Euler Angle Rotations (Body rotation)

For KDE singularity at $\frac{\pi}{2} + \pi n$ *for asymmetric sequence,* πn *for symmetric sequence at second angle*

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\theta) & s(\theta) \\ 0 & -s(\theta) & c(\theta) \end{bmatrix}, \quad R_2(\theta) = \begin{bmatrix} c(\theta) & 0 & -s(\theta) \\ 0 & 1 & 0 \\ s(\theta) & 0 & c(\theta) \end{bmatrix}, \quad R_3(\theta) = \begin{bmatrix} c(\theta) & s(\theta) & 0 \\ -s(\theta) & c(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Directed Cosine Matrix

$(\theta_k, \theta_j, \theta_i)$ about (k-j-i) = $C = [\boldsymbol{\mathcal{BN}}] = R_i(\theta_i)R_j(\theta_j)R_k(\theta_k)$ | $\det(C) = a_{1i}a_{2j}a_{3k}\epsilon_{ijk}$=1 | $C^T = C^{-1}$

$$\begin{bmatrix} \hat{\boldsymbol{b}}_1^T \\ \hat{\boldsymbol{b}}_2^T \\ \hat{\boldsymbol{b}}_3^T \end{bmatrix} = [\mathcal{BN}(\theta)] \begin{bmatrix} \hat{\boldsymbol{n}}_1^T \\ \hat{\boldsymbol{n}}_2^T \\ \hat{\boldsymbol{n}}_3^T \end{bmatrix}, \quad {}^{\mathcal{B}}\bar{\bar{\boldsymbol{I}}}_{\boldsymbol{O}} = [\mathcal{BN}(\theta)]\,{}^{\mathcal{N}}\bar{\bar{\boldsymbol{I}}}_{\boldsymbol{O}}[\mathcal{BN}(\theta)]^T$$

Differentiation:

$$\frac{{}^{\mathcal{S}}d}{dt}\boldsymbol{r}_{\mathcal{C}} = \boldsymbol{v}_{\mathcal{C}/\mathcal{S}} = \frac{{}^{\mathcal{N}}d}{dt}\boldsymbol{r}_{\mathcal{C}} + \boldsymbol{\omega}_{\mathcal{N}/\mathcal{S}} \times \boldsymbol{r}_{\mathcal{C}}, \quad \frac{d}{dt}\boldsymbol{C} = \dot{C} = -\tilde{\omega}C = -\boldsymbol{\omega} \times C, \quad \tilde{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Principle Axis

$$C = \begin{bmatrix} \lambda_1^2(1-c(\theta)) + c(\theta) & \lambda_1\lambda_2(1-c(\theta)) + \lambda_3 s(\theta) & \lambda_1\lambda_3(1-c(\theta)) - \lambda_2 s(\theta) \\ \lambda_1\lambda_2(1-c(\theta)) - \lambda_3 s(\theta) & \lambda_2^2(1-c(\theta)) + c(\theta) & \lambda_2\lambda_3(1-c(\theta)) + \lambda_1 s(\theta) \\ \lambda_3\lambda_1(1-c(\theta)) + \lambda_2 s(\theta) & \lambda_3\lambda_2(1-c(\theta)) - \lambda_1 s(\theta) & \lambda_3^2(1-c(\theta)) + c(\theta) \end{bmatrix}$$

$$(\boldsymbol{3-1-3}), (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}), \quad \alpha = \tan^{-1}\left(\frac{C_{31}}{-C_{32}}\right), \quad \beta = \cos^{-1}(C_{33}), \quad \gamma = \tan^{-1}\left(\frac{C_{13}}{C_{23}}\right)$$

$$(\boldsymbol{2-3-1}), (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}), \quad \alpha = \tan^{-1}\left(\frac{C_{12}}{C_{11}}\right), \quad \beta = \sin^{-1}(C_{13}), \quad \gamma = \tan^{-1}\left(\frac{C_{23}}{C_{33}}\right)$$

$$\theta = \operatorname{acos}\left(\frac{1}{2}(C_{11} + C_{22} + C_{33} - 1)\right), \quad \hat{\boldsymbol{\lambda}} = \frac{1}{2\sin(\theta)}\begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix}$$

Euler Parameters

$$\epsilon_4 = \pm\frac{1}{2}\sqrt{1 + C_{11} + C_{22} + C_{33}}, \quad \epsilon_1 = \frac{C_{23} - C_{32}}{4\epsilon_4}, \quad \epsilon_2 = \frac{C_{31} - C_{13}}{4\epsilon_4}, \quad \epsilon_2 = \frac{C_{31} - C_{13}}{4\epsilon_4}$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{1:3} \\ \epsilon_4 \end{bmatrix} = \begin{bmatrix} {}^{\mathcal{B}}\hat{\boldsymbol{\lambda}} s(\theta/2) \\ c(\theta/2) \end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{bmatrix} = \begin{bmatrix} \epsilon_4'' & \epsilon_3'' & -\epsilon_2'' & \epsilon_1'' \\ -\epsilon_3'' & \epsilon_4'' & \epsilon_1'' & \epsilon_2'' \\ \epsilon_2'' & -\epsilon_1'' & \epsilon_4'' & \epsilon_3'' \\ -\epsilon_1'' & -\epsilon_2'' & -\epsilon_3'' & \epsilon_4'' \end{bmatrix}\begin{bmatrix} \epsilon_1' \\ \epsilon_2' \\ \epsilon_3' \\ \epsilon_4' \end{bmatrix} = \begin{bmatrix} \epsilon_4' & -\epsilon_3' & \epsilon_2' & \epsilon_1' \\ \epsilon_3' & \epsilon_4' & -\epsilon_1' & \epsilon_2' \\ -\epsilon_2' & \epsilon_1' & \epsilon_4' & \epsilon_3' \\ -\epsilon_1' & -\epsilon_2' & -\epsilon_3' & \epsilon_4' \end{bmatrix}\begin{bmatrix} \epsilon_1'' \\ \epsilon_2'' \\ \epsilon_3'' \\ \epsilon_4'' \end{bmatrix}$$

$$\begin{bmatrix} \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \dot{\epsilon}_3 \\ \dot{\epsilon}_4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \epsilon_4 & -\epsilon_3 & \epsilon_2 & \epsilon_1 \\ \epsilon_3 & \epsilon_4 & -\epsilon_1 & \epsilon_2 \\ -\epsilon_2 & \epsilon_1 & \epsilon_4 & \epsilon_3 \\ -\epsilon_1 & -\epsilon_2 & -\epsilon_3 & \epsilon_4 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{bmatrix}, \quad \frac{{}^{\mathcal{B}}d}{dt}\boldsymbol{\epsilon}_{1:3} = \frac{1}{2}(\epsilon_4\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\epsilon}_{1:3} \times \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}), \quad \dot{\epsilon}_4 = -\frac{1}{2}\boldsymbol{\epsilon}_{1:3} \times \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{bmatrix} = 2\begin{bmatrix} \epsilon_4 & \epsilon_3 & -\epsilon_2 & -\epsilon_1 \\ -\epsilon_3 & \epsilon_4 & \epsilon_1 & -\epsilon_2 \\ \epsilon_2 & -\epsilon_1 & \epsilon_4 & -\epsilon_3 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \end{bmatrix}\begin{bmatrix} \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \dot{\epsilon}_3 \\ \dot{\epsilon}_4 \end{bmatrix}, \quad \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \frac{{}^{\mathcal{B}}d}{dt}\boldsymbol{\epsilon}_{1:3} - \boldsymbol{\epsilon}_{1:3} \times \frac{{}^{\mathcal{B}}d}{dt}\boldsymbol{\epsilon}_{1:3} - \dot{\epsilon}_4\boldsymbol{\epsilon}_{1:3}$$

$$C = \begin{bmatrix} 1 - 2\epsilon_2^2 - 2\epsilon_3^2 & 2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4) & 2(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4) \\ 2(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_3^2 & 2(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4) \\ 2(\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4) & 2(\epsilon_2\epsilon_3 - \epsilon_1\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_2^2 \end{bmatrix}$$

Classical Rodrigues Parameters

$$\boldsymbol{\rho} = \hat{\boldsymbol{\lambda}} \tan\left(\frac{\theta}{2}\right), \qquad \rho_i = \frac{\epsilon_i}{\epsilon_4}, \qquad \epsilon_i = \frac{\rho_i}{\sqrt{1+\boldsymbol{\rho}\cdot\boldsymbol{\rho}}}, \qquad \epsilon_4 = \frac{1}{\sqrt{1+\boldsymbol{\rho}\cdot\boldsymbol{\rho}}}, \qquad \boldsymbol{\rho} = \frac{1}{trace(C)+1}\begin{bmatrix} C_{23}-C_{32} \\ C_{31}-C_{13} \\ C_{12}-C_{21}\end{bmatrix}$$

$$\boldsymbol{\rho} = \boldsymbol{\rho}' \pm \boldsymbol{\rho}'' = \frac{\boldsymbol{\rho}'' \pm \boldsymbol{\rho}' \mp \boldsymbol{\rho}'' \times \boldsymbol{\rho}'}{\mathbf{1} \mp \boldsymbol{\rho}'' \cdot \boldsymbol{\rho}'}, \qquad \begin{bmatrix}\dot{\rho}_1 \\ \dot{\rho}_2 \\ \dot{\rho}_3\end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\rho_1^2 & \rho_1\rho_2-\rho_3 & \rho_1\rho_3+\rho_2 \\ \rho_2\rho_1+\rho_3 & 1+\rho_2^2 & \rho_2\rho_3-\rho_1 \\ \rho_3\rho_1-\rho_2 & \rho_3\rho_2+\rho_1 & 1+\rho_3^2\end{bmatrix}\begin{bmatrix}\omega_1 \\ \omega_2 \\ \omega_3\end{bmatrix}$$

$$C = \frac{1}{1+\rho\cdot\rho}\begin{bmatrix} 1+\rho_1^2-\rho_2^2-\rho_3^2 & 2(\rho_1\rho_2+\rho_3) & 2(\rho_1\rho_3-\rho_2) \\ 2(\rho_1\rho_2-\rho_3) & 1-\rho_1^2+\rho_2^2-\rho_3^2 & 2(\rho_2\rho_3+\rho_1) \\ 2(\rho_1\rho_3+\rho_2) & 2(\rho_2\rho_3-\rho_1) & 1-\rho_1^2-\rho_2^2+\rho_3^2\end{bmatrix}$$

Modified Rodrigues Parameters

$$\boldsymbol{\sigma} = \hat{\boldsymbol{\lambda}} \tan\left(\frac{\theta}{4}\right) = \frac{\boldsymbol{\epsilon}_{1:3}}{1+\epsilon_4}, \qquad \boldsymbol{\epsilon}_{1:3} = \frac{2\boldsymbol{\sigma}}{1+\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}}, \qquad \epsilon_4 = \frac{1-\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}}{1+\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}}, \qquad \boldsymbol{\sigma} = \frac{\boldsymbol{\rho}}{1+\sqrt{1+\boldsymbol{\rho}\cdot\boldsymbol{\rho}}}, \qquad \boldsymbol{\rho} = \frac{2\boldsymbol{\sigma}}{1-\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}}$$

$$\boldsymbol{\sigma}^S = \boldsymbol{\sigma}(-\boldsymbol{\epsilon}) = -\frac{\boldsymbol{\sigma}}{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}}, \qquad \boldsymbol{\sigma} = \frac{1}{d(d+1)}\begin{bmatrix} C_{23}-C_{32} \\ C_{31}-C_{13} \\ C_{12}-C_{21}\end{bmatrix}, d = \sqrt{Tr(C)+1},$$

$$\begin{bmatrix}\dot{\sigma}_1 \\ \dot{\sigma}_2 \\ \dot{\sigma}_3\end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1-\sigma^2+2\sigma_1^2 & 2(\sigma_1\sigma_2-\sigma_3) & 2(\sigma_1\sigma_2+\sigma_2) \\ 2(\sigma_2\sigma_1+\sigma_3) & 1-\sigma^2+2\sigma_2^2 & 2(\sigma_2\sigma_3-\sigma_1) \\ 2(\sigma_3\sigma_1-\sigma_2) & 2(\sigma_3\sigma_2+\sigma_1) & 1-\sigma^2+2\sigma_3^2\end{bmatrix}\begin{bmatrix}\omega_1 \\ \omega_2 \\ \omega_3\end{bmatrix}, \sigma^2 = \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}$$

Add for small θ, no singularity by switching at $\|\boldsymbol{\sigma}\| = 1$

$$\boldsymbol{\sigma} = \hat{\boldsymbol{\lambda}}\tan(\theta/4) \quad \to \quad \|\boldsymbol{\sigma}\|_2 = \sqrt{\tan^2(\theta/4)}, \qquad \boldsymbol{\sigma}^{\boldsymbol{s}} = -\hat{\boldsymbol{\lambda}}\cot(\theta/4) \quad \to \quad \|\boldsymbol{\sigma}^{\boldsymbol{s}}\|_2 = \sqrt{\cot^2(\theta/4)}$$

$$C = \frac{1}{(1+\sigma^2)^2}\begin{bmatrix} 4(\sigma_1^2-\sigma_2^2-\sigma_3^2)+(1-\sigma^2)^2 & 8\sigma_1\sigma_2+4\sigma_3(1-\sigma^2) & 8\sigma_1\sigma_3-4\sigma_2(1-\sigma^2) \\ 8\sigma_2\sigma_1-4\sigma_3(1-\sigma^2) & 4(-\sigma_1^2+\sigma_2^2-\sigma_3^2)+(1-\sigma^2)^2 & 8\sigma_2\sigma_3+4\sigma_1(1-\sigma^2) \\ 8\sigma_3\sigma_1+4\sigma_2(1-\sigma^2) & 8\sigma_3\sigma_2-4\sigma_1(1-\sigma^2) & 4(-\sigma_1^2-\sigma_2^2+\sigma_3^2)+(1-\sigma^2)^2 \end{bmatrix}$$

$$\sigma^2 = \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}$$

Axisymmetric Rigid Body Motion

$$T = \frac{1}{2}m\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}} + \frac{1}{2}\boldsymbol{\omega}^{\mathcal{B}}\bar{\bar{\mathbf{I}}}\,\boldsymbol{\omega}, \qquad {}^{\mathcal{B}}\boldsymbol{H}_c = {}^{\mathcal{B}}\bar{\bar{\mathbf{I}}}\,\boldsymbol{\omega}, \qquad {}^{\mathcal{B}}\bar{\bar{\mathbf{I}}}\,\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega}\times{}^{\mathcal{B}}\bar{\bar{\mathbf{I}}} + \boldsymbol{L_c} = -\boldsymbol{\omega}\times{}^{\mathcal{B}}\bar{\bar{\mathbf{I}}}$$

$$\dot{\omega}_1 = K_1\omega_2\omega_3, \qquad \dot{\omega}_2 = K_2\omega_3\omega_1, \qquad \dot{\omega}_3 = K_3\omega_1\omega_2$$

$$\omega_1(t) = A_1\cos(K\omega_3 t) + B_1\sin(K\omega_3 t) \qquad K = \frac{I_T - J}{I_T} = -1$$

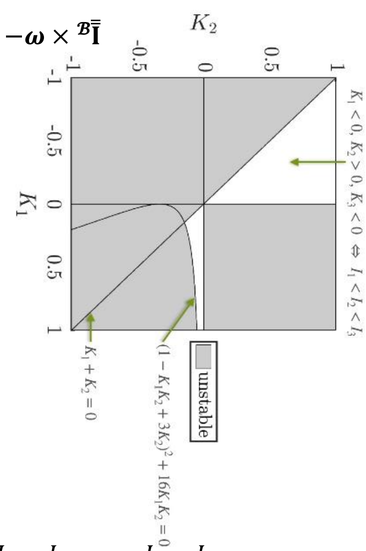
$$\omega_2(t) = A_2\cos(K\omega_3 t) + B_2\sin(K\omega_3 t)'$$

Body Spin: $\boldsymbol{\omega_{\mathcal{B}/\mathcal{C}}} = s\hat{\boldsymbol{c}}_{\boldsymbol{3}}$**,** Precession: $\boldsymbol{\omega_{\mathcal{C}/\mathcal{N}}} = p$

$$p\hat{\boldsymbol{h}} = \frac{H}{I_T}\hat{\boldsymbol{h}} = \boldsymbol{\omega_{\mathcal{C}/\mathcal{N}}} = \boldsymbol{\omega_{\mathcal{B}/\mathcal{N}}} - \boldsymbol{\omega_{\mathcal{B}/\mathcal{C}}}, \qquad \cos(\phi) = \frac{Js}{(I_T - J)p}$$

$$\{\hat{\boldsymbol{\lambda}}_{\mathcal{CN}}, \theta_{\mathcal{CN}}\} = \{\hat{\boldsymbol{h}}, p\cdot t\}, \qquad \{\hat{\boldsymbol{\lambda}}_{\mathcal{BC}}, \theta_{\mathcal{BC}}\} = \{\hat{\boldsymbol{c}}_{\boldsymbol{3}}, s\cdot t\}$$

$$\boldsymbol{\omega_{\mathcal{B}/\mathcal{N}}} = s\hat{\boldsymbol{c}}_{\boldsymbol{3}} + p\hat{\boldsymbol{h}}, \qquad p = \frac{H}{I_T} > 0, \qquad s = K\omega_3 = \frac{(I_T - J)\omega_3}{I_T}$$



s>0, p>0, outside roll direct precession, s<0, p>0, inside retrograde precession

Motion under Gravity Gradient

$$H^2 = H_1^2 + H_2^2 + H_3^2, \qquad 1 = \frac{H_1^2}{2I_1T_{rot}} + \frac{H_2^2}{2I_2T_{rot}} + \frac{H_3^2}{2I_3T_{rot}}$$

$$\boldsymbol{R} = {}^{\mathcal{B}}\boldsymbol{R}, \qquad \boldsymbol{\omega}' = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}}, \qquad \Omega = \sqrt{\frac{\mu}{R^3}}, \qquad K_1 = \frac{I_2 - I_3}{I_1}, K_1 = \frac{I_3 - I_1}{I_2}, K_1 = \frac{I_1 - I_2}{I_3}$$

$$\begin{bmatrix}\dot{\sigma}_1 \\ \dot{\sigma}_2 \\ \dot{\sigma}_3\end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1-\sigma^2+2\sigma_1^2 & 2(\sigma_1\sigma_2-\sigma_3) & 2(\sigma_1\sigma_2+\sigma_2) \\ 2(\sigma_2\sigma_1+\sigma_3) & 1-\sigma^2+2\sigma_2^2 & 2(\sigma_2\sigma_3-\sigma_1) \\ 2(\sigma_3\sigma_1-\sigma_2) & 2(\sigma_3\sigma_2+\sigma_1) & 1-\sigma^2+2\sigma_3^2\end{bmatrix}\begin{bmatrix}\omega_1' \\ \omega_2' \\ \omega_3'\end{bmatrix}, \qquad \begin{bmatrix}\dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3\end{bmatrix} = \begin{bmatrix}K_1\omega_2\omega_3 \\ K_2\omega_1\omega_3 \\ K_3\omega_1\omega_2\end{bmatrix} - \frac{3\Omega^2}{\|\boldsymbol{R_c}\|_2^2}\begin{bmatrix}K_1R_3R_2 \\ K_2R_1R_3 \\ K_3R_2R_1\end{bmatrix}$$

$$\lambda^2 - 3K_3\Omega^2 = 0, \qquad \lambda^4 + (1 - K_1K_2 + 3K_2)\Omega^2\lambda^2 - 4K_1K_2\Omega^4 = 0$$