Problem Set 2

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1. Exponential growth in cancer

The growth of the cancer cell follows the formula for exponential growth:

$$N_{t+1} = 2N_t,$$

where t is a timestep of 109 days. If $N_0 = 1$, then the above equation simplifies to $N_{t+1} = 2^t$ and we must solve

$$10^{12} = 2^{t_{crit}},$$

or

$$t_{crit} = \log_2(10^{12}),$$

which is 39.8631371. Since a time-step is 109 days, the number of days until the critical value is roughly $109 \cdot 40 = 4360$.

2. Sterile insect release

Let

$$F(N) = b\frac{N}{N+M} - d.$$

Throughout this section, I will suppose b, d, and M are non-negative. Further, population levels of N below zero we consider to be simply extinct (stuck at 0).

Equilibrium points and stability

I will write the equilibrium points as N^*, i.e., we seek solutions to the equation $F(N^*) = 0$. Solving this for N^*, we see that

$$N^* = \begin{cases} 0, & \text{when } b = d \\ \frac{d}{b-d}M & \text{otherwise.} \end{cases}$$

Note that these cases do not indicate there are two equilibrium points.

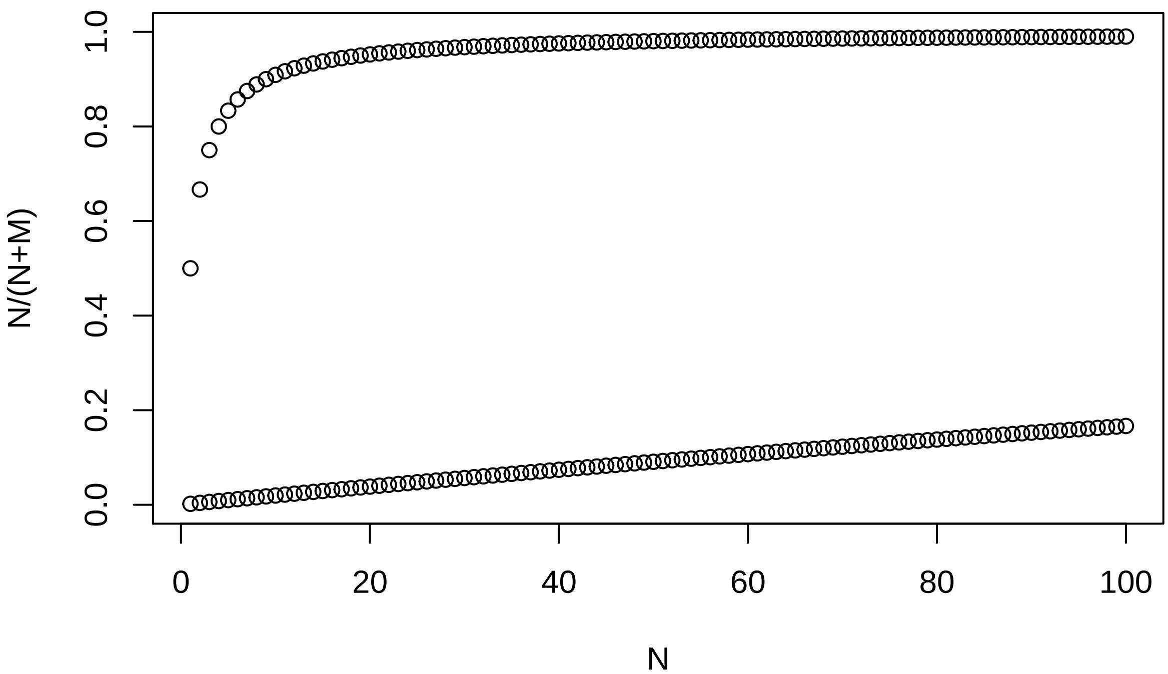
Further, since applying the quotient rule for $f(N) = N$ and $g(N) = N + M$ quickly gives $\frac{\mathrm{d}F}{\mathrm{d}N} = b\frac{M}{(N+M)^2} > 0$, and we know from class that N^* is stable if and only if $\frac{\mathrm{d}F}{\mathrm{d}N} < 0$, N^* is unstable for any parameter setting.

Finally, for completeness I should point out that by assuming that if N has crashed to less than or equal to 0, N will stay at 0, I have effectively insisted that $N* = 0$ is a stable equilibrium point.

Critical threshold for population extinction

Since from the previous section we know values of N above N^* will continue to grow away from N^*, to demonstrate that N^* is the critical threshold it remains to show values of N below N^* will be attracted to 0. However, this immediate after noting $N/(N+M)$, and hence F, is strictly monotone increasing:

```
X = 1:100
Y1 = X/(X + 1)
Y2 = X/(X + 500)
plot(X, Y1, ylim=c(0, 1), ylab="N/(N+M)", xlab="N")
points(X, Y2)
```



This can be proven after noting the first derivative of $N/(N+M)$ is clearly non-negative.

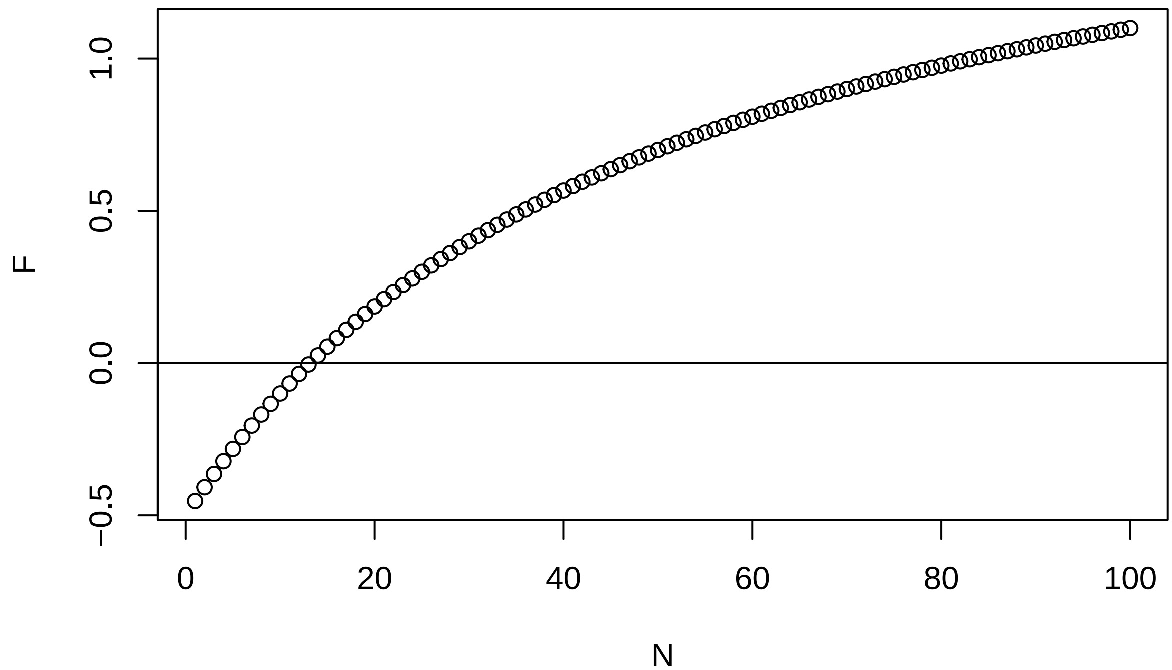
Since F is strictly monotone, this means

$$F(N) < F(N^*) = 0$$

for $0 \leq N < N^*$, so N will always decrease toward 0 in this case.

In summary, this model leads to a strong Allee effect:

```
Y3 = 2.4 * X/(X + 50) - .5
plot(X, Y3, xlab="N", ylab="F")
abline(h=0)
```



As can be seen, F scales with population size N such that if N is below the critical value, the population will crash, but past this value, growth will increase further as N increases.

3. Sterile insect release with competition

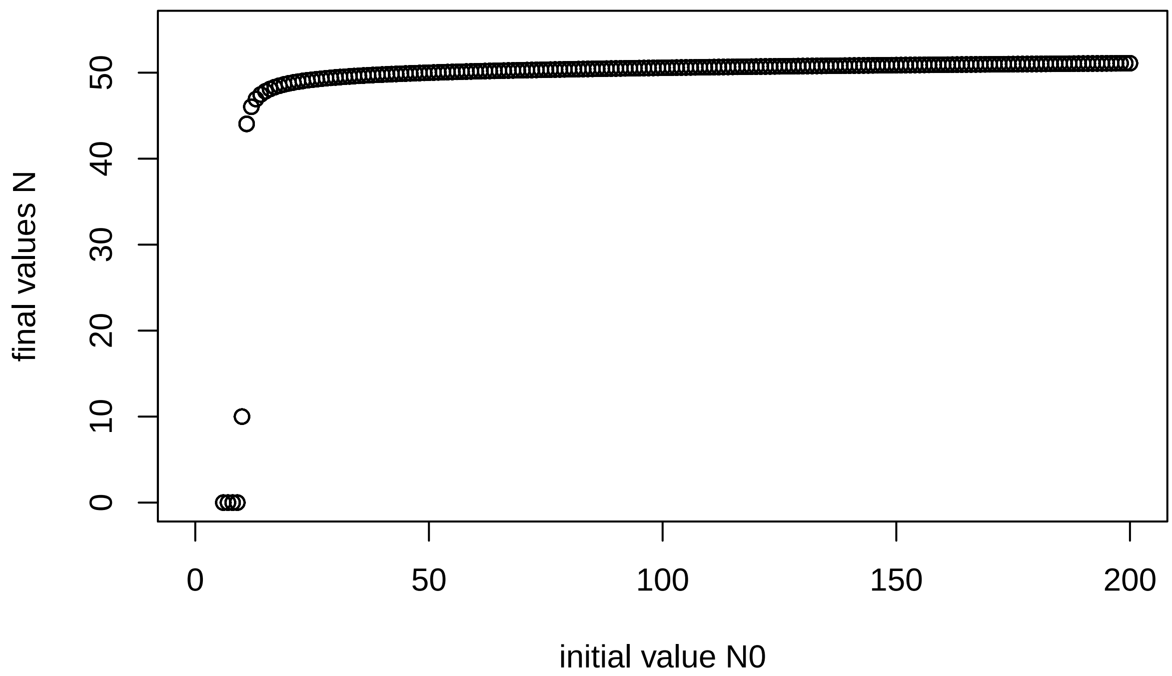
```
require(deSolve)
times <- seq(0,500,by=0.1)

# Parameters
params <- c(b = 2.4,c=0.02,M=50,d=0.2)

# Model
sterile_insect <- function(t,state,parameters){
  with(as.list(c(state,parameters)),{
    if (N <= 0) {
      dN = 0
    }
    else {
      dN <- b*N/(N+M) - d - c*N
    }
    return(list(c(dN)))})
}
```

To answer this question, I will first vary the initial N_0 and then plot the last 10 values of N after 5,000 timesteps.

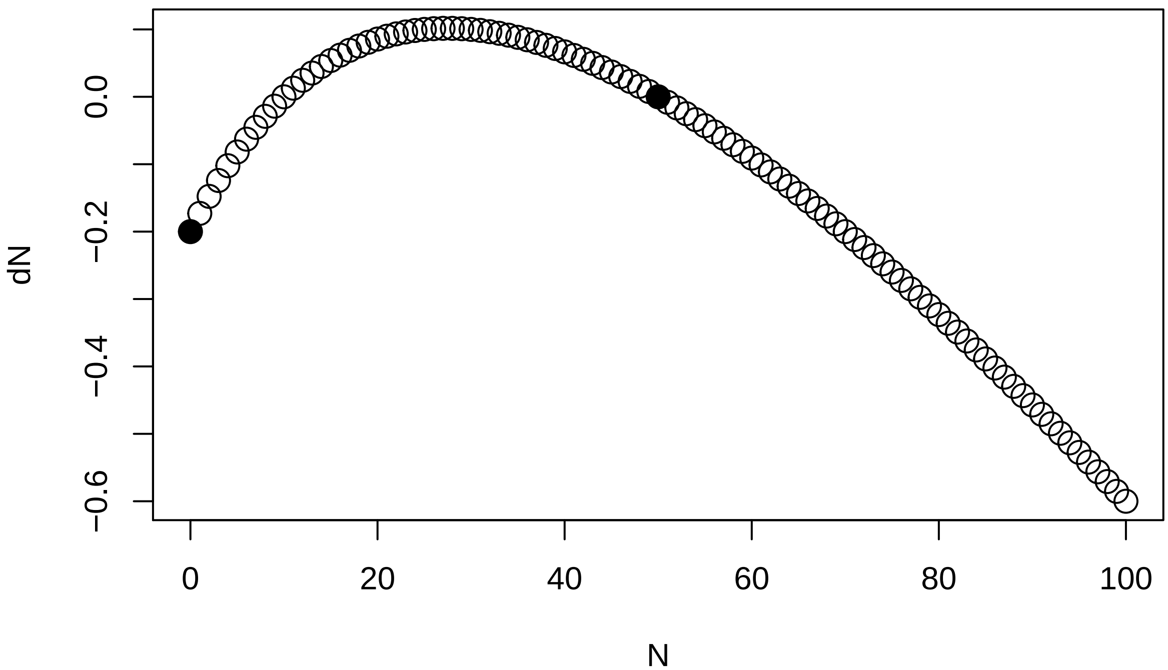
```r
plot(1, type="n", xlab="initial value N0", ylab="final values N", xlim=c(0, 200), ylim=c(0, 55))
for (n in seq(6, 200, by=1)) {
  out <- ode(y = c(N=n), times=times, func=sterile_insect, parms=params)
  points(x=rep(n, 10), y=tail(out[,2], 10))
}
```



This suggests there are three equilibrium points at $N^* = 0, 10, 50$ for these particular parameter settings. Further, since for $N_0 < 10$ and $N_0 > 50$ the model was attracted to the points 0 and 50 respectively, the points 0 and 50 are stable, while 10 is not.

```r
diff_eq = function(N) {
  return(params[1] * (N/(N + params[3])) - params[4] - params[2] * N)
}

X = 0:100
Y = diff_eq(0:100)
plot(X, Y,
     xlim=c(0, 100),
     xlab="N",
     ylab="dN",
     pch=ifelse(X %in% c(0, 50), 19, 1),
     cex=1.6)
```


0.0
−0.2
−0.4
−0.6
dN
0
20
40
60
80
100
N