Formulations and extensive comparisons of 3D frictional contact solvers based on performance profiles

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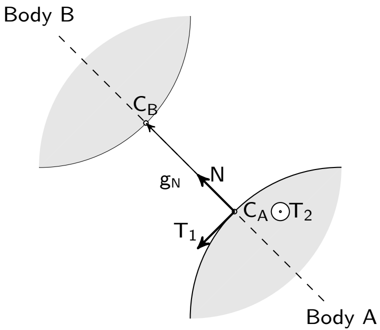
Signorini's condition and Coulomb's friction



- gap function $g_N = (C_B - C_A)N$.
- reaction forces velocities

$$r = r_N N + r_T, \quad \text{with } r_N \in \boldsymbol{R} \text{ and } r_T \in \boldsymbol{R}^2.$$

$$u = u_N N + u_T, \quad \text{with } u_N \in \boldsymbol{R} \text{ and } u_T \in \boldsymbol{R}^2.$$

- Signorini conditions

$$\text{position level}: 0 \leqslant g_N \perp r_N \geqslant 0.$$

$$\text{velocity level}: \begin{cases} 0 \leqslant u_N \perp r_N \geqslant 0 & \text{if } g_N \leqslant 0 \\ r_N = 0 & \text{otherwise.} \end{cases}$$

Signorini's condition and Coulomb's friction

Modeling assumption

Let μ be the coefficient of friction. Let us define the Coulomb friction cone K which is chosen as the isotropic second order cone

$$K = \{r \in \mathbb{R}^3 \mid \|r_{\mathrm{T}}\| \leqslant \mu r_n\}. \tag{1}$$

The Coulomb friction states

- for the sticking case that
$$u_{\mathrm{T}} = 0, \quad r \in K \tag{2}$$
- and for the sliding case that
$$u_{\mathrm{T}} \neq 0, \quad r \in \partial K, \exists\, \alpha > 0, r_{\mathrm{T}} = -\alpha u_{\mathrm{T}}. \tag{3}$$

Disjunctive formulation of the frictional contact behavior

$$\begin{cases} r = 0 & \text{if } g_{\mathrm{N}} > 0 \quad \text{(no contact)} \\ r = 0, u_{\mathrm{N}} \geqslant 0 & \text{if } g_{\mathrm{N}} \leqslant 0 \quad \text{(take–off)} \\ r \in K, u = 0 & \text{if } g_{\mathrm{N}} \leqslant 0 \quad \text{(sticking)} \\ r \in \partial K, u_{\mathrm{N}} = 0, \exists\, \alpha > 0, u_{\mathrm{T}} = -\alpha r_{\mathrm{T}} & \text{if } g_{\mathrm{N}} \leqslant 0 \quad \text{(sliding)} \end{cases} \tag{4}$$

Signorini's condition and Coulomb's friction

Second Order Cone Complementarity (SOCCP) formulation [?]

- Modified relative velocity $\hat{u} \in \boldsymbol{R}^3$ defined by

$$\hat{u} = u + \mu \|u_{\mathrm{T}}\| \mathrm{N}. \tag{5}$$

- Second-Order Cone Complementarity Problem (SOCCP)

$$K^\star \ni \hat{u} \perp r \in K \tag{6}$$

if $g_{\mathrm{N}} \leqslant 0$ and $r = 0$ otherwise. The set $K^\star$ is the dual convex cone to K defined by

$$K^\star = \{u \in \boldsymbol{R}^3 \mid r^\top u \geqslant 0, \quad \text{for all } r \in K\}. \tag{7}$$

Signorini's condition and Coulomb's friction

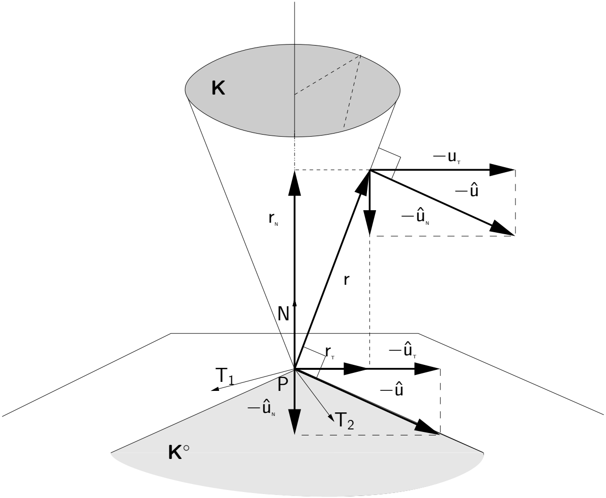


Figure: Coulomb's friction and the modified velocity $\hat{u}$. The sliding case.

3D frictional contact problem

Multiple contact notation

For each contact $\alpha \in \{1, \dots n_c\}$, we have

- the local velocity : $u^\alpha \in \boldsymbol{R}^3$, and

$$u = [[u^\alpha]^\top, \alpha = 1 \dots n_c]^\top$$

- the local reaction vector $r^\alpha \in \boldsymbol{R}^3$

$$r = [[r^\alpha]^\top, \alpha = 1 \dots n_c]^\top$$

- the local Coulomb cone

$$K^\alpha = \{r^\alpha, \|r^\alpha_{\mathrm{T}}\| \leqslant \mu^\alpha |r^\alpha_{\mathrm{N}}|\} \subset \boldsymbol{R}^3$$

  and the set K is the cartesian product of Coulomb's friction cone at each contact, that

$$K = \prod_{\alpha=1\dots n_c} K^\alpha \tag{8}$$

  and $K^\star$ is dual.

3D frictional contact problems

Problem 1 (General discrete frictional contact problem)

Given

- *a symmetric positive definite matrix* $M \in \mathbb{R}^{n\times n}$,
- *a vector* $f \in \mathbb{R}^n$,
- *a matrix* $H \in \mathbb{R}^{n\times m}$,
- *a vector* $w \in \boldsymbol{R}^m$,
- *a vector of coefficients of friction* $\mu \in \boldsymbol{R}^{n_c}$,

find three vectors $v \in \mathbb{R}^n$, $u \in \boldsymbol{R}^m$ *and* $r \in \boldsymbol{R}^m$, *denoted by* $\mathrm{FC/I}(M, H, f, w, \mu)$ *such that*

$$\begin{cases} Mv = Hr + f \\ u = H^\top v + w \\ \hat{u} = u + g(u) \\ K^\star \ni \hat{u} \perp r \in K \end{cases} \tag{9}$$

with $g(u) = [[\mu^\alpha \| u_T^\alpha \| \mathrm{N}^\alpha]^\top, \alpha = 1 \ldots n_c]^\top$. □

3D frictional contact problems

Problem 2 (Reduced discrete frictional contact problem)

Given

- *a symmetric positive semi–definite matrix* $W \in \mathbb{R}^{m\times m}$,
- *a vector* $q \in \mathbb{R}^{m}$,
- *a vector* $\mu \in \boldsymbol{R}^{n_c}$ *of coefficients of friction,*

find two vectors $u \in \boldsymbol{R}^m$ *and* $r \in \boldsymbol{R}^m$, *denoted by* $\mathrm{FC/II}(W, q, \mu)$ *such that*

$$\begin{cases} u = Wr + q \\ \hat{u} = u + g(u) \\ K^\star \ni \hat{u} \perp r \in K \end{cases} \tag{10}$$

with $g(u) = [[\mu^\alpha \| u^\alpha_T \| \mathrm{N}^\alpha]^\top, \alpha = 1 \dots n_c]^\top$. □

Relation with the general problem

$W = H^\top M^{-1} H$ and $q = H^\top M^{-1} f + w$.

From the mathematical programming point of view

Nonmonotone and nonsmooth problem

$$K^\star \ni Wr + q + g(Wr + q) \perp r \in K \tag{11}$$

Possible reformulation

- Variational inequality or normal cone inclusion

$$-\left(Wr + q + g(Wr + q)\right) \stackrel{\Delta}{=} -F(r) \in N_K(r). \tag{12}$$

- Nonsmooth equations $G(r) = 0$
  - The natural map F^{nat} associated with the VI (12) $F^{\text{nat}}(z) = z - P_X(z - F(z))$.
  - Variants of this map (Alart-Curnier formulation, ...)
  - one of the SOCCP-functions. (Fisher-Bursmeister function)
- and many other ...

VI based methods

Standard methods

- Basic fixed point iterations with projection [FP-VI]

$$z_{k+1} \leftarrow P_X(z_k - \rho_k F(z_k))$$

- Extragradient method [EG-VI]

$$z_{k+1} \leftarrow P_X(z_k - \rho_k F(P_X(z_k - \rho_k F(z_k))))$$

With fixed ρ, we get the Uzawa Algorithm of De Sacxé-Feng

Self-adaptive procedure for ρ_k

[UPK]

$$\text{Armijo-like : } m_k \in \boldsymbol{N} \quad \text{such that} \begin{cases} \rho_k = \rho 2^{m_k}, \\ \rho_k \|F(z_k) - F(\bar{z}_k)\| \leqslant \|z_k - \bar{z}_k\| \end{cases}$$

Nonsmooth Equations based methods

Nonsmooth Newton on $G(z) = 0$

$$z_{k+1} = z_k - \Phi^{-1}(z_k)(G(z_k)), \qquad \Phi(z_k) \in \partial G(z_k)$$

- Alart–Curnier Formulation [?] [NSN-AC]

$$\begin{cases} r_{\mathrm{N}} - P_{\mathbb{R}^{n_c}_{+}}(r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) = 0, \\ r_{\mathrm{T}} - P_{D(\mu, r_{\mathrm{N},+} + \rho u_{\mathrm{N}})}(r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}) = 0, \end{cases}$$

- Jean–Moreau Formulation [NSN-MJ]

$$\begin{cases} r_{\mathrm{N}} - P_{\mathbb{R}^{n_c}_{+}}(r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) = 0, \\ r_{\mathrm{T}} - P_{D(\mu, r_{\mathrm{N},+})}(r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}) = 0, \end{cases}$$

- Direct normal map reformulation [NSN-NM]

$$r - P_K\left(r - \rho(u + g(u))\right) = 0$$

- Extension of Fischer-Burmeister function to SOCCP [NSN-FB]

$$\phi_{\mathrm{FB}}(x, y) = x + y - (x^2 + y^2)^{1/2}$$

Matrix block-splitting and projection based algorithms [Moreau(1994), Jean and Touzot(1988)]

Block splitting algorithm with $W^{\alpha\alpha} \in \boldsymbol{R}^3$ [NSGS-*]

$$
\begin{cases}
u^{\alpha}_{i+1} - W^{\alpha\alpha} P^{\alpha}_{i+1} = q^{\alpha} + \sum\limits_{\beta<\alpha} W^{\alpha\beta} r^{\beta}_{i+1} + \sum\limits_{\beta>\alpha} W^{\alpha\beta} r^{\beta}_{i} \\[2ex]
\widehat{u}^{\alpha}_{i+1} = \left[u^{\alpha}_{\mathrm{N},i+1} + \mu^{\alpha}\, ||u^{\alpha}_{\mathrm{T},i+1}||, u^{\alpha}_{\mathrm{T},i+1} \right]^T \\[2ex]
\mathbf{K}^{\alpha,*} \ni \widehat{u}^{\alpha}_{i+1} \perp r^{\alpha}_{i+1} \in \mathbf{K}^{\alpha}
\end{cases}
\tag{13}
$$

for all $\alpha \in \{1 \dots m\}$.

Over-Relaxation [PSOR-*]

One contact point problem

- closed form solutions
- Any solver listed before.

Proximal point technique [Moreau(1962), Moreau(1965), Rockafellar(1976)]

Principle

We want to solve

$$\min_x f(x) \tag{14}$$

We define the approximation problem for a given x_k

$$\min_x f(x) + \rho\|x - x_k\|^2 \tag{15}$$

with the optimal point $x^\star$.

$$x^\star \triangleq \mathrm{prox}_{f,\rho}(x_k) \tag{16}$$

Proximal point algorithm [PPA-*]

$$x_{k+1} = \mathrm{prox}_{f,\rho_k}(x_k)$$

Special case for solving $G(x) = 0$

$$f(x) = \frac{1}{2} G^\top(x) G(x)$$

Optimization based methods

- Alternating optimization problems (Panagiotopoulos et al.) [PANA-*]
- Successive approximation with Tresca friction (Haslinger et al.) [TRESCA-*]

$$\begin{cases} \theta = h(r_{\mathrm{N}}) \\ \min \ \frac{1}{2} r^\top W r + r^\top q \\ \quad \text{s.t.} \qquad r \in C(\mu, \theta) \end{cases} \tag{17}$$

where $C(\mu, \theta)$ is the cylinder of radius $\mu\theta$.

- Fixed point on the norm of the tangential velocity [A., Cadoux, Lemaréchal, Malick(2011)] [ACLM-*].

$$\begin{cases} s = \|u_{\mathrm{T}}\| \\ \min \ \frac{1}{2} r^\top W r + r^\top (q + \alpha s) \\ \quad \text{s.t.} \qquad r \in K \end{cases} \tag{18}$$

Fixed point or Newton Method on $F(s) = s$

Siconos/Numerics

Siconos

Open source software for modelling and simulation of nonsmooth systems

Siconos/Numerics

Collection of C routines to solve FC3D problem

- NSGS : VI based projection/splitting algorithm
- TrescaFixedPoint : fixed point algorithm on Tresca fixed point
- LocalAlartCurnier : semi–smooth newton method of Alart–Curnier formulation
- ProximalFixedPoint : proximal point algorithm
- VIFixedPointProjection : VI based fixed-point projection
- VIExtragradient : VI based extra-gradient method
- ...

http://siconos.gforge.inria.fr

use and contribute ...

Measuring errors

Full error criteria

$$\text{error} = \frac{\|F^{\text{nat}}_{\text{vi-2}}(r)\|}{\|q\|}. \tag{19}$$

Cheap error

$$\text{error}_{\text{cheap}} = \frac{\|r_{k+1} - r_k\|}{\|r_k\|}. \tag{20}$$

The tolerance of solver is then self-adapted in the loop to meet the required tolerance based on the error given by (19).

Performance profiles [Dolan and Moré(2002)]

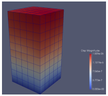
- Given a set of problems $\mathcal{P}$
- Given a set of solvers $\mathcal{S}$
- A performance measure for each problem with a solver $t_{p,s}$ (cpu time, flops, ...)
- Compute the performance ratio

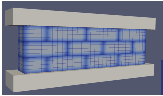
$$\tau_{p,s} = \frac{t_{p,s}}{\min\limits_{s\in\mathcal{S}} t_{p,s}} \geqslant 1 \tag{21}$$

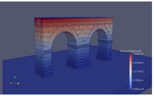
- Compute the performance profile $\rho_s(\tau) : [1, +\infty] \rightarrow [0, 1]$ for each solver $s \in \mathcal{S}$

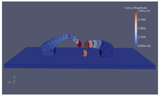
$$\rho_s(\tau) = \frac{1}{|\mathcal{P}|} \Big| \{ p \in \mathcal{P} \mid \tau_{p,s} \leqslant \tau \} \Big| \tag{22}$$

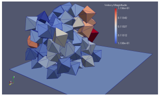
The value of $\rho_s(1)$ is the probability that the solver s will win over the rest of the solvers.

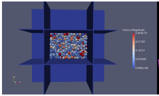

(a) Cubes_H8

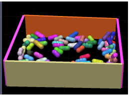

(b) LowWall_FEM


(c) Aqueduct_PR

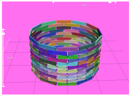

(d) Bridge_PR

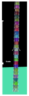

(e) 100_PR_Periobox


(f) 945_SP_Box_PL


(g) Capsules


(h) Chain


(i) KaplasTower


(j) BoxesStack

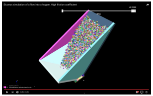
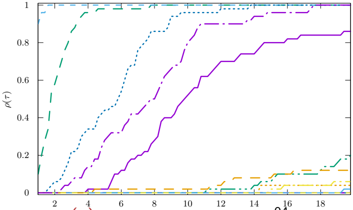

(k) Chute_1000, Chute_4000, Chute_local_problems

Figure: Illustrations of the FClib test problems

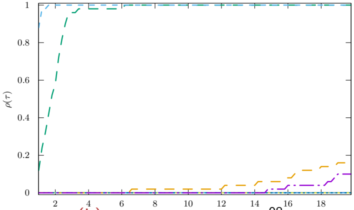
Test set	code	friction coefficient μ	# of problems	# of d.o.f.	# of contacts	contact density c	rank ratio(W)
Cubes_H8_2	LMGC90	0.3	15	162	[3 : 5]	[0.02 : 0.09]	1
Cubes_H8_5	LMGC90	0.3	50	1296	[17 : 36]	[0.02 : 0.09]	1
Cubes_H8_20	LMGC90	0.3	50	55566	[361 : 388]	[0.019 : 0.021]	1
LowWall_FEM	LMGC90	0.83	50	{7212}	[624 : 688]	[0.28 : 0.29]	1
Aqueduct_PR	LMGC90	0.8	10	{1932}	[4337 : 4811]	[6.81 : 7.47]	[6.80 : 7.46]
Bridge_PR	LMGC90	0.9	50	{138}	[70 : 108]	[1.5 : 2.3]	[2.27 : 2.45]
100_PR_Periobox	LMGC90	0.8	106	{606}	[14 : 578]	[0.2 : 3]	[1.76 : 3.215]
945_SP_Box_PL	LMGC90	0.8	60	{5700}	[2322 : 5037]	[1.22 : 2.65]	[1.0 : 2.66]
Capsules	Siconos	0.7	249	[96:600]	[17 : 304]	[0.53 : 1.52]	[1.08 : 1.55]
Chain	Siconos	0.3	242	{60}	[8 : 28]	[0.5 : 1.3]	[1.05 : 1.6]
KaplasTower	Siconos	0.7	201	[72 : 792]	[48 : 933]	[3.0 : 3.6]	[2.0 : 3.53]
BoxesStack	Siconos	0.7	255	[6 : 300]	[1 : 200]	[1.86 : 2.00]	[1.875 : 2.0]
Chute_1000	Siconos	1.0	156	[276 : 5508]	[74 : 5056]	[0.69 : 2.95]	[1.0 : 2.95]
Chute_4000	Siconos	1.0	40	[17280 : 20034]	[15965 : 19795]	[2.51 : 3.06]	–
Chute_local_problems	Siconos	1.0	834	3	1	1	1

Table: Description of the test sets of FCLib library (v1.0)

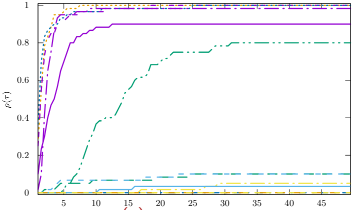
Comparisons by families of solvers



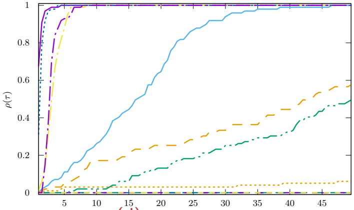
(a) LowWall_FEM II (tol = 1e−04)



(b) LowWall_FEM (tol = 1e−08)



(c) 945_SP_Box_PL



(d) 100_PR_PerioBox

NSGS-AC
NSN-AC-GP
NSN-AC
TRESCA-NSGS-FP-VI-UPK
EG-VI-UPK
PPA-NSN-AC-GP $\alpha_0 = 10^{+04}, \nu = 1, \sigma = 5.0$
PPA-NSGS-NSN-AC $\alpha_0 = 10^{+04}, \nu = 1, \sigma = 5.0$

ACLM-NSGS-FP-VI
ACLM-VI-EG
PPA-NSN-AC-GP adaptive $\alpha_0 = 10^{+03}$
PPA-NSN-AC-GP $\alpha_0 = 10^{+04}, \nu = 1.0, \sigma = 0.5$
NSGS-FP-VI-UPK $(\text{tol}_{\text{local}} = 10^{-06})$
NSGS-FP-VI-UPK $(\text{tol}_{\text{local}} = 10^{-14})$

Conclusions & Perspectives

Conclusions

1. A bunch of articles in the literature
2. No "Swiss–knife" solution : choose efficiency OR robustness
3. Newton–based solver solves efficiently the problems but robustness issues
4. First order iterative methods solves all the problems but very slowly
5. The rank of the H matrix (ratio number of contacts unknows/number of d.o.f) plays an important role.

Perspectives

1. Develop new algorithm and compare other algorithm in the literature. (issues with standard optimization software.)
2. Study the influence of the friction coefficient, the size of problem, the conditionning of the problem , . . .
3. Set up a collection of benchmarks ➜ FCLIB

FCLIB : a collection of discrete 3D Frictional Contact (FC) problems

Our inspiration: MCPLIB or CUTEst

What is FCLIB ?

- A open source collection of Frictional Contact (FC) problems stored in a specific HDF5 format
- A open source light implementation of Input/Output functions in C Language to read and write problems (Python and Matlab coming soon)

Goals of the project

Provide a standard framework for testing available and new algorithms for solving discrete frictional contact problems share common formulations of problems in order to exchange data

Call for contribution

http://fclib.gforge.inria.fr

Thank you for your attention.

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