Introduction to Analysis Notes

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1 Boolean Algebra

Boolean algebra is the algebra dealing exclusively with the values *true* and *false*.

The primary operations of Boolean algebra are *negation* (also called *not*) denoted by $\neg$, *conjunction* (also called *and*) denoted by $\wedge$, and *disjunction* (also called *or*) denoted by $\vee$.

Since Boolean algebra has only two elements, it is possible to enumerate all variable combinations for a function. This is often done in the form of a truth table — a table listing the values of variables and the corresponding function value as rows. For example, Table 1 gives a combined truth table for negation, conjunction, and disjunction. It also serves as the definition of these operations.

Table 1: Truth table of primary operations

X	Y	$\neg X$	$X \wedge Y$	$X \vee Y$
True	True	False	True	True
True	False	False	False	True
False	True	True	False	True
False	False	True	False	False

Definition 1.1. A statement P *implies* (also $\implies$) statement Q if Q is true any time that P is true. When P is false, Q can be true or false. If P is always false, then it implies all statements. $P \implies Q$ is equivalent to $Q \impliedby P$.

Definition 1.2. Statement P *if and only if* (also *iff* and $\iff$) statement Q if $P \implies Q$ and $Q \implies P$.

Theorem 1.1.

Conjunction and disjunction are commutative:

1. $x \wedge y = y \wedge x$
2. $x \vee y = y \vee x$

Conjunction and disjunction are associative:

1. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
2. $(x \vee y) \vee z = x \vee (y \vee z)$

Conjunction and disjunction are distributive:

1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

2. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

True is the identity element for conjunction and false is the identity element for disjunction:

1. $x \wedge true = x$
2. $x \vee false = x$

DeMorgan's Laws:

1. $\neg(x \wedge y) = (\neg x) \vee (\neg y)$
2. $\neg(x \vee y) = (\neg x) \wedge (\neg y)$

Additional properties:

1. $\neg(\neg x) = x$
2. $x \wedge x = x$
3. $x \vee x = x$
4. $x \wedge \neg x = false$
5. $x \vee \neg x = true$

Proof. All properties in Theorem 1.1 can be proved by simply writing out the corresponding truth tables. □

2 Sets and Lists

Definition 2.1. A set is an unordered group of distinct elements.

Example 2.1. $\{1, 2, 3\}$ is a set containing three elements: 1, 2, and 3.

Note. $\{1, 2, 3, 3\}$ is also set containing three elements, since the elements of a set are distinct.

Definition 2.2. The empty set (denoted $\varnothing$) is the unique set having no elements.

One of the most fundamental operations of sets is the "element of" operation, denoted by $\in$. $x \in X$ is true precisely when x is an element of the set X. Note that sets can be elements of other sets. $x \notin X$ is used to denote "not an element of".

Example 2.2. $\{\{1, 2\}, \{\}\}$ is a set containing two elements: the set $\{1, 2\}$, and the empty set.

Set comprehensions, or set builder notation, is a method of precisely defining a set. It can take various forms, such as enumerating all (or implying such) the elements of a set (e.g. $\{1, 2\}$ or $\{1, 2, \ldots, 5\}$). Or it can be used to build a set from another, such as $\{2n \mid n \in \mathbb{N}\}$, which says make a set by taking every natural number and doubling it — these are, of course, the even natural numbers. Set comprehensions can be made more complicated by including a predicate, for example $\{n \in \mathbb{N} \mid n \neq n^2\}$ — all natural numbers which are not their own square.

Two sets are equal when they contain precisely the same elements. For example, if we let $A = \{1, 1, 5, 2\}$ and $B = \{2, 2, 2, 1, 5\}$, then $A = B$ since for every element x in A, x is also in B and vice versa.

Definition 2.3. A set T is a *subset* of a set S when every element of T is also an element of S. This relationship is denoted $T \subseteq S$. S is also referred to as a *super set* of T.

Example 2.3. $\{1, 2, 3\}$ is a subset of $\{1, 2, 3\}$.

Remark. If a set A is a subset of set B, and B is a subset of A, then the sets must be equal. Showing that $A \subseteq B$ and $B \subseteq A$ is a common way to prove that two sets are equal.

Definition 2.4. A set T is a *proper subset* of a set S when every element of T is also an element of S, but not vice versa — that is, the sets are not equal. This relationship is denoted $T \subset S$.

Example 2.4. $\{1\}$ is a proper subset of $\{1, 2, 3\}$.

Definition 2.5. The intersection of sets A and B is the set $\{x \mid x \in A \wedge x \in B\}$. It is denoted by $A \cap B$.

Example 2.5. $\{1, 2, 3, 4\} \cap \{3, 4, 5\} = \{3, 4\}$.

Definition 2.6. The union of sets A and B is the set $\{x \mid x \in A \vee x \in B\}$. It is denoted by $A \cup B$.

Example 2.6. $\{1, 2, 3, 4\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$.

Remark. If A and B are sets, then $(A \cap B) \subseteq (A \cup B)$. $A = B$ if and only if $(A \cap B) = (A \cup B)$.

Definition 2.7. The complement of set A with respect to some super set U is the set $\{x \in U \mid x \notin A\}$. The complement is sometimes denoted A'.

Example 2.7. Let $U = \{1, 2, 3, 4, 5\}$, and $A = \{1, 2\}$. Then $A' = \{3, 4, 5\}$.

Example 2.8. Let $U = \mathbb{Z}$, and A the even numbers. Then A' is the set of the odd numbers.

Remark. $A' \cup A = U$. $A' \cap A = \varnothing$.

Definition 2.8. The set difference of sets A and B, denoted $A \setminus B$, is the set containing all elements of A which are not elements of B. $A \setminus B = \{x \in A \mid x \notin B\}$.

Definition 2.9. The symmetric difference of sets A and B, denoted $A \triangle B$, is defined to be the set $(A \setminus B) \cup (B \setminus A)$.

Remark. $(A \triangle B)' = (A \cap B)$ when $U = A \cup B$.

While sets are unordered groups of distinct elements, lists (also called n-tuples) are ordered groups of elements which are not necessarily distinct. An ordered pair (a, b) is a list of a length two (a tuple), where a, and b are elements of some set.

Definition 2.10. An ordered pair (a, b) is a tuple of elements of some set.

Ordered pairs (and n-tuples more generally) can be represented as sets themselves — the pair (a, b) can be represented as the set $\{a, \{a, b\}\}$.

Definition 2.11. The Cartesian product of two sets A and B is denoted $A \times B$. It is equal to $\{(a, b) \mid a \in A, b \in B\}$.

3 Binary Operation

Definition 3.1. A *binary operation* is a mathematical operation of arity two, where both domains and the codomain are the same set.

Example 3.1. Addition on $\mathbb{R}$ is a binary operation, as are multiplication and subtraction.

Definition 3.2. A binary operation $\circ$ on a set S is commutative if $x \circ y = y \circ x$ for all $x, y \in S$.

Definition 3.3. A binary operation $\circ$ on a set S is associative if $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in S$.

Definition 3.4. Let $\circ$ be a binary operation on a set S. Then an element e of S is called a left identity if $e \circ a = a$ for all a in S, and a right identity if $a \circ e = a$ for all a in S. If e is both a left and a right identity, then it is simply an identity.

Theorem 3.1. *Let $\circ$ be a binary operation on a set S. If $\circ$ has both a left identity and a right identity, then those identities are the same.*

Proof. Let e_1 be a left identity for $\circ$ and e_2 be a right identity. Then $e_1 = e_1 \circ e_2 = e_2$, so $e_1 = e_2$. □

Corollary 3.1.1. *If a binary operation has both a left identity and a right identity, it has only a single unique identity.*

Definition 3.5. Let $\circ$ be a binary operation on a set S, and e be an identity element. An element x of a set S is *invertible* if there exists some $x' \in S$ such that $x \circ x' = e$. x' is the *inverse* of x.

Theorem 3.2. *Let $\circ$ be a binary associative operation on a set S, and $u \in S$ be an invertible element. Then for all $x, y \in S$, $(x \circ u = y \circ u) \implies (x = y)$.*

Proof. Let u' be the inverse of the invertible element u. Then $(x \circ u) = (y \circ u) \implies (x \circ u) \circ u' = (y \circ u) \circ u'$. Since $\circ$ is associative, this implies that $x \circ (u \circ u') = y \circ (u \circ u')$. Since $u \circ u' = e$, we have $x \circ e = y \circ e$, and so $x = y$. $\square$

Corollary 3.2.1. *Let $\circ$ be a binary associative operation on a set S, and $u \in S$ be an invertible element. Then for all $x, y \in S$, $(u \circ x = u \circ y) \implies (x = y)$.*