1. (10 pts) Use truth tables (worlds) to show that the following pairs of sentences are equivalent:
   - $P \Rightarrow \neg Q$, $Q \Rightarrow \neg P$

These are equivalent, since $P \Rightarrow \neg Q \equiv \neg P \vee \neg Q$ and $Q \Rightarrow \neg P \equiv \neg Q \vee \neg P$, and $\vee$ is commutative.

| P | Q | $P \Rightarrow \neg Q \equiv \neg P \vee \neg Q$ | $Q \Rightarrow \neg P \equiv \neg Q \vee \neg P$ |
|---|---|---|---|
| T | T | F | F |
| T | F | T | T |
| F | T | T | T |
| F | F | T | T |

   - $P \Leftrightarrow \neg Q$, $((P \wedge \neg Q) \vee (\neg P \wedge Q))$

$P \Leftrightarrow \neg Q \equiv (P \Rightarrow \neg Q) \wedge (\neg Q \Rightarrow P) \equiv (\neg P \vee \neg Q) \wedge (Q \vee P)$

| P | Q | $P \Leftrightarrow \neg Q \equiv ((\neg P \vee \neg Q) \wedge (Q \vee P))$ | $((P \wedge \neg Q) \vee (\neg P \wedge Q))$ |
|---|---|---|---|
| T | T | F | F |
| T | F | T | T |
| F | T | T | T |
| F | F | F | F |

2. (20 pts) Consider the following sentences and decide for each whether it is valid, unsatisfiable, or neither:
   - $(Smoke \Rightarrow Fire) \Rightarrow (\neg Smoke \Rightarrow \neg Fire)$
   - $(Smoke \Rightarrow Fire) \Rightarrow ((Smoke \vee Heat) \Rightarrow Fire)$
   - $((Smoke \wedge Heat) \Rightarrow Fire) \Leftrightarrow ((Smoke \Rightarrow Fire) \vee (Heat \Rightarrow Fire))$

   Justify your answer using truth tables (worlds).

   a. $(Smoke \Rightarrow Fire) \Rightarrow (\neg Smoke \Rightarrow \neg Fire)$

| Smoke | Fire | Smoke $\Rightarrow$ Fire | $\neg$Smoke $\Rightarrow$ $\neg$Fire | (Smoke $\Rightarrow$ Fire) $\Rightarrow$ ($\neg$Smoke $\Rightarrow$ $\neg$Fire) |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | T |
| F | T | T | F | F |

| | | | | |
|---|---|---|---|---|
| F | F | T | T | T |

The sentence is **neither valid nor unsatisfiable**. 3 of the 4 worlds entail the sentence (w1, w2, w4 are satisfiable, but not all are), so the sentence is not valid.

b. $(\text{Smoke} \Rightarrow \text{Fire}) \Rightarrow ((\text{Smoke} \vee \text{Heat}) \Rightarrow \text{Fire})$

| Smoke | Fire | Heat | Smoke ⇒ Fire | (Smoke ∨ Heat) ⇒ Fire | (Smoke ⇒ Fire) ⇒ ((Smoke ∨ Heat) ⇒ Fire) |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | T | T | T |
| T | F | T | F | F | T |
| T | F | F | F | F | T |
| F | T | T | T | T | T |
| F | T | F | T | T | T |
| F | F | T | T | F | F |
| F | F | F | T | T | T |

This sentence is **neither valid nor unsatisfiable**. 7 of the 8 worlds entail the sentence (w1-w6, w8 are satisfiable, but not all are), so the sentence is not valid.

c. $((\text{Smoke} \wedge \text{Heat}) \Rightarrow \text{Fire}) \Leftrightarrow ((\text{Smoke} \Rightarrow \text{Fire}) \vee (\text{Heat} \Rightarrow \text{Fire}))$

| Smoke | Fire | Heat | (Smoke ∧ Heat) ⇒ Fire | Smoke ⇒ Fire | Heat ⇒ Fire | ((Smoke ∧ Heat) ⇒ Fire) ⇔ ((Smoke ⇒ Fire) ∨ (Heat ⇒ Fire)) |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | T | F | T | T | T | T |
| T | F | T | F | F | F | T |
| T | F | F | T | F | T | T |
| F | T | T | T | T | T | T |

| | | | | | | |
|---|---|---|---|---|---|---|
| F | T | F | T | T | T | T |
| F | F | T | T | T | F | T |
| F | F | F | T | T | T | T |

Because the set of worlds that entail the sentence is the set of all worlds, the sentence is valid and satisfiable.

3. (30 pts) Consider the following:

*If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.*

(a) Represent the above information using a propositional logic knowledge base (set of sentences in propositional logic).
(b) Convert the knowledge base into CNF.
(c) Can you use the knowledge base to prove that the unicorn is mythical? How about magical? Horned?

Justify your answers using resolution by providing corresponding resolution derivations. Make sure to clearly define all propositional symbols (variables) first, then define your knowledge base, and finally give your derivations.

a. Mythical ⇒ Immortal
¬Mythical ⇒ (¬Immortal ∧ Mammal)
(Immortal ∨ Mammal) ⇒ Horned
Horned ⇒ Magical

b. KB into CNF

Mythical ⇒ Immortal ≡ **(¬Mythical** ∨ **Immortal)**
¬Mythical ⇒ (¬Immortal ∧ Mammal) ≡ Mythical ∨ (¬Immortal ∧ Mammal) ≡ **(Mythical** ∨ **¬Immortal)** ∧ **(Mythical** ∨ **Mammal)**
(Immortal ∨ Mammal) ⇒ Horned ≡ ¬(Immortal ∨ Mammal) ∨ Horned ≡ (¬Immortal ∧ ¬Mammal) ∨ Horned ≡ **(¬Immortal** ∨ **Horned)** ∧ **(¬Mammal** ∨ **Horned)**
Horned ⇒ Magical ≡ **¬Horned** ∨ **Magical**

KB: **(¬Mythical** ∨ **Immortal)** ∧ **(Mythical** ∨ **¬Immortal)** ∧ **(Mythical** ∨ **Mammal)** ∧ **(¬Immortal** ∨ **Horned)** ∧ **(¬Mammal** ∨ **Horned)** ∧ **(¬Horned** ∨ **Magical)**

c.
Mythical? α: Mythical
1. ¬Mythical ∨ Immortal
2. Mythical ∨ ¬Immortal
3. Mythical ∨ Mammal
4. ¬Immortal ∨ Horned

5. ¬Mammal ∨ Horned
6. ¬Horned ∨ Magical
7. ¬Mythical ¬α

8. 2,7 ¬Immortal
9. 3,7 Mammal
10. 5,9 Horned
11. 6,10 Magical
Resolution is now exhausted, ¬Mythical ∧ ¬Immortal ∧ Mammal ∧ Horned ∧ Magical satisfies the KB

With these rules, we see that we can satisfy KB, so KB ∧ ¬Mythical is satisfiable, so **it is inconclusive whether the unicorn is mythical**.

Magical? α: Magical
1. ¬Mythical ∨ Immortal
2. Mythical ∨ ¬Immortal
3. Mythical ∨ Mammal
4. ¬Immortal ∨ Horned
5. ¬Mammal ∨ Horned
6. ¬Horned ∨ Magical
7. ¬Magical ¬α

8. 6,7 ¬Horned
9. 4,8 ¬Immortal
10. 1,9 ¬Mythical
11. 3,10 Mammal
12. 5,11 Horned
Contradiction: 8, 12 ¬Horned ∧ Horned
**So KB ∧ ¬Magical is unsat, so we prove that the unicorn is magical.**

Horned? α: Horned
1. ¬Mythical ∨ Immortal
2. Mythical ∨ ¬Immortal
3. Mythical ∨ Mammal
4. ¬Immortal ∨ Horned
5. ¬Mammal ∨ Horned
6. ¬Horned ∨ Magical
7. ¬Horned ¬α

8. 4,7 ¬Immortal
9. 1,8 ¬Mythical
10. 3,9 Mammal
11. 5,10 Horned
Contradiction: 7, 11 ¬Horned ∧ Horned

**So KB ∧ ¬Horned is unsat, so we prove that the unicorn is horned.**

4. (20 pts) Consider the two NNF circuits in Figure 1 and Figure 2. Identify whether they are decomposable, deterministic, smooth and why.

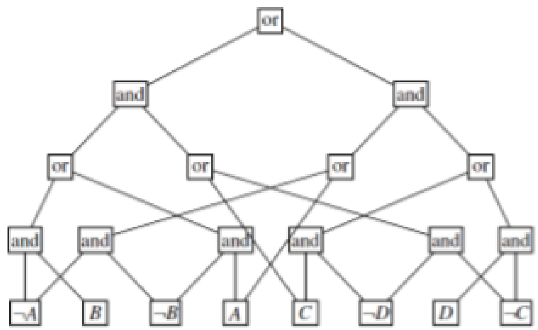



Figure 1

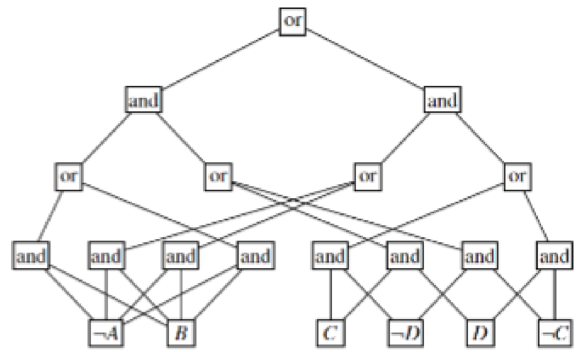



Figure 2

a. Figure 1

Figure 1 is **decomposable**, as none of the AND gate subcircuits share variables. The top two AND gates have AB on one side and CD on the other. The bottom left 3 ANDs have A on one side and B on the other, while the bottom right 3 ANDs have C on one side and D on the other side.

Figure 1 is **not deterministic**, as the top OR gate has more than 1 high input for the circuit input A, ¬B, C, ¬D. The two inputs to the top OR gate are:

((¬A and B) or **(A and ¬B)**) and (**C** or (¬C and ¬D))

((¬A and ¬B) or **A**) and (**(C and ¬D)** or (¬C and D))

With the circuit input A, ¬B, C, ¬D, we get ((A and ¬B) and C) on the left side of the top OR, which is high, and on the right side, we get (A and (¬C and D)), which is also high.

Figure 1 is **not smooth**, as the 2 middle OR gates (the second and third OR gates on the third level of the circuit) do not have exactly the same inputs. The middle left OR gate has inputs C and C,D; the middle right OR gate has inputs A,B and A, so the circuit is not smooth.

b. Figure 2

Figure 2 is **decomposable**, as none of the AND gate subcircuits share variables. The bottom left ANDs have variable A on one side and B on the other; the bottom right ANDs have C on one side and D on the other. The top left and right ANDs have AB on the left subcircuit and CD on the right subcircuit..

Figure 2 is **not deterministic**; the OR gates that have AB variables have the same inputs. The left OR gate is (¬A and B) or (¬A and B), and the middle right OR gate is also (¬A and B) or (¬A and B). So, for circuit input ¬A and B, there will be more than 1 high input for these OR gates.

Figure 2 is **smooth**, as the bottom OR gates have either exactly AB or CD subcircuits, and the top OR gate has ABCD inputs on both sides.

5. (20 pts) Given a propositional formula, where each literal has a weight ω in [0,1], the weight of a truth assignment is defined as the product of its literals weights. For example, ω(A, ¬B, C) = ω(A) ω(¬B) ω(C). The Weighted Model Count (WMC) of a propositional formula is defined as the added weight of its satisfying assignments (i.e., models).

Suppose we have the following literal weights: ω(A)=0.1, ω(¬A)=0.9, ω(B)=0.3, ω(¬B)=0.7, ω(C)=0.5, ω(¬C)=0.5, ω(D)=0.7, ω(¬D)=0.3.

(a) Compute the Weighted Model Count for formula (¬A ∧ B) ∨ (¬B ∧ A) by enumerating its models, computing their weights, then adding them up.

| A | B | ¬A ∧ B | ¬B ∧ A | (¬A ∧ B) ∨ (¬B ∧ A) |
|---|---|---|---|---|
| T | T | F | F | F |
| T | F | F | T | T |
| F | T | T | F | T |
| F | F | F | F | F |

Models: ¬B, A and ¬A, B
ω(¬B, A) = 0.7*0.1 = 0.07
ω(¬A, B) = 0.9*0.3 = 0.27
Added weight of models = **0.34**

(b) Consider the decomposable, deterministic and smooth NNF circuit in Figure 3. If we assign the weights of literals to all the leaf nodes, the count of each ∧ node is computed as the product of the counts of its children, and the count of each ∨ node is computed as the sum of the counts of its children. What is the relation between the count on the root with the Weighted Model Count for the formula?

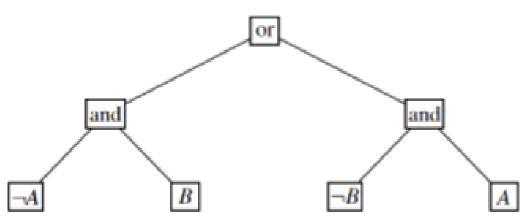



Figure 3

The counts for the count on the root and the Weighted Model Count are the same; if we have a decomposable, deterministic, smooth NNF circuit for a formula, we can easily find the WMC just by computing the product of counts at AND gates and computing the sum of counts at OR gates.

(c) Compute the Weighted Model Count for the formula associated with the decomposable, deterministic and smooth NNF circuit in Figure 4.

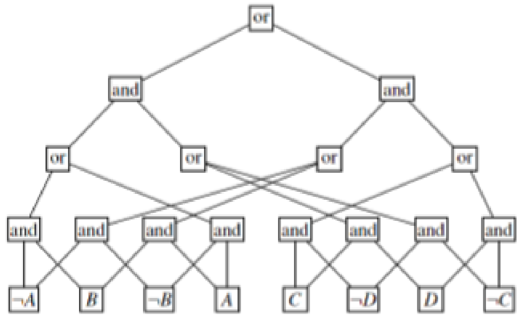
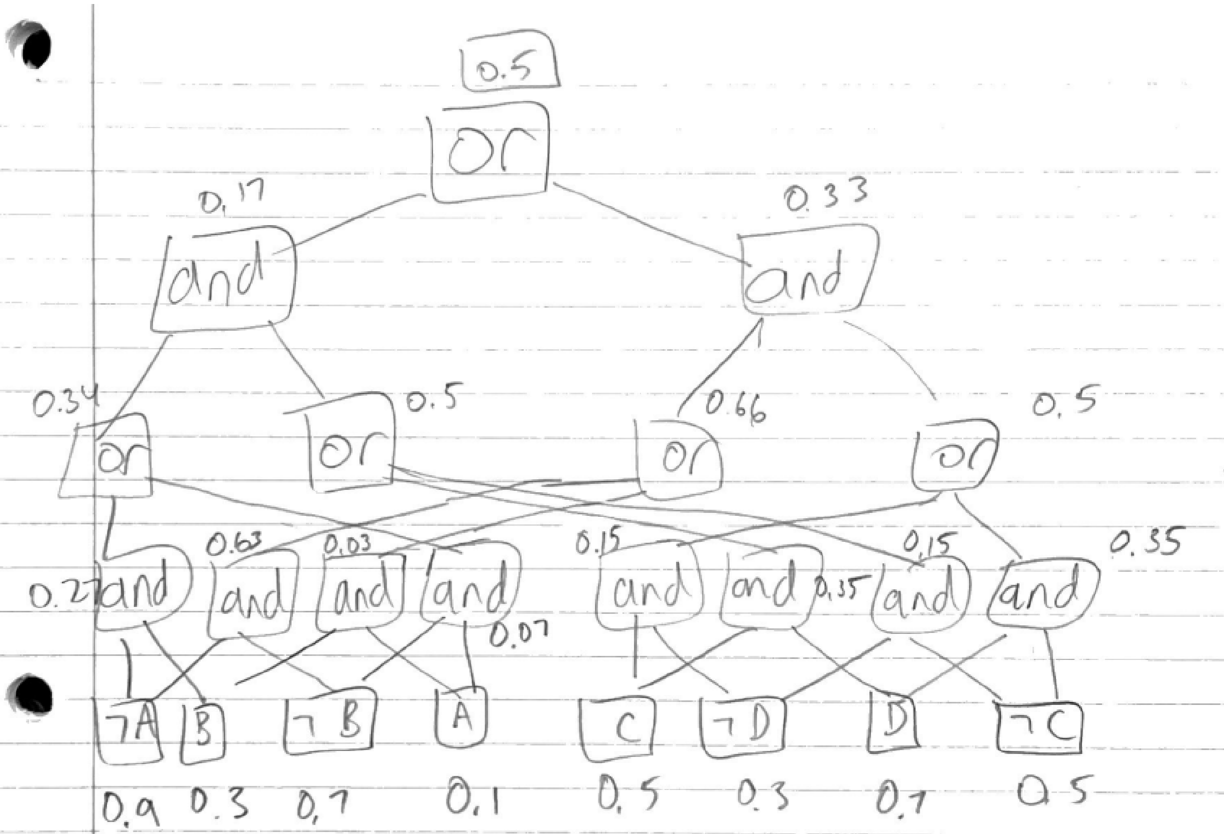



Figure 4

Suppose we have the following literal weights: ω(A)=0.1, ω(¬A)=0.9, ω(B)=0.3, ω(¬B)=0.7, ω(C)=0.5, ω(¬C)=0.5, ω(D)=0.7, ω(¬D)=0.3.




WMC = 0.5