The Double Pendulum

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Introduction

My project was to simulate and analyze the double pendulum. Since this kind of pendulum is a chaotic system, I wanted to see how two different numerical methods would calculate the path. I chose Euler's method and the fourth order Runge-Kutta method. In fact, Euler's method was just recently taught and was explained to be one of the easiest methods to numerically integrate ordinary differential equations. But due to its simplicity, it also has a higher degree of error. I wanted to see how it would perform in a chaotic system. To determine whether it is indeed a viable solution for the equations of motion of the double pendulum, I analyzed the mechanical energy of the system. The only force acting on the pendulum in my simulation is gravity, so the total energy must be conserved. My hypothesis was that Euler's method would be too simplistic as an ODE solver and that this would manifest in a change of mechanical energy (whether decrease or increase, but not stay constant).

Description of Numerical Method

Before applying the ODE solvers, the equations of motion of the pendulum must be obtained. The only assumption made is that the lengths of both rods are equal. All other values can be freely manipulated. The first step is to find the Lagrangian of the pendulum, given by

$$L = T - V$$

where T is the kinetic energy and V is the potential energy. We can then obtain the canonical momentum equations, by deriving the Lagrangian with respect to the angular velocity of each bob:

$$p_1 = \frac{\partial L}{\partial \dot{\theta}_1}$$

$$p_2 = \frac{\partial L}{\partial \dot{\theta}_2}$$

These are needed in the next step which is to convert the Lagrangian equation into the Hamiltonian equation. The relationship of the two is expressed by the following:

$$H = \sum_{i=1}^{2} \left(\dot{\theta}_i p_i \right) - L$$

Finally, the Hamilton equations can be obtained by deriving the Hamiltonian with respect to both momenta and both angles.

$$\dot{\theta}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p_i} = -\frac{\partial H}{\partial \theta_i}$$

The final equations of motion are the following:

$$\dot{\theta}_1 = \frac{lp_1 - lp_2 \cos(\theta_1 - \theta_2)}{l^3 \left[m_1 + m_2 \sin^2(\theta_1 - \theta_2) \right]}$$

$$\dot{\theta}_2 = \frac{l(m_1 + m_2) p_2 - lm_2 p_1 \cos(\theta_1 - \theta_2)}{l^3 m_2 \left[m_1 + m_2 \sin^2(\theta_1 - \theta_2) \right]}$$

$$\dot{p_1} = -(m_1 + m_2) gl \sin\theta_1 - A_1 + A_2$$

$$\dot{p_2} = -m_2 gl \sin\theta_2 + A_1 - A_2$$

where

$$A_1 = \frac{p_1 p_2 \sin(\theta_1 - \theta_2)}{l^2 \left[m_1 + m_2 sin^2 (\theta_1 - \theta_2) \right]}$$

$$A_2 = \frac{l^2 m_2 p_1^2 + l^2 (m_1 + m_2) p_2^2 - l^2 m_2 p_1 p_2 \cos(\theta_1 - \theta_2)}{2l^4 \left[m_1 + m_2 \sin^2(\theta_1 - \theta_2) \right]^2} \sin[2(\theta_1 - \theta_2)]$$

At this point there are four first order differential equations. These can now be solved using the numerical methods in question, namely Euler's and the fourth order Runge-Kutta.

Euler's:

$$y_{n+1} = y_n + hf'(t_n, y_n)$$

RK4:

$$k_1 = hf'(t_n, y_n)$$

$$k_2 = hf'\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf'\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_4 = hf'(t_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O\left(h^5\right)$$

By plugging the values obtained from each iteration into the energy equations, we can obtain the mechanical energy of the system as it evolves with time:

$$K = 0.5mv^2$$

$$U = mgh$$

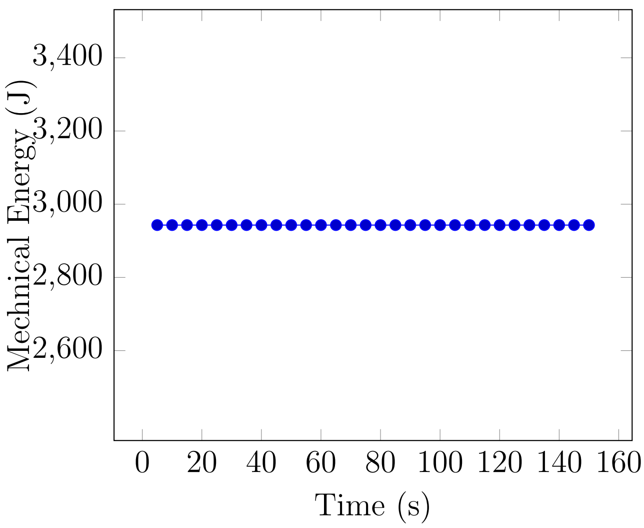
$$E_{mechanical} = K + U$$

where K is the kinetic energy and U is the potential energy.

Results

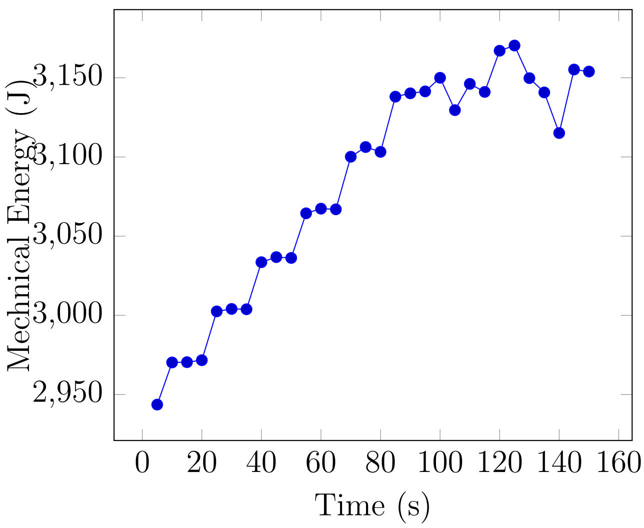
In the simulations, I set the length to 100 m, both masses to 1 kg, the time step to 0.01 s and the pendulum to an exact horizontal position. Running the simulation using the fourth order Runge-Kutta method gave these results:

Evolution of Energy using RK4



This serves as validation for the code. Since the energy is conserved at 2943 J, I know that there are no logical errors in the code. Running the simulation using Euler's method results in the following:

Evolution of Energy using Euler's Method



The energy of the system is increasing. It begins at the proper value of 2943 J, but from there it continues to increase indefinitely. The rate of increase does seem to decrease, but it still displays a fluctuating behaviour, which is inappropriate for the evolution of the system.

Euler's method clearly has a major weakness. Each iteration has too much error for it even to properly describe the path of the double pendulum in the first 5 seconds. It cannot adequately represent the evolution of the double pendulum, let alone of other chaotic systems.

References

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