

When $G'_i(\tau_i) = 0$, $a_i V'_i(\tau_i) = -\Pi'_i(\tau_i)$. Note also $\Pi'_i(\tau_i) = (\sigma - 1)\Pi_i(\tau_i)A(\tau_i)$ and $A'(\tau_i) = A(\tau_i)B(\tau_i)$.

$$\begin{aligned} G''_i(\tau_i) &= a_i [P_i(\tau_i)^{-\alpha} r''_i(\tau_i) - \alpha r'_i(\tau_i) P_i(\tau_i)^{-\alpha} A(\tau_i) - \alpha V'_i(\tau_i) A(\tau_i) - \alpha V_i(\tau_i) A'(\tau_i)] + \\ &\quad (\sigma - 1) \Pi'_i(\tau_i) A(\tau_i) + (\sigma - 1) \Pi_i(\tau_i) A'(\tau_i) \\ &= a_i [P_i(\tau_i)^{-\alpha} r''_i(\tau_i) - \alpha r'_i(\tau_i) P_i(\tau_i)^{-\alpha} A(\tau_i) - \alpha V'_i(\tau_i) A(\tau_i) - \alpha V_i(\tau_i) A'(\tau_i)] - \\ &\quad (\sigma - 1) a_i V'_i(\tau_i) A(\tau_i) - a_i V'_i(\tau_i) B(\tau_i) \end{aligned}$$

Consider first the case where $a_i = \infty$. Then $V'_i = 0 \implies \alpha V_i(\tau_i) A(\tau_i) = r'_i(\tau_i) P_i(\tau_i)^{-\alpha}$. Then,

$$\begin{aligned} V''_i(\tau_i) &= r''_i(\tau_i) P_i(\tau_i)^{-\alpha} - \alpha r'_i(\tau_i) P_i(\tau_i)^{-\alpha} A(\tau_i) - \alpha V'_i(\tau_i) A(\tau_i) - \alpha V_i(\tau_i) A'(\tau_i) \\ &= r''_i(\tau_i) P_i(\tau_i)^{-\alpha} - \alpha r'_i(\tau_i) P_i(\tau_i)^{-\alpha} A(\tau_i) - \alpha V_i(\tau_i) A'(\tau_i) \\ &= r''_i(\tau_i) P_i(\tau_i)^{-\alpha} - \alpha r'_i(\tau_i) P_i(\tau_i)^{-\alpha} A(\tau_i) - \alpha V_i(\tau_i) A(\tau_i) B(\tau_i) \\ &= r''_i(\tau_i) P_i(\tau_i)^{-\alpha} - \alpha^2 V_i(\tau_i) A(\tau_i)^2 - \alpha V_i(\tau_i) A(\tau_i) B(\tau_i) \\ &= r''_i(\tau_i) P_i(\tau_i)^{-\alpha} - \alpha A(\tau_i) V_i(\tau_i) (\alpha A(\tau_i) + B(\tau_i)) \end{aligned}$$

Now examine $r''_i(\tau_i) P_i(\tau_i)^{-\alpha}$

$$\begin{aligned} P_i(\tau_i)^{-\alpha} r''_i(\tau_i) &= P_i(\tau_i)^{-\alpha} \left(r'_i(\tau_i) B(\tau_i) + r_i(\tau_i) B'(\tau_i) + p x_{ij}^*(\tau_i) B(\tau_i) + \lambda'(\tau_i) r'_i(\tau_i) + \lambda(\tau_i) r''_i(\tau_i) + p x_{ij}^*(\tau_i) (\alpha I_i(\tau_i) \right. \\ &\quad \left. = P_i(\tau_i)^{-\alpha} (r'_i(\tau_i) B(\tau_i) + r_i(\tau_i) B'(\tau_i) + p x_{ij}^*(\tau_i) B(\tau_i) + \lambda'(\tau_i) r'_i(\tau_i) + \lambda(\tau_i) r''_i(\tau_i) + A(\tau_i) r'_i(\tau_i)) \right) \\ &= P_i(\tau_i)^{-\alpha} r_i(\tau_i) B'(\tau_i) + P_i(\tau_i)^{-\alpha} r'_i(\tau_i) (A(\tau_i) + B(\tau_i)) + \alpha I(\tau_i) P_i(\tau_i)^{-\alpha} A(\tau_i) B(\tau_i) + \\ &\quad P_i(\tau_i)^{-\alpha} (\lambda'(\tau_i) r'_i(\tau_i) + \lambda(\tau_i) r''_i(\tau_i)) \\ &= P_i(\tau_i)^{-\alpha} r_i(\tau_i) B'(\tau_i) + \alpha V(\tau_i) A(\tau_i) (A(\tau_i) + B(\tau_i)) + \alpha V(\tau_i) B(\tau_i) + \\ &\quad P_i(\tau_i)^{-\alpha} (\lambda'(\tau_i) r'_i(\tau_i) + \lambda(\tau_i) r''_i(\tau_i)) \\ &= P_i(\tau_i)^{-\alpha} r_i(\tau_i) B'(\tau_i) + \alpha V(\tau_i) A(\tau_i) (A(\tau_i) + B(\tau_i)) + \alpha V(\tau_i) B(\tau_i) + \\ &\quad \alpha V(\tau_i) A(\tau_i) I(\tau_i)^{-1} (1 - \lambda(\tau_i)) + P_i(\tau_i)^{-\alpha} \lambda(\tau_i) r''_i(\tau_i) \end{aligned}$$

Finally note $B'(\tau_i) < -B(\tau_i)$ and

$$r_i(\tau_i) = B(\tau_i)^{-1} (r'_i(\tau_i) - p x_{ij}^*(\tau_i))$$

Then,

$$\begin{aligned} P_i(\tau_i)^{-\alpha} r_i(\tau_i) B(\tau_i) &= P_i(\tau_i)^{-\alpha} r'_i(\tau_i) - P_i(\tau_i)^{-\alpha} p x_{ij}^*(\tau_i) \\ &= \alpha V_i(\tau_i) A(\tau_i) - P_i(\tau_i)^{-\alpha} A(\tau_i) \alpha I(\tau_i) \\ &= 0 \end{aligned}$$