# Uncapacitated Facility Location Problem (UFLP)

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## 1 Introduction

Location problems involve determining optimal positions for facilities to minimize costs or maximize service efficiency. In the **Uncapacitated Facility Location Problem (UFLP)**, a set of potential facility locations and a set of customers are given. Each facility has a fixed cost of opening, and a transportation cost is incurred when serving a customer from a facility. (See figure 1)

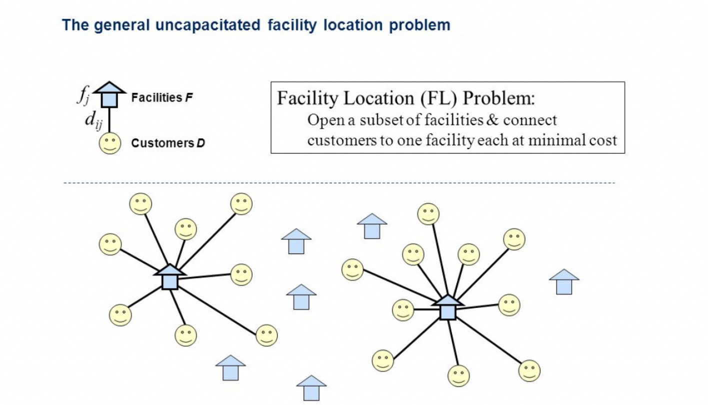


Figure 1: Uncapacitated Facility Location Problem (UFLP)

Figure 1: Visualization of the UFLP: Open facilities and customer assignments

The objective is to decide:

- Which facilities to open
- How to assign each customer to an open facility

so as to minimize the total cost.

## 2 Mathematical Model

Given:

- $N$: Set of potential facilities $\{1, 2, \ldots, n\}$
- $M$: Set of customers $\{1, 2, \ldots, m\}$
- $f_i$: Fixed cost to open facility $i$

- $c_{ij}$: Cost to serve customer $j$ from facility $i$
- $d_j$: Demand of customer $j$

**Decision variables:**

$$z_i = \begin{cases} 1 & \text{if facility } i \text{ is opened} \\ 0 & \text{otherwise} \end{cases}$$

$$x_{ij} = \begin{cases} 1 & \text{if customer } j \text{ is assigned to facility } i \\ 0 & \text{otherwise} \end{cases}$$

**Objective function:**

$$\min \quad \sum_{i \in N} f_i z_i + \sum_{i \in N} \sum_{j \in M} d_j c_{ij} x_{ij}$$

**Subject to:**

$$\sum_{i \in N} x_{ij} = 1 \qquad \forall j \in M \; x_{ij} \le z_i \qquad \forall i \in N, j \in M \; x_{ij}, z_i \in \{0,1\} \qquad \forall i, j$$

# 3 Solution Methods

Three approaches are implemented in C:

1. A MIP model solved with IBM CPLEX
2. A constructive heuristic with local search
3. A Lagrangian relaxation using subgradient optimization

# 4 Lagrangian Relaxation Method

The Lagrangian relaxation approach involves dualizing the demand constraints:

$$\sum_{i \in N} x_{ij} = 1, \quad \forall j \in M$$

using Lagrange multipliers $\lambda_j$. The relaxed problem becomes:

$$\begin{aligned} \min \quad & \sum_{i \in N} \sum_{j \in M} (d_j c_{ij} + \lambda_j) x_{ij} + \sum_{i \in N} f_i z_i - \sum_{j \in M} \lambda_j \\ \text{s.t.} \quad & x_{ij} \le z_i \quad \forall i \in N, j \in M \\ & x_{ij}, z_i \in \{0,1\} \end{aligned}$$

This problem decomposes into independent subproblems for each facility $i$. For each facility, we determine whether it is profitable to open it given the current values of the multipliers $\lambda_j$. A customer $j$ is assigned to facility $i$ only if:

$$d_j c_{ij} + \lambda_j < 0$$

and the facility is opened if the total contribution from these customers exceeds the setup cost:

$$f_i + \sum_{j \in M} \min(0, d_j c_{ij} + \lambda_j) < 0$$

The subgradient method is used to update the multipliers $\lambda$ iteratively. At each iteration $t$, we compute a subgradient:

$$s_j = \sum_{i \in N} x_{ij} - 1$$

and update:

$$\lambda_j^{t+1} = \lambda_j^t + \alpha_t s_j$$

where $\alpha_t$ is the step size:

$$\alpha_t = \epsilon \cdot \frac{UB - L(\lambda)}{\|s\|^2}$$

The upper bound (UB) is obtained using a heuristic that reassigns customers to the closest open facilities based on the solution $z_i$ from the relaxed problem.