

Let

$$C(\hat{\tau}) = \frac{1}{\hat{c}^{-1} \left(\hat{G}_j(\hat{\tau}_i^{j*}) - \hat{G}_j(\hat{\tau}) \right) - 1}$$

and

$$Y_{ji}(\hat{\tau}; \mathbf{b}) = \hat{G}_j(\hat{\tau}) - \hat{G}_j(\hat{\tau}_i^{j*})$$

Note first that

$$\begin{aligned} \chi_{ji}(\epsilon_{ji}^*) &= \frac{e^{-\alpha^T \mathbf{W}_{ji} + \epsilon_{ji}^* m_{ji}}}{e^{-\alpha^T \mathbf{W}_{ji} + \epsilon_{ji}^* m_{ji}} + m_{ii}} \\ &= \frac{e^{-\ln\left(\frac{m_{ji}}{m_{ii}}\right) + \ln(C(\hat{\tau}))} m_{ji}}{e^{-\ln\left(\frac{m_{ji}}{m_{ii}}\right) + \ln(C(\hat{\tau}))} m_{ji} + m_{ii}} \\ &= \frac{\frac{m_{ii}}{m_{ji}} C(\hat{\tau}) m_{ji}}{\frac{m_{ii}}{m_{ji}} C(\hat{\tau}) m_{ji} + m_{ii}} \\ &= \frac{m_{ii} C(\hat{\tau})}{m_{ii} C(\hat{\tau}) + m_{ii}} \\ &= \frac{C(\hat{\tau})}{C(\hat{\tau}) + 1} \end{aligned}$$

and that

$$\chi_{ji}(\bar{\epsilon}_{ji}) = 1$$

Now the expected utility for government i in stage 1 is

$$\begin{aligned} E[L(\hat{\tau}, \mathbf{m})] &= E \left[G_i(\hat{\tau}) - \sum_{j \neq i} \eta_{ij} \hat{G}_j(\hat{\tau}; b_j) - \sum_{j \neq i} \lambda_{ji}^X(\alpha, \epsilon) (Y_{ji}(\hat{\tau}; \mathbf{b}) + \hat{c} \chi_{ji}(\mathbf{m}; \alpha, \epsilon_{ji})^{-1}) \right] \\ &= G_i(\hat{\tau}) - \sum_{j \neq i} \int_{\eta_{ij}} \eta_{ij} \hat{G}_j(\hat{\tau}; b_j) f_{\eta}(\eta_{ij}) d\eta_{ij} - \sum_{j \neq i} \int_{\epsilon_{ji}} \lambda_{ji}^X(\alpha, \epsilon) (Y_{ji}(\hat{\tau}; \mathbf{b}) + \hat{c} \chi_{ji}(\mathbf{m}; \alpha, \epsilon_{ji})^{-1}) f_{\epsilon}(\epsilon_{ji}) d\epsilon_{ji} \\ &= G_i(\hat{\tau}) - \sum_{j \neq i} \int_{\epsilon_{ji}} \lambda_{ji}^X(\alpha, \epsilon) (Y_{ji}(\hat{\tau}; \mathbf{b}) + \hat{c} \chi_{ji}(\mathbf{m}; \alpha, \epsilon_{ji})^{-1}) f_{\epsilon}(\epsilon_{ji}) d\epsilon_{ji} \\ &= G_i(\hat{\tau}) - \sum_{j \neq i} \int_{\epsilon_{ji}^*}^{\bar{\epsilon}_{ji}} \lambda_{ji}^X(\alpha, \epsilon) (Y_{ji}(\hat{\tau}; \mathbf{b}) + \hat{c} \chi_{ji}(\mathbf{m}; \alpha, \epsilon_{ji})^{-1}) f_{\epsilon}(\epsilon_{ji}) d\epsilon_{ji} \end{aligned}$$

because $\lambda_{ji}^X(\epsilon_{ji}) = 0$ whenever $\epsilon_{ji} \leq \epsilon_{ji}^*$.

Note that I'm assuming multipliers don't depend on \mathbf{m} for epsilon of interest... can I show this?

A few facts:

$$Y_{ji}(\hat{\tau}^*; \mathbf{b}) + \hat{c} \chi_{ji}(\mathbf{m}; \alpha, \epsilon_{ji})^{-1} = 0$$

THIS GIVES US THE VALUE OF $C(\hat{\tau}^*)$. And pins down lower bound on induced chi distribution as a function of \mathbf{m} and parameters. So if we do change of variables and integrate over distribution of χ maybe this gives us a closed form for the integral?

Also,

$$\begin{aligned}\frac{\partial \hat{c}_{\chi_{ji}}(\mathbf{m}; \boldsymbol{\alpha}, \epsilon_{ji})^{-1}}{\partial m_{ii}} &= -\hat{c}_{\chi_{ji}}(\mathbf{m}; \boldsymbol{\alpha}, \epsilon_{ji})^{-2} \frac{\partial \chi_{ji}(\mathbf{m}; \boldsymbol{\alpha}, \epsilon_{ji})}{\partial m_{ii}} \\ &= \hat{c} \left(\frac{\rho_{ji} m_{ji} + m_{ii}}{\rho_{ji} m_{ji}} \right)^2 \frac{\rho_{ji} m_{ji}}{(\rho_{ji} m_{ji} + m_{ii})^2} \\ &= \hat{c} (\rho_{ji} m_{ji})^{-1}\end{aligned}$$

Then,

$$\frac{\partial E[L(\hat{\tau}^*, \mathbf{m})]}{\partial m_{ii}} = \frac{\hat{c}}{m_{ji}} \sum_{j \neq i} \int_{\epsilon_{ji}^*(\hat{\tau}^*, \mathbf{m})}^{\bar{\epsilon}} \lambda_{ji}^{\chi}(\boldsymbol{\alpha}, \boldsymbol{\epsilon}) \rho_{ji}(\boldsymbol{\alpha}, \epsilon_{ji})^{-1} f(\epsilon_{ji}) d\epsilon_{ji}$$

and the integral can be calculated by simulating over the government's choice problem. *Where to fix \mathbf{m} values here? Does this choice matter?*

Change of variables to integrate over values of χ ?

If we know the distribution of χ then we don't even have to calculate ϵ^* ... just simulate distribution of χ from epsilons. Then draw from this induced distribution and solve the problem many times.

Still not sure where to fix \mathbf{m} values... bounds of integration depend indirectly on these through effect on $\boldsymbol{\tau}^*$

If we drop allocation stage altogether does this help? Just assume all wars are bilateral and all or nothing (war stage is Schlieffen's dream). Then m s are data (strengths) and we only have to iterate on stages 2 and 3. I think try this for first cut.

Alternative model that preserves some of what I have now assumes that all troops return home to fight defensive wars... interpretation is a little more strained though I think.