

Introduction to Analysis Notes

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1 Sets and Lists

Definition 1.1. A set is an unordered group of distinct elements.

Example 1.1. $\{1, 2, 3\}$ is a set containing three elements: 1, 2, and 3.

Note. $\{1, 2, 3, 3\}$ is also set containing three elements, since the elements of a set are distinct.

Definition 1.2. The empty set (denoted \emptyset) is the unique set having no elements.

One of the most fundamental operations of sets is the “element of” operation, denoted by \in . $x \in X$ is true precisely when x is an element of the set X . Note that sets can be elements of other sets. $x \notin X$ is used to denote “not an element of”.

Example 1.2. $\{\{1, 2\}, \{\}\}$ is a set containing two elements: the set $\{1, 2\}$, and the empty set.

Set comprehensions, or set builder notation, is a method of precisely defining a set. It can take various forms, such as enumerating all (or implying such) the elements of a set (e.g. $\{1, 2\}$ or $\{1, 2, \dots, 5\}$). Or it can be used to build a set from another, such as $\{2n \mid n \in \mathbb{N}\}$, which says make a set by taking every natural number and doubling it — these are, of course, the even natural numbers. Set comprehensions can be made more complicated by including a predicate, for example $\{n \in \mathbb{N} \mid n \neq n^2\}$ — all natural numbers which are not their own square.

Two sets are equal when they contain precisely the same elements. For example, if we let $A = \{1, 1, 5, 2\}$ and $B = \{2, 2, 2, 1, 5\}$, then $A = B$ since for every element x in A , x is also in B and vice versa.

Definition 1.3. A set T is a *subset* of a set S when every element of T is also an element of S . This relationship is denoted $T \subseteq S$. S is also referred to as a *super set* of T .

Example 1.3. $\{1, 2, 3\}$ is a subset of $\{1, 2, 3\}$.

Remark. If a set A is a subset of set B , and B is a subset of A , then the sets must be equal. Showing that $A \subseteq B$ and $B \subseteq A$ is a common way to prove that two sets are equal.

Definition 1.4. A set T is a *proper subset* of a set S when every element of T is also an element of S , but not vice versa — that is, the sets are not equal. This relationship is denoted $T \subset S$.

Example 1.4. $\{1\}$ is a proper subset of $\{1, 2, 3\}$.

Definition 1.5. The intersection of sets A and B is the set $\{x \mid x \in A \wedge x \in B\}$. It is denoted by $A \cap B$.

Example 1.5. $\{1, 2, 3, 4\} \cap \{3, 4, 5\} = \{3, 4\}$.

Definition 1.6. The union of sets A and B is the set $\{x \mid x \in A \vee x \in B\}$. It is denoted by $A \cup B$.

Example 1.6. $\{1, 2, 3, 4\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$.

Remark. If A and B are sets, then $(A \cap B) \subseteq (A \cup B)$. $A = B$ if and only if $(A \cap B) = (A \cup B)$.

Definition 1.7. The complement of set A with respect to some super set U is the set $\{x \in U \mid x \notin A\}$. The complement is sometimes denoted A' .

Example 1.7. Let $U = \{1, 2, 3, 4, 5\}$, and $A = \{1, 2\}$. Then $A' = \{3, 4, 5\}$.

Example 1.8. Let $U = \mathbb{Z}$, and A the even numbers. Then A' is the set of the odd numbers.

Remark. $A' \cup A = U$. $A' \cap A = \emptyset$.

Definition 1.8. The set difference of sets A and B , denoted $A \setminus B$, is the set containing all elements of A which are not elements of B . $A \setminus B = \{x \in A \mid x \notin B\}$.

Definition 1.9. The symmetric difference of sets A and B , denoted $A \triangle B$, is defined to be the set $(A \setminus B) \cup (B \setminus A)$.

Remark. $(A \triangle B)' = (A \cap B)$ when $U = A \cup B$.

While sets are unordered groups of distinct elements, lists (also called n -tuples) are ordered groups of elements which are not necessarily distinct. An ordered pair (a, b) is a list of a length two (a tuple), where a , and b are elements of some set.

Definition 1.10. An ordered pair (a, b) is a tuple of elements of some set.

Ordered pairs (and n -tuples more generally) can be represented as sets themselves — the pair (a, b) can be represented as the set $\{a, \{a, b\}\}$.

Definition 1.11. The Cartesian product of two sets A and B is denoted $A \times B$. It is equal to $\{(a, b) \mid a \in A, b \in B\}$.

2 Boolean Algebra

Boolean algebra is the algebra dealing exclusively with the values *true* and *false*.

The primary operations of Boolean algebra are *negation* (also called *not*) denoted by \neg , *conjunction* (also called *and*) denoted by \wedge , and *disjunction* (also called *or*) denoted by \vee .

Since Boolean algebra has only two elements, it is possible to enumerate all variable combinations for a function. This is often done in the form of a truth table — a table listing the values of variables and the corresponding function value as rows. For example, Table 1 gives a combined truth table for negation, conjunction, and disjunction. It also serves as the definition of these operations.

Table 1: Truth table of primary operations

X	Y	$\neg X$	$X \wedge Y$	$X \vee Y$
True	True	False	True	True
True	False	False	False	True
False	True	True	False	True
False	False	True	False	False