Computational Problem Set 4 Solution

AAE440/590: Spacecraft Attitude Dynamics
Spring 2022
Due Date: 13, April 2022

Professor Oguri

Contents

Relevant Notation

Direction Cosine Matrices

Direction cosine matrices are indicated using the following convention:

$$^{X}C^{Y}$$

The "C" indicates a DCM, while the superscripts stand for "from X to Y" (of course, the order matters).

Inertia Characteristics

When dealing with axisymmetric bodies, the inertia component which is along the axis of symmetry is denoted as "J", while that for the other two axes (which are equivalent in the axisymmetric condition) is denoted "I".

Principal Inertia Characteristics:

When dealing with principal moments of inertia, that is, the moments of inertia corresponding to a reference frame aligned with the principal axes of the body, related quantities will be distinguished by an asterisk right-superscript. For example, the principal inertia tensor may be expressed like so:

$$\bar{\bar{I}}^{*}$$

Whereas an arbitrary inertia matrix (which may or may not be principal) will be lacking the superscript, like so:

$$\bar{\bar{I}}$$

Problem 1: Problem Statement

Consider a satellite moving in a force-free, torque-free environment. The inertial frame and satellite body-fixed frame are represented by $\mathcal{N}$-frame and $\mathcal{B}$-frame, where $\{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$ and $\{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$ are right-handed vector bases fixed in $\mathcal{N}$-frame and $\mathcal{B}$-frame, respectively. The inertia tensor of the body is given by:

$$\bar{\bar{\mathbf{I}}}_c = \frac{3}{4}\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_1 + \hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_2 + \frac{3}{2}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_3 \quad \text{(dimensionless)} \tag{1}$$

We consider the same initial satellite angular velocity as used in the Problem 2 of CPS3, i.e.,:

$$\boldsymbol{\omega}(t=0) = -0.1\ \hat{\boldsymbol{b}}_1 + 0.05\ \hat{\boldsymbol{b}}_2 + 0.1\ \hat{\boldsymbol{b}}_3 \quad \text{(dimensionless)} \tag{2}$$

(a): Using the function `dwdt_torqueFree` implemented in CPS3, **compute** the time history of $\boldsymbol{\omega}$ over a time span from $t = 0$ to $t = 200$ with integration tolerance 1×10^{-10}. **Show** the plots of $\omega_i(t)$ over time.

(b): To compare the obtained numerical result against an axisymmetric case, numerically integrate the same system as in the Problem 2 of CPS3 from $t = 0$ to $t = 200$ with integration tolerance 1×10^{-10}. Denote by $\bar{\bar{\mathbf{I}}}_{c,\text{axisym}}$ the inertia tensor assumed in the Problem 2 of CPS3, that is,

$$\bar{\bar{\mathbf{I}}}_{c,\text{axisym}} = \hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_1 + \hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_2 + \frac{3}{2}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_3 \quad \text{(dimensionless)} \tag{3}$$

Show the plots of $\omega_i(t)$ for the system with $\bar{\bar{\mathbf{I}}}_{c,\text{axisym}}$ over time.

(c): **Discuss** the qualitative differences and similarities of $\omega_i(t)$ obtained in (a) and (b) above, addressing the following points: overall behaviors (periodic? constant?), maximum/minimum values, and periods (if periodic).

(d): The two constant integrals of motion, namely the angular momentum magnitude, $\|\boldsymbol{H}\|_2$, and the rotational Kinetic energy, T_{rot}, are still useful tools to check the validity of numerical simulations for general inertia bodies. Regarding the result of (a), **compute** $\|\boldsymbol{H}(t)\|_2$ and $T_{\text{rot}}(t)$. **Show** the plots of $\|\boldsymbol{H}(t)\|_2$ and $T_{\text{rot}}(t)$, and **confirm** that they are indeed constant over time.

(e): We learned in class that there are three additional constant integrals of motion that can be found from the analogy to undamped Duffing equations. Repeating the equations in the lecture slides, the constants are given by:

$$C_i = \dot{\omega}_i^2 + A_i\omega_i^2 + \frac{B_i}{2}\omega_i^4 \quad \text{for } i = 1, 2, 3, \tag{4}$$

where A_i and B_i are constants that depend on the initial condition and $\bar{\bar{\mathbf{I}}}_c$, given by:

$$A_1 = \frac{(I_1 - I_2)(2I_3T_{\text{rot}} - \|\boldsymbol{H}\|_2^2) + (I_3 - I_1)(\|\boldsymbol{H}\|_2^2 - 2I_2T_{\text{rot}})}{I_1I_2I_3} \qquad B_1 = \frac{2(I_1 - I_2)(I_1 - I_3)}{I_2I_3}$$

$$A_2 = \frac{(I_2 - I_3)(2I_1T_{\text{rot}} - \|\boldsymbol{H}\|_2^2) + (I_1 - I_2)(\|\boldsymbol{H}\|_2^2 - 2I_3T_{\text{rot}})}{I_1I_2I_3} \qquad B_2 = -\frac{2(I_1 - I_2)(I_2 - I_3)}{I_1I_3}$$

$$A_3 = \frac{(I_3 - I_1)(2I_2T_{\text{rot}} - \|\boldsymbol{H}\|_2^2) + (I_2 - I_3)(\|\boldsymbol{H}\|_2^2 - 2I_1T_{\text{rot}})}{I_1I_2I_3} \qquad B_3 = \frac{2(I_1 - I_3)(I_2 - I_3)}{I_1I_2}$$

Compute the constants C_i in Eq. (4) over time for the result of (a), **show** the plots of C_i as functions of t, and **confirm** that they are indeed constant over time.
(Hint: $\dot{\omega}_i$ in Eq. (4) can be computed by using the Euler's equations of motion given $\omega_i(t)$)

(f): Another way to look at the torque-free attitude motion of general inertia bodies is the Poinsot construction. According to the Poinsot construction, the angular velocity vector $\boldsymbol{\omega}(t)$ moves in a plane perpendicular to the angular momentum vector $\boldsymbol{H}$ in the inertial frame (called *invariable plane*).
Focusing on the case with the unsymmetric inertia tensor given in Eq. (1), let us numerically verify this argument by visualizing $\boldsymbol{\omega}(t)$ and $\boldsymbol{H}$ in the inertial frame, as follows:

(f.1): In order to visualize these quantities in the inertial frame, we also need information of the satellite orientation over time to map $\boldsymbol{\omega}(t)$ and $\boldsymbol{H}$ from $\mathcal{B}$-frame to $\mathcal{N}$-frame. Here we use the modified Rodrigues parameter (MRP) to represent the orientation of $\mathcal{B}$-frame relative to $\mathcal{N}$-frame, and suppose that the MRP vector at $t = 0$ is evaluated as:

$$\boldsymbol{\sigma}(t=0) = \frac{1}{3}\hat{\boldsymbol{b}}_1 + \frac{1}{3}\hat{\boldsymbol{b}}_2 + \frac{1}{3}\hat{\boldsymbol{b}}_3 \tag{5}$$

Compute $\boldsymbol{\sigma}(t)$ over a time span from $t = 0$ to $t = 200$ by numerically integrating `KDE_MRP` and `dwdt_torqueFree` simultaneously, with integration tolerance 1×10^{-10}, where the shadow-set switching is not necessary (you may perform the switching at $\|\boldsymbol{\sigma}\|_2 = 1$ if you want too). **Show** the plots of $\sigma_i(t)$ over time.

(f.2): **Compute** ${}^{\mathcal{N}}\boldsymbol{H}(t)$, i.e., time history of $\boldsymbol{H}$ in $\mathcal{N}$-frame, and **show** the plots of ${}^{\mathcal{N}}H_i(t)$ to confirm that ${}^{\mathcal{N}}H_i(t)$ are indeed constant over time.

(f.3): **Compute** ${}^{\mathcal{N}}\boldsymbol{\omega}(t)$, and **show** the three-dimensional plot of $\boldsymbol{\omega}(t)$ and $\boldsymbol{H}$ in $\mathcal{N}$-frame, where represent $\boldsymbol{H}$ by a line that connects the origin and the point given by ${}^{\mathcal{N}}\boldsymbol{H}(t=0)$ (showing $\boldsymbol{H}$ only at $t = 0$ is sufficient since ${}^{\mathcal{N}}\boldsymbol{H}(t)$ is time-invariant). Visually **confirm** that $\boldsymbol{\omega}(t)$ is indeed confined in a plane perpendicular to $\boldsymbol{H}$ in $\mathcal{N}$-frame, where include at least two different views of the three-dimensional plot: a view that illustrates the perpendicularity of the invariable plane to $\boldsymbol{H}$ and a view that projects the locus of $\boldsymbol{\omega}(t)$ on the invariable plane (called *herpolhode*).

(g): Let us compare the herpolhode curve obtained in (f) against the one for an axisymmetric inertia case. To do this, **repeat** the procedure in (f.1)-(f.3) using $\bar{\bar{\mathbf{I}}}_{c,\text{axisym}}$ given in Eq. (3) with the same initial conditions Eqs. (2) and (5). **Discuss** the comparison of the two herpolhode curves obtained in (f) and in this question (e.g., are the curves circular? closed?).

Problem 1: Solution

Part (a):

We can use the same set of equations used in CPS3 in order to integrate the angular velocity components over time; the relevant equations are provided in (1) through (3) for reference. Since this is no longer the axisymmetric case, we cannot simplify the third equation to be zero, so we will have a variable third angular velocity component.

$$\dot{\omega}_1 = -\frac{I_3 - I_2}{I_1}\omega_2\omega_3 \tag{1}$$

$$\dot{\omega}_2 = -\frac{I_1 - I_3}{I_2}\omega_1\omega_3 \tag{2}$$

$$\dot{\omega}_3 = -\frac{I_2 - I_1}{I_3}\omega_1\omega_2 \tag{3}$$

If $I_1 = 3/4$, $I_2 = 1$, $I_1 = 3/2$, and the initial conditions for the angular velocity are such that $\omega_{1o} = -0.1$, $\omega_{2o} = 0.05$, $\omega_{3o} = 0.1$, we may integrate these equations using MATLAB's ode45 numerical integrator from times 0 to 200 in order to get the time history displayed in figure 1.

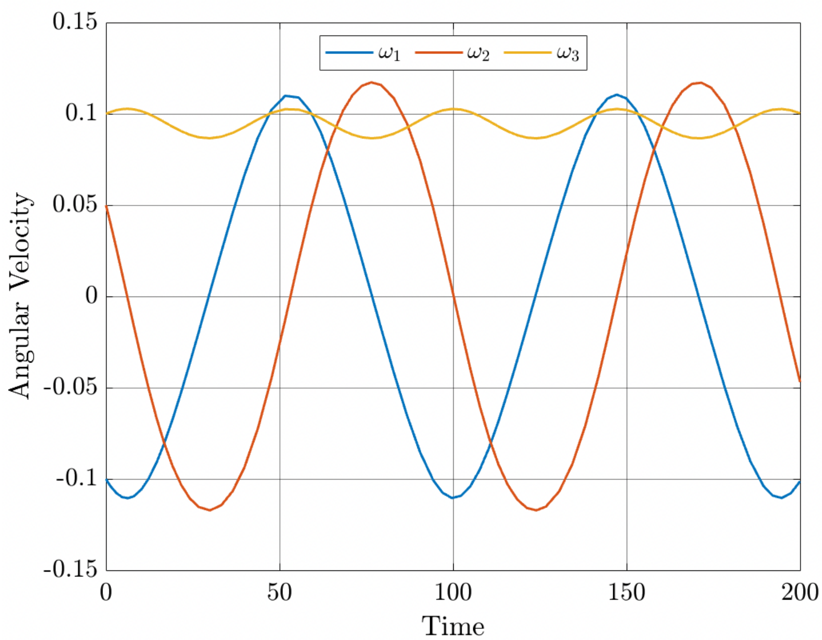


Fig. 1. Time history of $\vec{\omega}$ components for non-axisymmetric body

We can observe that the third component is indeed non-constant, but more on the behavior later.

Part (b):

We may repeat the same process, where now $I_1 = 1$, $I_2 = 1$, $I_1 = 3/2$, which only has the first inertia component changing by a small amount in order to match the second (a slight difference numerically, but more importantly, now axisymmetric). Using the same process as before, we arrive at the behavior shown in figure 2.

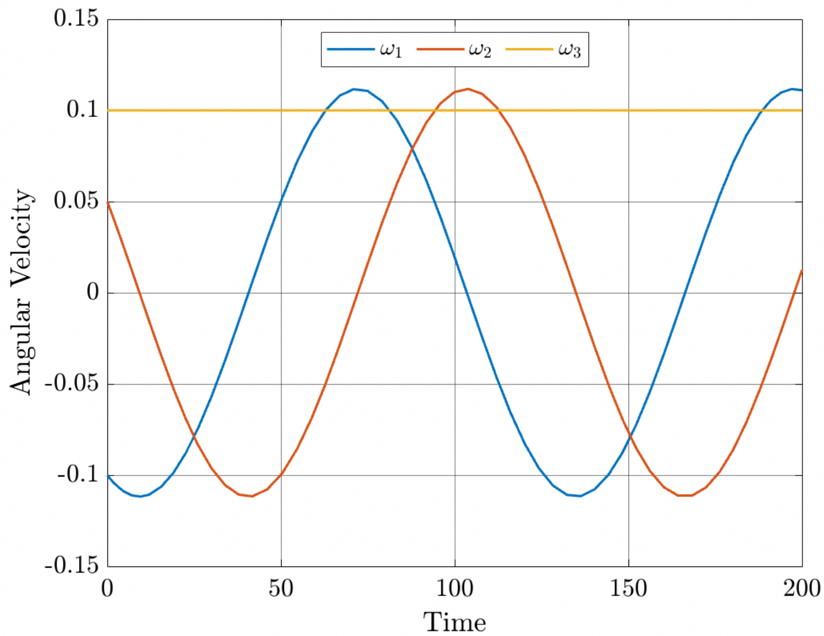


Fig. 2. Time history of $\vec{\omega}$ components for the axisymmetric body

We may notice immediately that, as expected, ω_3 is zero. There are more subtle differences as well which will be compared to figure 1 in the next part.

Part (c):

Now, to compare the two curves shown in figures 1 and 2, we mentioned previously that one of the primary differences between the two of them is that ω_3 is constant in the axisymmetric case (figure 2) while variable in the non-axisymmetric (figure 1). There are also other subtle differences, like the amplitudes of the other two curves being different in figure 1, while the same in the other. This is less due to the dynamics and more a byproduct of the initial conditions used. A more dynamical discussion is that of the periodicity of these solutions.

From observation, we can see that all of the curves displayed (aside from the flat line in figure 2) are periodic, oscillatory curves. Moving forward, it would be necessary to note that periodic means perfectly repeating, where as oscillatory does not have this constraint (it is a more qualitative description of sinusoidal behavior). Of course, there is little means to tell if there is *perfect* repetition of a numerically calculated curve (as there is latent error in numerical calculations), so estimation would be up to the discretion of the analyst. However, we do know that an asymmetric body has an aperiodic angular velocity; noting that aperiodic does not mean decidedly not repetitive, just that it does not perfectly repeat (but may come close to doing so). Analysis may be done with the data cursor tool, but is largely inconclusive since the functions are discrete, not continuous. One of the key takeaways is that there is a difference between "oscillatory" and "periodic", where the former simply describes a pattern of coming-and-going, while the latter is more contingent on perfect repetition.

As for the axisymmetric case, both oscillatory curves have a period of about 125 time units, which is significantly longer than the asymmetric body. The amplitudes, as hinted towards earlier, are also different. In the axisymmetric case, both non-constant curves have an amplitude of about 0.112 (identically), while in the asymmetric case, there is a minor difference, where the first curve has an amplitude of about 0.110, and the second 0.117. Again, these numbers are approximate.

Part (d):

Previously in CPS2, we found that we can find the angular momentum vector and rotational kinematic energy by using the following:

$$\vec{H}\ \{\hat{b}\} = \bar{\bar{I}}_c \vec{\omega} \tag{4}$$

$$T_{rot} = \frac{1}{2} \vec{\omega}^T \bar{\bar{I}}_c \vec{\omega} \tag{5}$$

We may apply these equations to the conditions at each time step for the asymmetric case to find the time histories of the angular momentum (magnitude) and rotational kinetic energy, as shown in figures 3 and 4 respectively.

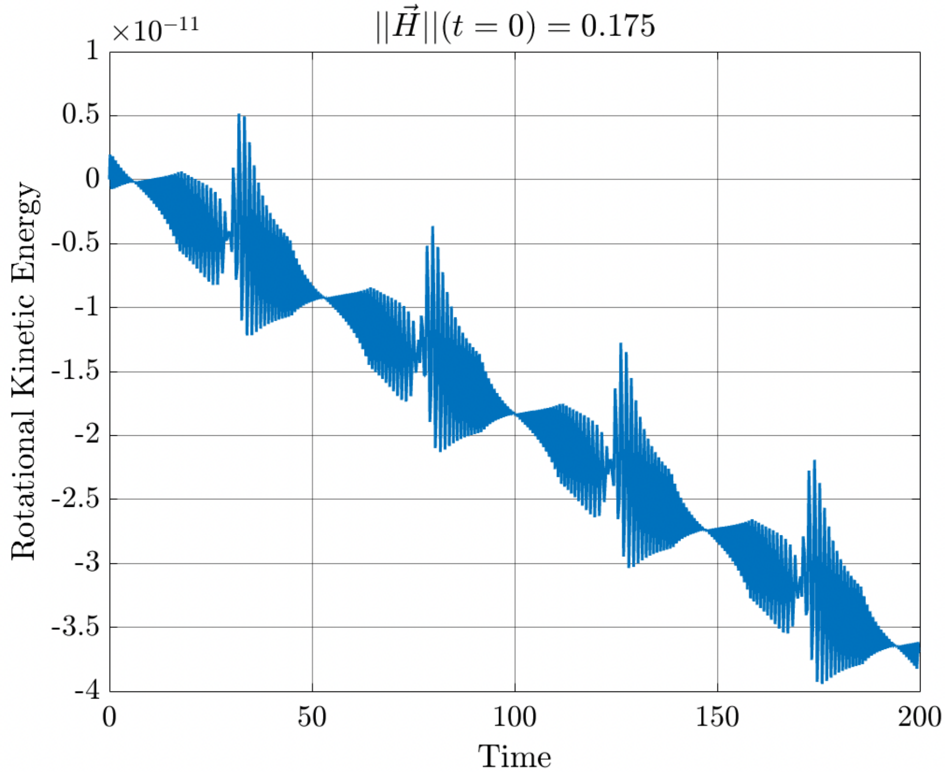


Fig. 3. Angular momentum magnitude over time

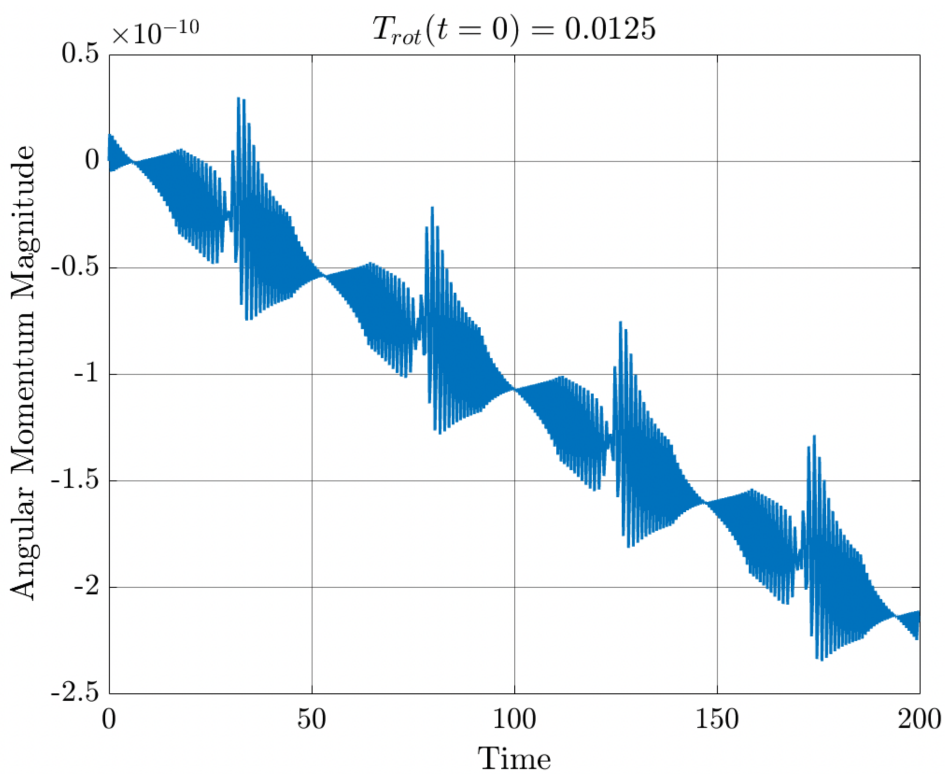


Fig. 4. Rotational kinetic energy over time

Part (e):

We will now look at the Duffing equations, given below:

$$C_i = \dot{\omega}_i^2 + A_i \omega_i^2 + \frac{B_i}{2}\omega_i^4 \tag{6}$$

$$A_1 = \frac{(I_1 - I_2)\left(2I_3T_{rot} - \left\|\vec{H}\right\|^2\right) + (I_3 - I_1)\left(\left\|\vec{H}\right\| - 2I_2T_{rot}\right)}{I_1I_2I_3} \tag{7}$$

$$A_1 = \frac{(I_2 - I_3)\left(2I_1T_{rot} - \left\|\vec{H}\right\|^2\right) + (I_1 - I_2)\left(\left\|\vec{H}\right\| - 2I_3T_{rot}\right)}{I_1I_2I_3} \tag{8}$$

$$A_1 = \frac{(I_3 - I_1)\left(2I_2T_{rot} - \left\|\vec{H}\right\|^2\right) + (I_2 - I_3)\left(\left\|\vec{H}\right\| - 2I_1T_{rot}\right)}{I_1I_2I_3} \tag{9}$$

$$B_1 = \frac{2(I_1 - I_2)(I_1 - I_3)}{I_2I_3} \tag{10}$$

$$B_2 = \frac{2(I_1 - I_2)(I_3 - I_2)}{I_1I_3} \tag{11}$$

$$B_3 = \frac{2(I_1 - I_3)(I_2 - I_3)}{I_1I_2} \tag{12}$$

We will use the results previously found as well as the first values of the rotational kinetic energy and angular momentum magnitude (as these were influenced by numerical inaccuracies). If you would like to be more accurate, you could re-run the code with a much shorter integration time and then take the values for T_{rot} and $\left\|\vec{H}\right\|$ from there. We may note that the coefficients are found using the values for $\vec{\omega}$ over time, thus providing the opportunity to show whether C_i are constant with respect to time or not. The results are shown in figure 5.

We may note that the results display constant behavior for the coefficients, which is what was expected. Of course, should we zoom in on any of the curves, the behavior would not be perfectly constant due to the latent numerical error present in the problem. Regardless, not having any significant visual deviations at a scale this small is an expected (and quite a pleasant) outcome.

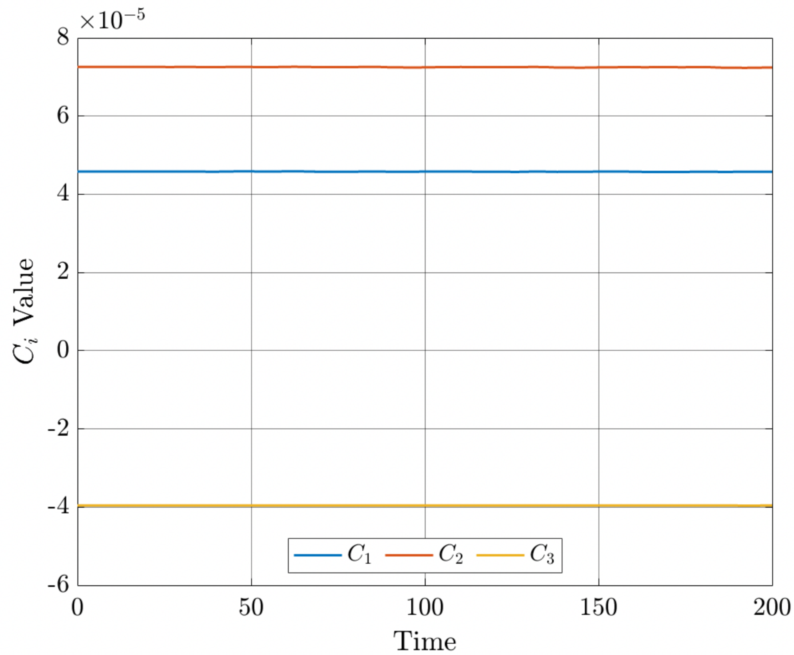


Fig. 5. Values of Duffing equation constants over time

Part (f):

Sub-Part (f.1):

For simultaneously integrating both the MRP and angular velocity components, we would need to augment our differential equations in the function called into ode45. The order does not matter in particular so long as that of your initial conditions match. We may augment the differential equations in the function like so:

$$\text{(Output)} = \begin{bmatrix} \frac{d\vec{\sigma}}{dt} \\ \frac{d\vec{\omega}}{dt} \end{bmatrix}$$

Where both are 3 by 1 column matrices found, where the first is found using (13) and the second is found using the previously defined differential equations, displayed in (1) through (3).

$$\frac{d\vec{\sigma}}{dt} = \frac{1}{4}((1 - \vec{\sigma} \bullet \vec{\sigma})\vec{\omega} + 2\vec{\sigma} \times \vec{\omega} + 2\vec{\omega} \bullet \vec{\sigma}\vec{\sigma}) \tag{13}$$

Performing the integration with the initial conditions for the MRP provided through the problem statement to be $\sigma_1 = \sigma_2 = \sigma_3 = 1/3$, we get the MRP time history, as shown in figure 6.

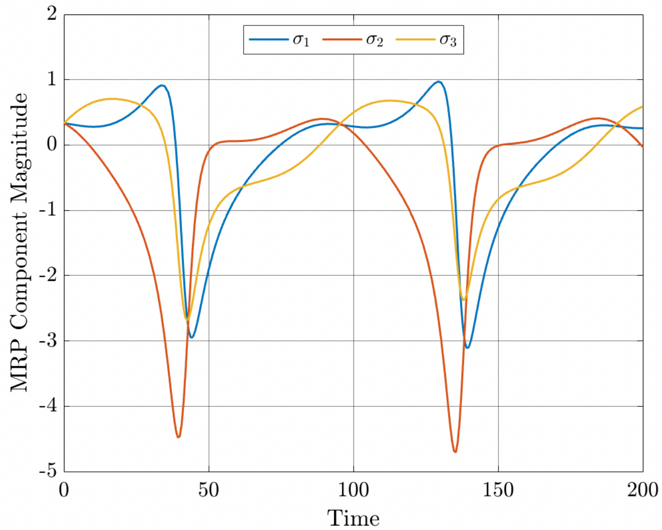


Fig. 6. MRP history for asymmetric body

Sub-Part (f.2):

Much like in previous assignments, we may go about converting the angular momentum vector to its inertial form by finding the MRP vector at each time step (done in the previous problem), then converting each MRP vector to its corresponding DCM using MRPtoDCM, then using the *transpose* of this DCM to convert $\vec{H}$ from the body frame to the inertial frame. It will likely be necessary to re-compute the body-fixed angular momentum vector as the time steps may have changed, but this can be accomplished in the same manner as before.

Doing all of this provides the resulting angular momentum measure numbers as expressed in the N frame, as shown in figure 7.

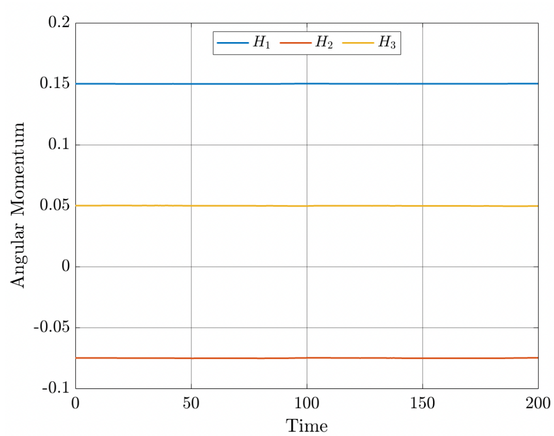


Fig. 7. Angular momentum measure numbers in the inertial frame

We can see that the angular momentum vector is indeed constant in the inertial reference frame. This makes sense as the model, while no longer axisymmetric, is still not being acted upon by and external force or torque.

Sub-Part (f.3):

In order to find the inertially-expressed angular velocity vector, we may return to the loop created in the previous problem and insert a line for using the DCM calculated at each time step (transposed) to convert the corresponding angular velocity vectors into their inertial form. This will take the same approach as that used for the angular momentum vector.

Plotting the spatial history of the tip of the angular velocity vector may be done by using the "plot3" command and plotting each of the components of the inertial angular velocity against one another (i.e. plot3(all of x components, all of y components, all of z components)). In addition to this, we may insert the angular momentum vector by making another matrix with the first row being zeros, and the second being the first value of $\vec{H}$ in N (this is fine since the vector is constant). Then plotting this using the same method discussed for the angular velocity. Doing so yields the completed plot as shown by the two views given in figure 8.

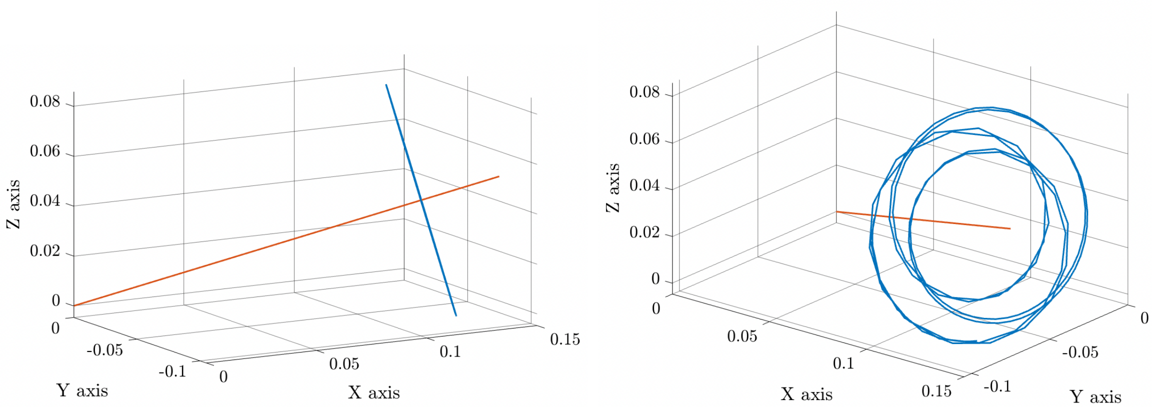


Fig. 8. Asymmetric Body: Angular velocity in blue, angular momentum in orange

The left image shows that the angular velocity spans a plane from which the angular momentum vector is normal, while the right image shows the herpolhode curve (jagged edges due to the decently large steps taken by the integrator).

<u>Part (g)</u>:

We may repeat the same process as that detailed in part (f), using the same initial conditions, with an axisymmetric body (conditions given previously in the problem). The resulting herpolhode plot is given in figure 9.

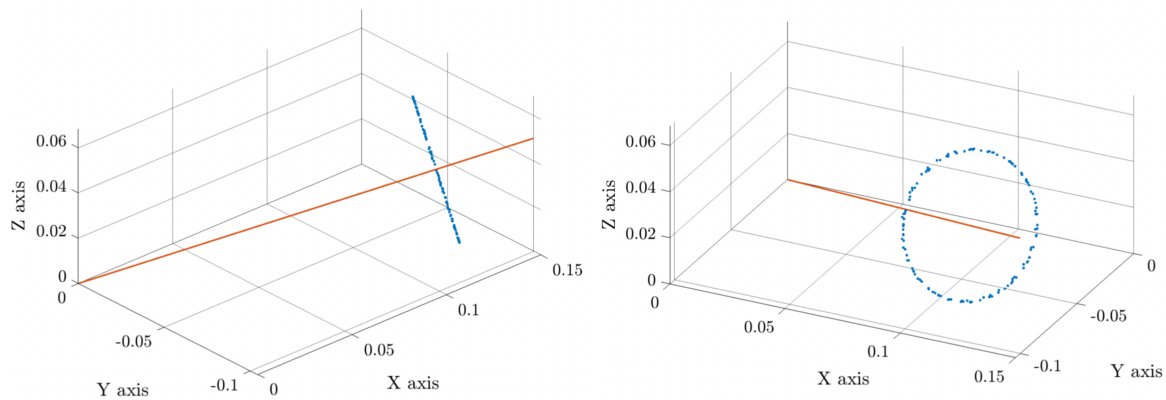


Fig. 9. Axisymmetric Body: Angular velocity in blue, angular momentum in orange

Points were plotted as lines would be a bit misleading, as the jumps from point to point are significant enough to make the curve look less periodic than it truly is. We may note that there are discrepancies where points do drift a bit away from the circle traced out. This is due to error from the lengthy integration duration, rather than the system dynamics. Let's discuss why both curves are planar.

First, we know that the rotational kinetic energy is constant as the model is force and torque free. From that, we can look at the formulation originally given in (5), which may be rewritten as such:

$$T_{rot} = \frac{1}{2}\vec{\omega} \bullet \bar{\bar{I}}_c \bullet \vec{\omega} = \frac{1}{2}\vec{\omega}\{\hat{n}\} \bullet \vec{H}\{\hat{n}\} = \text{constant}$$

Simplifying this, we get:

$$\vec{\omega}\{\hat{n}\} \bullet \vec{H}\{\hat{n}\} = \text{constant} \tag{14}$$

We can see that if the dot product is constant, this means that the projection of the angular velocity onto the angular momentum is a fixed value. This mathematically means that the component of the angular velocity vector in the direction of the angular momentum vector (both in the N frame) is constant over time. This does *not* mean that the angular velocity is constant, but it will vary only in the component orthogonal to the angular momentum vector, as shown in figure 10. Since the rotational kinetic energy is constant for both the axisymmetric and asymmetric cases, we know that this planar behavior is true for both, though this model is for the general asymmetric case.

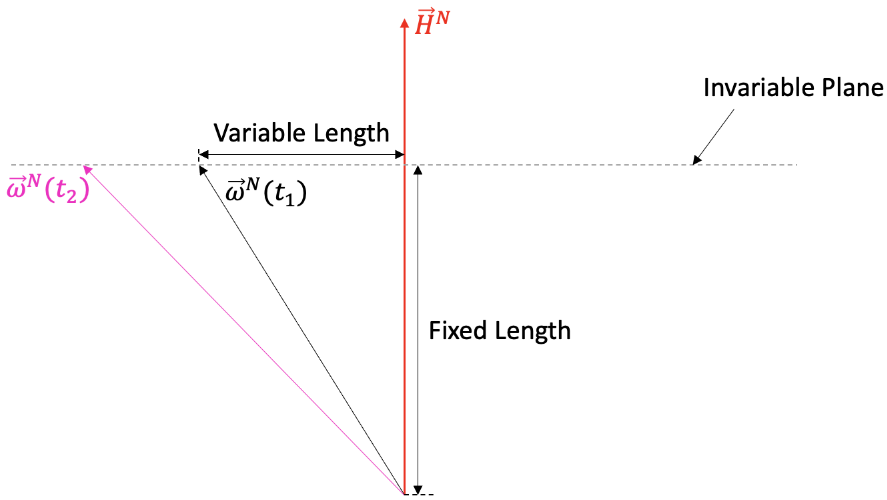


Fig. 10. Relationship between $\vec{\omega}$ and $\vec{H}$ in the N frame

One difference between the two models is that the axisymmetric case does not have a variable length *along* the invariable plane, but rather this is a fixed quantity as well. This is because in the axisymmetric case, $\frac{d}{dt}\|\vec{\omega}\| = 0$, meaning that if the H direction leg is of fixed magnitude and the overall magnitude is fixed, then the other leg must also have fixed length. Thus as the angular velocity precesses around the angular momentum, it will have a fixed distance both along and normal to the angular momentum, forming a perfect circle.

So, in closing, the axisymmetric solution is periodic with a closed circular path, while the asymmetric body has an aperiodic, planar but non-circular, not closed herpolhode.

Problem 2: Problem Statement

This problem is concerned with numerically verifying the instability of the rotation about the intermediate moment axis. In class, we learned that the Euler's rotational equations of motion without torque (Eq. (7)) have the following three particular solutions:

$$\begin{cases} \omega_2(t) = \omega_3(t) = 0, & \omega_1(t) = \omega_{1_0}(\neq 0) & \cdots \text{Solution 1} \\ \omega_3(t) = \omega_1(t) = 0, & \omega_2(t) = \omega_{2_0}(\neq 0) & \cdots \text{Solution 2} \\ \omega_1(t) = \omega_2(t) = 0, & \omega_3(t) = \omega_{3_0}(\neq 0) & \cdots \text{Solution 3} \end{cases} \tag{6}$$

Recall that the Euler's rotational equations of motion in a torque-free environment is given as follows:

$$\begin{cases} \dot{\omega}_1 = -\frac{I_3 - I_2}{I_1}\omega_2\omega_3 \\ \dot{\omega}_2 = -\frac{I_1 - I_3}{I_2}\omega_3\omega_1 \\ \dot{\omega}_3 = -\frac{I_2 - I_1}{I_3}\omega_1\omega_2 \end{cases} \tag{7}$$

Answer the following questions assuming the inertia tensor given by Eq. (1).

(a): **Show** that each of the particular solutions in Eq. (6) indeed satisfies the differential equations Eq. (7).

(b): **Discuss** which of the particular solutions in Eq. (6) corresponds to the maximum-energy, minimum-energy, and intermediate-energy rotational states, respectively.

(c): Let us also numerically confirm that Eq. (6) are the particular solutions of Eq. (7). Consider the following three cases for the initial angular velocity vectors:

$$\begin{cases} \boldsymbol{\omega}(t=0) = 0.1\ \hat{\boldsymbol{b}}_1 & \cdots \text{Case 1} \\ \boldsymbol{\omega}(t=0) = 0.1\ \hat{\boldsymbol{b}}_2 & \cdots \text{Case 2} \\ \boldsymbol{\omega}(t=0) = 0.1\ \hat{\boldsymbol{b}}_3 & \cdots \text{Case 3} \end{cases} \quad \text{(dimensionless)} \tag{8}$$

For each case in Eq. (8), **compute** $\boldsymbol{\omega}(t)$ by numerically integrating Eq. (7) from $t = 0$ to $t = 200$ with integration tolerance 1×10^{-10}. **Show** the plots of $\omega_i(t)$ for each of the three cases, and **confirm** that the results are consistent with Eq. (6).

(d): Now, let us consider a small perturbation in $\boldsymbol{\omega}(t=0)$ for each of the three cases in Eq. (8). Specifically, we consider the following perturbed angular velocity at $t = 0$:

$$\begin{cases} \boldsymbol{\omega}(t=0) = 0.1\ \hat{\boldsymbol{b}}_1 + 0.001\ \hat{\boldsymbol{b}}_2 + 0.001\ \hat{\boldsymbol{b}}_3 & \cdots \text{Case 1}' \\ \boldsymbol{\omega}(t=0) = 0.001\ \hat{\boldsymbol{b}}_1 + 0.1\ \hat{\boldsymbol{b}}_2 + 0.001\ \hat{\boldsymbol{b}}_3 & \cdots \text{Case 2}' \\ \boldsymbol{\omega}(t=0) = 0.001\ \hat{\boldsymbol{b}}_1 + 0.001\ \hat{\boldsymbol{b}}_2 + 0.1\ \hat{\boldsymbol{b}}_3 & \cdots \text{Case 3}' \end{cases} \quad \text{(dimensionless)} \tag{9}$$

(d.1): For each case in Eq. (9), **compute** $\boldsymbol{\omega}(t)$ over a time span from $t = 0$ to $t = 200$ with integration tolerance 1×10^{-10}. **Show** the plots of $\omega_i(t)$ over time.

(d.2): **Discuss** the numerical results obtained in the previous question, addressing the following points:

- how $\omega_i(t)$ behaves over time for each case
- how $\omega_i(t)$ compares against the unperturbed results (i.e., results obtained in (b)) for each case; does the perturbation grow over time or stay around the same level? why?
- which of the maximum-energy, minimum-energy, and intermediate-energy rotational states corresponds to the unstable case

(d.3): **Optional** (extra credit for *both* AAE 440 and 590 sections):
Show the three-dimensional herpolhode plots for each of the perturbed cases in Eq. (9), and **discuss** the results.

Problem 2: Solution

Part (a):

When looking at each of the solutions given by equation 6, we get the following results when inputting into Euler's equations, being those given in equation 7:

$$\text{Solution 1: } \dot{\omega}_1 = 0 \ \rightarrow \ \omega_1 = \text{constant} = \omega_{1o}$$
$$\text{Solution 2: } \dot{\omega}_2 = 0 \ \rightarrow \ \omega_2 = \text{constant} = \omega_{2o}$$
$$\text{Solution 3: } \dot{\omega}_3 = 0 \ \rightarrow \ \omega_3 = \text{constant} = \omega_{3o}$$

We may note that the constant solutions are not necessarily equal to zero, and could rather be any constant value.

Part (b):

When determining the energy, we first refer to the following equation for the energy level at any particular state:

$$\frac{H_1^2}{2I_1T} + \frac{H_2^2}{2I_2T} + \frac{H_3^2}{2I_3T} = 1 \tag{15}$$

Where we make the assumption that $I_1 < I_2 < I_3$. We also know that:

$$H_i = I_i\omega_i \tag{16}$$

From here, in solution 1, the second and third angular momentum terms are zero, thus we get:

$$\frac{H_1^2}{2I_1T} = 1 \ \rightarrow \ T = \frac{H^2}{2I_1}$$

Since I_1 is the smallest value, this means that T is maximized. By similar logic, when $\omega_1 = \omega_2 = 0$, T has the third inertia value in the denominator, which would minimize the value. From this, we can get the finalized results:

Maximum: Solution 1
Intermediate: Solution 2
Minimum: Solution 3

Note: several submissions have gone with an approach of fixing the angular velocity rather than the angular momentum, which would lead to the opposite answers. Although that wasn't the expected method when this problem was first made, we can allow this to be accepted without penalty.

Part (c):

We can numerically evaluate the results by using the inertia characteristics from the previous problem (it does not matter which you choose). Next, we can use the three cases provided where one of the three angular velocity components is equal to 0.1, while the other two are zero. Integrating this using "dwdt_torquefree" and ode45, we get the results shown in figure 11.

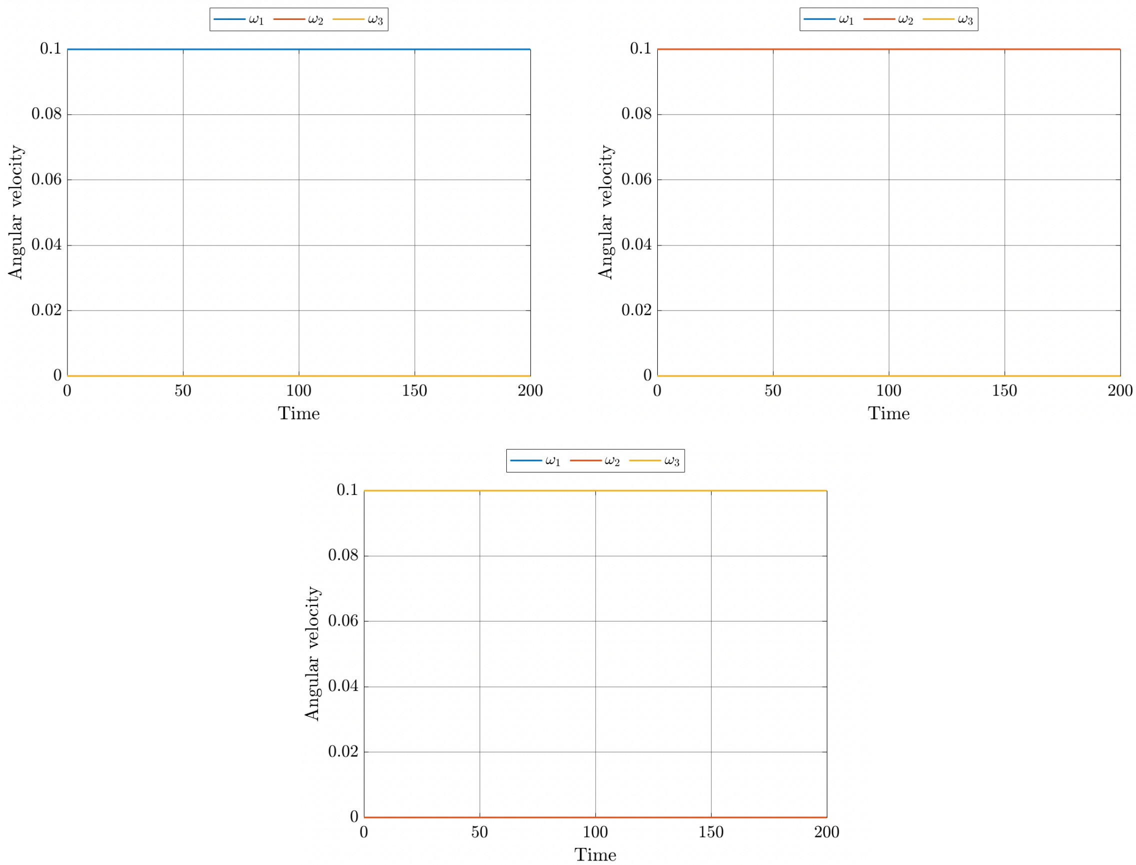


Fig. 11. (Top-Left) Case 1, (Lower) Case 2, (Top-Right) Case 3

We may note that in each of the graphs, the top line is alone, while the bottom line is an overlap of the remaining two. This is the case for each of the plots as shown in figure 11. We can see that each of these confirm the solutions given in equation 6.

Part (d.1):

For seeing the effect of minor perturbations (primed cases), we can use the same code as that for part (c) with differing values. In this case, the values used for the inertia matrix will pose more of an impact to the results. For consistency, we will use the inertia characteristics from problem 1. Doing so, we arrive in the results displayed in figures 12 through 14, labelled accordingly.

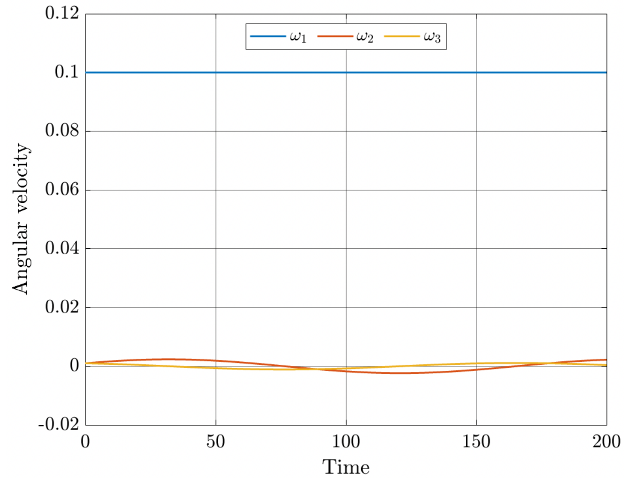


Fig. 12. Case 1’

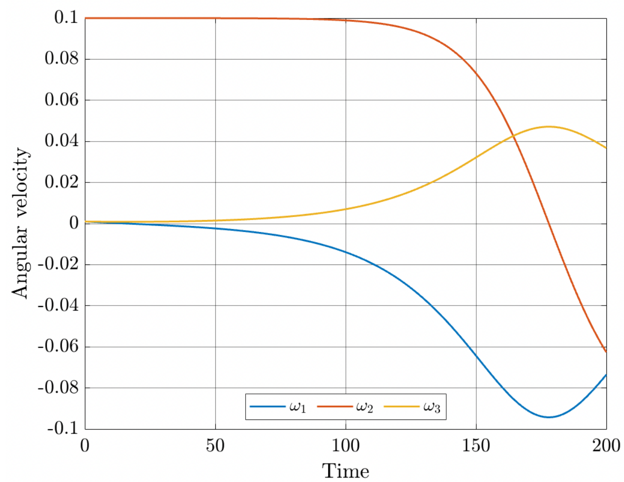


Fig. 13. Case 2’

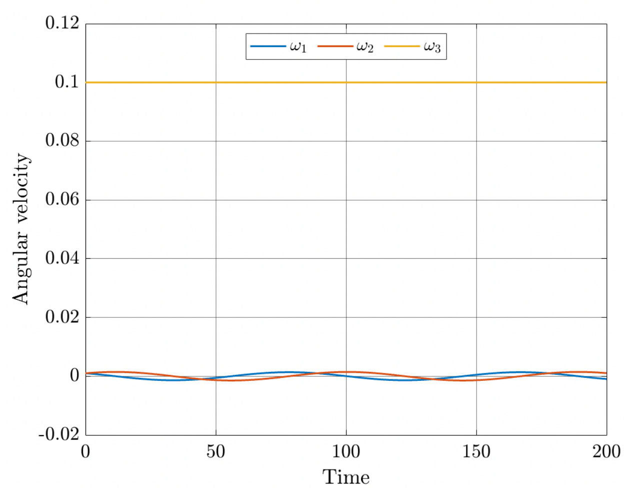


Fig. 14. Case 3’

Part (d.2):

Looking at the behavior of the angular velocity components in case 1', we can see that the component which is largely non-zero is quite constant while the other two, while more visibly variable, are still quite well-behaved (not necessarily converging back, but definitely bounded). The same can be said for case 3', which sees the same behavior for differing components. Case 2', on the other hand, sees a rather rapid divergence from the baseline solution for *each* of the coefficients. The non-zero component (ω_2) diverges from its baseline value at about 100 time units, while the other two by about 25. For cases 1' and 3', we can see some deviation from the zero line by a maximum of 0.002 units, which is quite well bounded considering the propagation is for 200 time units.

Now, it is evident that we do not see any asymptotic stability present in any of the measure numbers, as this would be indicated by rather rapid convergence on a baseline solution from which it would never deviate. Rather, we do see examples of Lyapunov stability, that is, bounded motion which does not converge, but does not diverge either, rather remaining in a bounded region about the baseline solution. This can be seen in cases 1' and 2', the most stable (linearly) of which is seen by ω_1 for case 1', and ω_3 for case 3'. The outlier is the second case, which shows clearly unstable behavior from each of the parameters as they drift further away from the baseline solution. Understanding the correlation will lead us to find that the $\boxed{\text{intermediate-energy}}$ state is that which is destabilizing. This correlates with the intermediate axis theorem, which states that any object spun about its intermediate axis will experience unstable behavior.

We should also note that the magnitude of the trends depend on the inertia characteristics chosen, so the submitted results may differ, but the trends should generally be the same.

Part (d.3):

We may also look at the curves traces out by the angular velocity vector. This may be easily accomplished by looking at the angular velocity components in the body-fixed frame in three-dimensional position space (recall the usage of "plot3" in problem 1). Doing so for each of the three cases gives the plots shown in figures 15 through 17. While the curves may look substantial, first take note of the scale of each of the axes, as they may be more minuscule than a first impression would let on.

We can see that there is quite bounded motion about the axis in question (primary motion about ω_3 would put the curve about the z axis), essentially reducing to motion *nearly* along the point, but more importantly, not further deviating from it over time. As to be expected, the second case does not align with this conclusion, and almost immediately deviates from its initial point. The path immediately taken is about aligned with a diagonal line, which is to be expected since a slight deviation from the intermediate axis would put it on motion near the diagonal separatrix.

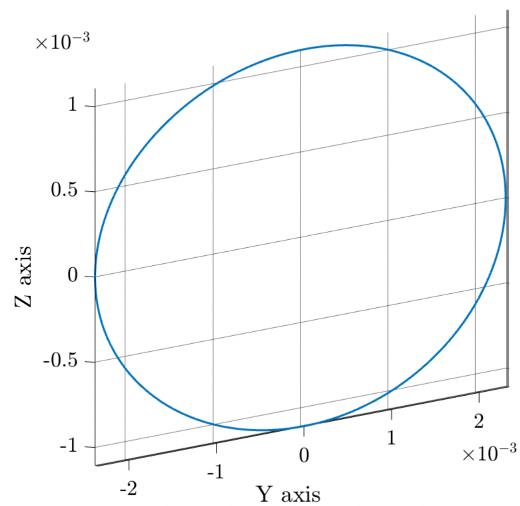


Fig. 15. Case 1' polhode curve

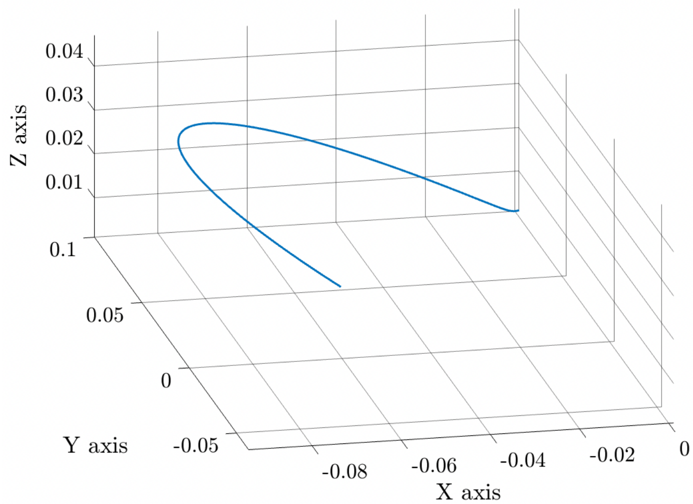


Fig. 16. Case 2' polhode curve

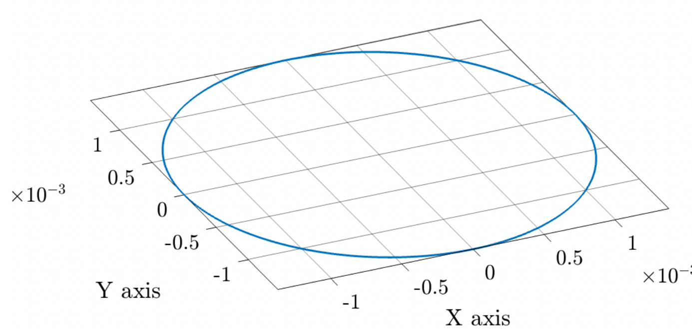
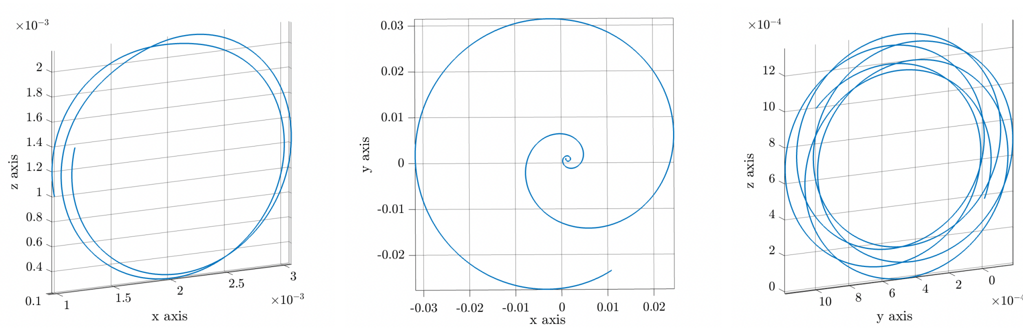


Fig. 17. Case 3' polhode curve

We can thus conclude visually that cases 1' and 3' depict linearly stable motion (marginally stable at best), while the second case is most certainly unstable.

Recall that the class convention defines marginal stability as "stable", but many in the outside world consider "stable" to mean "asymptotically stable", which cannot be the case with attitude dynamics as a whole. Rather, in practice, linearized models are often used in order to predict the stability of systems given a certain perturbation. We know that linearization is somewhat representative of reality, but is also removed from it as well. This means that while the linear model may state that a motion is "marginally stable" (meaning it will not converge nor diverge), this much is not guaranteed in the real world. Rather, linear modeling a marginally stable body means that the real model is simply not guaranteed to be unstable for the time it has been tested for. Reality could see it diverging far more rapidly than modeled, but a marginally stable linear model at least has the potential for this to also not be the case. On the other hand, a linearly unstable model is absolutely sure to be unstable in the nonlinear model. Since our analysis cannot allow for asymptotic stability (a byproduct of the mathematics and dynamics), we really have two outcomes: possibly unstable or definitely unstable. We may lastly note that while this seems bleak, more mild degrees of instability are not so extreme that we cannot easily remedy them with corrective measures.

Note that the wording of the question is may be interpreted to look for the polhode or herpolhode. Either will be accepted for full credit but the polhode was shown earlier for novelty. We may note that the herpolhode may be found the same as it has been before and the results for either case are shown in the below.



(Left) Case 1', (Center) Case 2', (Right) Case 3'

It is clear which is unstable and which isn't, and the same conclusion may be drawn from looking at either set of curves.

Problem 3: Problem Statement

Students in AAE 590 should solve this problem for full score*; students in AAE 440 who complete this problem with correct answers will receive extra credit.*

In class, we also learned that the angular momentum vector of the torque-free attitude motion of a general inertia body must always lie on an intersection of the angular momentum ellipsoid and energy ellipsoid.

(a): Answer the following questions to visually verify the argument, where consider a system with the inertia tensor and the initial angular velocity given by Eqs. (1) and (2), respectively.

(a.1): Recall from class that the analysis is simplified by using the angular momenta coordinates, ${}^{\mathcal{B}}H_i$, which makes the angular momentum ellipsoid a sphere. **Determine** the radius of the angular momentum sphere and the semi-axes of the energy ellipsoid in the angular momenta coordinates.

(a.2): **Show** the angular momentum sphere and the energy ellipsoid in a three-dimensional space with the angular momenta coordinates, overlaid with ${}^{\mathcal{B}}\boldsymbol{H}(t)$. Use different colors for the two surfaces for better illustration. Include at least three different views of the plot, each showing the projection of the plot on the ${}^{\mathcal{B}}H_1$-${}^{\mathcal{B}}H_2$ plane, ${}^{\mathcal{B}}H_1$-${}^{\mathcal{B}}H_3$ plane, and ${}^{\mathcal{B}}H_2$-${}^{\mathcal{B}}H_3$ plane. **Confirm** that ${}^{\mathcal{B}}\boldsymbol{H}(t)$ is indeed on a intersection of the two surfaces (called *polhode* curve).
Hint: if you are using Matlab, a surface can be visualized by using the `surf` function (for python users, `plot_surface` function in the `matplotlib` library). Data points for spheres and ellipsoids can be generated by using `sphere` and `ellipsoid` functions in Matlab.

(b): **Optional** (extra credit for *both* AAE 440 and 590 sections):

(b.1): **Repeat** the procedure in (a) for an axisymmetric inertia case, where use $\bar{\bar{\mathbf{I}}}_{c,\text{axisym}}$ given in Eq. (3) with the same initial condition Eq. (2). Again, include at least three different views of the plot, each showing the projection onto the three planes.

(b.2): **Discuss** the comparison of the two polhode curves obtained in (a) and in the previous question.

Problem 3: Solution

Part (a.1):

In simplifying the model to be in terms of a spherical angular momentum distribution, we should first look into the derivation of the ellipsoid model. In looking at the magnitude of the angular momentum vector, we can formulate it as such:

$$H^2 = \vec{H} \bullet \vec{H} = H_1^2 + H_2^2 + H_3^2 = (I_1\omega_1)^2 + (I_2\omega_2)^2 + (I_3\omega_3)^2 \tag{17}$$

Now, we know that the angular momentum is constant in the inertial reference frame in a torque-free and force-free model (and its magnitude is constant in *any* frame), thus we can equate the expression shown in (17) to a constant value (or simply bear in mind that H^2 is constant). We can then look at the left-half of (17) while eliminating an erroneous equation to get:

$$H^2 = H_1^2 + H_2^2 + H_3^2 = \text{constant} \tag{18}$$

If we treat each of the components of the angular momentum vector (body fixed) as components of interest (i.e. x, y, and z), then we can look at this equation in a new light. We may notice that the equation takes the form of an equation of a sphere, which is expressed in general as such:

$$R^2 = x^2 + y^2 + z^2 = \text{constant} \tag{19}$$

Where x, y, and z are the coordinates of choice and R is the radius of the sphere. This formulation has the same structure as that shown in (18), where we are now in the coordinate system spanned by $\{H_1, H_2, H_3\}$. Thus this (18) leads to a sphere in the angular momentum space of radius H.

Well, we can refer back to (17) to see how we can find the magnitude of the angular momentum vector given information about the inertia and initial angular momentum. Using it with the initial conditions provided from Problem 1, we arrive at:

$$\boxed{H = 0.175}$$

Now, let's define the energy ellipsoid. We know that the rotational kinetic energy is also a constant value for a body not acted upon by any external forces or torques. From its definition, we have:

$$T_{rot} = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \tag{20}$$

We can put this to be in terms of angular momentum by substituting in $H_i = I_i\omega_i$ (doing so in order to completely eliminate all angular velocity terms such that the expression is entirely in terms of angular momentum coordinates and constants) to get:

$$2T_{rot} = \frac{H_1^2}{I_1} + \frac{H_2^2}{I_2} + \frac{H_3^2}{I_3} \tag{21}$$

$$1 = \frac{H_1^2}{2T_{rot}I_1} + \frac{H_2^2}{2T_{rot}I_2} + \frac{H_3^2}{2T_{rot}I_3} \tag{22}$$

$$\tag{23}$$

We can see that, if the inertia characteristics are all not equal, that this is not an equation of a sphere, rather it is that of an ellipsoid, which takes the general form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{24}$$

Where a, b and c are the semi-diameters of the ellipsoid. In doing the same transition to the angular momentum space as we did with the sphere from earlier, we can characterize the energy ellipsoid with an ellipse centered at zero with semi-diameters:

$$a = \sqrt{2T_{rot}I_1}$$
$$b = \sqrt{2T_{rot}I_2}$$
$$c = \sqrt{2T_{rot}I_3}$$

Solving these for the system defined by the parameters given in Problem 1, we get:

$$\boxed{\begin{aligned} a &= 0.1369 \\ b &= 0.1581 \\ c &= 0.1936 \end{aligned}}$$

Part (a.2):

In order to plot a three-dimensional sphere with a particular radius, we can simply use the "sphere" command in MATLAB, which produces the data file for a unit sphere, which can then be multiplied about each dimension by the sphere's radius to get the appropriately proportioned sphere. Lastly we can use the "surf" command to plot a surface of the sphere.

Moreover, in order to plot the ellipsoid, MATLAB conveniently has an "ellipsoid" command which operates similarly to "sphere" but we input both the coordinates of the center of the shape and the semi-diameters. We will center it at zero along with the sphere, the plot both to overlap and modify the colors such that we get the following:

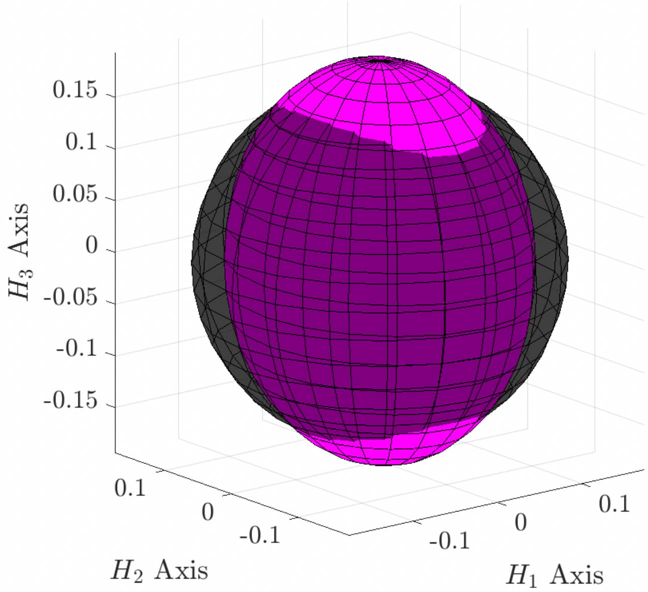


Fig. 18. (Black) Angular momentum sphere, (Magenta) Energy ellipsoid

The above is an isometric view of the surfaces, but we can look at each of the fundamental two-dimensional projections, as shown in figure 19.

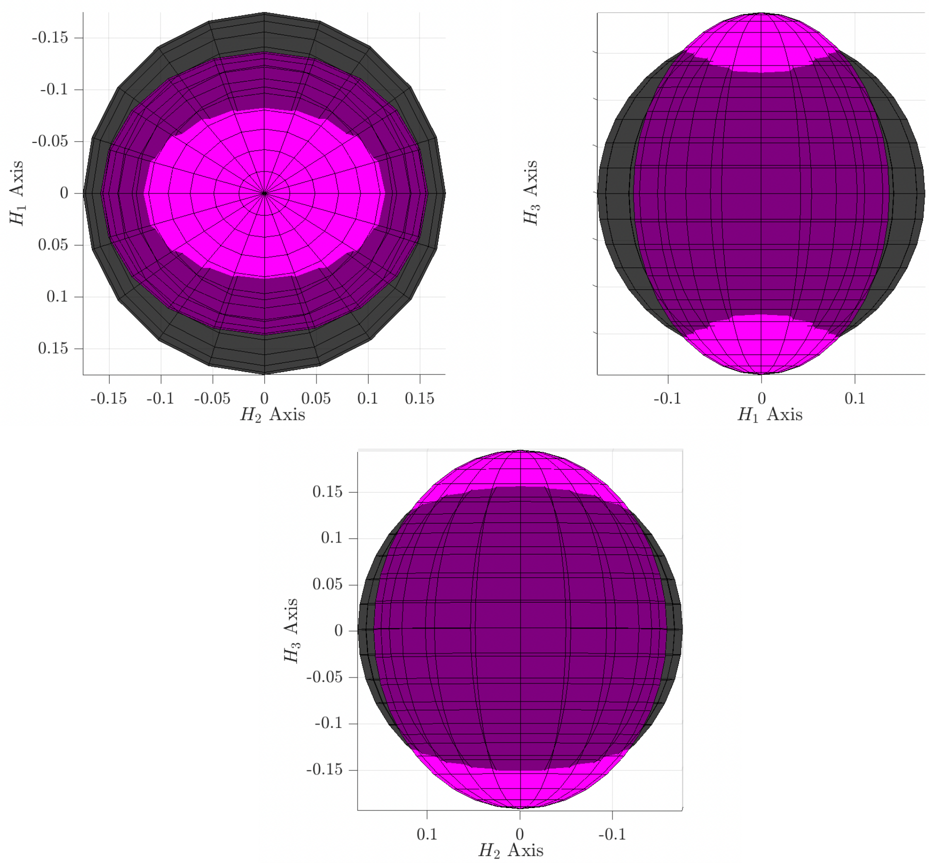


Fig. 19. (Black) Angular momentum sphere, (Magenta) Energy ellipsoid

Lastly, we can compute the angular momentum as a function of time by integrating it using ode45 paired with "dwdt_torquefree" for an arbitrary duration long enough to get a curve to wrap completely around. Doing so gives the yellow curve shown in figure 20, which is clearly along the boundary between the surfaces. This means that the intersection of the surfaces corresponds to a spatial time history of the angular momentum vector's components, or more simply is the polhode curve.

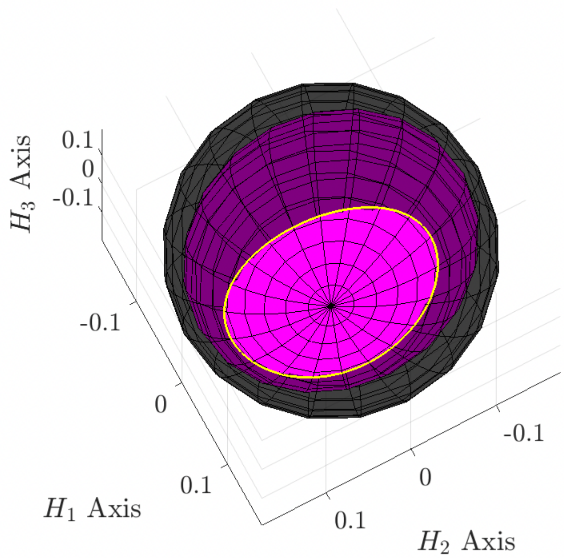


Fig. 20. (Black) Angular momentum sphere, (Magenta) Energy ellipsoid, (Yellow) Polhode curve

Part (b.1):

We can repeat the same process for the axisymmetric case (numbers also provided in Problem 1) to get the results given in figures 21 and 22 (all provided on the next page).

Part (b.2):

Now, we can observe from figures 20 and 22 that we have two polhode curves which both border the surfaces, but also have differing geometries between the two of them. This is a result of the dynamics of the problem, as well as geometrically visible from the change in the shape of the energy ellipsoid (thus changing the shape of the intersection, as one dimension is enlarging compared to the other). Due to the variant properties of the asymmetric case (discussed previously), we would expect its herpolhode (and thus polhode as well) to deviate from the circle as well. In fact, we determined versions of the polhode curves in Problem 2, so we have some intuition of the expected shape of these when projected onto the surfaces.

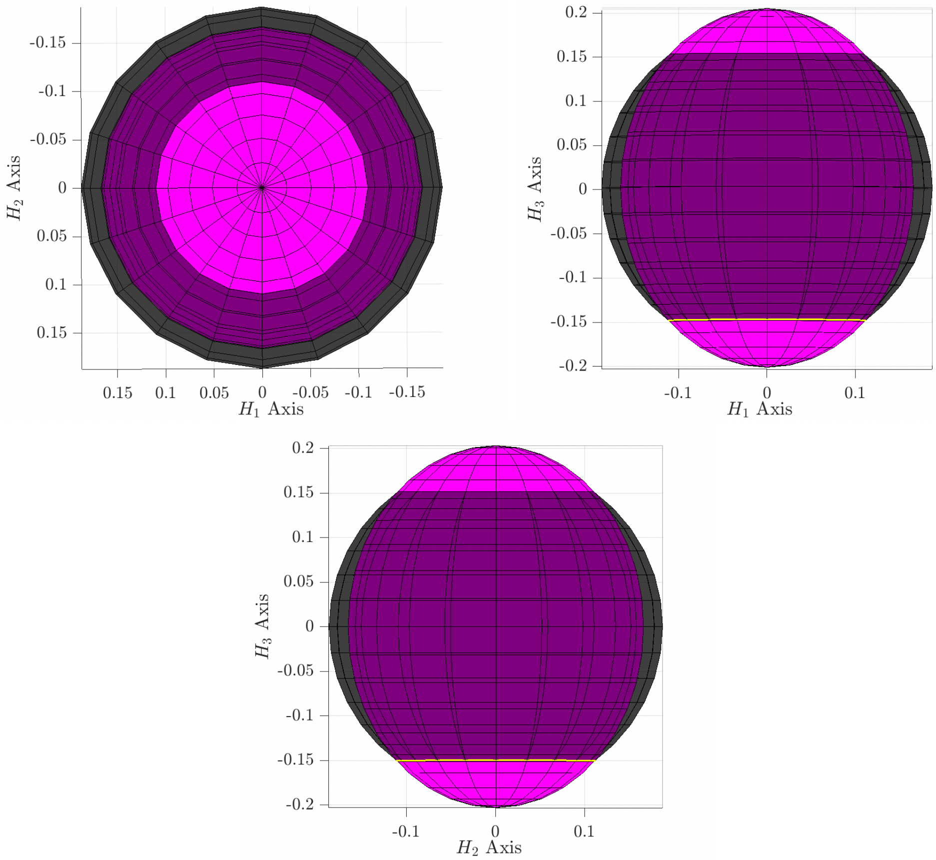


Fig. 21. (Black) Angular momentum sphere, (Magenta) Energy ellipsoid

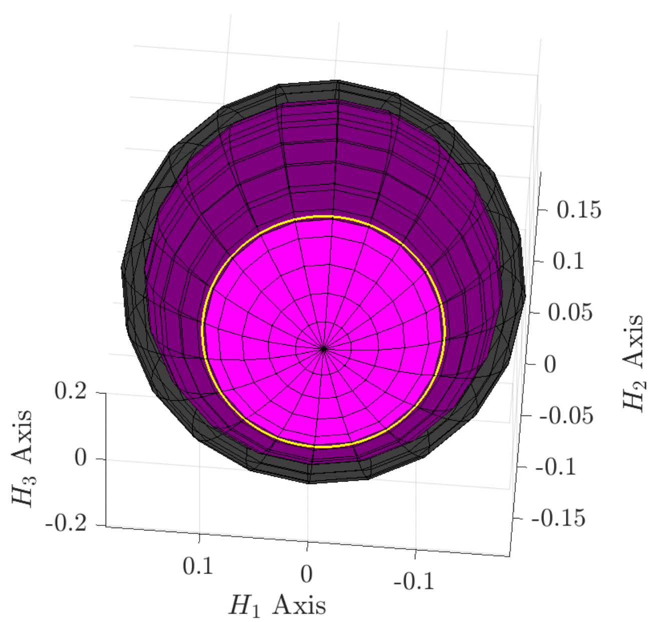


Fig. 22. (Black) Angular momentum sphere, (Magenta) Energy ellipsoid, (Yellow) Polhode curve

Grading Criteria:

Problem 1:

Part (a):

- (1 Point): Plot for angular velocity components over time

Part (b):

- (1 Point): Plot for angular velocity components over time

Part (c):

- (1 Point): Qualitative discussion highlighting if either plot is:
  - Periodic or constant
  - Amplitudes of curves (do not need to give exact numerical values, but qualitative description)
  - (-0.5 Points) For each topic in the above list which is not covered clearly (or not at all)

Part (d):

- (0.5 Points): Constant angular momentum plot
- (0.5 Points): Constant rotational kinetic energy plot

Part (e):

- (1 Point): Constant C_i plot

Part (f.1):

- (1 Point): Plot for MRP components over time
  - May perform MRP switching, but not required

Part (f.2):

- (1 Point): Constant angular momentum plot

Part (f.3):

- (1 Point): Plot of herpolhode with inertial angular momentum vector clearly perpendicular
  - (-0.2 Points): Only one view provided
  - (-0.5 Points): $\vec{H}$ clearly not perpendicular (but close to being so) to herpolhode plane

Part (g):

- (0.5 Points): Herpolhode plot for axisymmetric case
- (0.5 Points): Discussion about geometry of herpolhode (must mention which of the curves are closed, as well as which are circular)

Problem 2:

Part (a)

- Sufficient proof that solutions in equation 6 satisfy the differential equations in equation 7
  - (-0.7 Points): Proof significantly skips steps

Part (b):

- (1/3 Point each): Correctly matching which of the solutions correspond to the maximum energy, minimum energy and intermediate case
  - (-0.3 Points): No discussion/derivation as to *why* this is the case

Part (c):

- (1 Point): Plot for angular velocity components

Part (d.1):

- (1 Point): Plot for angular velocity components

Part (d.2):

- (1 Point): Qualitative discussion which covers the following points (-0.3 points for each not covered sufficiently)
  - General behavior of angular velocity components
  - Comparison with unperturbed case
  - Which energy state corresponds to unstable condition (must conclude intermediate)

Part (d.3):

- (0.5 Points): Plots for three-dimensional curves (either polhode or herpolhode curves)
- (0.5 Points): Discussion about behavior/geometry of results

Problem 3:

Part (a.1):

- (0.25 Points): Radius of momentum sphere
- (0.25 Points (each)): Semi-axes of energy ellipsoid

Part (a.2):

- (1 Point): Plot(s) showcasing geometry of both surfaces as well as polhode at intersection
  - (No credit): Plot is difficult to interpret

Part (b.1):

- (1 Point): Plot(s) showcasing geometry of both surfaces as well as polhode at intersection
  - (No credit): Plot is difficult to interpret

Part (b.2):

- (1 Point): Qualitative comparison of polhode curves found in parts (a) and (b)