Numerical Solutions of PDEs, Final Project

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Motivation

Robin boundary conditions describe the boundary behavior of a PDE as a linear combination of the function described by the PDE and its normal derivative at the boundary. In the context of the heat equation, Robin boundary conditions could describe convective heat transfer from a fluid to the walls of some container, where the temperature at the wall depends on how the temperature changes in the container's surrounding environment. In the context of electrostatics, Robin boundary conditions could indicate the existence of an electric field just beyond the boundary, such that the potential at the boundary depends on the gradient of the potential at the boundary, as influenced by the external field.

Heat Equation

Consider the one-dimensional heat equation, $u_t = u_{xx}$ for $x \in [0, 2\pi]$ with the following Robin boundary conditions:

$$u(x=0,t) = \frac{\partial u}{\partial x}\bigg|_{x=0}, \qquad u(x=2\pi,t) = \frac{\partial u}{\partial x}\bigg|_{x=2\pi}$$

We'll assume periodicity and use a grid of N points to approximate $u(x,t)$ over the interval $[0, 2\pi]$. Using second-order centered differences for our Robin boundary conditions:

$$u_0^{(n)} = u_{2\pi}^{(n)} = \frac{u_1^{(n)} - u_{N-1}^{(n)}}{2\Delta x}$$

Applying these conditions, we obtain the following system:

$$\frac{1}{\Delta x^2}\underbrace{\begin{bmatrix} \frac{1}{2\Delta x}-2 & 1 & 0 & \dots & -\frac{1}{2\Delta x} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2\Delta x} & \dots & 0 & 1 & -\frac{1}{2\Delta x}-2 \end{bmatrix}}_{\mathscr{D}} \underbrace{\begin{bmatrix} u(x_1,t) \\ \vdots \\ u(x_{N-1},t) \end{bmatrix}}_{\vec{u}} = \underbrace{\begin{bmatrix} u_t(x_1,t) \\ \vdots \\ u_t(x_{N-1},t) \end{bmatrix}}_{\vec{u_t}}$$

To verify the accuracy of this approach, we can solve the two-dimensional steady-state heat equation with a forcing function, $\frac{1}{\Delta x^2}(I \otimes \mathscr{D} + \mathscr{D} \otimes I)(\vec{u}(x,y)) = f(x,y)$ (Figures 1 and 2). We've used second-order finite differences in this approximation, so we expect second-order convergence.

Immersed Boundary

Consider the PDE from the electrostatics homework, $\nabla^2 u = S^{ib}(q)$, with homogeneous Dirichlet conditions at the spatial boundary (x,y) and Robin conditions rather than Dirichlet conditions on the immersed boundary x^{ib}, y^{ib}:

$$u(x^{ib}, y^{ib}) - \frac{\partial u(x^{ib}, y^{ib})}{\partial n} = g(x^{ib}, y^{ib}) \qquad \text{where } g(x^{ib}, y^{ib}) = \cos(2x^{ib})\cos(2y^{ib})$$

$$x^{ib} = \pi + \cos(\theta), \quad y^{ib} = \pi + \sin(\theta), \quad \theta \in [0, 2\pi]$$

We'll introduce the following variables to denote small displacements at the immersed boundary coordinates:

$$x^{inner} = \pi + (1-\Delta_{ib})\cos(\theta) \qquad y^{inner} = \pi + (1-\Delta_{ib})\sin(\theta)$$
$$x^{outer} = \pi + (1+\Delta_{ib})\cos(\theta) \qquad y^{outer} = \pi + (1+\Delta_{ib})\sin(\theta)$$

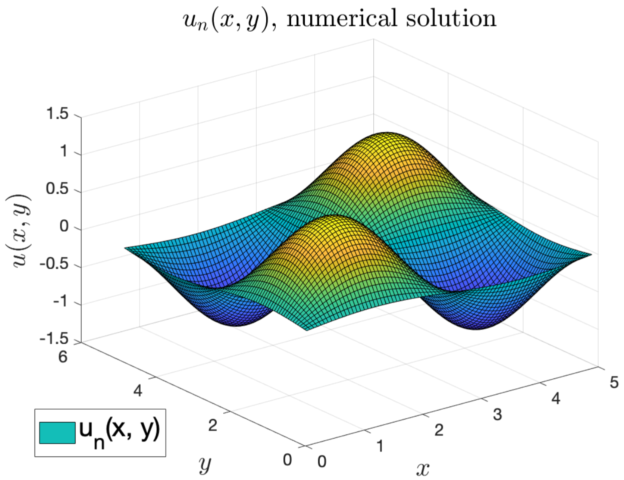

Figure 1: A plot of the numerical solution to $\nabla^2 u(x,y) = -2\sin(x)\sin(y)$ with the specified Robin boundary conditions and grid size $N = 80$.

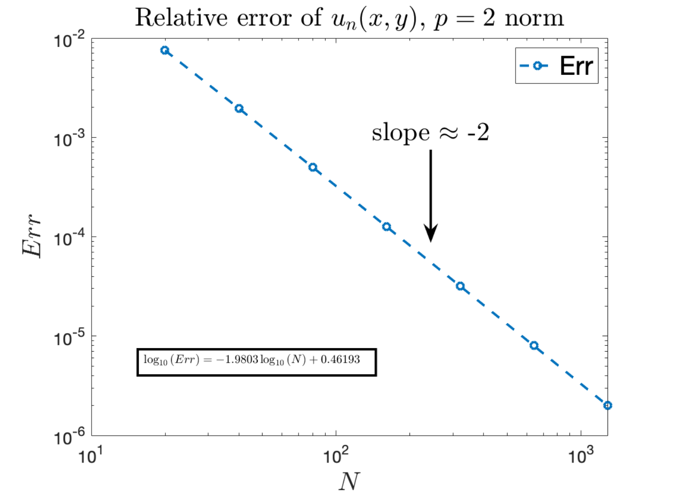

Figure 2: A plot of the relative error of the numerical estimate for $u(x,y)$ in Figure 1. $u_n(x,y)$ computed at grid sizes $N = [20, 40, 80, ..., 1280]$.

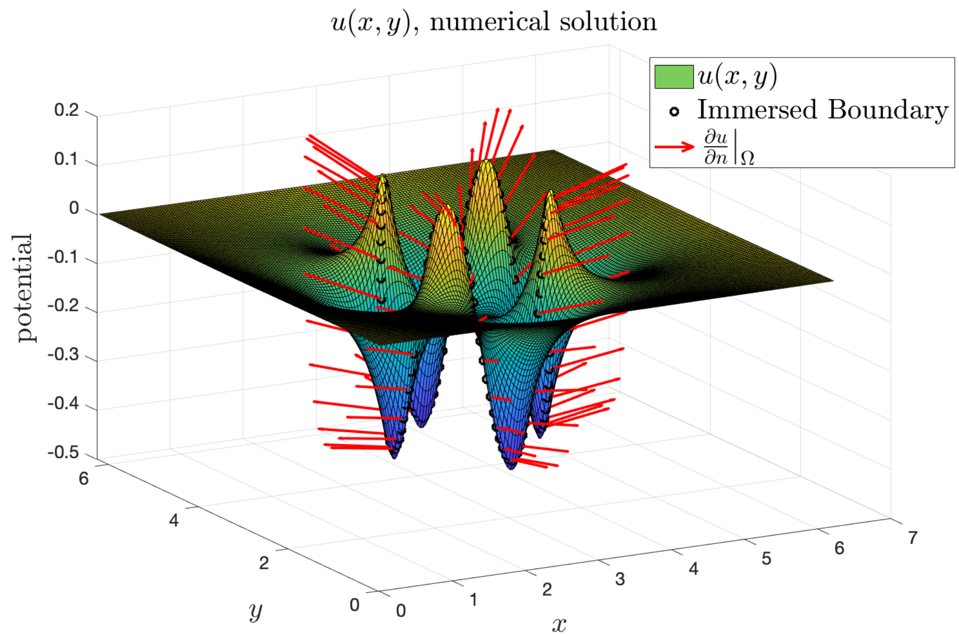

Figure 5: A plot of the numerical solution to $\nabla^2 u = S^{ib}(q)$ on the immersed boundary and the normal derivative of the solution computed at each immersed boundary point, using grid size $N = 160$.

where $\Delta_{ib} \sim \Delta x$. Using a second-order finite difference to approximate the normal derivative,

$$
\begin{aligned}
J^{ib}(u) &= \frac{u(x^{outer}, y^{outer}) - u(x^{inner}, y^{inner})}{2\Delta_{ib}} + g(x^{ib}, y^{ib}) \\
&= \frac{J^{outer}(u) - J^{inner}(u)}{2\Delta_{ib}} + g(x^{ib}, y^{ib}) \\
(2\Delta_{ib}J^{ib} + J^{inner} - J^{outer})(u) &= 2\Delta_{ib} g(x^{ib}, y^{ib})
\end{aligned}
$$

So, this system yields the following saddle point system:

$$
\begin{bmatrix} \nabla^2 & -S^{ib} \\ (2\Delta_{ib}J^{ib} + J^{inner} - J^{outer}) & 0 \end{bmatrix} \begin{bmatrix} u(x,y) \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 2\Delta_{ib} g(x^{ib}, y^{ib}) \end{bmatrix}
$$

After solving this system and plotting $u(x,y)$, we can solve for the normal derivative at each immersed boundary point (Figure 5). To get more accurate values for the normal derivatives, we could increase the grid size or set $\Delta_{ib} << \Delta x$. It would be interesting to consider Robin conditions for a more complicated boundary, or in more dimensions.