Let

$$C(\hat{\tilde{\boldsymbol{\tau}}}) = \frac{1}{\hat{c}^{-1}\left(\hat{G}_j(\hat{\tau}_i^{j\star}) - \hat{G}_j(\hat{\tilde{\boldsymbol{\tau}}})\right) - 1}$$

and

$$Y_{ji}(\hat{\tilde{\boldsymbol{\tau}}}; \boldsymbol{b}) = \hat{G}_j(\hat{\tilde{\boldsymbol{\tau}}}) - \hat{G}_j(\hat{\tau}_i^{j\star})$$

Note first that

$$\begin{aligned}
\chi_{ji}(\epsilon_{ji}^\star) &= \frac{e^{-\boldsymbol{\alpha}^T \boldsymbol{W}_{ji} + \epsilon_{ji}^\star} m_{ji}}{e^{-\boldsymbol{\alpha}^T \boldsymbol{W}_{ji} + \epsilon_{ji}^\star} m_{ji} + m_{ii}} \\
&= \frac{e^{-\ln\left(\frac{m_{ji}}{m_{ii}}\right) + \ln\left(C(\hat{\tilde{\boldsymbol{\tau}}})\right)} m_{ji}}{e^{-\ln\left(\frac{m_{ji}}{m_{ii}}\right) + \ln\left(C(\hat{\tilde{\boldsymbol{\tau}}})\right)} m_{ji} + m_{ii}} \\
&= \frac{\frac{m_{ii}}{m_{ji}} C(\hat{\tilde{\boldsymbol{\tau}}}) m_{ji}}{\frac{m_{ii}}{m_{ji}} C(\hat{\tilde{\boldsymbol{\tau}}}) m_{ji} + m_{ii}} \\
&= \frac{m_{ii} C(\hat{\tilde{\boldsymbol{\tau}}})}{m_{ii} C(\hat{\tilde{\boldsymbol{\tau}}}) + m_{ii}} \\
&= \frac{C(\hat{\tilde{\boldsymbol{\tau}}})}{C(\hat{\tilde{\boldsymbol{\tau}}}) + 1}
\end{aligned}$$

and that

$$\chi_{ji}(\bar{\epsilon}_{ji}) = 1$$

Now the expected utility for government i in stage 1 is

$$\begin{aligned}
\mathrm{E}[L(\hat{\tilde{\boldsymbol{\tau}}}, \boldsymbol{m})] &= \mathrm{E}\left[G_i(\hat{\tilde{\boldsymbol{\tau}}}) - \sum_{j \neq i} \eta_{ij} \hat{G}_j(\hat{\tilde{\boldsymbol{\tau}}}; b_j) - \sum_{j \neq i} \lambda_{ji}^{\chi}(\boldsymbol{\alpha}, \boldsymbol{\epsilon}) \left(Y_{ji}(\hat{\tilde{\boldsymbol{\tau}}}; \boldsymbol{b}) + \hat{c}\chi_{ji}(\boldsymbol{m}; \boldsymbol{\alpha}, \epsilon_{ji})^{-1}\right)\right] \\
&= G_i(\hat{\tilde{\boldsymbol{\tau}}}) - \sum_{j \neq i} \int_{\eta_{ij}} \eta_{ij} \hat{G}_j(\hat{\tilde{\boldsymbol{\tau}}}; b_j) f_\eta(\eta_{ij}) d\eta_{ij} - \sum_{j \neq i} \int_{\epsilon_{ji}} \lambda_{ji}^{\chi}(\boldsymbol{\alpha}, \boldsymbol{\epsilon}) \left(Y_{ji}(\hat{\tilde{\boldsymbol{\tau}}}; \boldsymbol{b}) + \hat{c}\chi_{ji}(\boldsymbol{m}; \boldsymbol{\alpha}, \epsilon_{ji})^{-1}\right) f_\epsilon(\epsilon_{ji}) \\
&= G_i(\hat{\tilde{\boldsymbol{\tau}}}) - \sum_{j \neq i} \int_{\epsilon_{ji}} \lambda_{ji}^{\chi}(\boldsymbol{\alpha}, \boldsymbol{\epsilon}) \left(Y_{ji}(\hat{\tilde{\boldsymbol{\tau}}}; \boldsymbol{b}) + \hat{c}\chi_{ji}(\boldsymbol{m}; \boldsymbol{\alpha}, \epsilon_{ji})^{-1}\right) f_\epsilon(\epsilon_{ji}) d\epsilon_{ji} \\
&= G_i(\hat{\tilde{\boldsymbol{\tau}}}) - \sum_{j \neq i} \int_{\epsilon_{ji}^\star}^{\bar{\epsilon}_{ji}} \lambda_{ji}^{\chi}(\boldsymbol{\alpha}, \boldsymbol{\epsilon}) \left(Y_{ji}(\hat{\tilde{\boldsymbol{\tau}}}; \boldsymbol{b}) + \hat{c}\chi_{ji}(\boldsymbol{m}; \boldsymbol{\alpha}, \epsilon_{ji})^{-1}\right) f_\epsilon(\epsilon_{ji}) d\epsilon_{ji}
\end{aligned}$$

because $\lambda_{ji}^{\chi}(\epsilon_{ji}) = 0$ whenever $\epsilon_{ji} \leq \epsilon_{ji}^\star$.

Note that I'm assuming multipliers don't depend on $\boldsymbol{m}$ for epsilon of interest... can I show this?

A few facts:

$$Y_{ji}(\hat{\tilde{\boldsymbol{\tau}}}^\star; \boldsymbol{b}) + \hat{c}\chi_{ji}(\boldsymbol{m}; \boldsymbol{\alpha}, \epsilon_{ji})^{-1} = 0$$

THIS GIVES US THE VALUE OF $C(\hat{\tilde{\boldsymbol{\tau}}}^{\star})$. And pins down lower bound on induced chi distribution as a function of $\boldsymbol{m}$ and parameters. So if we do change of variables and integrate over distribution of χ maybe this gives us a closed form for the integral?

Also,

$$\begin{aligned}
\frac{\partial \hat{c}\chi_{ji}(\boldsymbol{m}; \boldsymbol{\alpha}, \epsilon_{ji})^{-1}}{\partial m_{ii}} &= -\hat{c}\chi_{ji}(\boldsymbol{m}; \boldsymbol{\alpha}, \epsilon_{ji})^{-2} \frac{\partial \chi_{ji}(\boldsymbol{m}; \boldsymbol{\alpha}, \epsilon_{ji}}{\partial m_{ii}} \\
&= \hat{c} \left(\frac{\rho_{ji} m_{ji} + m_{ii}}{\rho_{ji} m_{ji}} \right)^2 \frac{\rho_{ji} m_{ji}}{\left(\rho_{ji} m_{ji} + m_{ii}\right)^2} \\
&= \hat{c} \left(\rho_{ji} m_{ji}\right)^{-1}
\end{aligned}$$

Then,

$$\frac{\partial \mathrm{E}[L(\hat{\tilde{\boldsymbol{\tau}}}^{\star}, \boldsymbol{m})]}{\partial m_{ii}} = \frac{\hat{c}}{m_{ji}} \sum_{j \neq i} \int_{\epsilon^{\star}_{ji}(\hat{\tilde{\boldsymbol{\tau}}}^{\star}, \boldsymbol{m})}^{\bar{\epsilon}} \lambda^{\chi}_{ji}(\boldsymbol{\alpha}, \boldsymbol{\epsilon}) \rho_{ji}(\boldsymbol{\alpha}, \epsilon_{ji})^{-1} f(\epsilon_{ji}) d\epsilon_{ji}$$

and the integral can be calculated by simulating over the government's choice problem. *Where to fix $\boldsymbol{m}$ values here? Does this choice matter?*

Change of variables to integrate over values of χ?

If we know the distribution of χ then we don't even have to calculate $\epsilon^{\star}$... just simulate distribution of χ from epsilons. Then draw from this induced distribution and solve the problem many times.

Still not sure where to fix $\boldsymbol{m}$ values... bounds of integration depend indirectly on these through effect on $\boldsymbol{\tau}^{\star}$