1. Prove the following identity using induction:

$$Pr(\alpha_1, \ldots, \alpha_n \mid \beta) = Pr(\alpha_1 \mid \alpha_2, \ldots, \alpha_n, \beta) Pr(\alpha_2 \mid \alpha_3, \ldots, \alpha_n, \beta) \ldots Pr(\alpha_n \mid \beta).$$

Base case.

Let $n=1$:

$Pr(\alpha_1 | \beta) = Pr(\alpha_1 | \beta)$ ✓

Let $n=2$:

$Pr(\alpha_1, \alpha_2 | \beta) = Pr(\alpha_1 | \alpha_2, \beta)\, Pr(\alpha_2 | \beta)$

$\frac{Pr(\alpha_1, \alpha_2, \beta)}{P(\beta)} = \frac{Pr(\alpha_1, \alpha_2, \beta)}{\cancel{P(\alpha_2, \beta)}} \cdot \frac{\cancel{Pr(\alpha_2 | \beta)}}{Pr(\beta)}$ ✓

Inductive step: Assume the identity holds true for $n=k$, where $k \in \mathbb{Z}^+$

$$Pr(\alpha_1, \ldots, \alpha_{k+1} | \beta) = \frac{Pr(\alpha_1, \ldots \alpha_{k+1}, \beta)}{P(\beta)}$$

$$= \frac{Pr(\alpha_1, \ldots, \alpha_k | \alpha_{k+1}, \beta)\, Pr(\alpha_{k+1}, \beta)}{P(\beta)}$$

$$= Pr(\alpha_1, \ldots, \alpha_k | \alpha_{k+1}, \beta)\, Pr(\alpha_{k+1} | \beta)$$

Since we assume the identity

$Pr(\alpha_1, \ldots \alpha_k | \beta) = Pr(\alpha_1 | \alpha_2, \ldots, \alpha_k, \beta) \ldots Pr(\alpha_k | \beta)$

is true, then

$$Pr(\alpha_1, \ldots, \alpha_{k+1} | \beta) = Pr(\alpha_1 | \alpha_2, \ldots \alpha_{k+1}, \beta)\, Pr(\alpha_2 | \alpha_3, \ldots \alpha_{k+1}, \beta) \cdot Pr(\alpha_k | \alpha_{k+1}, \beta) \cdot Pr(\alpha_{k+1} | \beta)$$

Thus, the identity holds for $n=k+1$, so by the principle of induction, the identity is proven.

2. A well is being drilled on a farm. Based on what has happened to similar farms, we judge the probability of oil being present to be 0.5, the probability of natural gas being present to be 0.2, and the probability of neither being present to be 0.3. Oil and natural gas cannot be present at the same time. If oil is present, a geological test will give a positive result with probability 0.9; if natural gas is present, it will give a positive result with probability 0.3; and if neither are present, the test will be positive with probability 0.1. Suppose the test comes back positive. What's the probability that oil is present?

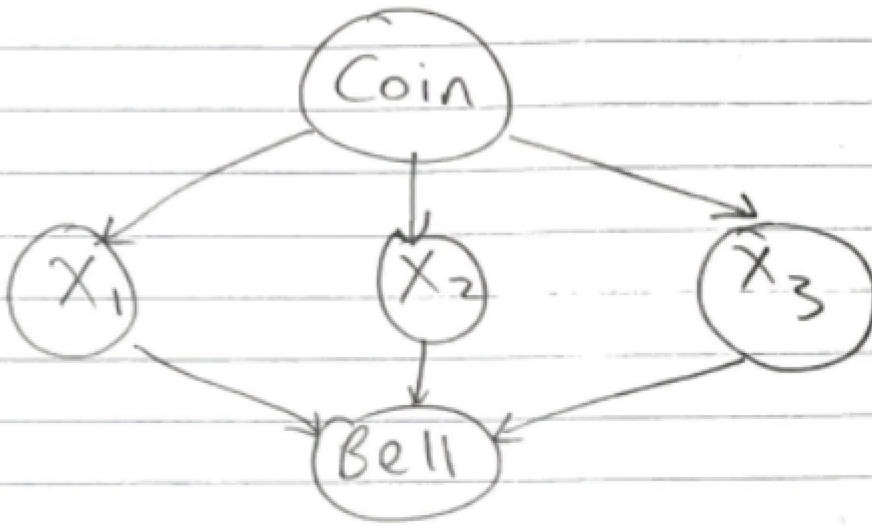
P(Oil) = 0.5
P(Gas) = 0.2
P(Neither) = 0.3
P(Both) = 0
P(Positive | Oil) = 0.9
P(Positive | Gas) = 0.3
P(Positive | Neither) = 0.1
P(Oil | Positive) = ?
P(Oil | Positive) = P(Positive | Oil) * P(Oil) / P(Positive)

P(Positive) = P(Positive | Oil) * P(Oil) + P(Positive | Gas) * P(Gas) + P(Positive | Neither) * P(Neither) + P(Positive | Both) * P(Both) = 0.9*0.5 + 0.3*0.2 + 0.1*0.3 + 0 = 0.54

P(Oil | Positive) = 0.9 * 0.5 / 0.54 = 0.83

3. We have a bag of three biased coins a, b, and c with probabilities of coming up heads of 20%, 40%, and 80%, respectively. One coin is drawn randomly from the bag (with equal likelihood of drawing each of the three coins), and then the coin is flipped three times to generate the outcomes X_1, X_2, and X_3. A bell will ring "on" if all coins flips come out the same. Draw the Bayesian network corresponding to this setup and define the necessary CPTs (Conditional Probability Tables).

Bayesian Network:



let C denote Coin

$C = \{a, b, c\}$

For the coin, H is for heads, T is for tails

let B denote Bell

$B = \{on, off\}$

C	θ_C
a	1/3
b	1/3
c	1/3

C	X_1	$\theta_{X_1 \mid C}$
a	H	0.2
a	T	0.8
b	H	0.4
b	T	0.6
c	H	0.8
c	T	0.2

C	X_2	$\theta_{X_2 \mid C}$
a	H	0.2
a	T	0.8
b	H	0.4
b	T	0.6
c	H	0.8
c	T	0.2

C	X_3	$\theta_{X_3 \mid C}$
a	H	0.2
a	T	0.8
b	H	0.4
b	T	0.6
c	H	0.8
c	T	0.2

X_1	X_2	X_3	B	$\theta_{B \mid x_1, x_2, x_3}$
H	H	H	on	1
H	H	H	off	0
H	H	T	on	0
H	H	T	off	1
H	T	H	on	0
H	T	H	off	1
H	T	T	on	0
H	T	T	off	1
T	H	H	on	0
T	H	H	off	1
T	H	T	on	0
T	H	T	off	1
T	T	H	on	0
T	T	H	off	1
T	T	T	on	1
T	T	T	off	0

4. Consider the DAG in Figure 1:

   (a) List the Markovian assumptions asserted by the DAG.

   (b) True or false? Why?

   - $d\_separated(A, F, E)$
   - $d\_separated(G, B, E)$
   - $d\_separated(AB, CDE, GH)$

   (c) Express $Pr(a, b, c, d, e, f, g, h)$ in factored form using the chain rule for Bayesian networks.

   (d) Compute $Pr(A = 1, B = 1)$ and $Pr(E = 0 \mid A = 0)$. Justify your answers.

$Pr(A = 0)$	$Pr(A = 1)$
.8	.2

$Pr(B = 0)$	$Pr(B = 1)$
.3	.7

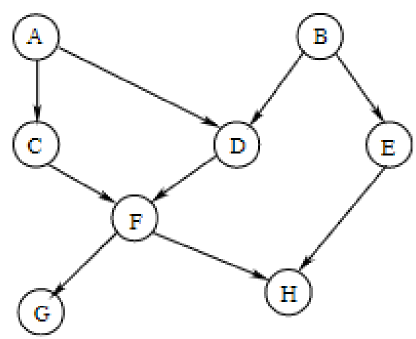
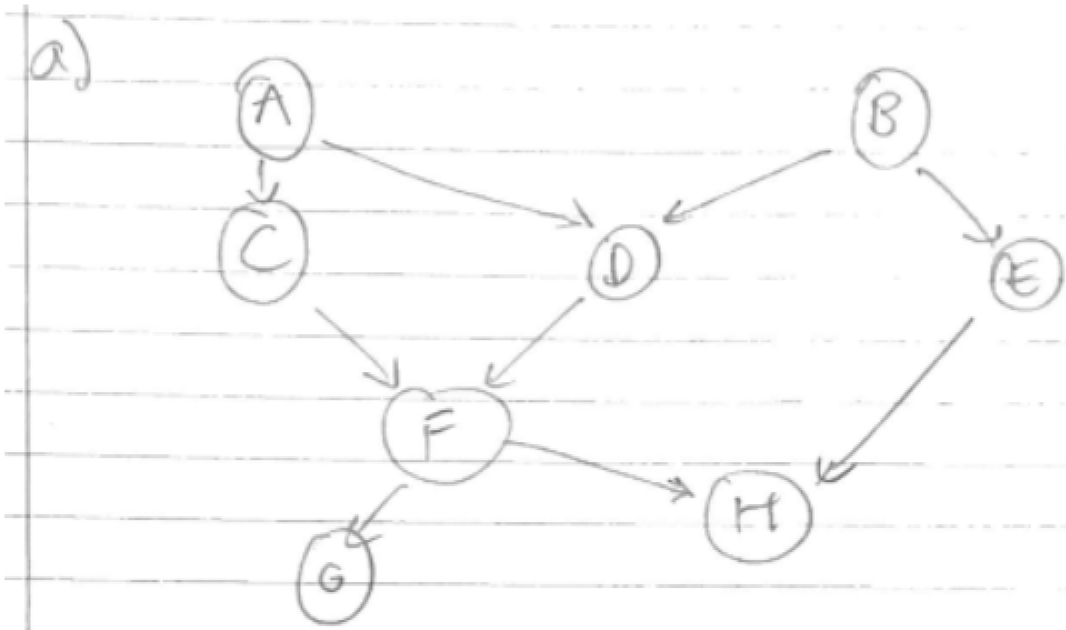


Figure 1: The DAG of a Bayesian network.

	$Pr(E = 0 \mid B)$	$Pr(E = 1 \mid B)$
$B = 0$	.1	.9
$B = 1$	.9	.1

	$Pr(D = 0 \mid A, B)$	$Pr(D = 1 \mid A, B)$
$A = 0, B = 0$	.2	.8
$A = 0, B = 1$	.9	.1
$A = 1, B = 0$	.4	.6
$A = 1, B = 1$	.5	.5



a)

I(A, ∅, BE)
I(B, ∅, AC)
I(C, A, BDE)
I(D, AB, CE)
I(E, B, ACDFG)
I(F, CD, ABE)
I(G, F, ABCDEH)
I(H, EF, ABCDG)

b) d_sep(A, F, E) = ?   Z = {F}
(all) Paths between A, E must be blocked by Z
A-D-B-E: not blocked by Z = {F}
D: convergent - open (descendant F is in Z)
B: divergent - open (B not in Z)

Since A-D-B-E is not blocked by F, not all paths between A, E are blocked, so A, E are not d-separated by F
NO, False

b)

d_separated(A,F,E) is False, since the path A-D-B-E is not blocked by F.

• d-sep (G,B,E) = ?   Z = {B}
3 paths: G-F-C-A-D-B-E, G-F-D-B-E, GF-H-E
G-F-C-A-D-B-E: blocked by Z = {B}
B: divergent - closed (B in Z)
G-F-D-B-E: blocked by Z
B: divergent - closed (B in Z)
G-F-H-E: blocked by Z
H: convergent - closed (H not in Z, has no descendants)

All paths between G, E are blocked by Z = {B}, so G, E are d-separated by B

YES, True

d_separated(G,B,E) is True, since all paths from G to E are blocked by B.

• d-sep (AB, CDE, GH)   Z = {C, D, E}
X = {A, B}
Y = {G, H}

All paths from a node in X to a node in Y go through C, D, or E, which are sequential and in Z, so each path is blocked by Z, so

AB and GH are d-separated by CDE YES True

For A→G:
A-C-F-G: blocked by Z
C: sequential - closed (C in Z)
A-D-F-G: blocked by Z
D: sequential - closed (D in Z)
A-D-B-E-H-F-G: blocked by Z
E: Sequential - closed (E in Z)

For A→H:
A-C-F-H: blocked by Z
C: sequential - closed (C in Z)
A-D-F-H. blocked by Z
D: Sequential - closed (D in Z)
A-D-B-E-H: blocked by Z
E: Sequential - closed (E in Z)
A-C-F-D-B-E-H: blocked by Z
C: Sequential - closed (C in Z)

For B→G:
B-D-A-C-F-G: blocked by Z
C: sequential - closed (C in Z)
B-D-F-G: blocked by Z
D: sequential - closed (D in Z)
B-E-H-F-G: blocked by Z
E: Sequential - closed (E in Z)

B→H:
B-D-A-C-F-H: blocked (C is closed sequential)
B-D-F-H: blocked (D is closed, sequential)
B-E-H: blocked (E closed, sequential)

d_separated(AB, CDE, GH) is True, since all paths from A to G, A to H, B to G, and B to H are blocked by CDE.

c) $Pr(a,b,c,d,e,f,g,h) = ?$

$$Pr(a,b,c,d,e,f,g,h) = Pr(a) \cdot Pr(b) \cdot Pr(c|a) \cdot Pr(d|a,b) \cdot Pr(e|b) \cdot Pr(f|c,d) \cdot Pr(g|f) \cdot Pr(h|e,f)$$

c)

d) $Pr(A=1, B=1) = ?$

We know A and B are independent (given $\emptyset$) from the Markovian assumption: $I(A, \emptyset, BE)$, so

$$Pr(A=1, B=1) = Pr(A=1) \cdot Pr(B=1) = 0.2 \cdot 0.7 = \boxed{0.14}$$

$Pr(E=0|A=0) = ?$

We know A and E are independent (given $\emptyset$) from the Markovian assumption $I(A, \emptyset, BE)$, so

$$Pr(E=0|A=0) = Pr(E=0)$$

$$Pr(E=0) = Pr(E=0|B=0)\,Pr(B=0) + Pr(E=0|B=1)\,Pr(B=1)$$

$$= 0.1 \cdot 0.3 + 0.9 \cdot 0.7$$

$$\boxed{Pr(E=0|A=0) = 0.66}$$

d)

5. Consider the joint probability distribution in Table 1 and the propositional sentence $\alpha : A \Rightarrow B$.

(a) List the models of α.

(b) Compute the probability $Pr(\alpha)$.

(c) Compute the conditional probability distribution $Pr(A, B \mid \alpha)$ as in Table 1.

(d) Compute the probability $Pr(A \Rightarrow \neg B \mid \alpha)$.

	A	B	$Pr(A, B)$
w_0	T	T	0.3
w_1	T	F	0.2
w_2	F	T	0.1
w_3	F	F	0.4

Table 1: A joint probability distribution.

a)

	A	B	α: A => B
w_0	T	T	T
w_1	T	F	F
w_2	F	T	T
w_3	F	F	T

The models of α: M(α) = {w_0, w_2, w_3}

b) Pr(α) = sum of probabilities of worlds that entail α

Pr(α) = Pr(w_0) + Pr(w_2) + Pr(w_3) = 0.3 + 0.1 + 0.4 = **0.8**

c) Pr(A,B | α) = ?

Pr(w | α) = 0 if w entails ¬α; Pr(w | α) = Pr(w) / Pr(α) if w entails α

A	B	Pr(A,B)	Pr(A,B | α)
T	T	0.3	0.3 / 0.8 = **0.375**
T	F	0.2	**0**
F	T	0.1	0.1 / 0.8 = **0.125**
F	F	0.4	0.4 / 0.8 = **0.5**

d) Pr(A => ¬B | α) = ?

A	B	α	A => ¬B
T	T	T	F
T	F	F	T
F	T	T	T
F	F	T	T

M(A => ¬B ∧ α) = $\{w_2, w_3\}$
Pr(A => ¬B | α) = Pr(A => ¬B ∧ α) / Pr(α) = Pr(w_2) + Pr(w_3) = 0.125 + 0.5 = **0.625**