \perp r_{\mathrm{N}} \geqslant 0, \tag{12}$$
{eq:signo}

and models the unilateral contact. The condition (12), written at the *position level*, can also be defined at the *velocity level*. To this end, the relative velocity $u \in \mathbb{R}^3$ of the point C_B with respect to C_A is also decomposed in the local frame as

$$u \coloneqq u_{\mathrm{N}}\mathsf{N} + u_{\mathrm{T}_1}\mathsf{T}_1 + u_{\mathrm{T}_2}\mathsf{T}_2 \quad \text{with } u_{\mathrm{N}} \in \mathbb{R} \text{ and } u_{\mathrm{T}} = [u_{\mathrm{T}_1}, u_{\mathrm{T}_2}]^\top \in \mathbb{R}^2. \tag{13}$$

At the velocity level, the Signorini condition is written

$$\begin{cases} 0 \leqslant u_{\mathrm{N}} \perp r_{\mathrm{N}} \geqslant 0 & \text{if } g_{\mathrm{N}} \leqslant 0 \\ r_{\mathrm{N}} = 0 & \text{otherwise.} \end{cases} \tag{14}$$
{eq:signo-vel

The Moreau's viability Lemma [Moreau, 1988] ensures that (14) implies (12) if $g_{\mathrm{N}} \geqslant 0$ holds in the initial configuration.

Coulomb's friction models the frictional behavior of the contact force law in the tangent plane spanned by (T_1, T_2). Let us define the Coulomb friction cone K which is the isotropic second order cone (Lorentz or ice–cream cone)

$$K = \{r \in \mathbb{R}^3 \mid \|r_{\mathrm{T}}\| \leqslant \mu r_{\mathrm{N}}\}, \tag{15}$$
{eq:CoulombCo

where μ is the coefficient of friction. The Coulomb friction states for the sticking case that

$$u_{\mathrm{T}} = 0, \quad r \in K, \tag{16}$$
{eq:Coulom-st

and for the sliding case that

$$u_{\mathrm{T}} \neq 0, \quad r \in \partial K, \quad \text{and} \quad \exists\, \alpha > 0 \text{ such that } r_{\mathrm{T}} = -\alpha u_{\mathrm{T}}. \tag{17}$$
{eq:Coulom-sl

With the Coulomb friction model, there are two relations between u_{T} and r_{T}. The distinction is based on the value of the relative velocity u_{T} between the two bodies. If $u_{\mathrm{T}} = 0$ (sticking case), we have $r_{\mathrm{T}} \leqslant \mu r_{\mathrm{N}}$. On the other hand, we get the sliding case.

Disjunctive formulation of the Signorini-Coulomb model If we consider the velocity-level Signorini condition (14) together with the Coulomb friction (16)–(17) which is naturally expressed in terms of velocity, we obtain a disjunctive formulation of the frictional contact behavior as

$$\begin{cases} r = 0 & \text{if } g_{\mathrm{N}} > 0 \quad \text{(no contact)} \\ r = 0, u_{\mathrm{N}} \geqslant 0 & \text{if } g_{\mathrm{N}} \leqslant 0 \quad \text{(take–off)} \\ r \in K, u = 0 & \text{if } g_{\mathrm{N}} \leqslant 0 \quad \text{(sticking)} \\ r \in \partial K, u_{\mathrm{N}} = 0, \exists\, \alpha > 0, u_{\mathrm{T}} = -\alpha r_{\mathrm{T}} & \text{if } g_{\mathrm{N}} \leqslant 0 \quad \text{(sliding)} \end{cases} \tag{18}$$
{eq:contact-d

In the computational practice, the disjunctive formulation is not suitable for solving the Coulomb problem as it suggests the use of enumerative solvers, with an exponential complexity. In the sequel, alternative formulations of the Signorini-Coulomb model suitable for numerical applications are delineated. The core idea is to translate the cases in (18) into complementarity relations.

Inclusion into normal cones The Signorini condition (12) and (14), in their complementarity forms can be equivalently written as an inclusion into a normal cone to $\mathbb{R}_+$

$$-\,g_{\rm N} \in N_{\mathbb{R}_+}(r_{\rm N}) \qquad \text{and} \qquad -\,u_{\rm N} \in N_{\mathbb{R}_+}(r_{\rm N}), \tag{19}$$

{eq:signo-inc

if $g_{\rm N} \leqslant 0$ and $r_{\rm N} = 0$ otherwise. An inclusion form of the Coulomb friction for the tangential part can be also proposed: let $D(\cdot)$ be the Coulomb disk:

$$D(c) \coloneqq \{x \in \mathbb{R}^2 \mid \|x\| \leqslant c\}. \tag{20}$$

{eq:diskR}

For the Coulomb friction, we get

$$-\,u_{\rm T} \in N_{D(\mu r_{\rm N})}(r_{\rm T}). \tag{21}$$

{eq:Coulomb-i

Since $D(\mu r_{\rm N})$ is not a cone, the inclusion (21) is not a complementarity problem, but a variational inequality. The formulation (21) is often related to Moreau's maximum dissipation principle of the frictional behavior:

$$r_{\rm T} \in \underset{\|z\| \leqslant \mu r_{\rm N}}{\arg\max}\; z^\top u_{\rm T}. \tag{22}$$

{eq:moreau-ma

This means that the couple $(r_{\rm T}, u_{\rm T})$ maximizes the energy lost through dissipation.

SOCCP formulation of the Signorini-Coulomb model In [Acary and Brogliato, 2008, Acary et al., 2011], another formulation is proposed inspired by the so-called bipotential [De Saxcé, 1992, De Saxcé and Feng, 1991, Saxcé and Feng, 1998]. The goal is to form a complementarity problem out of (19) and (21) To this end, we introduce the modified relative velocity $\hat{u} \in \mathbb{R}^3$ defined by

$$\hat{u} = u + [\mu \|u_{\rm T}\|, 0, 0]^\top. \tag{23}$$

{eq:modified-

The entire contact model (18) can be put into a Second-Order Cone Complementarity Problem (SOCCP) as

$$K^\star \ni \hat{u} \perp r \in K \tag{24}$$

{eq:contact-S

if $g_{\rm N} \leqslant 0$ and $r = 0$ otherwise.

2.2 Frictional contact discrete problems

We assume that a finite set of n_c contact points and their associated local frames has been defined. In general, this task is not straightforward and amounts to correctly discretizing the contact surfaces. For more details, we refer to [Wriggers, 2006, Laursen, 2003]. For each contact $\alpha \in \{1, \ldots, n_c\}$, the local velocity is denoted by $u^\alpha \in \mathbb{R}^3$, the normal velocity by $u^\alpha_{\rm N} \in \mathbb{R}$ and the tangential velocity by $u^\alpha_{\rm T} \in \mathbb{R}^2$ with $u^\alpha = [u^\alpha_{\rm N}, (u^\alpha_{\rm T})^\top]^\top$. The vectors $u, u_{\rm N}, u_{\rm T}$ respectively collect all the local velocity $u = [(u^\alpha)^\top, \alpha = 1 \ldots n_c]^\top$, all the normal velocity $u_{\rm N} = [u^\alpha_{\rm N}, \alpha = 1 \ldots n_c]^\top$, and all the tangential velocity $u_{\rm T} = [(u^\alpha_{\rm T})^\top, \alpha = 1 \ldots n_c]^\top$. For a contact α, the modified local velocity, denoted by $\hat{u}^\alpha$, is defined by

$$\hat{u}^\alpha = u^\alpha + g^\alpha(u) \qquad \text{where} \qquad g^\alpha(u^\alpha) = [\mu^\alpha \|u^\alpha_{\rm T}\|, 0, 0]^\top. \tag{25}$$

{eq:modified}

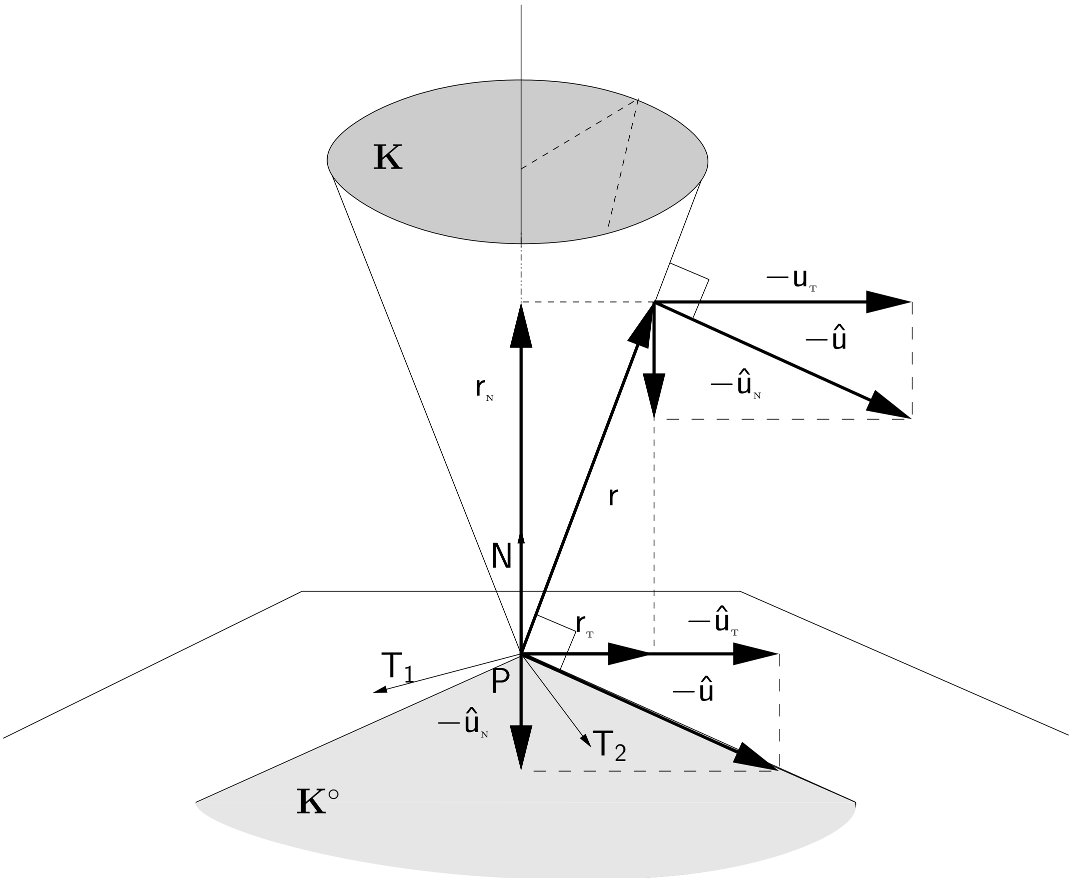

Figure 2: Coulomb's friction law in the sliding case.

The vector $\hat{u}$ and the function g collect all the modified local velocity at each contact $\hat{u} = [\hat{u}^\alpha, \alpha = 1 \dots n_c]^\top$ and the function $g(u) = [[\mu^\alpha \|u^\alpha_{\mathrm{T}}\|, 0, 0]^\top, \alpha = 1 \dots n_c]^\top$.

For each contact α, the reaction vector $r^\alpha \in \mathbb{R}^3$ is also decomposed in its normal part $r^\alpha_{\mathrm{N}} \in \mathbb{R}$ and the tangential part $r^\alpha_{\mathrm{T}} \in \mathbb{R}^2$ as $r^\alpha = [r^\alpha_{\mathrm{N}}, (r^\alpha_{\mathrm{T}})^\top]^\top$ The Coulomb friction cone for a contact α is defined by $K^\alpha = \{r^\alpha \in \mathbb{R}^3 \mid \|r^\alpha_{\mathrm{T}}\| \leqslant \mu^\alpha |r^\alpha_{\mathrm{N}}|\}$ and the set $K^{\alpha,\star}$ is its dual. The set K is the cartesian product of Coulomb's friction cone at each contact, that is

$$K = \prod_{\alpha=1,\dots,n_c} K^\alpha \quad \text{and} \quad K^\star \text{ is its dual.} \tag{26}$$ {eq:CC_bis}

In this article, we investigate the case where the problem is given in its *reduced* form. We consider that the discretized dynamics are of the form

$$Mv = Hr + f, \tag{27}$$ {eq:global_dy

with M an symmetric positive-definite matrix. The local velocities at the point of contact are given by

$$u = H^\top v + w. \tag{28}$$ {eq:local_v}

More information on the term w is given later in this section. The (global) velocities v can be substitute in (28) by using a Schur-complement technique. This yields

$$u = H^\top M^{-1} H r + H^\top M^{-1} f + w. \tag{29}$$

Let us define W, often called the *Delassus matrix*, as

$$W := H^\top M^{-1} H \tag{30}$$ {eq:Delassus}

and the vector q as

$$q := H^\top M^{-1} f + w. \tag{31}$$ {eq:qq}

We are now ready to define the mathematical problem that we solve.

Problem FC (Discrete frictional contact problem). *Given*

- *a symmetric positive semi–definite matrix* $W \in \mathbb{R}^{m \times m}$,
- *a vector* $q \in \mathbb{R}^m$,
- *a vector* $\mu \in \mathbb{R}^{n_c}$ *of coefficients of friction,*

find a vector $r \in \mathbb{R}^m$, *such that*

$$\begin{cases} K^\star \ni \hat{u} \perp r \in K \\ u = Wr + q \\ \hat{u} = u + g(u) \end{cases} \tag{32}$$ {eq:soccp2}

with $g(u) = [[\mu^\alpha \|u^\alpha_T\|, 0, 0]^\top, \alpha = 1 \dots n_c]^\top$. *An instance of the problem is denoted by* $\mathrm{FC}(W, q, \mu)$ □

Redaction note V.A. 2.1.

Is the assumption on the symmetry of the matrix M is mandatory ? I don't think so but it has to be checked carefully in the sequel.

2.3 Existence of solutions

The question of the existence of solution for the Problem FC have been studied in [Klarbring and Pang, 1998] and [Acary et al., 2011] with different analysis techniques. The key assumption for existence of solutions in both articles is as follows

$$\exists v \in \mathbb{R}^m \; : \; H^\top v + w \in \operatorname{int} K^\star, \tag{33}$$ {ass}

or equivalently

$$w \in \operatorname{im} H + \operatorname{int} K^\star. \tag{34}$$ {asseq}

Under the assumption, the Problem FC have a solution. Therefore, it makes sense to design a procedure to solve the problem. In the sequel, we will compare numerical methods only when this assumption is satisfied.

This assumption is easy verified in numerous applications. For applications in nonsmooth dynamics where the unknown v is a relative contact velocity, the term w vanishes if we have only scleronomic constraints. For $w \in \operatorname{im}(H^\top)$ (and especially $w = 0$), the assumption is trivially satisfied. As it is explained in [Acary and Cadoux, 2013], the term w has several possible sources. If the constraints are formulated at the velocity level, an input term of w is given in dynamics by the impact law. In the case of the Newton impact law, it holds that $w \in \operatorname{im}(H^\top)$. For other impact law, this is not clear. Another input in w is given by constraints that depends explicitly on time. In that case, we can have $w \notin \operatorname{im}(H^\top)$ and non existence of solutions. If the constraints are written at the position level, w can be given by initial of terms that comes from velocity discretization. In that cases, the existence is also not ensured.

The assumption is also satisfied whenever $\operatorname{im} H = \mathbb{R}^m$ or in other words if $H^\top$ has full row rank. Unfortunately, in large number of applications $H^\top$ is rank deficient. From the mechanical point of view, the rank deficiency of H and the amount of friction seems to play a fundamental role in the question of the existence (and uniqueness) of solutions. In the numerical comparisons, we will attempt to get a deeper understanding on the role of these assumptions on the convergence of the algorithms. The rank deficiency of H is related to the number of constraints that are imposed to the system with respect to the number of degrees of freedom in the system. It is closely related to the concept of hyperstaticity in overconstrained systems. In the most favorable cases, it yields indeterminate Lagrange multipliers but also to unfeasible problems and then to the lost of solutions in the worth cases. The second assumption on the amount of friction is also well–know. The frictionless problem is easy to solve if it is feasible. It is clear that large friction coefficients prevent from sliding and therefore increase the degree of hyperstaticity of the system.

3 Alternative formulations

In this section, various equivalent formulations of Problem FC are given. Our goal is to show that such problems can be recast into several well-known problems in the mathematical programming and optimization community. These formulations will serve as a basis for numerical solution procedures that we develop in later sections.

3.1 Variational Inequalities (VI) formulations

Let us recall the definition a finite-dimensional VI(X, F): find $z \in X$ such that

$$F^\top(z)(y-z) \geqslant 0 \quad \text{for all } y \in X, \tag{35}$$ {eq:vi}

with X a nonempty subset of $\mathbb{R}^n$ and F a mapping from $\mathbb{R}^n$ into itself. We refer to [Harker and Pang, 1990, Facchinei and Pang, 2003] for the standard theory of finite–dimensional variational inequalities. The easiest way to state equivalent VI formulations of Problem FC is to use the following equivalences:

$$K^\star \ni \hat{u} \perp r \in K \quad \Longleftrightarrow \quad -\hat{u} \in N_K(r) \quad \Longleftrightarrow \quad \hat{u}^\top(s-r) \geqslant 0, \text{ for all } s \in K. \tag{36}$$ {eq:SOCCP-1}

For Problem FC, the following equivalent formulation in VI is directly obtained from

$$-\left(Wr + q + g(Wr+q)\right) \in N_K(r). \tag{37}$$ {eq:inclusion

The resulting VI is denoted by VI$(F_{\rm vi}, X_{\rm vi})$ with

$$F_{\rm vi}(r) \coloneqq Wr + q + g(Wr+q) \quad \text{and} \quad X_{\rm vi} \coloneqq K. \tag{38}$$ {eq:vi-II}

Uniqueness properties. In the general case, it is difficult to prove uniqueness of solutions to (38). If the matrix H has full rank and the friction coefficients are “small”, a classical argument for the uniqueness of solution of VIs can be satisfied. Note that the full rank hypothesis on H implies that W is positive-definite. Therefore, we have $(x-y)^\top W(x-y) \geqslant C_W\|x-y\|^2$ with $C_W > 0$. Using this relation (38) yields

$$\begin{array}{rcl}(F_{\rm vi}(x) - F_{\rm vi}(y))^\top(x-y) & = & (x-y)^\top W(x-y) \\ & & \quad + \sum_{\alpha=1}^{n_c} \mu^\alpha (x^\alpha_{\rm N} - y^\alpha_{\rm N})[\|[Wx+q]^\alpha_{\rm T}\| - \|[Wy+q]^\alpha_{\rm T}\|] \\ & \geqslant & C_W\|x-y\|^2 + \sum_{\alpha=1}^{n_c} \mu^\alpha (x^\alpha_{\rm N} - y^\alpha_{\rm N})[\|[Wx+q]^\alpha_{\rm T}\| - \|[Wy+q]^\alpha_{\rm T}\|].\end{array} \tag{39}$$ {eq:mono-II}

Note that for small values of the coefficients of friction the first term in the right-hand side dominates the second one. Hence, the mapping $F_{\rm vi}$ is strictly monotone and this ensures that the VI has at most one solution [Facchinei and Pang, 2003, Theorem 2.3.3]. The fact that H is full rank also implies that the Assumption (33) for the existence of solutions is trivially satisfied. Hence, there exists a unique solution to the VI$(F_{\rm vi}, X_{\rm vi})$.

3.2 Quasi-Variational Inequalities (QVI)

Let us recast Problem FC into the QVI framework. A QVI is a generalization of the VI, where the feasible set is allowed to depend on the solution. Let us define this precisely: let X be a multi-valued mapping $\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and let F be a mapping from $\mathbb{R}^n$ into itself. The quasi-variational inequality problem, denoted by QVI(X, F), is to find a vector $z \in X(z)$ such that

$$F^\top(z)(y - z) \geqslant 0, \forall y \in X(z). \quad (40) \quad \texttt{\{eq:qvi\}}$$

The QVI formulation of the frictional contact problems is obtained by considering the inclusions (19) and (21). We get

$$u^\top(s - r) \geqslant 0, \text{ for all } s \in C(\mu, r_{\rm N}), \quad (41) \quad \texttt{\{eq:qvi-1\}}$$

where $C(\mu, r_{\rm N})$ is the Cartesian product of the semi–cylinders of radius $\mu^\alpha r^\alpha_{\rm N}$ defined as

$$C(\mu, r_{\rm N}) \coloneqq \prod_{\alpha=1}^{n_c} \left\{ s \in \mathbb{R}^3 \mid s_{\rm N} \geqslant 0, \|s_{\rm T}\| \leqslant \mu^\alpha r^\alpha_{\rm N} \right\}. \quad (42) \quad \texttt{\{eq:cylinder\}}$$

Note that the QVI (41) involves only u and not $\hat{u}$: this is the main interest of this formulation. The price to pay is the dependence on r of the set $C(\mu, r_{\rm N})$. Problem FC can be expressed as a QVI by substituting the expression of u, which yields

$$(Wr + q)^\top(s - r) \geqslant 0, \text{ for all } s \in C(\mu, r_{\rm N}). \quad (43) \quad \texttt{\{eq:qvi-IIbis}$$

This expression is compactly rewritten as QVI$(F_{\rm qvi}, X_{\rm qvi})$, with

$$F_{\rm qvi}(r) \coloneqq Wr + q \quad \text{and} \quad X_{\rm qvi}(r) \coloneqq C(\mu, r_{\rm N}). \quad (44) \quad \texttt{\{eq:qvi-II\}}$$

Since W is assumed to be positive semi-definite matrix, $F_{\rm qvi}$ is monotone. Thus we get an affine monotone QVI$(F_{\rm qvi}, X_{\rm qvi})$ for Problem FC.

3.3 Nonsmooth Equations

In this section, we expose a classical approach to solving a VI or a QVI, based on a reformulation of the inclusion as an nonsmooth equation. The term nonsmooth equation highlights that the mapping we consider fails to be differentiable. This is the price to pay for this reformulation. We can apply fixed-point and Newton-like algorithms to solve the resulting equation. Given the nonsmooth nature of the problem, applying Newton's method appears challenging, but it can still be done for some reformulations. More precisely for Problem FC, we search for an equation of the type

$$G(r) = 0 \quad (45) \quad \texttt{\{eq:ne-1\}}$$

where G is generally only locally Lipschitz continuous. The mapping G is such that the zeroes (r) of (45) are the solutions of (32).

Natural and normal maps for the VI formulations A general-purpose reformulation of VI is obtained by using the normal and natural maps, see [Facchinei and Pang, 2003] for details. The natural map $F^{\mathsf{nat}}\colon \mathbb{R}^n \to \mathbb{R}^n$ associated with the VI (35) is defined by

$$F^{\mathsf{nat}}(z) := z - P_X(z - F(z)), \tag{46}$$ {eq:naturalma

where P_X is the Euclidean projector on the set X. A well-known result (see [Facchinei and Pang, 2003]) states that the solutions of a VI are related to the zeroes of the natural map :

$$z \text{ solves } \mathrm{VI}(X, F) \quad \Longleftrightarrow \quad F^{\mathsf{nat}}(z) = 0, \tag{47}$$ {eq:VI-normal

Using (35), it is easy to see that if z solves $\mathrm{VI}(X, F)$, then it is also a solution to $\mathrm{VI}(X, \rho F)$ for any $\rho > 0$. Therefore, we can define a parametric variant of the natural map by

$$F^{\mathsf{nat}}_\rho(z) = z - P_X(z - \rho F(z)). \tag{48}$$ {eq:parametri

The relations given in (47) continue to hold for the parametric mapping. Using those equivalences, the frictional contact problem can be restated as zeroes of nonsmooth functions. With the natural map, Problem FC under the VI form (38) can be reformulated as

$$F^{\mathsf{nat}}_{\mathrm{vi}}(r) := \Big[\ r - P_K\left(r - \rho(Wr + q + g(Wr + q))\right)\ \Big] = 0. \tag{49}$$ {eq:natural-I

Following the same lines, the normal map may also be used to derive algorithms. The normal map $F^{\mathsf{nor}}\colon \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$F^{\mathsf{nor}}(x) := F(P_X(x)) + x - P_X(x), \tag{50}$$ {eq:normalmap

and its parametric variant

$$F^{\mathsf{nor}}_\rho(x) = \rho F(P_X(x)) + x - P_X(x). \tag{51}$$ {eq:normalmap

An equivalent results apply

$$z \text{ solves } \mathrm{VI}(X, F) \quad \Longleftrightarrow \quad z = P_X(x) \text{ for some } x \text{ such that } F^{\mathsf{nor}}(x) = 0. \tag{52}$$ {eq:VI-normal

The normal map based formulation of VI are also obtained in the same way.

In the seminal work of [Sibony, 1970], iterative methods for solving monotone VIs are based on the natural map and fixed point iterations. The role of ρ is recognized to be very important for the rate of convergence. To improve the methods, Sibony [1970] proposes to use "skewed" projector based on a non-Euclidean metric. Given a positive definite matrix $R \in R^{n\times n}$, a skewed projector $P_{X,R}$ onto X is defined as follows: $z = P_{X,R}(x)$ is the unique solution of the convex programm

$$\begin{cases} \min \dfrac{1}{2}(y - x)^\top R(y - x), \\ \\ s.t. \quad y \in X. \end{cases} \tag{53}$$ {eq:opt-proj-

The skew natural map can be also defined and yield the following nonsmooth equation

$$F^{\mathsf{nat}}_R(z) = z - P_{X,R}(z - R^{-1}F(z)). \tag{54}$$ {eq:skew-natu

The zeros of $F^{\mathsf{nat}}_R(z)$ are also solution of the VI(X, F). Considering the skew natural map, we obtain for Problem FC under the VI form (38),

$$F^{\mathsf{nat}}_{\mathrm{vi},R}(r) := \left[\ r - P_{K,R}\left(r - R^{-1}(Wr + q + g(Wr + q))\right)\ \right]. \tag{55}$$ {eq:natural-t

The previous case is retrieved by choosing $R = \rho^{-1} I_{n\times n}$.

Redaction note V.A. 3.1.

It can be interesting to use something R as a preconditionner of the problem ? $R = diag(W)$ or incomplete LU. Warning, W is only SPR, so we cannot have $R = W$.

Jean–Moreau's and Alart-Curnier's functions Using the alternative inclusions formulations (19)–(21) with a given set of parameters $\rho_{\mathrm{N}}, \rho_{\mathrm{T}}$ such that

$$\begin{cases} -\rho_{\mathrm{N}} u_{\mathrm{N}} \in N_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N}}), & \rho_{\mathrm{N}} > 0, \\ -\rho_{\mathrm{T}} u_{\mathrm{T}} \in N_{D(\mu,(r_n)_+)}(r_{\mathrm{T}}), & \rho_{\mathrm{T}} > 0, \end{cases} \tag{56}$$ {eq:inclusion

we can replace P_K into $P_{\mathbb{R}^{n_c}_+}$ and $P_{D(\mu,(r_n)_+)}$ where

$$D(\mu, (r_n)_+) = \prod_{\alpha=1\ldots n_c} D(\mu^\alpha (r^\alpha_{\mathrm{N}})_+). \tag{57}$$ {eq:diskR-pro

defines the Cartesian product of the Coulomb disks for each contact. The notation x_+ stands for $x_+ = \max(0, x)$. Using this procedure, Jean and Moreau [1987], Christensen et al. [1998] propose the following nonsmooth equation formulation of the frictional contact condition

$$\begin{cases} r_{\mathrm{N}} - P_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) = 0, \\ r_{\mathrm{T}} - P_{D(\mu,(r_{\mathrm{N}})_+)}(r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}) = 0. \end{cases} \tag{58}$$ {eq:Moreau-Je

The parameters $\rho_{\mathrm{N}}, \rho_{\mathrm{T}}$ may be also chosen contact by contact. Problem FC is then reformulated as

$$F_{\mathrm{mj}}(r) := \begin{bmatrix} r_{\mathrm{N}} - P_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N}} - \rho_{\mathrm{N}}(Wr + q)_{\mathrm{N}}) \\ r_{\mathrm{T}} - P_{D(\mu,(r_{\mathrm{N}})_+)}(r_{\mathrm{T}} - \rho_{\mathrm{T}}(Wr + q)_{\mathrm{T}}) \end{bmatrix} = 0. \tag{59}$$ {eq:MJ-II}

In the seminal work of Alart & Curnier [Curnier and Alart, 1988, Alart and Curnier, 1991], the augmented Lagrangian approach is invoked (see Remark 2) to obtain a similar formulation motivated by the development of nonsmooth (or generalized) Newton methods (see Section 5.2). To be accurate, the original Alart–Curnier function is given by

$$\begin{cases} r_{\mathrm{N}} - P_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) = 0, \\ r_{\mathrm{T}} - P_{D(\mu,(r_{\mathrm{N}} - \rho u_{\mathrm{N}})_+)}(r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}) = 0. \end{cases} \tag{60}$$ {eq:AC-1}

The difference between (58) and (60) is in the radius of the disk: $D(\mu, (r_{\mathrm{N}} - \rho u_{\mathrm{N}})_+)$ rather than $D(\mu, (r_{\mathrm{N}})_+)$. Problem FC can be also reformulated as in (59) using (60). This yields

$$F_{\mathrm{ac}}(r) := \begin{bmatrix} r_{\mathrm{N}} - P_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N}} - \rho_{\mathrm{N}}(Wr+q)_{\mathrm{N}}) \\ r_{\mathrm{T}} - P_{D(\mu,(r_{\mathrm{N}}-\rho_{\mathrm{N}} u_{\mathrm{N}})_+)}(r_{\mathrm{T}} - \rho_{\mathrm{N}}(Wr+q)_{\mathrm{T}}) \end{bmatrix} = 0. \tag{61}$$ {eq:AC-II}

Remark 1. *From the QVI formulation* (41)*, the following nonsmooth equation can also be written*

$$r = P_{C(\mu, r_N)}(r - \rho u) \tag{62}$$ {eq:qvi-proj}

which corresponds to (58)*.*

Remark 2. *In the literature of computational mechanics [Curnier and Alart, 1988, Simo and Laursen, 1992, Alart and Curnier, 1991], very similar expressions are obtained using the concept of augmented Lagrangian functions. This concept introduced in the general framework of Optimization by [Hestenes, 1969] and developed and popularized by [Rockafellar, 1974, 1993] is a strong theoretical tool for analyzing existence and regularity of solutions of constrained optimization problems. Its numerical interest is still a subject of intense debate in the mathematical programming community. In the nonconvex nonsmooth context of problems of frictional contact problems, its invocation is not so clear, but it has enabled the design of robust numerical techniques. Nevertheless, it is worth to note that some of these methods appear as variants of the methods developed to solve variational inequalities in other contexts. The method developed by [Simo and Laursen, 1992] is a dedicated version of fixed point with projection for VI (see Section 1) and the method of [Alart and Curnier, 1991] is a tailored version of semi–smooth Newton methods (see Section 5). Nevertheless, the concept of augmented Lagrangian have never been used in the optimization literature for this purpose.*

Xuewen-Soh-Wanji functions Following earlier work of [Park and Kwak, 1994] and [Leung et al., 1998], the following function is proposed in [Xuewen et al., 2000] as follows

$$F_{\mathrm{xsw}}(r) := \begin{bmatrix} \min(u_{\mathrm{N}}, r_n) \\ \min(\|u_{\mathrm{T}}\|, \mu r_{\mathrm{N}} - \|r_{\mathrm{T}}\|) = 0 \\ |u_{\mathrm{T}_1} r_{\mathrm{T}_2} - u_{\mathrm{T}_2} r_{\mathrm{T}_1} + \max(0, u_{\mathrm{T}_1} r_{\mathrm{T}_1}) = 0 \end{bmatrix} = 0. \tag{63}$$ {eq:XSW-II}

In [Xuewen et al., 2000], the system is solved by a generalized Newton method with a line-search procedure.

Hüeber–Stadler–Wolhmuth functions In [Stadler, 2004, Hüeber et al., 2008], and subsequently in [Koziara and Bićanić, 2008], another function is used to reformulate the problem FC:

$$F_{\mathrm{hsw}}(r) := \begin{bmatrix} r_{\mathrm{N}} - P_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N}} - \rho_{\mathrm{N}}(Wr+q)_{\mathrm{N}}) \\ \max(\mu(r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}), \|r_t - \rho_t u_{\mathrm{T}}\|) r_{\mathrm{T}} - \mu \max(0, r_n - \rho_{\mathrm{N}} u_{\mathrm{N}})(r_t - \rho_{\mathrm{T}} u_{\mathrm{T}}) \end{bmatrix} = 0. \tag{64}$$ {eq:HSW-II}

In [Hüeber et al., 2008], this function is used considering the problem written at the velocity level rather than in [Koziara and Bićanić, 2008] is stated at the velocity level.

General SOCC-functions More generally, a large family of reformulations of the SOCCP (24) in terms of equations can be obtained by using a so-called Second Order Cone Complementarity (SOCC) function. Let us consider the following SOCCP over a symmetric cone $K^\star = K$. A SOCC-function ϕ is defined by

$$K \ni x \perp y \in K \quad \Longleftrightarrow \quad \phi(x,y) = 0. \tag{65}$$ {eq:SOCC-func

The frictional contact problem can be written as a SOCCP over symmetric cones by applying the following transformations

$$x = T_x \hat{u} = \begin{bmatrix} \hat{u}_{\mathrm{N}} \\ \mu \hat{u}_{\mathrm{T}} \end{bmatrix} \text{ and } y = T_y r \begin{bmatrix} \mu r_{\mathrm{N}} \\ r_{\mathrm{T}} \end{bmatrix}. \tag{66}$$ {eq:SOCC-func

Clearly, the nonsmooth equations of the previous sections provides several examples of SOCC-functions and the natural map offers the simplest one. In [Fukushima et al., 2001], the standard complementarity functions for Nonlinear Complementarity Problems (NCP) such as the celebrated Fischer-Burmeister function are extended to the SOCCP by means of Jordan algebra. Smoothing functions are also given with theirs Jacobians and they studied their properties in view of the application of Newton's method. For the second order cone, the Jordan algebra can be defined with the following non-associative Jordan product

$$x \cdot y = \begin{bmatrix} x^\top y \\ y_{\mathrm{N}} x_{\mathrm{T}} + x_{\mathrm{N}} y_{\mathrm{T}} \end{bmatrix} \tag{67}$$ {eq:Jordan-pr

and the usual componentwise addition $x+y$. The vector x^2 denotes $x \cdot x$ and there exists a unique vector $x^{1/2} \in K$, the square root of $x \in K$, defined as

$$(x^{1/2})^2 = x^{1/2} \cdot x^{1/2} = x. \tag{68}$$ {eq:Jordan-sq

A direct calculation for the SOC in $\mathbb{R}^3$ yields

$$x^{1/2} = \begin{bmatrix} s \\ \dfrac{x_{\mathrm{T}}}{2s} \end{bmatrix}, \qquad \text{where } s = \sqrt{(x_{\mathrm{N}} + \sqrt{x_{\mathrm{N}}^2 - \|x_{\mathrm{T}}\|^2})/2}. \tag{69}$$ {eq:Jordan-sq

We adopt the convention that $0^{1/2} = 0$. The vector $|x| \in K$ denotes $(x^2)^{1/2}$. Thanks to this algebra and its associated operator, the projection onto K can be written as

$$P_K(x) = \frac{x + |x|}{2}. \tag{70}$$ {eq:Jordan-pr

This formula provides a new expression for the natural map and its associated nonsmooth equations. This is exactly what is done in [Hayashi et al., 2005] where the natural map (46) is used together with an expression of the projection operator based on the Jordan algebra calculus. The resulting SOCCP is then solved with a semi–smooth Newton method, and a smoothing parameter can be added.

Most of the calculus in Jordan algebra are based on the spectral decomposition, a basic concept in Jordan algebra, see [Fukushima et al., 2001] for more details. For $x = (x_{\mathrm{N}}, x_{\mathrm{T}}) \in \mathbb{R} \times \mathbb{R}^2$, the spectral decomposition is defined by

$$x = \lambda_1 u_1 + \lambda_2 u_2, \tag{71}$$ {eq:Jordan-sp

where $\lambda_1, \lambda_2 \in \mathbb{R}$ and $u_1, u_2 \in \mathbb{R}^3$ are the spectral values and the spectral vectors of x given by

$$\lambda_i = x_{\mathrm{N}} + (-1)^i \|x_{\mathrm{T}}\|, \quad u_i = \begin{cases} \frac{1}{2}\begin{bmatrix} 1 \\ (-1)^i \dfrac{x_{\mathrm{T}}}{\|x_{\mathrm{T}}\|} \end{bmatrix}, \text{ if } x_{\mathrm{T}} \neq 0 \\ \frac{1}{2}\begin{bmatrix} 1 \\ (-1)^i w \end{bmatrix}, \text{ if } x_{\mathrm{T}} = 0 \end{cases} \qquad i = 1,2 \tag{72}$$

{eq:Jordan-sp

with $w \in \mathbb{R}^2$ any unit vector. Note that the decomposition is unique whenever $x_{\mathrm{T}} \neq 0$. The spectral decomposition enjoys very nice properties that simplifies the computation of basic functions such that

$$\begin{aligned} x^{1/2} &= \sqrt{\lambda_1} u_1 + \sqrt{\lambda_2} u_2, \text{ for any } x \in K, \\ P_K(x) &= \max(0, \lambda_1) u_1 + \max(0, \lambda_2) u_2. \end{aligned} \tag{73}$$

{eq:Jordan-sp

More interestingly, general SOCC-functions can also be extended and smoothed version of this function can be also developed (see [Fukushima et al., 2001]). Let us start with the Fischer-Burmeister function

$$\phi_{\mathsf{FB}}(x,y) = x + y - (x^2 + y^2)^{1/2}. \tag{74}$$

{eq:Jordan-FB

It can show that the zeroes of ϕ_{FB} are solutions of the SOCCP (65) using the Jordan algebra associated with K. Using the spectral decomposition, the Fischer-Burmeister function can be easily computed as

$$\phi_{\mathsf{FB}}(x,y) = x + y - (\sqrt{\bar\lambda_1}\bar u_1 + \sqrt{\bar\lambda_2}\bar u_2) \tag{75}$$

{eq:Jordan-FB

where $\bar\lambda_1, \bar\lambda_2 \in \mathbb{R}$ and $\bar u_1, \bar u_2 \in \mathbb{R}^3$ are the spectral values and the spectral vectors of $x^2 + y^2$ that is

$$\begin{aligned} \bar\lambda_i &= \|x\|^2 + \|y\|^2 + 2(-1)^i \|x_{\mathrm{N}} x_{\mathrm{T}} + y_{\mathrm{N}} y_{\mathrm{T}}\| \\ \bar u_i &= \begin{cases} \frac{1}{2}\begin{bmatrix} 1 \\ (-1)^i \dfrac{x_{\mathrm{N}} x_{\mathrm{T}} + y_{\mathrm{N}} y_{\mathrm{T}}}{\|x_{\mathrm{N}} x_{\mathrm{T}} + y_{\mathrm{N}} y_{\mathrm{T}}\|} \end{bmatrix}, \text{ if } x_{\mathrm{N}} x_{\mathrm{T}} + y_{\mathrm{N}} y_{\mathrm{T}} \neq 0 \\ \frac{1}{2}\begin{bmatrix} 1 \\ (-1)^i w \end{bmatrix}, \text{ if } x_{\mathrm{N}} x_{\mathrm{T}} + y_{\mathrm{N}} y_{\mathrm{T}} = 0 \end{cases} \end{aligned} \quad , i = 1,2. \tag{76}$$

{eq:Jordan-FB

Finally, Problem FC is then reformulated as

$$F_{\mathsf{FB}}(u,r) \coloneqq \begin{bmatrix} u - Wr - q \\ \Phi_{\mathsf{FB}}\left(\begin{bmatrix} \mu r_{\mathrm{N}} \\ r_{\mathrm{T}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\mu}(u_{\mathrm{N}} + \mu\|u_{\mathrm{T}}\|) \\ u_{\mathrm{T}} \end{bmatrix}\right) \end{bmatrix} = 0. \tag{77}$$

{eq:FB-II}

where the mapping $\Phi_{\mathsf{FB}} : \mathbb{R}^{3n_c} \times \mathbb{R}^{3n_c} \to \mathbb{R}^{3n_c}$ is defined as

$$\Phi_{\mathsf{FB}}(x,y) = \left[(\phi(x^\alpha, y^\alpha), \alpha = 1 \ldots n_c)^\top\right]. \tag{78}$$

{eq:FB-1}

3.4 Optimization problems

In this section, several optimization-based formulations are proposed. The quest for an efficient optimization formulation of the frictional problem is a hard task. Since the problem is nonsmooth and nonconvex, the use of an associated optimization problem is interesting from the numerical point of view if we want to improve the robustness and the stability of the numerical methods.

A straightforward optimization problem can be written whose cost function to minimize is the scalar product $r^\top \hat{u}$. Indeed, this product is always positive and vanishes at the solution. Let us consider this first optimization formulation

$$\begin{cases} \min\, r^\top \hat{u} = r^\top u + \sum_{\alpha=1}^{n_c} \mu^\alpha r^\alpha_{\mathrm{N}} \|u^\alpha_{\mathrm{T}}\| \\ s.t. \quad \hat{u} \in K^\star, \\ \qquad\quad r \in K, \end{cases} \tag{79}$$ {eq:opt-1}

which amounts to minimizing the DeSaxcé's bipotential function [De Saxcé, 1992] over $K^\star \times K$. A first simplification can be made by noting that

$$\hat{u} \in K^\star \Longleftrightarrow u_{\mathrm{N}} \geqslant 0, \tag{80}$$ {eq:equiv-con

which leads to

$$\begin{cases} \min\, r^\top u + \sum_{\alpha=1}^{n_c} \mu^\alpha r^\alpha_{\mathrm{N}} \|u^\alpha_{\mathrm{T}}\| \\ s.t. \quad u_{\mathrm{N}} \geqslant 0 \\ \qquad\quad r \in K. \end{cases} \tag{81}$$ {eq:opt-2}

Starting from Problem FC, a direct substitution of $u = Wr + q$ yields

$$\begin{cases} \min\, r^\top (Wr+q) + \sum_{\alpha=1}^{n_c} \mu^\alpha r^\alpha_{\mathrm{N}} \|(Wr+q)^\alpha_{\mathrm{T}}\| \\ s.t. \quad (Wr+q)\mathsf{N} \geqslant 0, \\ \qquad\quad r \in K. \end{cases} \tag{82}$$ {eq:opt-3}

which is a nonlinear optimization problem with a nonsmooth and nonconvex cost function. From the numerical point of view this problem may be very difficult and we have to ensure that the cost function have to be zero at the solution which is not guaranteed of some local minima are reached in the minimization process.

Other optimization-based formulations have been proposed in the literature. They are not direct optimization formulation but they try to identify an optimization sub-problem which is well-posed and for which efficient numerical methods are available. Three approaches can be listed in three categories: a) the *alternating optimization* problems, b) the *successive approximation* method and c) the *convex SOCP* approach.

The Panagiotopoulos alternating optimization approach aims at solving the frictional contact problem by alternatively solving the Signorini condition for a fixed value of the tangential reaction $r_{\rm T}$, and solving the Coulomb friction model for a fixed value of the normal reaction $r_{\rm N}$. Let us split the matrix W and the vector q in the following way:

$$u = Wr + q \quad\iff\quad \begin{bmatrix} u_{\rm N} \\ u_{\rm T} \end{bmatrix} = \begin{bmatrix} W_{\rm NN} & W_{\rm NT} \\ W_{\rm TN} & W_{\rm TT} \end{bmatrix} \begin{bmatrix} r_{\rm N} \\ r_{\rm T} \end{bmatrix} + \begin{bmatrix} q_{\rm N} \\ q_{\rm T} \end{bmatrix}. \tag{83}$$
{eq:W-split}

Two sub-problems can therefore be identified: the first one is to find $u_{\rm N}$ and $r_{\rm N}$ such that

$$\begin{cases} u_{\rm N} = W_{\rm NN} r_{\rm N} + \tilde q_{\rm N}, \\ 0 \leqslant u_{\rm N} \perp r_{\rm N} \geqslant 0, \end{cases} \tag{84}$$
{eq:AO-1}

where $\tilde q_{\rm N} = q_{\rm N} + W_{\rm NT} r_{\rm T}$. The second problem is to find $u_{\rm T}$ and $r_{\rm T}$ such that

$$\begin{cases} u_{\rm T} = W_{\rm TT} r_{\rm T} + \tilde q_{\rm T}, \\ -u_{\rm T} \in N_{D(\mu,\tilde r_{\rm N})}(r_{\rm T}), \end{cases} \tag{85}$$
{eq:AO-2}

where $\tilde r_{\rm N}$ is fixed and $\tilde q_{\rm T} = q_{\rm T} + W_{\rm TN} r_{\rm N}$. Since W is a symmetric positive semi–definite matrix, $W_{\rm NN}$ and $W_{\rm TT}$ are also symmetric semi–definite positive matrices. Therefore, two convex optimization problems can be formulated:

$$\begin{cases} \min \ \dfrac{1}{2} r_{\rm N}^\top W_{\rm NN} r_{\rm N} + r_{\rm N}^\top \tilde q_{\rm N} \\ \text{s.t.} \quad r_{\rm N} \geqslant 0, \end{cases} \tag{86}$$
{eq:AO-3}

and

$$\begin{cases} \min \ \dfrac{1}{2} r_{\rm T}^\top W_{\rm TT} r_{\rm T} + r_{\rm T}^\top \tilde q_{\rm T} \\ \text{s.t.} \quad r_{\rm T} \in D(\mu, \tilde r_{\rm N}). \end{cases} \tag{87}$$
{eq:AO-4}

This approach has been proposed by [Panagiotopoulos, 1975] for two–dimensional applications in soil foundation computing. It has also been used in other finite element applications in [Barbosa and Feijóo, 1985, Tzaferopoulos, 1993] and studied from the mathematical point of view in [Haslinger and Panagiotopoulos, 1984, Haslinger et al., 1996].

The successive approximation method identifies a single optimization problem by introducing a function that maps the normal reaction to itself (or the friction threshold) such that

$$h(r_{\rm N}) = r_{\rm N}. \tag{88}$$
{eq:Haslinger

Using this artifact, we can define a new problem from Problem FC such that

$$\begin{cases} \theta = h(r_{\rm N}) \\ u = Wr + q \\ -u_{\rm N} \in N_{{\rm I\!R}^{nc}_+}(r_{\rm N}) \\ -u_{\rm T} \in N_{D(\mu,\theta)}(r_{\rm T}). \end{cases} \tag{89}$$
{eq:Haslinger

Since W is a symmetric positive semi-definite matrix, the last three lines are equivalent to a convex optimization problem over the product of semi–cylinders $C(\mu, \theta)$, that is

$$\begin{cases} \theta = h(r_{\mathrm{N}}) \\ \min \ \frac{1}{2} r^\top W r + r^\top q \\ \ \text{s.t.} \quad r \in C(\mu, \theta) \end{cases} \tag{90}$$

{eq:Haslinger

The method of successive approximation has been extensively used for proving existence and uniqueness of solutions to the discrete frictional contact problems. We refer to [Haslinger et al., 1996] which summarizes the seminal work of the Czech school [Nečas et al., 1980, Haslinger, 1983, 1984]. We will see in the sequel that this approach also provides us with very efficient numerical solvers in Section 7.2.

The convex SOCP approach is in the same vein as the previous one, with the difference that a SOCQP sub-problem is identified. To this aim, we augment the problem by introduction an auxiliary variable s, the image of $g(u)$ introduced in (25). We then obtain

$$\begin{cases} s = g(u) \\ \hat{u} = Wr + q + s \\ K^\star \ni \hat{u} \perp r \in K. \end{cases} \tag{91}$$

{eq:ACLM-2}

Since W is a positive semi-definite matrix, a new convex optimization sub-problem can be defined

$$\begin{cases} s = g(u) \\ \begin{cases} \min & \frac{1}{2} r^\top W r + r^\top (q + s) \\ \text{s.t.} & r \in K. \end{cases} \end{cases} \tag{92}$$

{eq:ACLM-3}

This formulation introduced in [Cadoux, 2009] and developed in [Acary and Cadoux, 2013, Acary et al., 2011] has been used to give an existence criteria to the discrete frictional contact problems. Furthermore, this existence criteria can be numerically checked by solving a linear program of second-order cone (SOCLP).

4 Numerical methods for VIs

4.1 Fixed point and projection methods for VI

Starting from the VI formulations (35) or more precisely an associated nonsmooth equation through the natural map,

$$F^{\mathsf{nat}}_R(z) = z - P_{X,R}(z - R^{-1}F(z)). \tag{93}$$ {eq:skew-natu

The basic idea of the algorithm is to perform fixed point iterations on the mapping

$$z \mapsto P_{X,R}(z - R^{-1}F(z)), \tag{94}$$ {eq:skew-fixe

yielding to Algorithm 1 with the specific choice of $R = \rho_k^{-1} I$. The choice of the updating rule of ρ_k is detailed in Section 4.2.

Algorithm 1 Fixed point iterations for the VI (35)

Require: F, X Data of VI (35)
Require: z_0 initial values
Require: tol > 0 a tolerance value and $\mathsf{iter}_{\max} > 0$ the max number of iterations
Require: ρ_0 initial value for ρ
Ensure: z solution of VI (35)
  $\mathsf{k} \leftarrow 0$
  while error $>$ tol and $\mathsf{k} < \mathsf{iter}_{\max}$ **do**
    Update the value of ρ_k
    $\mathsf{z}_{\mathsf{k}+1} \leftarrow \mathsf{P}_{\mathsf{X}}(\mathsf{z}_{\mathsf{k}} - \rho_{\mathsf{k}}\, \mathsf{F}(\mathsf{z}_{\mathsf{k}}))$
    Evaluate error.
    $\mathsf{k} \leftarrow \mathsf{k} + 1$
  end while
  $\mathsf{z} \leftarrow \mathsf{z}_{\mathsf{k}}$

For the formulation (38), the following iterations are performed

$$r_{k+1} \leftarrow P_{K,R}(r_k - R^{-1}(Wr_k + q + g(Wr_k + q))). \tag{95}$$ {eq:FP-vi-II}

In the sequel when a parameter ρ is specified, it is assumed that $R = \rho^{-1} I$.

The convergence of such methods are generally shown for strongly monotone VI. In our case, this assumption is not satisfied, but we will see in the sequel that such methods can converge in practice.

Remark 3. *Algorithm 1 with the iteration rule* (95) *and a fixed value of ρ_k has been originally proposed in [De Saxcé and Feng, 1991, 1998]. The algorithm is called Uzawa's algorithm by reference to the algorithm due to Uzawa in computing the optimal values of convex program by primal-dual techniques[Glowinski et al., 1976, Fortin and Glowinski, 1983]. Note that the algorithm in [Simo and Laursen,* *1992] is similar to the fixed point algorithm with projection though based on augmented Lagrangian concept (see Remark 2).*

Extragradient methods The extragradient method [Korpelevich, 1976] is also a well-known method for VI which improves the previous projection method. It can be described as

$$\begin{array}{lcl} \bar{z}_k & \leftarrow & P_X(z_k - \rho F(z_k)) \\ z_{k+1} & \leftarrow & P_X(z_k - \rho\, F(\bar{z}_k)) \end{array} \tag{96}$$ {eq:vi-ge5}

and formally defined in Algorithm 2. The convergence of this method is guaranteed under the following

Algorithm 2 Extragradient method for the VI (35)

Require: F, X Data of VI (35)
Require: z_0 initial values
Require: tol > 0 a tolerance value and iter$_{\max} > 0$ the max number of iterations
Ensure: z solution of VI (35)
  k $\leftarrow 0$
  while error $>$ tol and k $<$ iter$_{\max}$ **do**
    Update the value of ρ_k
    $\bar{\mathsf{z}}_\mathsf{k} \leftarrow \mathsf{P}_\mathsf{X}(\mathsf{z}_\mathsf{k} - \rho_\mathsf{k}\mathsf{F}(\mathsf{z}_\mathsf{k}))$
    $\mathsf{z}_{\mathsf{k}+1} \leftarrow \mathsf{P}_\mathsf{X}(\mathsf{z}_\mathsf{k} - \rho_\mathsf{k}\,\mathsf{F}(\bar{\mathsf{z}}_\mathsf{k}))$
    Evaluate error.
    k $\leftarrow$ k $+ 1$
  end while
  z $\leftarrow \mathsf{z}_\mathsf{k}$

assumptions: there exists a solution and the function F is Lipschitz–continuous and pseudo–monotone.

4.2 Self-adaptive step-size rules

A key ingredient in this efficiency and the convergence of the numerical methods for VI presented above is the choice of the sequence $\{\rho_k\}$. A sensible work has been done in the literature mainly motivated by some convergence proofs under specific assumption. Besides the relaxation of the assumption for the convergence, we are interesting in improving the numerical efficiency and robustness. We present in this section, the most popular approach for choosing the sequence $\{\rho_k\}$.

In [Khobotov, 1987], a method is proposed to improve the extragradient method of Korpelevich [1976] by adapting ρ_k in the following way. The goal is the find ρ_k that satisfies

$$0 < \rho_k \leqslant \min\left\{\bar{\rho}, L\frac{\|z_k - \bar{z}_k\|}{\|F(z_k) - F(\bar{z}_k)\|}\right\} \text{ with } L \in (0,1) \tag{97}$$ {eq:khobotov1

where $\bar{\rho}$ is the maximum value of ρ_k which is chosen in the light of the specific problem. The objective is to find a coefficient that is bounded by the local Lipschitz constant. The standard way to do that is to use an Armijo–type procedure by successively trying value of $\rho_k = \bar{\rho}\nu^m$ with $m \in \mathbb{N}$ and $\nu \in (0, 1)$, with a typical value of 2/3. In the original article of [Khobotov, 1987], there is no procedure to size $\bar{\rho}$ or to update it. In [He and Liao, 2002] and in the context of prediction–correction, the authors propose to use the rule $\rho_k = \rho_{k-1}\nu^m$ and if the criteria (97) is largely satisfied for ρ_k, the value is increased. In [Han and Lo, 2002], a similar procedure is used for the extragradient method by adding an increasing step of ρ_k, which is done after the correction as in [He and Liao, 2002]. The criteria (97) is verified by computing the ratio

$$r_k \leftarrow \frac{\rho_k \|F(z_k) - F(\bar{z}_k)\|}{\|z_k - \bar{z}_k\|}. \tag{98}$$ {eq:Khobotov-

In [Solodov and Tseng, 1996], similar Armijo–like technique is used, and the ratio r_k is computed as follows:

$$r_k \leftarrow \frac{\rho_k (z_k - \bar{z}_k)^\top (F(z_k) - F(\bar{z}_k))}{\|z_k - \bar{z}_k\|^2}. \tag{99}$$ {eq:SolodovTs

The approach is summarized in Algorithm 3. The parameter L typically chosen around 0.9 is a safety coefficient in the evaluation of ρ_k. The parameter $\mathsf{L_{min}}$ that triggers an increase of ρ_k is chosen around 0.3. In Algorithm 3, a Boolean option is added to the standard Armijo approach. The approximation $\bar{z}_k$ is updated within the self–adaptive loop. This trick is not justifies by any theoretical argument and is most of the articles this operation is not performed but in practice (see Section ??) it appears to improve the convergence speed. The update of the Armijo rule $\rho_k \leftarrow \nu\,\rho_k$ can also be replaced by $\rho_k \leftarrow \nu\,\rho_k \min\{1, 1/r_k\}$ but it appears that this trick does improve the self–adaptive procedure. Other more evolved step–lengths strategies can be found in [Wang et al., 2010] that have been tried in this study.

Redaction note O.H. 4.1.

The last paragraph is hard to follow. Maybe we could put some informations in the algorithm description, like the typical values.

4.3 Nomenclature

A nomenclature for the algorithms based on the VI formulation is given in Table 1.

Algorithm 3 Updating rule for ρ_k

Require: F, X

Require: Search and safety parameters. $\mathsf{L} \in (0,1), 0 < \mathsf{L}_{\mathsf{min}} < \mathsf{L}, \nu \in (0,1)$

Require: Initial values $\mathsf{z_k} \in \mathsf{X}, \rho_{\mathsf{k-1}} > 0$

  $\rho_{\mathsf{k}} \leftarrow \rho_{\mathsf{k-1}}$

  $\bar{\mathsf{z}}_{\mathsf{k}} \leftarrow \mathsf{P_X}(\mathsf{z_k} - \rho_{\mathsf{k}}\mathsf{F}(\mathsf{z_k}))$

  Evaluate r_k with (98) (or (99))

  while $\mathsf{r_k} > \mathsf{L}$ **do**

    $\rho_{\mathsf{k}} \leftarrow \nu\, \rho_{\mathsf{k}}$

    $\bar{\mathsf{z}}_{\mathsf{k}} \leftarrow \mathsf{P_X}(\mathsf{z_k} - \rho_{\mathsf{k}}\mathsf{F}(\mathsf{z_k}))$

    Evaluate r_k with (98) (or (99))

  end while

  Perform the correction step of extragradient or prediction–correction method.

  if $\mathsf{r_k} < \mathsf{L}_{\mathsf{min}}$ **then**

    $\rho_{\mathsf{k}} = \frac{1}{\nu}\rho_{\mathsf{k}}$

  end if

Name	Algo.	Additional informations
FP-DS	1	iteration rule (95) and fixed ρ
FP-VI-UPK	1 and 3	iteration rule (95) and updating rule (98)
FP-VI-UPTS	1 and 3	iteration rule (95) and updating rule (99)
EG-VI-UPK	2 and 3	iteration rule (96) and updating rule (98)
EG-VI-UPTS	2 and 3	iteration rule (96) and updating rule (99)

Table 1: Naming convention for the algorithms based on VI formulations.

5 Newton based methods

5.1 Principle of the nonsmooth Newton methods

In Section 3.3, several formulations of the frictional contact problem by means of nonsmooth equations have been presented. These nonsmooth equations call for the use of nonsmooth Newton's methods. Remember that the standard Newton method is to solve

$$G(z) = 0 \tag{100}$$ {eq:NSN1}

by performing the following Newton iteration

$$z_{k+1} = z_k - J^{-1}(z_k)G(z_k). \tag{101}$$ {eq:NSN2}

If the mapping G is smooth enough, the matrix J is the Jacobian matrix of G with respect to z, that is $J(z) = \nabla_z^\top G(z)$. When the G is nonsmooth but locally Lipschitz continuous, the Jacobian matrix is replaced by an element of the generalized Jacobian at z denoted by $\Phi(z) \in \partial G(z)$. Let us recall the definition of the generalized Jacobian. By Rademacher's Theorem, if G is ocally Lipschitz continuous, then G is almost everywhere differentiable and let us define the set D_G by

$$D_G \coloneqq \{z \mid G \text{ is differentiable at } z\}. \tag{102}$$ {eq:NSN4}

The generalized Jacobian of G at z can be defined by

$$\partial G(z) = \text{conv}\partial_B G(z), \tag{103}$$ {eq:NSN5}

with

$$\partial_B G(z) = \{\lim_{\bar{z}\to z, \bar{z}\in D_G} \nabla G(\bar{z})\}. \tag{104}$$ {eq:NSN6}

If $\Phi(z)$ is nonsingular, then an iteration of the nonsmooth Newton method is given by

$$z_{k+1} = z_k - \Phi^{-1}(z_k)(G(z_k)). \tag{105}$$ {eq:NSN3}

The resulting nonsmooth Newton method is detailed in Algorithm 4.

The convergence of nonsmooth Newton methods is based on the assumption of semi–smoothness of the nonsmooth function in (100). For this reason they are often called semi–smooth Newton methods (see [Facchinei and Pang, 2003, Section 7.5] and references therein).

5.2 Application to the discrete frictional contact problem

Newton method based on the Jean–Moreau and Alart–Curnier functions Let us consider now the Alart–Curnier function $F_{\text{ac}}(u, r)$ in (61) or the Jean–Moreau function $F_{\text{mj}}(u, r)$ (59) for Problem FC. Algorithm 4 is applied with

$$\Phi(r) \in \partial F_{\text{ac}}(r) \quad \text{or} \quad \Phi(r) \in \partial F_{\text{mj}}(r). \tag{106}$$ {eq:phiphi-ac

The details of a possible computation of Φ can be found in Appendix B.2.

Algorithm 4 Nonsmooth Newton method for (100)

Require: G data of Problem (100)
Require: z_0 initial values
Require: tol > 0 a tolerance value and $\text{iter}_{\max} > 0$ the max number of iterations
Ensure: z solution of Problem (100)

```
k ← 0
while error > tol and k < iter_max do
    compute (select) Φ(z_k) ∈ ∂G(z_k)
    z_{k+1} ← z_k − Φ^{-1}(z_k)(G(z_k))
    Evaluate error.
    k ← k + 1
end while
z ← z_k
```

Newton method based on SOCC-function Let us consider now the Fischer-Burmeister function $F_{\mathsf{FB}}(u, r)$ in (77) for Problem FC. Algorithm 4 is applied with

$$\Phi(r) \in \partial F_{\mathsf{FB}}(r). \tag{107}$$ {eq:phiphi-ac

The details of a possible computation of Φ can be found in Appendix **??**.

Nonsmooth newton based on the natural map Let us consider the natural map $F_{\text{vi}}^{\mathsf{nat}}$ in (49) that enables to write Problem FC as a nonsmooth equation. Algorithm 4 is applied with

$$\Phi(r) \in \partial F_{\text{vi}}^{\mathsf{nat}}(r). \tag{108}$$ {eq:phiphi}

The details of a possible computation of Φ can be found in Appendix B.1. Similar computations can also be found in [Joli and Feng, 2008] where a Newton method based on the formulation (49) is used contact by contact in a Gauss–Seidel loop.

Lipschitz continuity properties For the mapping $F_{\text{vi}}^{\mathsf{nat}}, F_{\text{ac}}, F_{\mathsf{FB}}, F_{\text{mj}}, F_{\text{xsw}}$ that are mainly generated by a composition of the Lipschitz functions $P_X, \min, \max$ and $\|.\|$; the local Lipschitz properties can be proven without difficulties. For the mapping F_{FB}, the proof of Lipschitz continuity of ϕ_{FB} can be found in [Sun and Sun, 2005] and references therein. This ensures the consistency of the definition of the generalized Jacobians.

5.3 Convergence and robustness issues.

The local convergence of the nonsmooth Newton methods is based on the semi-smoothness of the mapping G and the fact that all elements of the generalized Jacobian at the solution point $z^\star$, $\Phi(z^\star) \in \partial G(z^\star)$

are non singular (see [Qi and Sun, 1993] and Chapter 1 of [Qi et al., 2018] for a survey of mathematical results). For our application, the semi-smoothness of the mapping $F_{\mathrm{ac}}, F_{\mathrm{mj}},$ or F_{hsw}) is proven in several papers [Christensen and Pang, 1998, Hüeber et al., 2008]. The strong semi–smoothness of ϕ_{FB} can be found in [Sun and Sun, 2005].

On the contrary, the regularity of all elements of the generalized Jacobians is not necessarily ensured. The first reason is the possible rank deficiency of the matrix W, which is usual in rigid body applications as discussed in Section 2.3. Even if we consider a full rank matrix W, as in the standard one contact case for instance, the invertibility of all the elements of the generalized Jacobian at the solution point is not straightforward. For the mapping $F_{\mathrm{ac}}, F_{\mathrm{mj}}$, some results are given in [Alart, 1993, 1995, Jourdan et al., 1998]. Some of the results depends on the value of the coefficient of friction and the exact penalty parameters $\rho, \rho_{\mathrm{N}}, \rho_{\mathrm{T}}$ parameters. For the mapping F_{hsw}, some other results can be found in [Hüeber et al., 2008].

In the numerical practice and even if W is full-rank, it may happen that the elements of the generalized Jacobians are not regular or very badly conditioned even when we are far from the solution. This fact is reported in [Alart, 1993, 1995, Jourdan et al., 1998, Hüeber et al., 2008, Koziara and Bićanić, 2008]. Some divergence of the Newton algorithm can be encountered. A few work has been done to understand this problem. Among them, we cite [Hüeber et al., 2008] where some modifications of the elements of the generalized Jacobian are performed far form the solution to keep the Newton iteration matrix regular and well conditioned when the function F_{hsw} is chosen. This very interesting work opens new directions of research for the other mappings. In [Koziara and Bićanić, 2008], some other heuristics are developed to try avoid divergence of the Newton loop. In the two next sections, we present two complementary ways to partly solve this problem by choosing consistently the parameters $\rho, \rho_{\mathrm{N}}, \rho_{\mathrm{T}}$ and by applying some line–searches techniques to globalize the convergence.

5.4 Estimation of $\rho, \rho_{\mathrm{N}}, \rho_{\mathrm{T}}$ parameters

One of the key parameters in the efficiency of the nonsmooth Newton methods are the choice of the parameter ρ in the parameterized natural map in (48) and the parameters ρ_{N} and ρ_{T} in the Jean–Moreau and Alart-Curnier functions (59) and (61). The default choice is to set these parameters equal to 1 but the numerical practice shows that the convergence of the nonsmooth solvers is drastically deteriorated, especially if the norm or the conditioning of the matrix W is far from this unit value. There is no theoretical rules to size this parameters, but some heuristics may be found in the literature for a single contact problem that we expose in the sequel.

Inverse of a norm of W A first simple choice is to consider the inverse of a norm of the matrix W. With this heuristics, we set the ρ parameter before the Newton loop as follows:

$$\rho = \frac{1}{\|W\|}, \quad \rho_{\mathrm{N}} = \rho_{\mathrm{T}} = \frac{1}{\|W\|}. \tag{109}$$ {eq:rho-1}

This choice is mainly based on a guess of the inverse of the local Lipschitz constant of the operator $Wr+q$. In the case of the natural map, it amounts to neglecting the nonlinear contribution of g. Naturally, this choice depends on the choice of the matrix norm. If we assume that the matrix is a symmetric definite positive matrix, a possibility is to choose a 2-norm based on the spectral radius $\|W\|_2 = \rho(W) = \lambda_{\max}(W)$ of the matrix:

$$\rho = \frac{1}{\lambda_{\max}(W)}, \quad \rho_{\mathrm{N}} = \rho_{\mathrm{T}} = \frac{1}{\lambda_{\max}(W)}. \tag{110}$$ {eq:rho-2}

Estimation based on the splitting $W_{\mathbf{NN}}$ and $W_{\mathbf{TT}}$ A second possible choice for the map (59) and (61) is to use the fact that the problem is split with respect to the normal and the tangent directions. In that case, we compute a value of ρ_{N} that is based on the eigenvalues of W_{NN} and a value of ρ_{T} based on the eigenvalue of W_{TT}. For a single contact, we set

$$\rho_{\mathrm{N}} = \frac{1}{W_{\mathrm{NN}}}, \quad \rho_{\mathrm{T}} = \frac{1}{\lambda_{\max}(W_{\mathrm{TT}})} \tag{111}$$ {eq:rho-3}

A third option it also to take into account the conditioning of the matrix W_{TT} by choosing

$$\rho_{\mathrm{N}} = \frac{1}{W_{\mathrm{NN}}}, \quad \rho_{\mathrm{T}} = \frac{\lambda_{\min}(W_{\mathrm{NN}})}{\lambda^2_{\max}(W_{\mathrm{TT}})} \tag{112}$$ {eq:rho-4}

Adaptive estimation of the parameters In [Koziara and Bićanić, 2008], an adaptive way of updating ρ is proposed that has been implemented for our experiments.

Default choices By default, we use the rule (112) for the mapping (59) and (61) and the rule (110) for the natural. When other rules are chosen in the comparison, they are specified.

5.5 Damped Newton and line-search procedures

These line-searches algorithms follow the presentation done in chapter 3 of [Bonnans et al., 2003], where one may find notably all the mathematical explanations of why they terminate. However, despite the mathematical proofs, in practice, is is recommended that some emergency tests should be added during extrapolation and interpolation phases to avoid infinite loops.

The choice of the values for the parameters m_1, m_2 for the Goldstein-Price line-search and the parameter m_1 alone for the Armijo line-search is also discussed and it is advised to choose $m_1 < \frac{1}{2}$ and $m_2 > \frac{1}{2}$.

Merit functions

Goldstein–Price (GP) line search

Armijo line–search

Algorithm 5 Goldstein–Price (GP) line search

Require: x, the starting point of the line-search.

Require: d, the direction of search.

Require: t, an initial stepsize-value.

Require: $t \to q(t)$, for $t \geqslant 0$, with $q \in C^1$ bounded from below and $q'(0) < 0$, a merit function representing $f(x + td)$

Require: m_1, m_2, parameters with $0 < m_1 < m_2 < 1$

Require: a, with $a > 1$, parameter for extrapolation

Ensure: a finite line-search

$t_L \leftarrow 0$

$t_R \leftarrow 0$

$\Delta \leftarrow \frac{q(t)-q(0)}{t}$

while $m_2 q'(0) > \Delta$ or $\Delta > m_1 q'(0)$ **do**

  if $m_1 q'(0) < \Delta$ **then**

    $t_R \leftarrow t$

  end if

  if $\Delta < m_2 q'(0)$ **then**

    $t_L \leftarrow t$

  end if

  if $t_R = 0$ **then**

    $t \leftarrow at$

  else

    $t \leftarrow \frac{t_L + t_R}{2}$

  end if

  $\Delta \leftarrow \frac{q(t)-q(0)}{t}$

end while

Algorithm 6 Armijo(A) line search

Require: x, the starting point of the line-search.

Require: d, the direction of search.

Require: t, an initial stepsize-value.

Require: $t \to q(t)$, for $t \geqslant 0$, with $q \in C^1$ bounded from below and $q'(0) < 0$, a merit function representing $f(x + td)$

Require: m_1, a parameter with $0 < m_1 < 1$

Require: a, with $a > 1$, parameter for extrapolation

Ensure: a finite line-search

```
while m_1 q'(0) < (q(t) - q(0))/t do
  if t_R = 0 then
    t ← at
  else
    t_R ← t
    t ← t_R/2
  end if
end while
```

Non-monotone line–search see [Koziara and Bićanić, 2008] based on [Ferris and Lucidi, 1994, Grippo et al., 1986]

5.6 Nomenclature

A nomenclature for the algorithms based on the nonsmooth Newton methods is given listed in Table 2.

Name	Algo.	Additional informations
NSN-NM	4	Natural map formulation (49)
NSN-AC	4	Alart–Curnier formulation (61)
NSN-JM	4	Jean–Moreau formulation (59)
NSN-FB	4	Fischer-Burmeister formulation (77)
NSN-NM-GP	4 and 5	Natural map formulation (49) and the Goldstein–Price (GP) line search
NSN-AC-GP	4 and 5	Alart–Curnier formulation (61) and the Goldstein–Price (GP) line search
NSN-JM-GP	4 and 5	Jean–Moreau formulation (59) and the Goldstein–Price (GP) line search
NSN-FB-GP	4 and 5	Fischer-Burmeister formulation (77) and the Goldstein–Price (GP) line search
NSN-NM-A	4 and 6	Natural map formulation (49) and the Armijo(A) line search
NSN-AC-A	4 and 6	Alart–Curnier formulation (61) and the Armijo(A) line search
NSN-JM-A	4 and 6	Jean–Moreau formulation (59) and the Armijo(A) line search
NSN-FB-A	4 and 6	Fischer-Burmeister formulation (77) and the Armijo(A) line search

Table 2: Naming convention for the algorithms based on nonsmooth Newton (NSN) method

6 Splitting techniques and proximal point algorithm

Splitting techniques are standard techniques to solve $\mathrm{VI}(F,X)$ when the function F is affine that is $F(z) = Mz + q$ and the set X can decomposed in a Cartesian product of independent smaller sets $X = \Pi_i X_i$. Usually, a block splitting of the matrix M is performed and a Projected Successive Over Relaxation (PSOR) method is used to solve the VI. Since the cone K is a product of second-order cones in $\mathbb{R}^3$, a natural way to split the problem is to form sub-problems by using single contact as a building block. The sub-problems can be solved by any method for the VI that have been presented in the previous sections. In the same way, the proximal point algorithm can also be used which amounts to solving the original $\mathrm{VI}(F,X)$ by solving a sequence of $\mathrm{VI}(F_{c,x_k},X)$ problems such that $F_{c,x_k}(z) = z - x_k + cF(z), \quad c > 0$ and $\lim_{k\to+\infty} \|x_k - z\| = 0$.

6.1 Splitting techniques

The particular structure of the cone K as a product of second-order cone in $\mathbb{R}^3$ calls for a splitting of the problem contact by contact. For Problem FC, the relation

$$u = Wr + q \tag{113}$$

{eq:delassus-

is splitted along each contact as follows

$$u^\alpha = W^{\alpha\alpha} r^\alpha + \sum_{\beta\neq\alpha} W^{\alpha\beta} r^\beta + q^\alpha, \text{ for all } \alpha \in 1\ldots n_c, \tag{114}$$

{eq:delassus-

where the matrices α and β are used to label the variable for each contact. The matrices $W^{\alpha\beta}$ with $\alpha \in 1,\ldots,n_c$ and $\beta \in 1,\ldots,n_c$ are easily identified from (113). From (114), a projected Gauss–Seidel (PGS) method is obtained by using the following update rule at the k-th iterate:

$$u^\alpha_{k+1} = W^{\alpha\alpha} r^\alpha_{k+1} + \sum_{\beta<\alpha} W^{\alpha\beta} r^\beta_{k+1} + \sum_{\beta>\alpha} W^{\alpha\beta} r^\beta_k + q^\alpha, \text{ for all } \alpha \in 1\ldots n_c. \tag{115}$$

{eq:pgs-1}

A Projected Successive Over Relaxation (PSOR) scheme is derived by introducing a relaxation parameter $\omega > 0$ such that

$$u^\alpha_{k+1} = \frac{1}{\omega} W^{\alpha\alpha} r^\alpha_{k+1} - \frac{1}{\omega} W^{\alpha\alpha} r^\alpha_k + \sum_{\beta<\alpha} W^{\alpha\beta} r^\beta_{k+1} + \sum_{\beta\geqslant\alpha} W^{\alpha\beta} r^\beta_k + q^\alpha, \text{ for all } \alpha \in 1\ldots n_c. \tag{116}$$

{eq:psor-1}

At the k-th iteration, the following problem is solved for each contact α:

$$\begin{cases} u^\alpha_{k+1} = \bar{W}^{\alpha\alpha} r^\alpha_{k+1} + \bar{q}^\alpha_{k+1}, \\ \hat{u}^\alpha_{k+1} = u^\alpha_{k+1} + g(u^\alpha_{k+1}), \\ K^{\alpha,\star} \ni \hat{u}^\alpha_{k+1} \perp r^\alpha_{k+1} \in K^\alpha, \end{cases} \tag{117}$$

{eq:psor-3}

where

$$\begin{cases} \bar{W}^{\alpha\alpha} = \frac{1}{\omega} W^{\alpha\alpha} \\ \bar{q}^\alpha_{k+1} = -\frac{1}{\omega} W^{\alpha\alpha} r^\alpha_k + \sum_{\beta<\alpha} W^{\alpha\beta} r^\beta_{k+1} + \sum_{\beta\geqslant\alpha} W^{\alpha\beta} r^\beta_k + q^\alpha \end{cases}, \text{ for all } \alpha \in 1\ldots n_c. \tag{118}$$

{eq:psor-2}

The problem (117) has exactly the same structure as Problem FC, but is of lower size since it is only for one contact. It is solved by a *local solver*, which can be any of the algorithms presented in this article or even an analytical method (enumerating all the possible cases as in [Bonnefon and Daviet, 2011]).

The PSOR algorithm is summarized in Algorithm 7 and the NSGS can be recovered by setting $\omega = 1$.

Algorithm 7 PSOR algorithm for Problem FC

Require: $\mathsf{W}, \mathsf{q}, \mu$
Require: r_0 initial values
Require: $\mathsf{tol} > 0, \mathsf{tol}_{\mathsf{local}}$ tolerance values and $\mathsf{iter}_{\max} > 0, \mathsf{iter}_{\mathsf{local}\max} > 0$ the max number of local iterations
Require: ω a relaxation parameter
Ensure: r, u solution of Problem FC
  while error $>$ tol and $\mathsf{k} < \mathsf{iter}_{\max}$ **do**
    for $\alpha = 1 \dots \mathsf{n_c}$ **do**
      $\bar{\mathsf{W}}^{\alpha\alpha}_{\mathsf{k}+1} \leftarrow \frac{1}{\omega}\mathsf{W}^{\alpha\alpha}$
      $\bar{\mathsf{q}}^{\alpha}_{\mathsf{k}+1} \leftarrow -\frac{1}{\omega}\mathsf{W}^{\alpha\alpha}\mathsf{r}^{\alpha}_{\mathsf{k}} + \sum_{\beta<\alpha}\mathsf{W}^{\alpha\beta}\mathsf{r}^{\beta}_{\mathsf{k}+1} + \sum_{\beta\geqslant\alpha}\mathsf{W}^{\alpha\beta}\mathsf{r}^{\beta}_{\mathsf{k}} + \mathsf{q}^{\alpha}$
      Solve the single contact problem $\mathrm{FC}(\bar{\mathsf{W}}^{\alpha\alpha}, \bar{\mathsf{q}}^{\alpha}_{\mathsf{k}+1}, \mu)$ at accuracy $\mathsf{tol}_{\mathsf{local}}$ with a maximum of iteration $\mathsf{iter}_{\mathsf{local}\max} > 0$
    end for
    Evaluate error.
    $\mathsf{k} \leftarrow \mathsf{k} + 1$
  end while
  $\mathsf{r} \leftarrow \mathsf{r}_{\mathsf{k}}$
  $\mathsf{u} \leftarrow \mathsf{u}_{\mathsf{k}}$

Applications methods in frictional contact date back to the work of [Mitsopoulou and Doudoumis, 1988, 1987] for two-dimensional friction. In [Jourdan et al., 1998], this method is developed in the Gauss-Seidel configuration ($\omega = 1$) with a local Newton solver based on the Alart–Curnier formulation. If the local solver is only one iteration of the VI solver based on projection, we get a standard splitting technique for VI. In Table 3, the methods based on PSOR used in the comparison are summarized.

6.2 Proximal points techniques

Redaction note O.H. 6.1.

The following paragraph is a work in progress.

Proximal points methods, or more generally augmented Lagrangian methods, find their roots in the study of variational inequalities. They rely on rather theoretical tools (monotone operators and resolvents), introduced to study VI. However, they are also useful to device numerical methods as we shall see. The basic idea is to design an iterative algorithm to find a solution to the inclusion

$$0 \in F(x) + N_X(x) \qquad \text{also} \quad 0 \in T(x), \tag{119}$$

where T is a maximal monotone operator (see Definition 1). Remember that the first inclusion is a VI.

Redaction note V.A. 6.1.

Small introduction on proximal functions of Moreau, algo of Martinet and result of Rockafellar (see [Chen and Teboulle, 1993])

Without entering into theoretical aspects, the key idea of proximal points algorithms is to replace the original VI(F, X) by a sequence of VI(F_{ρ,x_k}, X) problems such that

$$F_{\rho,x}(z) = z - x_k + \alpha F(z), \quad \rho > 0 \tag{120}$$ {eq:prox-algo

with the property that $\lim_{k\to+\infty} \|x_k - z\| = 0$. This is usually performed by defining a sequence x_k such that

$$x_{k+1} = (1-\omega)x_k + \omega z_{k+1} \tag{121}$$ {eq:prox-algo

where ω is a relaxation parameter and z_{k+1} the solution of VI(F_{ρ,x_k}, X). The algorithm is described in algorithm 8

Algorithm 8 Proximal point algorithm for the VI (35)

Require: F, X Data of VI (35)
Require: ω relaxation parameter
Require: α proximal point parameter
Require: x_0 initial values
Require: tol > 0, $\mathrm{tol_{in}}$ tolerance values and $\mathrm{iter_{max}} > 0$ the max number of iterations
Ensure: z solution of VI (35)
  $k \leftarrow 0$
  while error $>$ tol and $k < \mathrm{iter_{max}}$ **do**
    Solve VI(F_{α,x_k}, X) for z_{k+1} at accuracy $\mathrm{tol_{int}}$
    $x_{k+1} \leftarrow (1-\omega)x_k + \omega z_{k+1}$
    Evaluate error.
    $k \leftarrow k+1$
  end while
  $z \leftarrow z_{k+1}$

For solving the sub-problem VI(F_{α,x_k}, X), any of the previous presented algorithms can be used. In this sense, the proximal point algorithm defined a general family of algorithm. The main interest of the proximal point algorithm is the regularization it introduces in the definition (120). For instance, let consider an affine VI, that is defined with $F(z) = Mz + q$. The function is the sub-problem is given by

$$F_{\alpha,x}(z) = (I + \alpha M)z - x_k + \rho q, \quad \rho > 0. \tag{122}$$ {eq:prox-algo

We can easily that for sufficiently small ρ, we get a monotone affine VI and even better a strongly monotone VI. For Problem FC with F_{vi}, the proximal point algorithm yields

$$F_{\mathrm{vi},\rho,x}(r) = (I + \alpha W)r - x_k + \alpha(q + g(Wr + q)), \quad \rho > 0. \tag{123}$$ {eq:prox-algo

and it should not be difficult to prove that with a small ρ parameter that we get a monotone VI. Naturally, there is a price to pay, smaller the parameter ρ is, easier the VI problem is, but the resulting solution of z_{k+1} is far from the solution.

The use of proximal point algorithm can be very interesting when the solving of the problem suffers for the lack of regularity of the operator. For instance, the Newton methods are in trouble when the Jacobian is not invertible. Thanks to the proximal point algorithm, we can retrieve invertible Jacobian.

Redaction note V.A. 6.2.

- *Prove it !*
- *Is there an analogy with choosing a small r ?*
- *Techniques of [Han, 2008] for sizing alpha*

In [He and Liao, 2002], the extragradient method is reinterpreted as a prediction-correction method applied to the proximal point algorithm. Indeed, the proximal point algorithm amounts to solving the following VI for a given z_k

$$((z - z_k) + \alpha_k F(z))^\top (y - z) \geqslant 0, \text{ for all } y \in X \text{ and } \alpha_k > 0 \tag{124}$$ {eq:PPA-VI1}

or equivalently

$$z = P_X(z - [z - z_k + \alpha_k F(z)]) = P_X(z_k - \alpha_k F(z)). \tag{125}$$ {eq:PPA-VI2}

Searching for a solution z_{k+1} of (125) yields

$$z_{k+1} = P_X(z_k - \alpha_k F(z_{k+1})) \tag{126}$$ {eq:PPA-VI3}

which can be viewed as an implicit version of the basic fixed point iteration based on (**??**). A prediction–correction method based may be written as

$$\begin{array}{lcl} \bar{z}_k & \leftarrow & P_X(z_k - \rho_k F(z_k)) \\ z_{k+1} & \leftarrow & P_X(z_k - \alpha_k\, F(\bar{z}_k)) \end{array} \tag{127}$$ {eq:PPA-VI4}

where the scalars α_k and ρ_k have to be updated at each iteration.

Algorithm	Description
NSGS-AC	Algorithm 7 with $\omega = 1$ with the local solver NSN-AC with tolerance $\mathsf{tol}_{\mathsf{local}}$
NSGS-JM	Algorithm 7 with $\omega = 1$ with the local solver NSN-JM with tolerance $\mathsf{tol}_{\mathsf{local}}$
NSGS-AC-GP	Algorithm 7 with $\omega = 1$ with the local solver NSN-AC-GP with tolerance $\mathsf{tol}_{\mathsf{local}}$
NSGS-JM-GP	Algorithm 7 with $\omega = 1$ with the local solver NSN-JM=GP with tolerance $\mathsf{tol}_{\mathsf{local}}$
NSGS-FP-DS-One	Algorithm 7 with $\omega = 1$ with one iteration of FP-DS for the local solver
NSGS-FP-VI-UPK	Algorithm 7 with $\omega = 1$ with FP-VI-UPK for the local solver with tolerance $\mathsf{tol}_{\mathsf{local}}$
NSGS-EXACT	Algorithm 7 with $\omega = 1$ with the exact local solver.
PSOR-AC	Algorithm 7 with the local solver NSN-AC with tolerance $\mathsf{tol}_{\mathsf{local}}$
PPA-NSN-AC	Algorithm 8 with the NSN-AC solver as internal solver.
PPA-NSGS-AC	Algorithm 8 with the NSGS-AC solver as internal solver.

Table 3: Naming convention for the algorithms based on splitting and proximal algorithms

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Redaction note O.H. 6.2.

I think that the analogy with the proximal point algorithm is premature. We only introduce it in Section 6.2.

I think we are presenting result with the extragradient. Maybe add a teaser at the end of the paragraph?

VA OK we can discuss this point. It is difficult to do a (European) perfect linear presentation.

6.3 Control of the tolerance of internal solvers $\mathsf{tol}_{\mathsf{int}}$ and $\mathsf{tol}_{\mathsf{local}}$ in the splitting and proximal approaches

In Algorithms 8 and 7, an internal tolerance is used to control the accuracy of the internal solver. It is generally not useful to solve the internal problem at the accuracy of the global one. In the comparison study, we set the internal tolerance $\mathsf{tol}_{\mathsf{int}}$ to a fraction of the error $\mathsf{error}/10.0$ for the Algorithm 8. For Algorithm 7, the local tolerance $\mathsf{tol}_{\mathsf{local}}$ is set by default to a very low value of 10^{-14}. When it is specified, an adaptive local tolerance is set to $\mathsf{error}/10.0$.

6.4 Nomenclature

A nomenclature for the algorithms based on the projection/splitting approach is given in Table 3.

7 Optimization based methods

7.1 Alternating optimization problem

The Panagiotopoulos approach described in Section 3.4 generates a family of solvers by choosing a particular solver for the normal contact problem (86) and the tangential contact problem (87). This method may be viewed as a two-block Gauss-Seidel method as it has been pointed out by [Tzaferopoulos, 1993]. More precisely, the following choices may be made for the normal and tangent problems.

The normal contact problem

$$\begin{cases} \min \dfrac{1}{2} r_{\mathrm{N}}^\top W_{\mathrm{NN}} r_{\mathrm{N}} + r_{\mathrm{N}}^\top \tilde{q}_{\mathrm{N}} \\ \text{s.t.} \quad r_{\mathrm{N}} \geqslant 0 \end{cases} \quad \text{with } \tilde{q}_{\mathrm{N}} = q_{\mathrm{N}} + W_{\mathrm{NT}} r_{\mathrm{T},k}, \tag{128}$$

{eq:AO-3-bis}

is a convex quadratic program with simple bound constraints. In the literature, a bunch of solvers has been developed to solve such problems. Among others, we might cite the active set strategy solvers [Fletcher, 1987, Nocedal and Wright, 1999] that are mainly dedicated to small–scale systems and the projected gradient [Calamai and More, 1987] and projected conjugate gradient methods [Moré and Toraldo, 1989, Moré and Toraldo, 1991] that are more dedicated to large–scale systems. Note that there exists also a bunch of methods in the literature that improves the methods of [Moré and Toraldo, 1991] for large–scale systems. For the reader interested in deeper details, we refer to the book of [Dostál, 2016]. (see especially [Dostál, 2016], Section 8.7 for a review of the different approaches). It is clear that we might also use semi-smooth Newton methods or interior point methods but our experience has shown that such methods are not efficient when $\ker(W_{\mathrm{NN}}) \neq \{0\}$. The optimality conditions of this quadratic reduced to a linear complementarity problem with a semi-definite matrix. In that case, it is also possible to solve the problem with PSOR techniques with line-searches. In order to keep the article in a reasonable size, we decided in this work to use the projected gauss-seidel (PGS) algorithm and the projected gradient algorithm of [Calamai and More, 1987] to solve the normal problem described. The projected gradient algorithm solved the following QP for a convex set C

$$\begin{cases} \min q(r) := \dfrac{1}{2} r^\top W r + r^\top \tilde{b} \\ \text{s.t.} \quad r \in C \end{cases} \tag{129}$$

{eq:ConvexQP}

with the algorithm described in Algorithm 9.

The tangential problem,

$$\begin{cases} \min \dfrac{1}{2} r_{\mathrm{T}}^\top W_{\mathrm{TT}} r_{\mathrm{T}} + r_{\mathrm{T}}^\top \tilde{q}_{\mathrm{T}} \\ \text{s.t.} \quad r_{\mathrm{T}} \in D(\mu, \tilde{r}_{\mathrm{N}}) \end{cases} \quad \text{with } \tilde{q}_{\mathrm{T}} = q_{\mathrm{T}} + W_{\mathrm{TN}} r_{\mathrm{N},k+1}, \tag{130}$$

{eq:AO-4-bis}

is also a convex program but with a more complex structure since the constraints are quadratic one. It exists also some dedicated methods to solve this specific problem as the method in [Dostál and Kozubek,

Algorithm 9 Projected gradient algorithm for QP 129

Require: W, b that defines $q(r)$

Require: C a convex set

Require: r_0 initial values

Require: tol > 0 a tolerance value and $\mathsf{iter}_{\max} > 0$ the max number of iterations

Require: $\rho_0 > 0$, $l, \sigma \in (0, 1)$

Require: $\mathsf{i}_{\mathsf{lsmax}}$ maximum number of line-search iterations

Ensure: r solution of Problem 129

  $\mathsf{r}_\mathsf{k} \leftarrow \mathsf{r}_0$; $\theta_0 \leftarrow \mathsf{q}(\mathsf{r}_0)$; $\mathsf{k} \leftarrow 0$

  while error $>$ tol and $\mathsf{k} < \mathsf{iter}_{\max}$ **do**

    Armijo like–search procedure

    $\mathsf{i}_{\mathsf{ls}} \leftarrow 0$

    while criterion > 0 and $i_{ls} < i_{lsmax}$ **do**

      $\rho \leftarrow \rho_0 \mathsf{l}^{\mathsf{i}_{\mathsf{ls}}}$

      $\mathsf{r} \leftarrow \mathsf{P}_\mathsf{C}(\mathsf{r}_\mathsf{k} - \rho(\mathsf{M}\mathsf{r}_\mathsf{k} + \mathsf{b}))$

      $\theta \leftarrow \mathsf{q}(\mathsf{r})$

      criterion $\leftarrow \theta - \theta_\mathsf{k} - \sigma(\mathsf{M}\mathsf{r}_\mathsf{k} + \mathsf{b})^\top(\mathsf{r} - \mathsf{r}_\mathsf{k})$

      $i_{ls} \leftarrow i_{ls} + 1$

    end while

    $\mathsf{r}_\mathsf{k} \leftarrow \mathsf{r}$; $\theta_\mathsf{k} \leftarrow \mathsf{q}(\mathsf{r})$;

    evaluate error.

  end while

2012] (an extension of [Moré and Toraldo, 1991]) which is an algorithm for solving QP over convex constraints (disk constraint for instance). Earlier application of projected gradient and projected gradient techniques for the frictionless problem can also be found in [Barbosa et al., 1997] with comparison with PSOR techniques.

In the article, we will use a) a reformulation of the optimality conditions of this problem as a variational inequality and we apply the fixed point algorithm and the extra gradient algorithm of Section 4 or b) an adaptation of one of the splitting techniques detailed in Section 6. The algorithm is described in Algorithm 10.

In Table 4, we detailed the algorithms we use in the present study.

Algorithm 10 Panagiotopoulos decomposition algorithm for Problem FC

Require: $\mathsf{W}, \mathsf{q}, \mu$
Require: r_0 initial values
Require: $\mathsf{tol} > 0, \mathsf{tol}_{\mathsf{int}}$ tolerance values and $\mathsf{iter}_{\max} > 0$ the max number of iterations
Ensure: r, u solution of Problem FC
  $\mathsf{r}_\mathsf{k} \leftarrow \mathsf{r}_0$; $\mathsf{k} \leftarrow 0$
  while error $>$ tol and $\mathsf{k} < \mathsf{iter}_{\max}$ **do**
    $\tilde{\mathsf{q}}_{\mathsf{N}} \leftarrow \mathsf{q}_{\mathsf{N}} + \mathsf{W}_{\mathsf{NT}}\mathsf{r}_{\mathsf{T},\mathsf{k}}$
    solve (128) for $r_{\mathsf{N},k+1}$ at accuracy $\mathsf{tol}_{\mathsf{int}}$
    $\tilde{\mathsf{q}}_{\mathsf{T}} \leftarrow \mathsf{q}_{\mathsf{T}} + \mathsf{W}_{\mathsf{TN}}\mathsf{r}_{\mathsf{N},\mathsf{k}+1}$
    solve (130) for $r_{\mathsf{T},k+1}$ at accuracy $\mathsf{tol}_{\mathsf{int}}$
    $\mathsf{k} \leftarrow \mathsf{k}+1$
    evaluate error.
  end while
  $\mathsf{r} \leftarrow \mathsf{r}_\mathsf{k}$
  $\mathsf{u} \leftarrow \mathsf{W}\mathsf{r} + \mathsf{q}$

7.2 Successive approximation method

The methods of successive approximation is a natural tool for the numerical realization of Problem FC. It is based on the Tresca approximation of the Coulomb cone as it is described in Section 3.4 and the work of the celebrated Czech school which summarizes the seminal work of the Czech school [Nečas et al., 1980, Haslinger, 1983, 1984, Haslinger et al., 1996]. Each iterative step is represented by an auxiliary contact problem with given friction described by quadratic program over a cylinder (90), that we recall there:

$$\begin{cases} \theta = h(r_{\rm N}) \\ \min\ \frac{1}{2} r^\top W r + r^\top q \\ \text{s.t.} \quad r \in C(\mu, \theta). \end{cases} \tag{131}$$

{eq:Haslinger

The radius of the cylinder is then updated in an iterative procedure. The algorithm is described in Algorithm 11

Algorithm 11 Tresca approximation algorithm for Problem FC

Require: $\mathsf{W}, \mathsf{q}, \mu$
Require: r_0 initial values
Require: $\mathsf{tol} > 0, \mathsf{tol}_{\mathsf{int}}$ tolerance values and $\mathsf{iter}_{\max} > 0$ the max number of iterations
Ensure: r, u solution of Problem FC
  $\mathsf{r_k} \leftarrow \mathsf{r}_0$; $\mathsf{k} \leftarrow 0$
  while error $>$ tol and $\mathsf{k} < \mathsf{iter}_{\max}$ **do**
    $\theta \leftarrow h(r_{\mathsf{N},k})$
    solve (131) for $r_{\mathsf{N},k+1}, r_{\mathsf{T},k+1}$ at accuracy $\mathsf{tol}_{\mathsf{int}}$
    $\mathsf{k} \leftarrow \mathsf{k} + 1$
    evaluate error.
  end while
  $\mathsf{r} \leftarrow \mathsf{r_k}$
  $\mathsf{u} \leftarrow \mathsf{Wr} + \mathsf{q}$

In the literature, the successive approximation technique has been used in the bidimensional case in [Haslinger et al., 2002] & [Dostál et al., 2002] with improved and dedicated QP solvers overs box-constraints. Two strategies are implemented : a) the classical Tresca iteration (called FPMI) and b) the Panagiotopoulos decomposition plus a Fixed point (called FPMII). They use a specific QP solver for box constraint [Dostál, 1997] that is an improvement of Moré–Toraldo method [Moré and Toraldo, 1991]. This technique has been directly extended in the three-dimensional case with a faceting of the cone in [Haslinger et al., 2004]. In the latter case, the problem is still a box constrained QP since it contains only polyhedral constraints. In [Haslinger et al., 2012] the authors propose a successive approximation technique in 3D with the special solver of [Kučera, 2007, 2008] which is itself an extension to disk constraints of the Polyak method (conjugate gradient with active set on the bounds constraint) and its improvements [Dostál, 1997, Dostál and Schöberl, 2005]. Other improvements of the method may be found in [Dostál and Kučera, 2010] with a last improvement of the method in [Dostál and Kozubek, 2012]. All this work is summarized and details in [Dostál, 2016]

7.3 ACLM approach

In the convex SOCCP approach described in Section 3.4, we have to solve for a given value the following problem

$$\begin{cases} \min & \frac{1}{2} r^\top W r + r^\top (q+s) \\ \text{s.t.} & r \in K. \end{cases} \qquad (132) \quad \texttt{\{eq:ACLM-4\}}$$

which is again a convex quadratic program over second–order cone. The approach listed below could again be used to solve this problem. In this work, we solve it by three different ways: a) an adaptation of one of the splitting techniques detailed in Section 6, b) using the projected gradient algorithm dedicated to convex QP described in 9 or c) the fixed point algorithm and the extra gradient algorithm of Section 4 . The algorithm is described in Algorithm 12 and we detailed the algorithms we use in the present study in Table 4.

Algorithm 12 ACLM approximation algorithm for Problem FC

Require: $\mathsf{W}, \mathsf{q}, \mu$
Require: r_0 initial values
Require: $\mathsf{tol} > 0, \mathsf{tol}_{\mathsf{int}}$ tolerance values and $\mathsf{iter}_{\max} > 0$ the max number of iterations
Ensure: r, u solution of Problem FC
  $\mathsf{u}_0 \leftarrow \mathsf{W}\mathsf{r}_0 + \mathsf{q}$; $\mathsf{k} \leftarrow 0$
  while error $> \mathsf{tol}$ and $\mathsf{k} < \mathsf{iter}_{\max}$ **do**
    $\mathsf{s} \leftarrow \mathsf{g}(\mathsf{u}_\mathsf{k})$
    solve (132) for r_{k+1} at accuracy $\mathsf{tol}_{\mathsf{int}}$
    $\mathsf{u}_{\mathsf{k}+1} \leftarrow \mathsf{W}\mathsf{r}_{\mathsf{k}+1} + \mathsf{q}$
    $\mathsf{k} \leftarrow \mathsf{k} + 1$
    evaluate error.
  end while
  $\mathsf{r} \leftarrow \mathsf{r}_\mathsf{k}$
  $\mathsf{u} \leftarrow \mathsf{u}_\mathsf{k}$

A nomenclature for the algorithms based on the optimisation approach is given in Table 4.

7.4 Convex relaxation and the SOCCP approach

Finally, we propose to compare the optimization based algorithm to a complete convex relaxation of the problem by solving the convex SOCCP (132) with $s = 0$. This procedure is very similar to the approach in [Tasora and Anitescu, 2009, Anitescu and Tasora, 2010, Tasora and Anitescu, 2011] where the convex problem is only solved.

Name	Algo.	Additional informations
PANA-PGS-FP-VI-UPK	10	The normal problem is solved by a PGS algorithm and the tangent problem is solved with the FP-VI-UPK algorithm
PANA-PGS-EG-VI-UPK	10	The normal problem is solved by a PGS algorithm and the tangent problem is solved with the EG-VI-UPK algorithm
PANA-PGS-CONVEXQP-PG	10 &9	The normal problem is solved by a PGS algorithm and the tangent problem is solved with Algorithm 9
PANA-CONVEXQP-PG	10&9	Both normal and tangent problems are solved with Algorithm 9
TRESCA-NSGS-FP-VI-UPK	11	The problem 131 is solved with the FP-VI-UPK algorithm
TRESCA-FP-VI-UPK	11 & 1	The problem 131 is solved with the FP-VI-UPK algorithm
TRESCA-EG-VI-UPK	11 & 2	The problem 131 is solved with the EG-VI-UPK algorithm
TRESCA-CONVEXQP-PG	11 &9	The problem 131 is solved with Algorithm 9
ACLM-NSGS-FP-VI-UPK	12	The problem 132 is solved with the NSGS-FP-VI-UPK algorithm
ACLM-FP-VI-UPK	12& 1	The problem 132 is solved with the FP-VI-UPK algorithm
ACLM-EG-VI-UPK & 2	12	The problem 132 is solved with the EG-VI-UPK algorithm
ACLM-CONVEXQP-PG	12	The problem 132 is solved with the Algorithm 9

Table 4: Naming convention for optimization based algorithms

7.5 Control of the tolerance of internal solvers $\mathsf{tol}_{\mathsf{int}}$ in optimization approach

In Algorithms 10, 11 and 12, an internal tolerance is used to control the accuracy of the internal solver. It is generally not useful to solve the internal problem at the accuracy of the global one. In the comparison study, we set the internal tolerance $\mathsf{tol}_{\mathsf{int}}$ to $error/10.0$.

8 Comparison framework

In this section, we present our comparison framework. Especially, we specify how we measure performance and how the performance profile are built.

8.1 Measuring errors

A key parameter in the measurement of performance of the solver is the definition of errors. The absolute error is given by the norm of the natural map. A relative error is computed with respect to the norm of the vector q. More precisely, the error is given by

$$\text{error} = \frac{\|F_{\text{vi}}^{\textsf{nat}}(r)\|}{\|q\|}. \tag{133}$$ {eq:error-1}

This error is used as the final error measurement for all solvers.

For some iterative solvers such as VI-FP, VI-EG, NSGS and PSOR, the computation of the error at each iteration penalizes the performance of the solver since it amounts to computing a matrix-vector product that costs more that one iteration. This is the reason why a cheaper error measurement is used inside the main loop in Algorithms 1, 2 and 7. This cheaper error measurement is based on

$$\text{error}_{\text{cheap}} = \frac{\|r_{k+1} - r_k\|}{\|r_k\|}. \tag{134}$$ {eq:error-2}

The tolerance of solver is then self-adapted in the loop to meet the required tolerance based on the error given by (133).

8.2 Performance profiles

The concept of performance profiles was introduced in [Dolan and Moré, 2002] for bench-marking optimization solvers. It enables to comparison of several solvers on a large set of problems. For a set P of n_p problems, and a set S of n_s solvers, we define a performance criterion for a solver s and a problem p by

$$t_{p,s} = \text{computing time required for solver } s \text{ to solve problem } p\,, \tag{135}$$ {eq:measure}

and a performance ratio over all the solvers

$$r_{p,s} = \frac{t_{p,s}}{\min\{t_{p,s},\ s \in S\}} \geqslant 1\,. \tag{136}$$

For $\tau \geqslant 1$, we define a distribution function ρ_s for the performance ratio for a solver s as

$$\rho_s(\tau) = \frac{1}{n_p}\text{size}\{p \in P,\ r_{p,s} \leqslant \tau\} \leqslant 1. \tag{137}$$

This distribution computes the number of problems p that are solved with a performance ratio below a given threshold τ. In other words, $\rho_s(\tau)$ represents the probability that the solver s has a performance ratio not larger than a factor τ of the best solver. It is worth noting that $\rho_s(1)$ represents the probability that the solver s beats the other solvers, and $\rho_s(\infty)$ characterizes the robustness of the method. The higher ρ_s is, the better the method is. The term *performance profile* will be used in the sequel for this graph of the functions $\rho_s(\tau), \tau \geqslant 1$.

In (135), the computational time is used to measure performance. Other criterion can be used. The number of floating point operations (flops) is a better measure of performance since it is independent of the computer. Unfortunately, it is usually difficult to measure automatically it in a robust over various platform. This is the reason why we prefer use the computational time. In our experiments, we decided to fix the required accuracy with the tolerance of each solver. Another performance criteria could be also used. For instance, we can fix a given computational time and measure to obtained error of the solver. This a way to measure the ability of solver to give a solution in a given time that may be interesting for real-time applications. Another way to measure performance may also to divide the computational time by the number of contacts in order to judge of the ability of the solver to be scalable. For the sake of conciseness, this has not been done in this article.

8.3 Benchmarks presentation

To perform the comparison of the solvers on a fair basis, we use a large set of problems that comes from various applications. This collection is FCLib (Frictional Contact libraries) which is an open collection of problems in a hdf5 format described in [Acary et al., 2014] and more information can be found at https://frictionalcontactlibrary.github.io. The whole collection of problems can be found at https://github.com/FrictionalContactLibrary/fclib-library and we used the version `v1.0` for the comparisons that contains 2368 problems.

The test sets are illustrated in Figure 3 ans some details on their contents are given in Table 5. All the problems has been generated thanks to the software codes LMGC90 and Siconos. In Table 5, the number of degrees of freedom n corresponds to the degrees of freedom of the systems before its condensation to local variables. In other words, the number of rows of the matrix M and H in (5). The contact density c is the ratio of the number of contact unknowns over the number of degrees of freedom:

$$c = \frac{3n_c}{n} = \frac{m}{n}. \qquad (138)$$ {eq:fclib-1}

The coefficient c corresponds also to the ratio between the number of rows of H over its number of columns. If this number is larger than 1, the matrix H can not be full row rank and then the matrix W is also rank deficient. In the case that $m > n$, we can observe in Table 5 that this number c is a good approximation of the rank ratio of the matrix W in our applications. The estimation of the rank of matrix W shows that is is very close to the number of degrees of freedom of the system when $c > 1$. For $c >> 1$, the contact density is really high and the system suffers from hyperstaticity as we discussed in Section 2.3. In Table 5, we also give an estimation of the conditioning of the matrix W. When is was possible from a computational point of view, we perform a singular value decomposition (SVD) of the matrix W to estimate the spectral radius and then the conditioning by cutting the small eigenvalues. This process has two drawbacks. Firstly, the computation of the SVD decomposition can ba really expensive for large dense matrix. Secondly, the value of the condition number of the matrix is very sensitive to the threshold for cutting off the small eigenvalues renders. This is we also use LSMR [Fong and Saunders, 2011] algorithm to give an better approximation of the condition number of rank deficient matrix.

The four first tests in 5 are examples that involve flexible elastic bodies meshed by element methods. The matrix W is that case is full rank. We will call this set of examples the flexible test set in the sequel.

Redaction note V.A. 8.1.

Can we say more on the description of the examples ?

8.4 Sofware & implementation details

All the solvers that are used in this article are implemented in standard C in the component of Siconos called Siconos/Numerics. The aim of Siconos is to provide a common platform for the modeling, simulation, analysis and control of general nonsmooth dynamical systems. More information on the software is available at http://siconos.gforge.inria.fr and the software can be downloaded at https://github.com/siconos/siconos. The algebraic manipulations are based on BLAS/LAPACK and we use SuiteSparse for the implementation of sparse matrices. The algorithm VI-FP, VI-EG, NSGS and PSOR uses the sparse block structure of the matrix W. The NSN solvers relies on a standard sparse implementation. The simulations are performed on the University of Grenoble-Alpes cluster CIMENT[3].

Redaction note V.A. 8.2.

something more ?

8.5 Simulation campaign

In Sections 9 and 10, we report the results for the simulation campaign. Since the large amount of dataand the numerous of performance profiles for each test sets that are generated, we do not report any profiles when a list of solvers fails or is not able to meet the required tolerance within the prescribed time defined by the timeout values. Nevertheless, the reader may have access to the complete list of performance profiles at https://github.com/siconos/faf/blob/master/TeX/Full-test/full-test_current.pdf.

Redaction note V.A. 8.3.

a permanent address would be better

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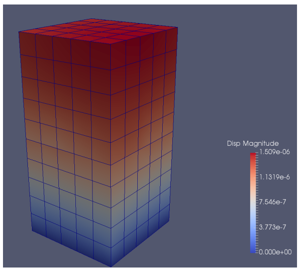
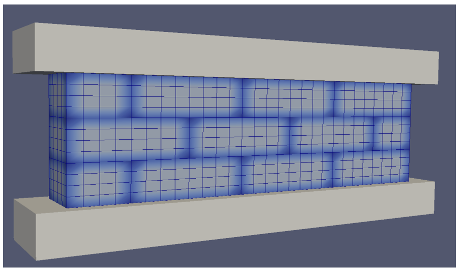
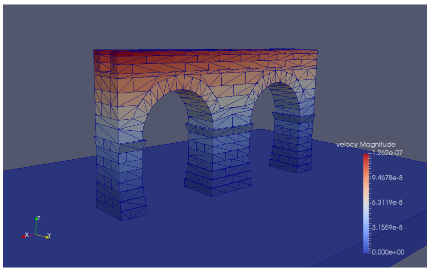
[3]https://ciment-grid.ujf-grenoble.fr/



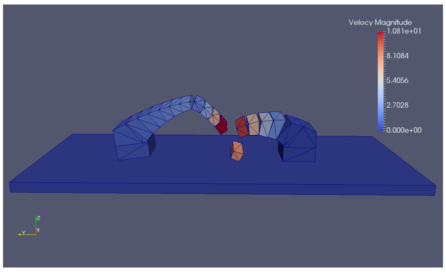
(a) Cubes_H8



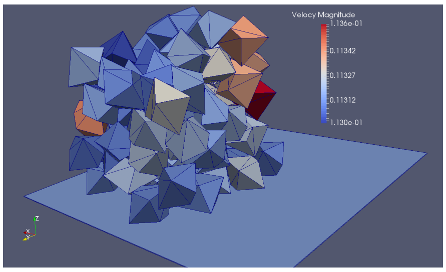
(b) LowWall_FEM



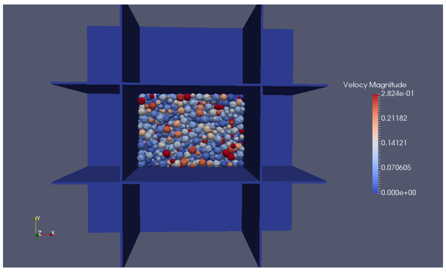
(c) Aqueduct_PR



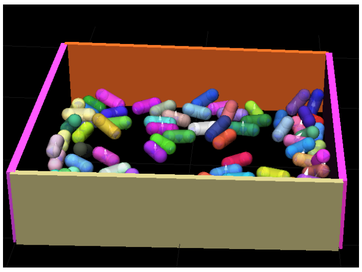
(d) Bridge_PR



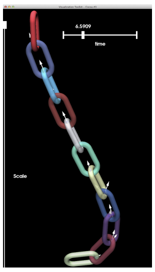
(e) 100_PR_Periobox



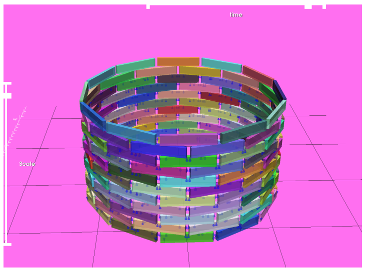
(f) 945_SP_Box_PL



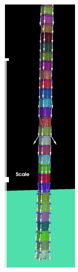
(g) Capsules



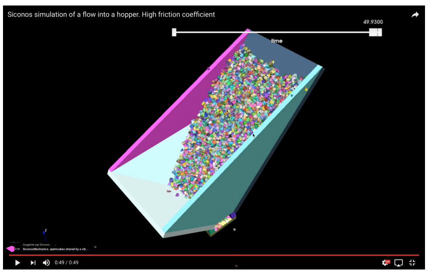
(h) Chain



(i) KaplasTower



(j) BoxesStack



(k) Chute_1000, Chute_4000, Chute_local_problems

Figure 3: Illustrations of the FClib test problems

Test set	code	friction coefficient μ	# of problems	# of d.o.f.	# of contacts	contact density c	rank ratio(W)	cond(W)	cond(W) LSMR
Cubes_H8_2	LMGC90	0.3	15	162	[3:5]	[0.02:0.09]	1	$[2.2.10^1, 1.3.10^3]$	$[8.1.10^5, 1.5.10^6]$
Cubes_H8_5	LMGC90	0.3	50	1296	[17:36]	[0.02:0.09]	1	$[3.3.10^4, 7.2.10^4]$	$[1.3.10^6, 3.1.10^6]$
Cubes_H8_20	LMGC90	0.3	50	55566	[361:388]	[0.019:0.021]	1	$[2.4.10^5, 2.5.10^5]$	$[1.3.10^6, 5.2.10^6]$
LowWall_FEM	LMGC90	0.83	50	{7212}	[624:688]	[0.28:0.29]	1	–	$[9.3.10^2, 5.0.10^5]$
Aqueduct_PR	LMGC90	0.8	10	{1932}	[4337:4811]	[6.81:7.47]	[6.80:7.46]	$[4.7.10^7, 3.4.10^8]$	$[6.7.10^1, 1.5.10^2]$
Bridge_PR	LMGC90	0.9	50	{138}	[70:108]	[1.5:2.3]	[2.27,2.45]	$[8.3.10^4, 1.1.10^5]$	$[1.9.10^3, 2.6.10^4]$
100_PR_Periobox	LMGC90	0.8	106	{606}	[14:578]	[0.2:3]	[1.76:3.215]	$[4.3.10^2, 1.0.10^6]$	$[6.3.10^5, 3.5.10^6]$
945_SP_Box_PL	LMGC90	0.8	60	{5700}	[2322:5037]	[1.22:2.65]	[1.0, 2.66]	$[2.2.10^4, 4.4.10^5]$	$[2.9.10^1, 9.2.10^2]$
Capsules	Siconos	0.7	249	{300}	[1:200]	[1.2:1.5]	[1.08:1.55]	$[1.2.10^6, 7.5.10^9]$	–
Chain	Siconos	0.3	242	{60}	[8:28]	[0.5:1.3]	[1.05:1.6]	$[7.4.10^4, 4.0.10^9]$	$[1.5.10^1, 4.7.10^5]$
KaplasTower	Siconos	0.7	201	[72:792]	[48:933]	[3.0:3.6]	[2.0:3.53]	[67 : 2174]	[8 : 67]
BoxesStack	Siconos	0.7	255	[6:300]	[1:200]	[1.86:2.00]	[1.875,2.0]	$[3.8.10^4, 2.5.10^7]$	$[9.0, 5.4.10^3]$
Chute_1000	Siconos	1.0	156	[276:5508]	[74:5056]	[0.69:2.95]	[1.0:2.95]	$[2.1.10^1, 1.9.10^3]$	
Chute_4000	Siconos	1.0	40	[17280:20034]	[15965:19795]	[2.51:3.06]	–	–	$[5.5.10^1, 9.0.10^3]$
Chute_local_problems	Siconos	1.0	834	3	1	1	1	[1.04:4.66]	$[2.6, 2.6.10^1]$

Table 5: Description of the test sets of FCLib library (v1.0)

Test set	precision	timeout (s)	mean performance of the fastest solver $\mu\{\min\{t_{p,s}, s \in S\}\}$	std. deviation performance of the fastest solver $\sigma(\min\{t_{p,s}, s \in S\})$	mean performance of the fastest solver by contact $\mu\{\min\{t_{p,s}/n_{c,p}, s \in S\}\}$	std. deviation performance of the fastest solver by contact $\sigma(\min\{t_{p,s}/n_{c,p}, s \in S\})$	# of unsolved problems
Cubes_H8_$\star$	10^{-08}	100	1.73	2.13	4.83^{-03}	5.78^{-03}	0
Cubes_H8_$\star$ II	10^{-04}	100	0.92	1.06	2.66^{-03}	2.83^{-03}	0
LowWall_FEM	10^{-04}	400	14.8	2.85	2.16^{-02}	4.54^{-03}	0
Aqueduct_PR	10^{-04}	200	5.80	6.36	4.90^{-04}	3.03^{-04}	0
Bridge_PR	10^{-08}	100	11.0	13.6	1.05^{-01}	1.33^{-01}	0
Bridge_PR II	10^{-04}	100	0.048	0.038	1.30^{-03}	1.42^{-03}	0
100_PR_Periobox	10^{-08}	100	34.0	34.5	1.78^{-01}	1.86^{-01}	69
100_PR_Periobox II	10^{-04}	100	0.064	0.062	1.56^{-04}	1.22^{-04}	0
945_SP_Box_PL	10^{-04}	100	3.20	1.71	6.45^{-04}	3.36^{-04}	0
Capsules	10^{-08}	100	0.010	0.012	4.03^{-05}	4.41^{-05}	0
Chain	10^{-08}	50	$6.19.10^{-04}$	$3.68.10^{-04}$	$3.15.10^{-05}$	$1.46.10^{-05}$	0
KaplasTower	10^{-08}	200	$1.27.10^{-01}$	$3.75.10^{-01}$	$1.84.10^{-04}$	$4.57.10^{-04}$	0
KaplasTower II	10^{-04}	200	$2.41.10^{-02}$	$1.27.10^{-01}$	$3.39.10^{-05}$	$1.55.10^{-04}$	0
BoxesStack	10^{-08}	100	$3.42.10^{-02}$	$8.87.10^{-02}$	$3.24.10^{-04}$	$9.77.10^{-04}$	0
Chute_1000	10^{-04}	200	2.62	3.06	6.76^{-04}	6.58^{-04}	0
Chute_4000	10^{-04}	200	10.52	7.88	5.71^{-04}	4.07^{-04}	0

Table 6: Parameters of the simulation campaign

9 Comparison of methods by family

In this section, we perform a comparison of the solvers by family in order to study the influence of the various possible parameters on the performance of the solvers.

9.1 Numerical methods for VI: FP-DS, FP-VI-$\star$ and FP-EG-$\star$

Evaluation of the influence of the self–adaptive procedure for step length In Figure 4, we compare the different numerical solver for VI. Only the test sets for which the solvers are reached the precicion before timeout are presented. For that reason, the results for the test sets LowWall_FEM, Cubes_H8, Bridge_PR, AqueducPR, 945_SP_Box_PL, 100_PR_PerioBox, Chute_4000 and Chute_1000 are not presented. The solvers are quite low in practice for a large number of contacts and they failed to meet the required tolerance in time for the examples that are not depicted in Figure 4. The main conclusions are as follows:

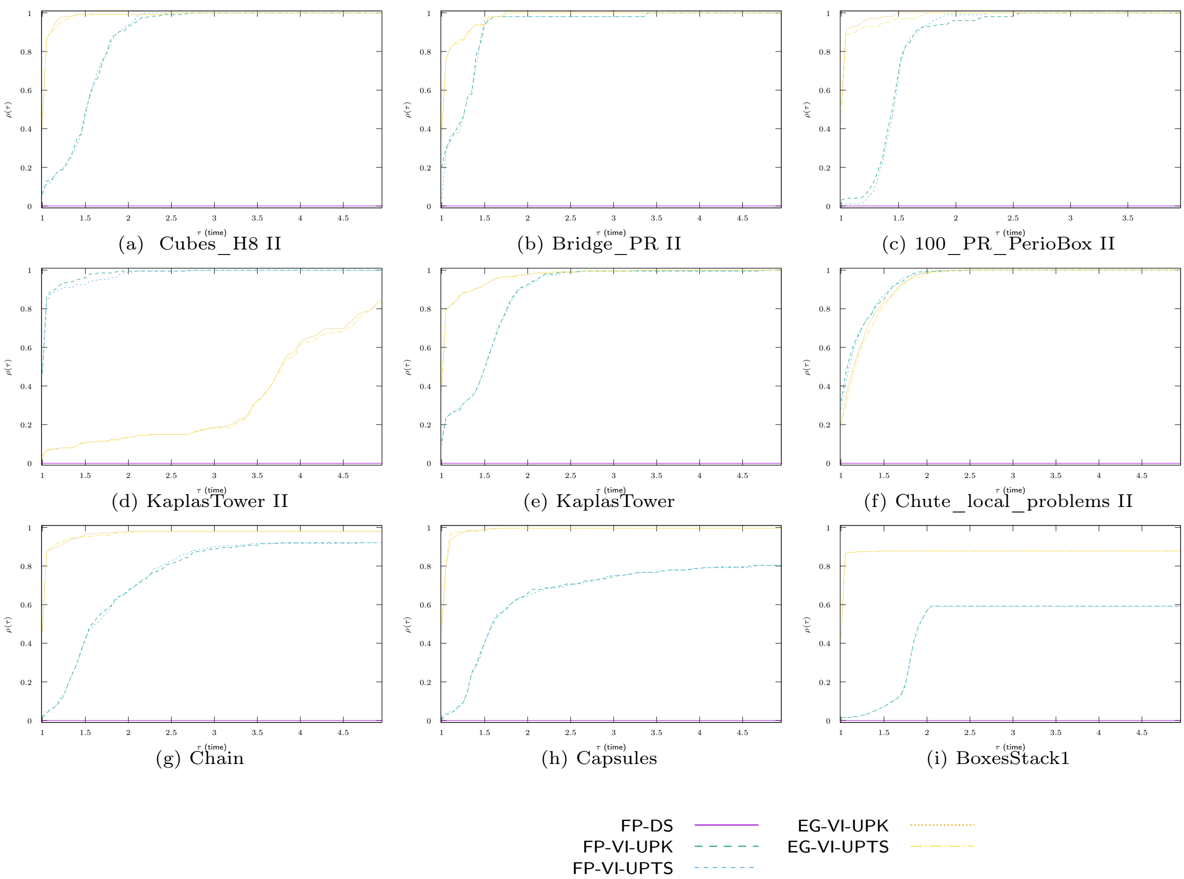
1. The solver FP-DS suffers from robustness problems and a lot of divergence has been observed in practice. The self-adaptive rule for size the paramater ρ_k is of utmost importance.
2. The solvers FP-VI-$\star$ and FP-EG-$\star$ are really robust but slow. They are able to solve all the problems but they require a lot of time and we didn't observe divergence issues.
3. Except for the test set KaplasTower II, the FP-EG-$\star$ performs better than FP-VI-$\star$.
4. The difference between the UPK and UPTS is very small in all the examples and the choice of the update rule is not fundamentally important.

9.2 Splitting based algorithms: NSGS-$\star$ and PSOR-$\star$

Again the results on the test sets Bridge_PR, 100_PR_PerioBox, Chute_local_problems are not reported since the NSGS-$\star$ are not able to meet the required tolerance for a sufficient percentage of the problems mainly due to a slow convergence rate. We can see that this problem also appears in the Cubes_H8 and the BoxesStack test sets.

Effect and influence of the local solver in NSGS-$\star$ algorithms In Figure 5, we report the performance profiles of the NSGS-$\star$ for the different local solvers. The main conclusions are as follows:

1. When the timeout value is sufficiently large, we observe the NSGS-$\star$ are quite robust solvers. In other words, the convergence is slow but robust. Indeed, we are able to find to local solver for each test sets that is able to give a solution at the required accuracy. One of the problem is the fact that there is no universal efficient local solver and the global efficiency depends on the local solver in various ways depending on the test sets.



(a) Cubes_H8 II (b) Bridge_PR II (c) 100_PR_PerioBox II (d) KaplasTower II (e) KaplasTower (f) Chute_local_problems II (g) Chain (h) Capsules (i) BoxesStack1

Figure 4: Comparison of numerical method for VI FP-DS, FP-VI-$\star$ and FP-EG-$\star$

2. Except for the test sets KaplasTower II and BoxesStack, the solver NSGS-EXACT behaves poorly. This is mainly due to the fact that the solver is not robust to find a solution for one contact when we are far from the solutions for all the contacts. This behavior was already reported in [Daviet et al., 2011] where another solver based a nonsmooth Newton technique was used when the exact solution was not satisfactory.

3. The solver NSGS-FP-DS-One is most efficient on the test sets Bridge_PR II, KaplasTower II, Chain and BoxesStack. These tests sets seem to be among the easiest one the solve. Nevertheless, the solver seems slow or suffers from robustness issues for other test sets.

4. On the flexible test sets and 945_SP_Box_PL, the best solver is NSGS-FP-VI-UPK for a relatively low required tolerance ($\mathsf{tol}_{\mathsf{local}} = 10^{-06}$). For these test sets, an approximate solution of the single contact problems seems sufficient to ensure a efficient convergence towards the solution without entailing robustness.

5. On the test sets 100_PR_PerioBox II, KaplasTower, Chain, Capsules, the solver NSGS-NSN-$\star$ are the best solvers and behave very well on Bridge_PR II. It seems that when a tight accuracy is required, the use of NSGS-NSN-$\star$ are useful.

6. For the Chute_1000, Chute_4000 test sets, we observe large difference between the local formulation of the nonsmooth equations for the Newton solvers. The solver $\mathsf{NSGS}-\mathsf{NSN}-\mathsf{JM}$ is the best solver and really better than $\mathsf{NSGS}-\mathsf{NSN}-\mathsf{AC}$ although their theoretical formulation are very close. Theses two test sets are characterized by difficult local problems where the matrix W is asymmetric.

7. `Comment on the line searches method`

Influence of the tolerance of the local solver tol_{local} **in NSGS-FP-VI-UPK algorithms** In this section, the tolerance of the local solver is varied and its effect on the global convergence of the solver is reported. In Figure 6, we report the performance profile of NSGS-FP-VI-UPK algorithms for the $\mathsf{tol}_{\mathsf{local}}$ in the range $[10^{-04}, 10^{-16}]$. We also report the efficiency of the adaptive value of the local tolerance. The main observations are

1. For the test sets that are solved a a quicker way, such as BoxesStack, Capsules, Chain, Chute_local_problems, KaplasTower, KaplasTower II, a tight tolerance on the local solver 10^{-16} improves the efficiency and the global convergence of the NSGS-FP-VI-UPK solver. It seems that these problems correspond to the problem that are solved quite quickly (see Table 6)

2. For the other problems that are longer to solve, that is when we expect more iterations of the NSGS-FP-VI-UPK solver, the adaptive rule or low local tolerances are better.

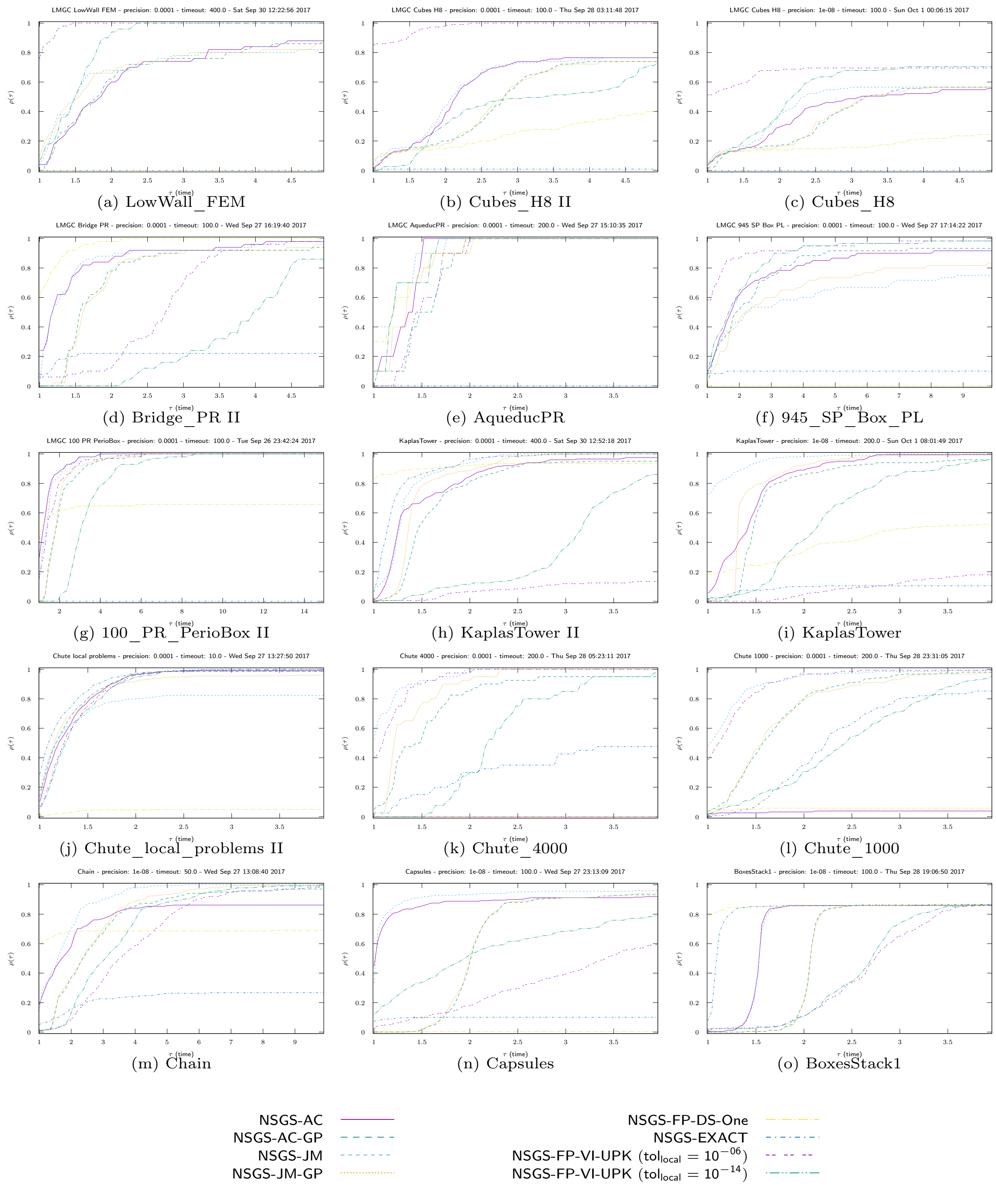


Figure 5: Influence of the local solver in NSGS-$\star$ algorithms.

From these results, it is quite difficult to guess in advance the internal dynamics of the solver. Note that the range of τ that we used in the graph is quite small so the difference in performance between the solvers are not crucial.

Influence of the tolerance of the local solver tol_{local} in NSGS-NSN-AC-GP algorithms In Figure 7, we report the performance profile of NSGS-NSN-AC-GP algorithms for the $\mathsf{tol}_{\mathsf{local}}$ in the range $[10^{-04}, 10^{-16}]$. We also report the efficiency of the adaptive value of the local tolerance. Except for the test set Chute_local_problems, the main observation is that the local tolerance does not change fundamentally the convergence of the solver. For the test set Chute_local_problems, the is no internal dynamics of convergence of the loop on the contact since we have only one contact. It is therefore normal to see that adaptive performs better that the other.

Influence of the choice of the parameters $\rho_{\mathrm{N}}, \rho_{\mathrm{T}}$ in the local solver of NSGS-NSN-$\star$ algorithms In Figure 8, we evaluate the influence of the choice of the parameters $\rho_{\mathrm{N}}, \rho_{\mathrm{T}}$ on the convergence of the solver. The main conclusion are:

1. For the test sets 945_SP_Box_PL, 100_PR_PerioBox, KaplasTower II, KaplasTower, Chute_local_problems, Chute_local_problems II, Chute_4000, Chute_1000, Capsules, a fixed value of $\rho_{\mathrm{N}} = \rho_{\mathrm{T}} = 1$ has a dramatic effect on the convergence of the algorithm. The scaling of ρ is of utmost importance for the efficiency and the robustness of the solver. Note that the rule (112) that takes into account the condition number of the local matrix w deteriorates the performance for $Chute\_local\_problems, Chute\_local\_problems1$ In this problem, the local matrix is asymmetric due to large Coriolis effects in the simulation.

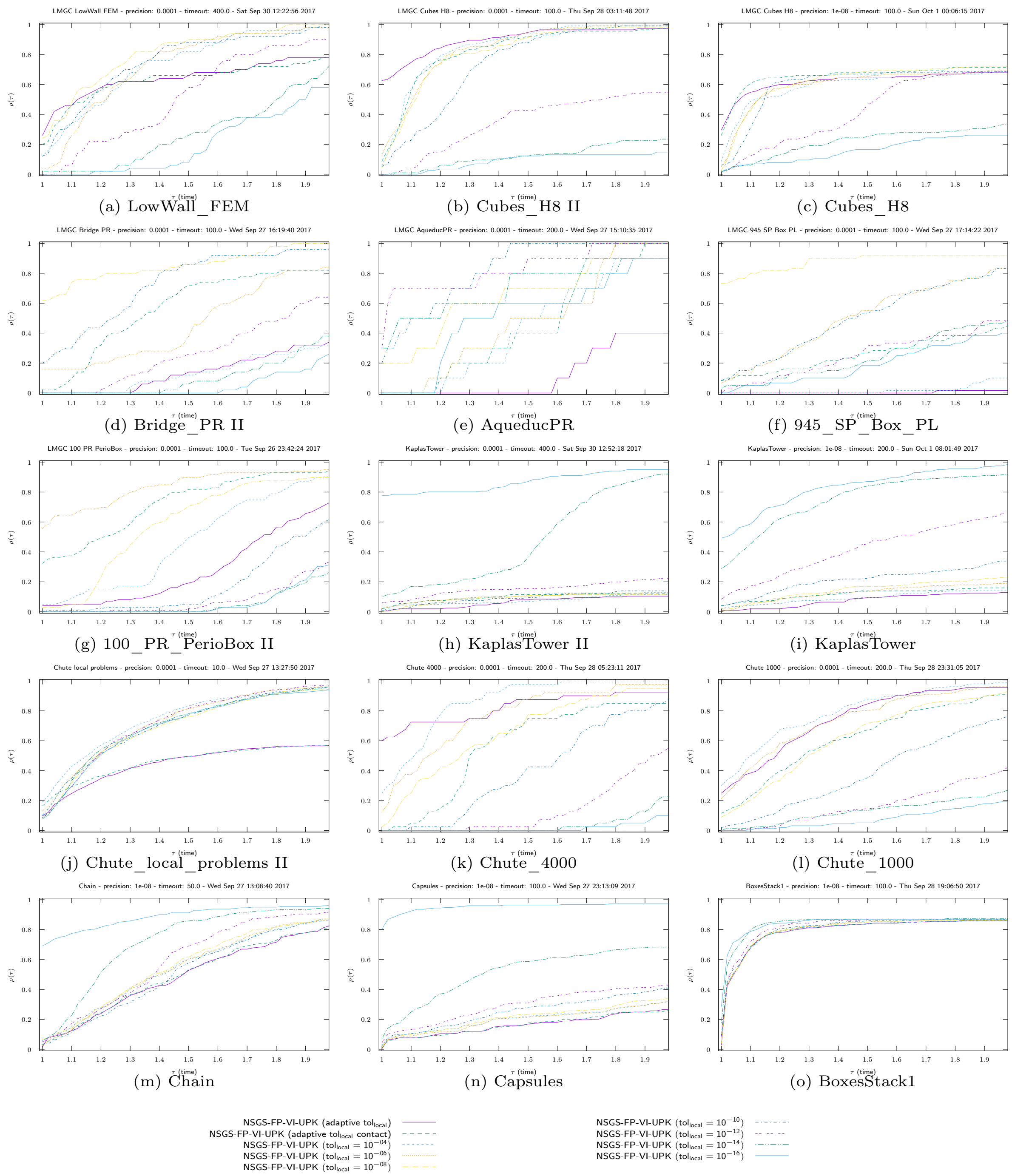
2. For the other tests, the choice of $\rho_{\mathrm{N}}, \rho_{\mathrm{T}}$ does not really change the results, mainly due the fact that the order of magnitude of the chosen ρ with the rules (110), (111) or (112) is around 1.

One of the conclusion of this study is as follows, the rules (110), (111) improves a lot some simulations without increasing the computational tie for the others. Therefore, we advise to use it.

Influence of the contacts order in NSGS algorithms In this section, we study the influence of the contact order within the loop of the NSGS-AC-QP solver. We reproduce in Figure **??** the result of the solvers with the original contact list of the problem NSGS-AC-GP with two other ways of iterating over the contacts. The solver NSGS-AC-GP Shuffled to a first randomization of the list of contacts at the beginning of the algorithm. In the solver NSGS-AC-GP Fully shuffled, the is shuffled at each iteration.

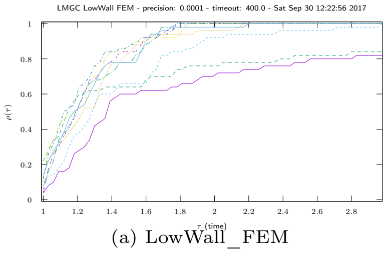
1. The solver NSGS-AC-GP Fully shuffled performs really better on the flexible test sets

2. For the rigid test sets, we reproduce here only the test set 100_PR_PerioBox II because the other test sets behave identically. The NSGS-AC-GP Fully shuffled has a really bad influence on the convergence of the solver. It seems that is modify the internal dynamics of the solver in a way the rate of convergence is really decreased.

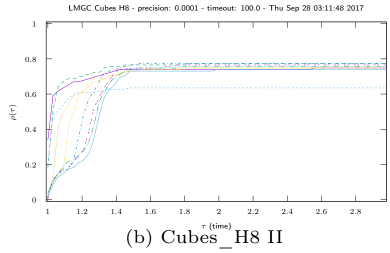


(a) LowWall_FEM (b) Cubes_H8 II (c) Cubes_H8
(d) Bridge_PR II (e) AqueducPR (f) 945_SP_Box_PL
(g) 100_PR_PerioBox II (h) KaplasTower II (i) KaplasTower
(j) Chute_local_problems II (k) Chute_4000 (l) Chute_1000
(m) Chain (n) Capsules (o) BoxesStack1

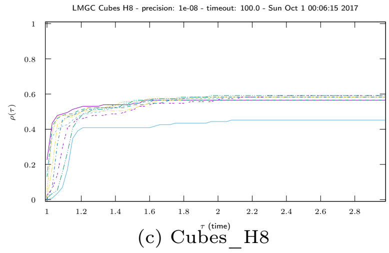
Figure 6: Influence of the tolerance of the local solver $\mathsf{tol_{local}}$ in NSGS-FP-VI-UPK algorithms.



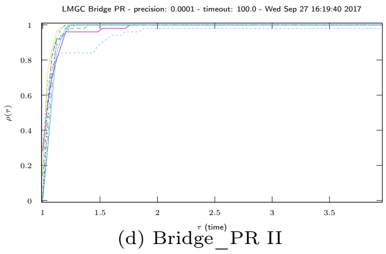
(a) LowWall_FEM



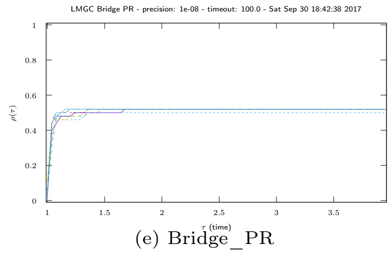
(b) Cubes_H8 II



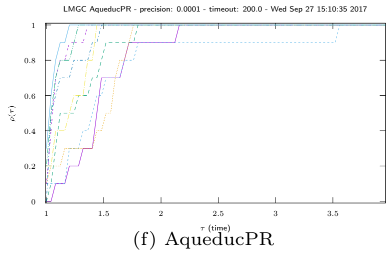
(c) Cubes_H8



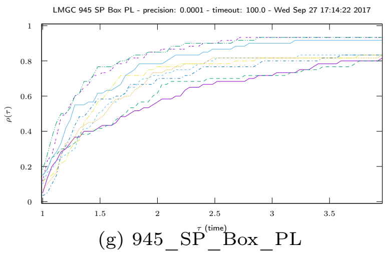
(d) Bridge_PR II



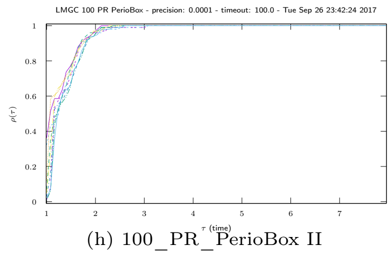
(e) Bridge_PR



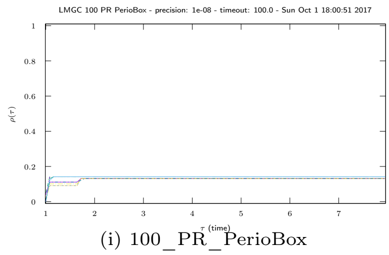
(f) AqueducPR



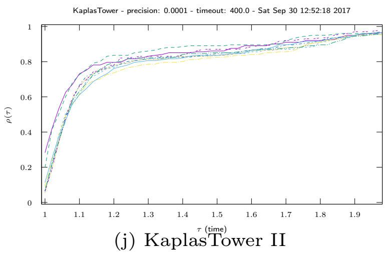
(g) 945_SP_Box_PL



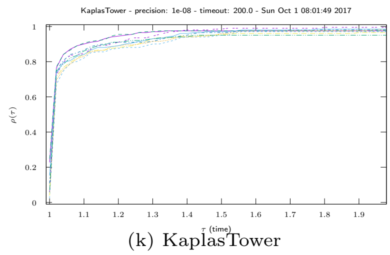
(h) 100_PR_PerioBox II



(i) 100_PR_PerioBox



(j) KaplasTower II



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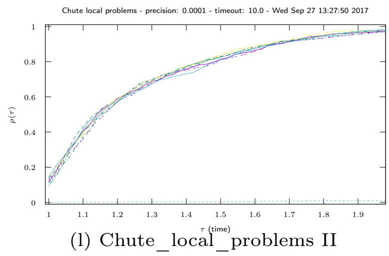
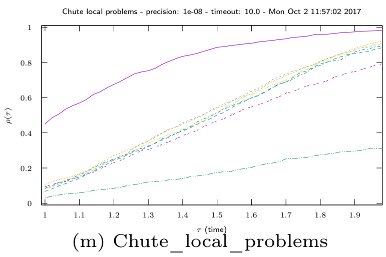
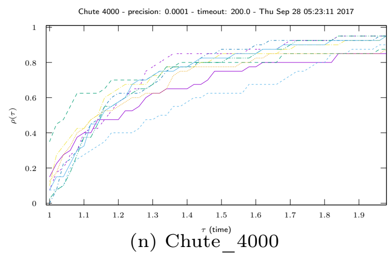


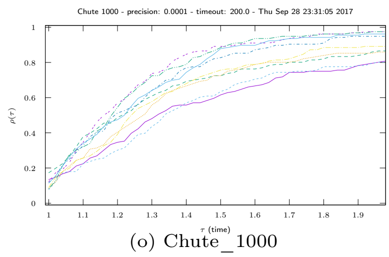
(l) Chute_local_problems II



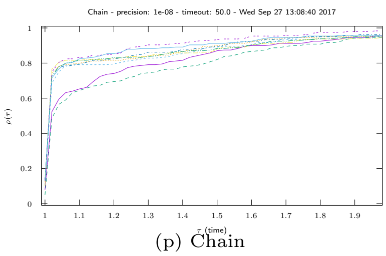
(m) Chute_local_problems



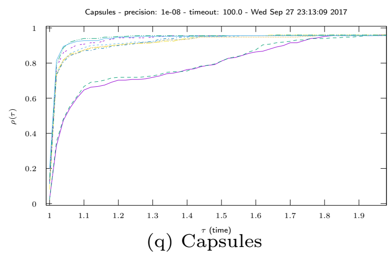
(n) Chute_4000



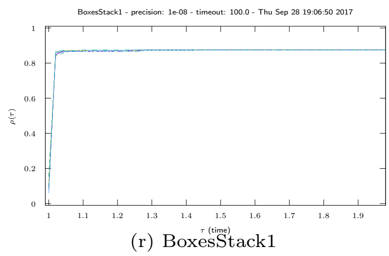
(o) Chute_1000



(p) Chain

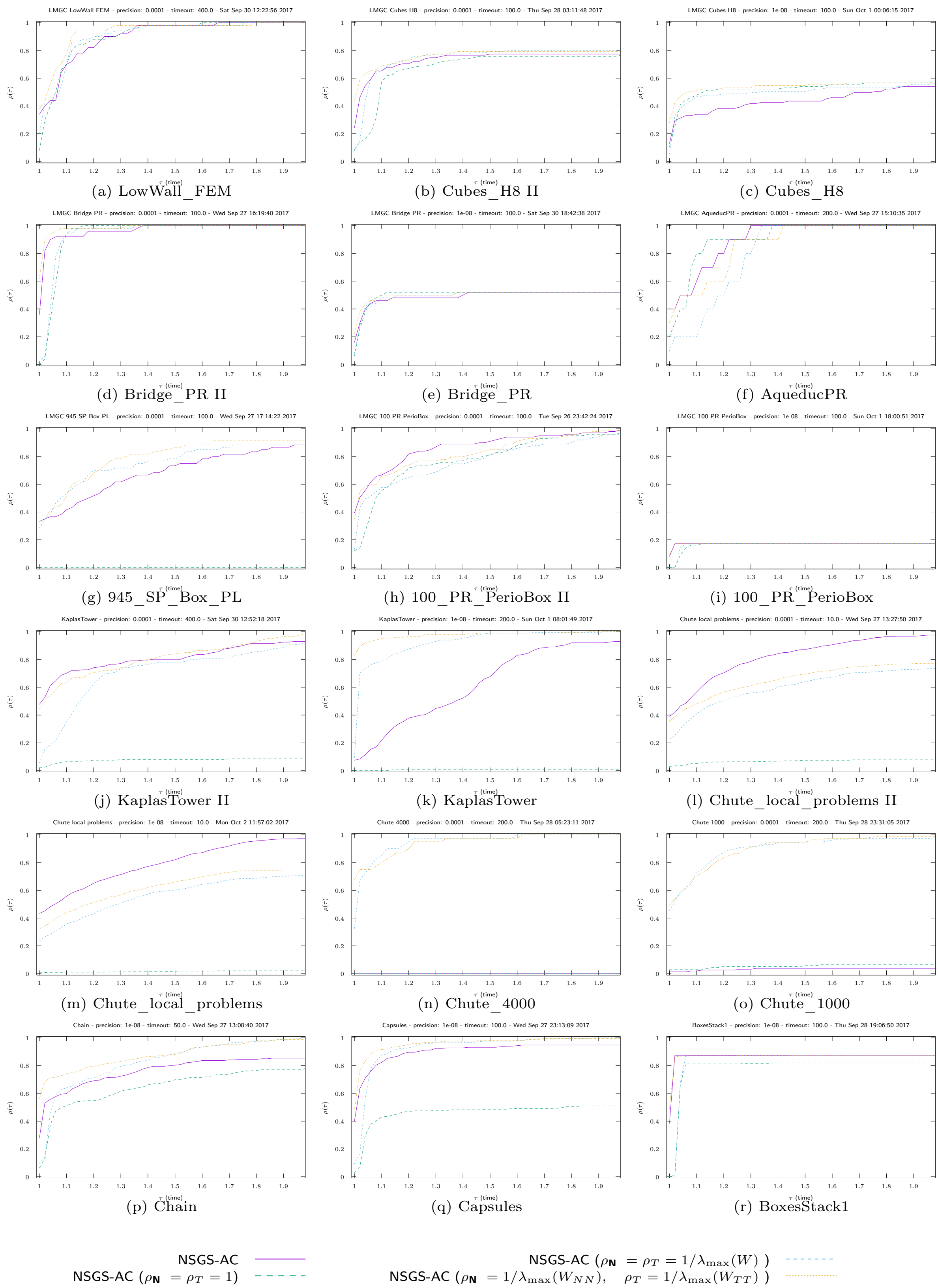


(q) Capsules



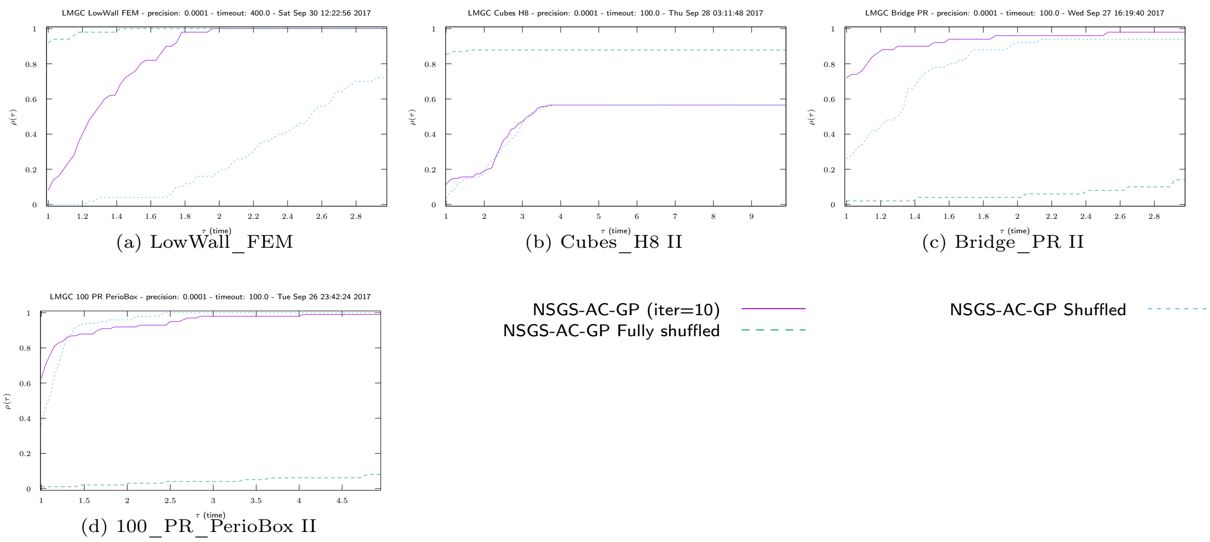
(r) BoxesStack1

Figure 7: Influence of the tolerance of the local solver $\mathsf{tol}_{\mathsf{local}}$ in NSGS-FP-NSN-AC-GP algorithms.



(a) LowWall_FEM (b) Cubes_H8 II (c) Cubes_H8
(d) Bridge_PR II (e) Bridge_PR (f) AqueducPR
(g) 945_SP_Box_PL (h) 100_PR_PerioBox II (i) 100_PR_PerioBox
(j) KaplasTower II (k) KaplasTower (l) Chute_local_problems II
(m) Chute_local_problems (n) Chute_4000 (o) Chute_1000
(p) Chain (q) Capsules (r) BoxesStack1

Figure 8: Influence of the tolerance of the local solver tol_{local} in NSGS-FP-VI-UPK algorithms.



(a) LowWall_FEM (b) Cubes_H8 II (c) Bridge_PR II

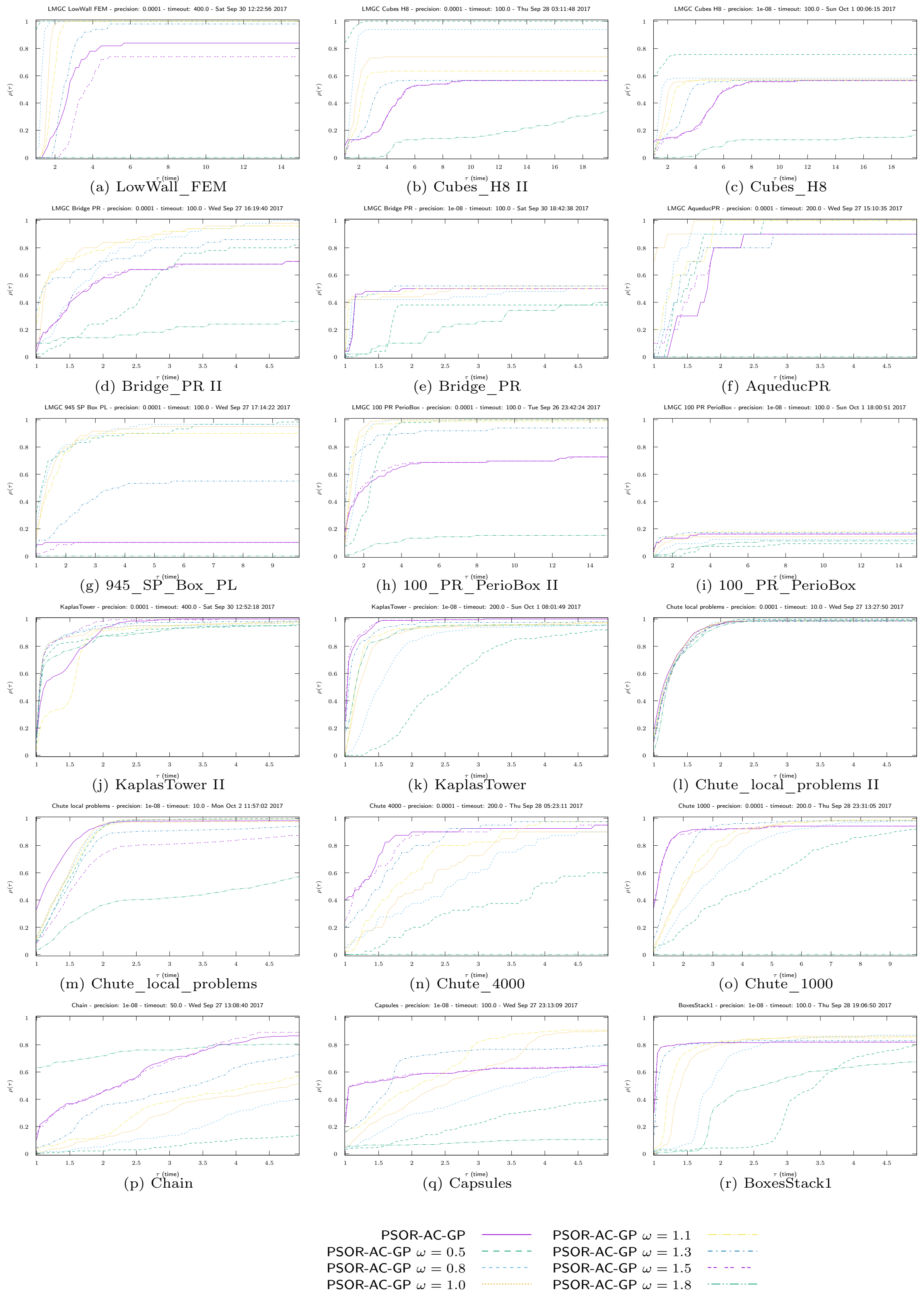
(d) 100_PR_PerioBox II

Figure 9: Influence of the contacts order in NSGS algorithms.

Comparison of PSOR algorithm with respect to the relaxation parameter ω In Figure 10, we study the effect of the relaxation parameter ω ranging from $[0.5, 1.8]$ on the performance. Two conclusions can be drawn

1. For the flexible tests, the efficiency of the solver is really improved as we decreased the value of ω. Moreover, this is done without destroyig the robustness of the solver.
2. for the rigid tests, the effect of the relaxation is not clear. For values of ω greater than 1.0, the efficiency is improved but the robustness deteriorates and we observe the contrary for the ω less than 1.0

To conclude, it is difficult to advice to use PSOR algorithm with $\omega \neq 1$. If it accelerates drastically the rate of convergence of the algorithm for some problems it deteriorates the convergence for other. Further studies would be needed to design self–adaptive schemes for the choice of ω.



(a) LowWall_FEM (b) Cubes_H8 II (c) Cubes_H8

(d) Bridge_PR II (e) Bridge_PR (f) AqueducPR

(g) 945_SP_Box_PL (h) 100_PR_PerioBox II (i) 100_PR_PerioBox

(j) KaplasTower II (k) KaplasTower (l) Chute_local_problems II

(m) Chute_local_problems (n) Chute_4000 (o) Chute_1000

(p) Chain (q) Capsules (r) BoxesStack1

Figure 10: Effect of relation coefficient ω in PSOR-AC-GP algorithm.

9.3 Comparison of NSN-$\star$ algorithms

9.4 Comparison of PPA-NSN-AC algorithm with respect to the step-size parameter σ, μ

Redaction note V.A. 9.1.

redo this comparison on a set of mixed examples

add some results with $\nu < 1$

---

10 Comparison of different families of solvers.

11 General conclusions

- NSGS-$\star$
  1. robust
  2. slow
  3. local solvers must preserve robutness.
- NSN-$\star$
  1. Choice of rho and robustness convergence. more studies

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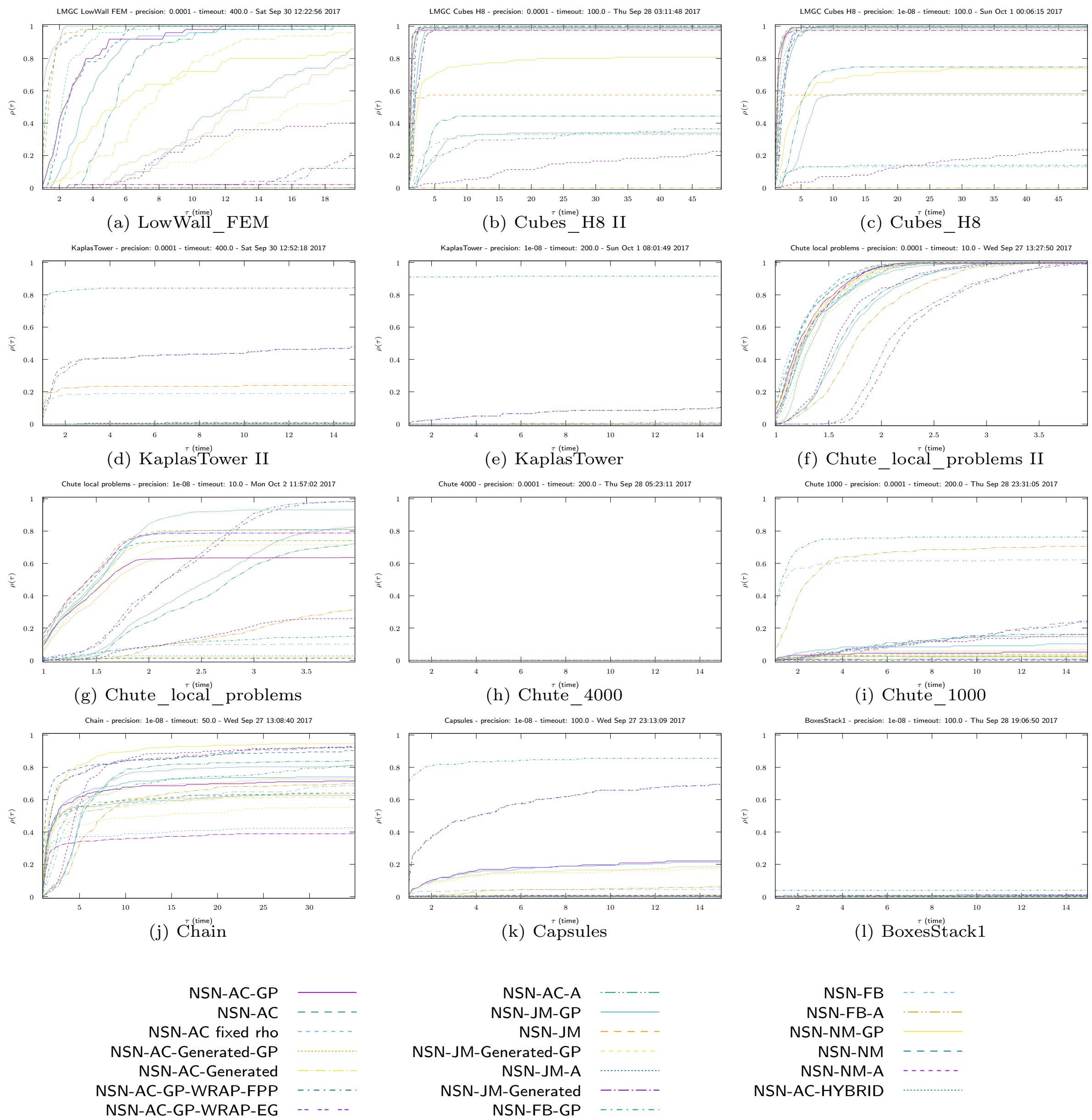
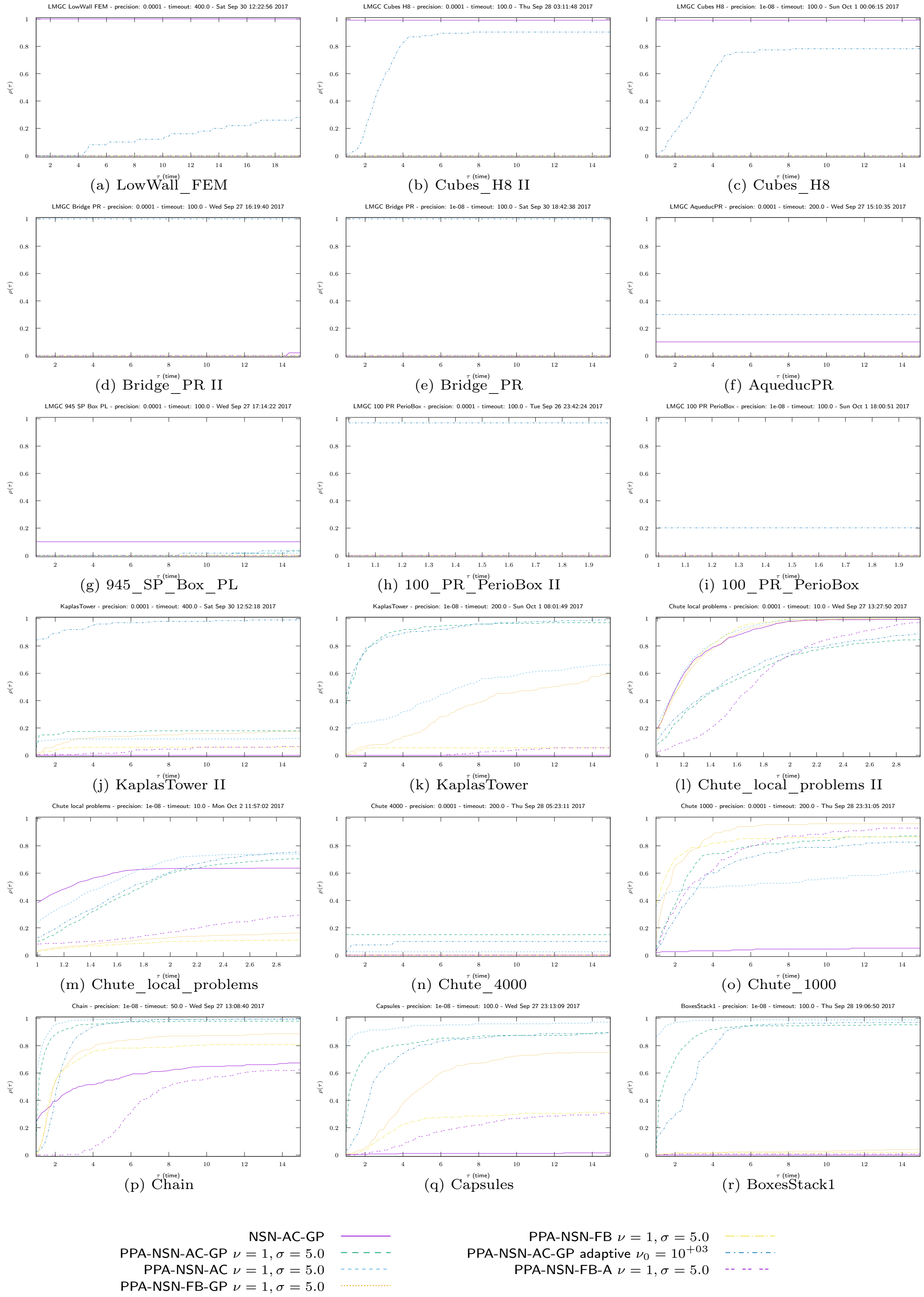


Figure 11: Comparison of NSN-$\star$ algorithms.



(a) LowWall_FEM (b) Cubes_H8 II (c) Cubes_H8
(d) Bridge_PR II (e) Bridge_PR (f) AqueducPR
(g) 945_SP_Box_PL (h) 100_PR_PerioBox II (i) 100_PR_PerioBox
(j) KaplasTower II (k) KaplasTower (l) Chute_local_problems II
(m) Chute_local_problems (n) Chute_4000 (o) Chute_1000
(p) Chain (q) Capsules (r) BoxesStack1

Figure 12: Comparison of internal solvers in PPA-$\star$ algorithms.

Figure 13: Effect of the step-size parameter σ, μ in PPA-NSN-AC algorithm

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A Basics in Convex Analysis

Definition 1 ([Rockafellar and Wets, 1997]). *Let $X \subseteq \mathbb{R}^n$. A multivalued (or point-to-set) mapping $T\colon X \rightrightarrows X$ is said to be (strictly) monotone if there exists $c(>) \geqslant 0$ such that for all $\hat{x}, \widetilde{x} \in X$*

$$(\hat{v} - \widetilde{v})^\top (\hat{x} - \widetilde{x}) \geqslant c\|\hat{x} - \widetilde{x}\| \qquad \text{with } \hat{v} \in T(\hat{x}), \widetilde{v} \in T(\widetilde{x}). \tag{139}$$

Moreover T is said to be maximal when it is not possible to add a pair (x, v) to the graph of T without destroying the monotonicity.

The Euclidean projector P_X onto a closed convex set X: for a vector $x \in \mathbb{R}^n$, the projected vector $z = P_X(x)$ is the unique solution of the convex quadratic programm

$$\begin{cases} \min \dfrac{1}{2}(y-x)^\top (y-x), \\ \\ \text{s.t.} \quad y \in X. \end{cases} \tag{140}$$ {eq:opt-proj}

$$\begin{aligned} y = P_K(x) &\Longleftrightarrow \begin{array}{ll} \min & \frac{1}{2}(y-x)^\top (y-x) \\ \text{s.t.} & y \in K \end{array} && (141) \\ &\Longleftrightarrow -(y-x) \in N_K(y) && (142) \\ &\Longleftrightarrow (x-y)^\top (y-z) \geqslant 0, \forall z \in K && (143) \end{aligned}$$ {eq:tutu}

$$\begin{aligned} -F(x) \in N_K(x) &\Longleftrightarrow -\rho F(x)^\top (y-x) \geqslant 0, \forall y \in K && (144) \\ &\Longleftrightarrow (x - (x - \rho F(x))^\top (y-x) \geqslant 0, \forall y \in K && (145) \\ &\Longleftrightarrow x = P_K(x - \rho F(x)) \text{ thanks to (143)} && (146) \end{aligned}$$

$$\partial \|z\|_2 = \begin{cases} \dfrac{z}{\|z\|} \\ \{x, \|x\| = 1\} \end{cases} \tag{147}$$ {eq:sub-norm}

A.1 Euclidean projection on the disk in $\mathbb{R}^2$.

Let $D = \{x \in \mathbb{R}^3, \|x\| \leqslant 1\}$. We have

$$P_D(z) = \begin{cases} z & \text{if} \quad z \in D \\ \dfrac{z}{\|z\|} & \text{if} \quad z \notin D \end{cases} \tag{148}$$ {eq:projD}

and

$$\partial P_D(z) = \begin{cases} I & \text{if} \quad z \in D \setminus \partial D \\ I + (s-1)zz^\top, s \in [0,1] & \text{if} \quad z \in \partial D \\ \dfrac{I}{\|z\|} - \dfrac{zz^\top}{\|z\|^3} & \text{if} \quad z \notin D \end{cases} \tag{149}$$ {eq:sub-projD

A.2 Euclidean projection on the second order cone of $\mathbb{R}^3$.

Let $K = \{x = [x_{\mathrm{N}} x_{\mathrm{T}}]^T \in \mathbb{R}^3, x_{\mathrm{N}} \in \mathbb{R}, \|x_{\mathrm{T}}\| \leqslant \mu x_{\mathrm{N}}\}$. We have

$$P_K(z) = \begin{cases} z & \text{if} \quad z \in K \\ 0 & \text{if} \quad -z \in K^* \\ \dfrac{1}{1+\mu^2}(z_{\mathrm{N}} + \mu\|z_{\mathrm{T}}\|) \begin{bmatrix} 1 \\ \mu\dfrac{z_{\mathrm{T}}}{\|z_{\mathrm{T}}\|} \end{bmatrix} & \text{if} \quad z \notin K \text{ and } -z \notin K^* \end{cases} \tag{150}$$ {eq:proj}

Direct computation of an element of the subdifferential The computation of the subdifferential are given as follows

- if $z \in K \setminus \partial K$, $\partial_z P_K(z) = I$,
- if $-z \in K^* \setminus \partial K^*$, $\partial_z P_K(z) = 0$,
- if $z \notin K$ and $-z \notin K^*$ and, $\partial_z P_K(z) = 0$, we get

$$\partial_{z_{\mathrm{N}}} P_K(z) = \frac{1}{1+\mu^2} \begin{bmatrix} 1 \\ \mu z_{\mathrm{T}} \end{bmatrix} \tag{151}$$ {eq:sub-proj_

and

$$\partial_{z_{\mathrm{T}}}[P_K(z)]_{\mathrm{N}} = \frac{\mu}{1+\mu^2}\frac{z_{\mathrm{T}}}{\|z_{\mathrm{T}}\|} \tag{152}$$ {eq:sub-proj_

$$\partial_{z_{\mathrm{T}}}[P_K(z)]_{\mathrm{T}} = \frac{\mu}{(1+\mu^2)}\left[\mu\frac{z_{\mathrm{T}}}{\|z_{\mathrm{T}}\|}\frac{z_{\mathrm{T}}^\top}{\|z_{\mathrm{T}}\|} + (z_{\mathrm{N}} + \mu\|z_{\mathrm{T}}\|)(\frac{I_2}{\|z_{\mathrm{T}}\|} - \frac{z_{\mathrm{T}} z_{\mathrm{T}}^\top}{\|z_{\mathrm{T}}\|^3})\right] \tag{153}$$ {eq:sub-proj_

that is

$$\partial_{z_{\mathrm{T}}}[P_K(z)]_{\mathrm{T}} = \frac{\mu}{(1+\mu^2)\|z_{\mathrm{T}}\|}\left[(z_{\mathrm{N}} + \mu\|z_{\mathrm{T}}\|)\, I_2 + z_{\mathrm{N}}\frac{z_{\mathrm{T}} z_{\mathrm{T}}^\top}{\|z_{\mathrm{T}}\|^2})\right] \tag{154}$$ {eq:sub-proj_

Redaction note V.A. A.1.

`to be checked carefully`

Computation of the subdifferential using the spectral decomposition In [Hayashi et al., 2005], the computation of the Clarke subdifferential of the projection operator is also done by inspecting the different cases using the spectral decomposition

$$\partial P_K(x) = \begin{cases} I & (\lambda_1 > 0, \lambda_2 > 0) \\ \dfrac{\lambda_2}{\lambda_1+\lambda_2} I + Z & (\lambda_1 < 0, \lambda_2 > 0) \\ 0 & (\lambda_1 < 0, \lambda_2 < 0) \\ \mathrm{co}\{I, I+Z\} & (\lambda_1 = 0, \lambda_2 > 0) \\ \mathrm{co}\{0, Z\} & (\lambda_1 < 0, \lambda_2 = 0) \\ \mathrm{co}\{0 \cup I \cup S\} & (\lambda_1 = 0, \lambda_2 = 0) \end{cases} \tag{155}$$ {eq:Jordan-pr

where

$$
\begin{aligned}
Z &= \frac{1}{2}\begin{bmatrix} -y_{\mathrm{N}} & y_{\mathrm{T}}^\top \\ y_{\mathrm{T}} & -y_{\mathrm{N}} y_{\mathrm{T}} y_{\mathrm{T}}^\top \end{bmatrix}, \\
S &= \left\{ \frac{1}{2}(1+\beta) I + \frac{1}{2}\begin{bmatrix} -\beta & w^\top \\ w & -\beta w w^\top \end{bmatrix} \mid -1 \leqslant \beta \leqslant 1, \|w\| = 1 \right\}
\end{aligned}
\tag{156}
$$ {eq:Jordan-pr

with $y = x/\|x_{\mathrm{T}}\|$. A simple verification shows that the previous computation is an element of the subdifferential.

Redaction note V.A. A.2.

to be checked carefully

B Computation of Generalized Jacobians for Nonsmooth Newton methods

B.1 Computation of components of subgradient of $F_{\mathrm{vi}}^{\mathsf{nat}}$

Let us introduce the following notation for an element of the sub–differential

$$
\Phi(u,r) = \begin{bmatrix} \rho I & -\rho W \\ \Phi_{ru}(u,r) & \Phi_{rr}(u,r) \end{bmatrix} \in \partial F_{\mathrm{vi}}^{\mathsf{nat}}(u,r) \tag{157}
$$ {eq:Phi-natur

where $\Phi_{xy}(u,r) \in \partial_x [F_{\mathrm{vi}}^{\mathsf{nat}}]_y(u,r)$. Since $\Phi_{uu}(u,r) = I$, a reduction of the system is performed in practise and Algorithm 4 is applied or $z = r$ with

$$
\begin{cases} G(z) = [F_{\mathrm{vi}}^{\mathsf{nat}}]_r(Wr+q, r) \\ \Phi(z) = \Phi_{rr}(r, Wr+q) + \Phi_{ru}(r, Wr+q) W \end{cases} \tag{158}
$$ {eq:phiphi_te

Redaction note V.A. B.1.

to be checked carefully. compare with [Hayashi et al., 2005]

Let us introduce the following notation for an element of the sub–differential with a obvious simplification

$$
\Phi(v,r) = \begin{bmatrix} \rho M & -\rho H & \\ -\rho H^\top & \rho I & 0 \\ 0 & \Phi_{ru}(v,u,r) & \Phi_{rr}(v,u,r) \end{bmatrix} \in \partial F_{\mathrm{vi}}^{\mathsf{nat}}(u,r) \tag{159}
$$ {eq:Phi-natur

where $\Phi_{xy}(v,u,r) \in \partial_x [F_{\mathrm{vi\text{-}1}}^{\mathsf{nat}}]_y(v,u,r)$. A possible computation of $\Phi_{ru}(v,u,r)$ and $\Phi_{rr}(v,u,r)$ is directly given by (161) and (160). In this case, the variable u can be also substituted.

For one contact, a possible computation of the remaining parts in $\Phi(u,r)$ is given by

$$
\Phi_{ru}(u,r) = \begin{cases} 0 & \text{if} \quad r - \rho(u + g(u)) \in K \\ I - \partial_r[P_K(r - \rho(u+g(u)))] & \text{if} \quad r - \rho(u+g(u)) \notin K \end{cases} \tag{160}
$$ {eq:Phi-natur

$$
\Phi_{ru}(u,r) = \begin{cases}
\rho\left(I + \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{u_{\mathrm{T}}}{\|u_{\mathrm{T}}\|} & 0 & 0 \end{bmatrix}\right) & \text{if } \begin{cases} r-\rho(u+g(u)) \in K \\ u_{\mathrm{T}} \neq 0 \end{cases} \\[2ex]
\rho\left(I + \begin{bmatrix} 0 & 0 & 0 \\ s & 0 & 0 \end{bmatrix}\right), s \in \mathbb{R}^2, \|s\| = 1 & \text{if } \begin{cases} r-\rho(u+g(u)) \in K \\ u_{\mathrm{T}} = 0 \end{cases} \\[2ex]
I + \rho\left(I + \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{u_{\mathrm{T}}}{\|u_{\mathrm{T}}\|} & 0 & 0 \end{bmatrix}\right)\partial_u[P_K(r-\rho(u+g(u)))] & \text{if } \quad r-\rho(u+g(u)) \notin K
\end{cases} \tag{161}
$$

{eq:Phi-natur

The computation of an element of ∂P_K is given in Appendix A.

B.2 Alart–Curnier function and its variants

For one contact, a possible computation of the remaining parts in $\Phi(u,r)$ is given by

$$
\Phi_{r_{\mathrm{N}}u_{\mathrm{N}}}(u,r) = \begin{cases} \rho_{\mathrm{N}} & \text{if } r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{162}
$$

{eq:Phi-AC-II

$$
\Phi_{r_{\mathrm{N}}r_{\mathrm{N}}}(u,r) = \begin{cases} 0 & \text{if } r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}} > 0 \\ 1 & \text{otherwise} \end{cases} \tag{163}
$$

{eq:Phi-AC-II

$$
\Phi_{r_{\mathrm{T}}u_{\mathrm{N}}}(u,r) = \begin{cases}
0 & \text{if } \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| \leqslant \mu\max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}}) \\
0 & \text{if } \begin{cases} \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| > \mu\max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}}) \\ r_{\mathrm{N}} - \rho_{\mathrm{N}}u_n \leqslant 0 \end{cases} \\
\mu\rho_{\mathrm{N}}\dfrac{r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}}{\|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\|} & \text{if } \begin{cases} \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| > \mu\max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}}) \\ r_{\mathrm{N}} - \rho_{\mathrm{N}}u_n > 0 \end{cases}
\end{cases} \tag{164}
$$

{eq:Phi-AC-II

$$
\Phi_{r_{\mathrm{T}}u_{\mathrm{T}}}(u,r) = \begin{cases}
\rho_{\mathrm{T}} & \text{if } \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| \leqslant \mu\max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}}) \\
\mu\rho_{\mathrm{T}}(r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}})_+\Gamma(r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}) & \text{if } \begin{cases} \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| > \mu\max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}}) \\ r_{\mathrm{N}} - \rho_{\mathrm{N}}u_n > 0 \end{cases}
\end{cases} \tag{165}
$$

{eq:Phi-AC-II

$$
\Phi_{r_{\mathrm{T}}r_{\mathrm{N}}}(u,r) = \begin{cases}
0 & \text{if } \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| \leqslant \mu\max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}}) \\
0 & \text{if } \begin{cases} \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| > \mu\max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}}) \\ r_{\mathrm{N}} - \rho_{\mathrm{N}}u_n \leqslant 0 \end{cases} \\
-\mu\dfrac{r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}}{\|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\|} & \text{if } \begin{cases} \|r_{\mathrm{T}} - \rho_{\mathrm{T}}u_{\mathrm{T}}\| > \mu\max(0, r_{\mathrm{N}} - \rho_{\mathrm{N}}u_{\mathrm{N}}) \\ r_{\mathrm{N}} - \rho_{\mathrm{N}}u_n > 0 \end{cases}
\end{cases} \tag{166}
$$

{eq:Phi-AC-II

$$\Phi_{r_\mathrm{T} r_\mathrm{T}}(u,r) = \begin{cases} 0 & \text{if } \|r_\mathrm{T} - \rho_\mathrm{T} u_\mathrm{T}\| \leqslant \mu \max(0, r_\mathrm{N} - \rho_\mathrm{N} u_\mathrm{N}) \\ I_2 - \mu(r_\mathrm{N} - \rho_\mathrm{N} u_\mathrm{N})_+ \Gamma(r_\mathrm{T} - \rho_\mathrm{T} u_\mathrm{T}) & \text{if } \begin{cases} \|r_\mathrm{T} - \rho_\mathrm{T} u_\mathrm{T}\| > \mu \max(0, r_\mathrm{N} - \rho_\mathrm{N} u_\mathrm{N}) \\ r_\mathrm{N} - \rho_\mathrm{N} u_n > 0 \end{cases} \end{cases} \tag{167}$$

{eq:Phi-AC-II

with the function $\Gamma(\cdot)$ defined by

$$\Gamma(x) = \frac{I_{2\times 2}}{\|x\|} - \frac{x\, x^\top}{\|x\|^3} \tag{168}$$

{eq:AC-L12}

If the variant (58) is chosen, the computation of $\Phi_{r_\mathrm{T}\bullet}$ simplify in

$$\Phi_{r_\mathrm{T} u_\mathrm{N}}(u,r) = 0 \tag{169}$$

{eq:Phi-CKPS-

$$\Phi_{r_\mathrm{T} u_\mathrm{T}}(u,r) = \begin{cases} \rho_\mathrm{T} & \text{if } \|r_\mathrm{T} - \rho_\mathrm{T} u_\mathrm{T}\| \leqslant \mu r_\mathrm{N} \\ -\mu \rho_\mathrm{T} r_{n,+} \Gamma(r_\mathrm{T} - \rho_\mathrm{T} u_\mathrm{T}) & \text{if } \|r_\mathrm{T} - \rho_\mathrm{T} u_\mathrm{T}\| > \mu r_\mathrm{N} \end{cases} \tag{170}$$

{eq:Phi-CKPS-

$$\Phi_{r_\mathrm{T} r_\mathrm{N}}(u,r) = \begin{cases} 0 & \text{if } \|r_\mathrm{T} - \rho_\mathrm{T} u_\mathrm{T}\| \leqslant \mu r_\mathrm{N} \\ 0 & \text{if } \begin{cases} \|r_\mathrm{T} - \rho_\mathrm{T} u_\mathrm{T}\| > \mu r_n \\ r_\mathrm{N} \leqslant 0 \end{cases} \\ -\mu \dfrac{r_\mathrm{T} - \rho_\mathrm{T} u_\mathrm{T}}{\|r_\mathrm{T} - \rho_\mathrm{T} u_\mathrm{T}\|} & \text{if } \begin{cases} \|r_\mathrm{T} - \rho_\mathrm{T} u_\mathrm{T}\| > \mu r_n \\ r_\mathrm{N} > 0 \end{cases} \end{cases} \tag{171}$$

{eq:Phi-CKPS-

$$\Phi_{r_\mathrm{T} r_\mathrm{T}}(u,r) = \begin{cases} 0 & \text{if } \|r_\mathrm{T} - \rho_\mathrm{T} u_\mathrm{T}\| \leqslant \mu r_\mathrm{N} \\ I_2 - \mu(r_\mathrm{N})_+ \Gamma(r_\mathrm{T} - \rho_\mathrm{T} u_\mathrm{T}) & \text{if } \|r_\mathrm{T} - \rho_\mathrm{T} u_\mathrm{T}\| > \mu r_\mathrm{N} \end{cases} \tag{172}$$

{eq:Phi-CKPS-

Redaction note V.A. B.2.

* *Is there a difference with the computation of Florent in his thesis?*
*

B.3 Fischer-Burmeister function

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