Computation Set 05 Solution

AAE 440: Spacecraft Attitude Dynamics
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Problem 01: Problem Statement

In class, we learned that the gravity force and torque acting on a rigid body satellite are expressed as follows:

$$
\begin{aligned}
\boldsymbol{F}_G &= -\frac{\mu}{R_c^3}\left[m\boldsymbol{R}_c + \underbrace{\frac{3}{R_c^2}\bar{\bar{\mathbf{I}}}_c\cdot\boldsymbol{R}_c + \frac{3}{2R_c^2}\mathrm{Tr}[\bar{\bar{\mathbf{I}}}_c]\boldsymbol{R}_c - \frac{15}{2R_c^4}(\boldsymbol{R}_c\cdot\bar{\bar{\mathbf{I}}}_c\cdot\boldsymbol{R}_c)\boldsymbol{R}_c}_{\mathcal{O}(\varepsilon^2)} + \mathcal{O}\left(\varepsilon^3\right)\right] \\
\boldsymbol{L}_G &= \frac{3\mu}{R_c^5}\boldsymbol{R}_c\times\bar{\bar{\mathbf{I}}}_c\cdot\boldsymbol{R}_c + \mathcal{O}\left(\varepsilon^3\right)
\end{aligned}
\tag{1}
$$

where m and $\bar{\bar{\mathbf{I}}}_c$ are the total mass and the inertia tensor of the satellite; μ is the gravitational parameter of the point-mass gravitational body (usually Earth in this class); $\boldsymbol{R}_c$ is the position of the satellite center of mass (CoM) measured from the gravitational body; and R_c is its magnitude, i.e., $R_c = \|\boldsymbol{R}_c\|_2$. $\varepsilon = \frac{r}{R_c}$ is a small parameter, where r is the distance between the satellite CoM and a satellite differential mass $\mathrm{d}m$.

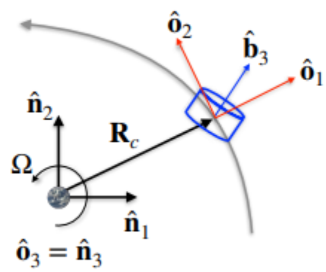


Figure 1: Inertial frame $\{\hat{\boldsymbol{n}}_i\}$, satellite body-fixed frame $\{\hat{\boldsymbol{b}}_i\}$, and orbit frame $\{\hat{\boldsymbol{o}}_i\}$

Consider a satellite in a circular low-Earth orbit (LEO), depicted in Fig. 1. The inertial frame, satellite body-fixed frame, and orbit frame are represented by $\mathcal{N}$-frame, $\mathcal{B}$-frame, and $\mathcal{O}$-frame, where $\{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$, $\{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$, and $\{\hat{\boldsymbol{o}}_1, \hat{\boldsymbol{o}}_2, \hat{\boldsymbol{o}}_3\}$ are right-handed vector bases fixed in $\mathcal{N}$-frame, $\mathcal{B}$-frame, and $\mathcal{O}$-frame, respectively. $\mathcal{N}$-frame and $\mathcal{O}$-frame are taken so that the orbit lies in the plane spanned by $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$, $\hat{\boldsymbol{n}}_3 = \hat{\boldsymbol{o}}_3$, and that the orbit radius vector $\boldsymbol{R}_c$ is aligned with $\hat{\boldsymbol{o}}_1$, i.e., $\boldsymbol{R}_c = R_c\hat{\boldsymbol{o}}_1$. Use $\mu = 3.9860\times10^5\ \mathrm{km}^3/\mathrm{s}^2$ for the gravitational parameter of the Earth.

Suppose that the satellite inertia tensor is given by:

$$
\bar{\bar{\mathbf{I}}}_c = \frac{5}{12}ml^2\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_1 + \frac{5}{6}ml^2\hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_2 + \frac{13}{12}ml^2\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_3 \tag{2}
$$

where l represents the characteristic length of the satellite. Also, the orientation of $\mathcal{B}$-frame relative to $\mathcal{O}$-frame in terms of a modified Rodrigues parameter (MRP) vector is measured at $t = 0$ as:

$$
\boldsymbol{\sigma}_{B/O}(t=0) = \frac{1}{3}\hat{\boldsymbol{b}}_1 + \frac{1}{4}\hat{\boldsymbol{b}}_2 + \frac{1}{5}\hat{\boldsymbol{b}}_3,
$$

(a): **Discuss** whether the body with the inertia tensor given in Eq. (2) is centrobaric. Why or why not?

(b): When the body is not centrobaric, the center of mass (CoM) and the center of gravity (CoG) do not coincide. **Show** that the position of the CoG measured from the Earth CoM is expressed as in Eq. (3):

$$
\boldsymbol{R}_{cg} = -R_{cg}\frac{\boldsymbol{F}_G}{\|\boldsymbol{F}_G\|_2},\quad R_{cg} = \sqrt{\frac{\mu m}{\|\boldsymbol{F}_G\|_2}} \tag{3}
$$

Hint: from the definition of CoG, the gravity force acting on a body can be expressed in terms of $\boldsymbol{R}_{cg}$:

$$
\boldsymbol{F}_G = -\frac{\mu m}{R_{cg}^3}\boldsymbol{R}_{cg},
$$

where R_{cg} is the magnitude of $\boldsymbol{R}_{cg}$, i.e., $R_{cg} = \|\boldsymbol{R}_{cg}\|_2$.

(c): Suppose that the properties of our satellite are $m = 100$ kg and $l = 70$ cm (i.e., $l = 7.0 \times 10^{-4}$ km) in an LEO of orbit radius $R_c = 6578$ km (~ 200 km altitude). Answer the following problems.

(c.1): Using Eq. (1) and ignoring the terms of the order ε^3 and higher, **determine** the values of $\boldsymbol{F}_G$ and $\boldsymbol{L}_G$ in $\mathcal{B}$-frame at $t = 0$.

(c.2): **Determine** the value of the first-order approximation of $\boldsymbol{F}_G$ (i.e., ignore ε^2 and higher order terms) in $\mathcal{B}$-frame at $t = 0$, and **report** the value of $\boldsymbol{F}_{G,\text{2nd}} - \boldsymbol{F}_{G,\text{1st}}$, where $\boldsymbol{F}_{G,\text{2nd}}$ denotes the second-order approximation of $\boldsymbol{F}_G$ while $\boldsymbol{F}_{G,\text{1st}}$ the first-order approximation.

(c.3): **Determine** the distance between the CoG and CoM of the satellite at $t = 0$, i.e., $\|\boldsymbol{R}_{cg} - \boldsymbol{R}_c\|_2$, where use the second-order approximation of $\boldsymbol{F}_G$ given in Eq. (1) to compute $\boldsymbol{R}_{cg}$.

(d): Next, consider a much larger body with $m = 4.5 \times 10^5$ kg and $l = 33$ m ($= 3.3 \times 10^{-2}$ km)—similar size as the international space station (ISS) but a different shape—in the same orbit as part (c). **Answer** the same questions as (c.1)-(c.3) for this system.

(e): Next, consider the large body assumed in part (d) with a different orbit radius $R_c = 3.58 \times 10^5$ km (close to the geosynchronous orbit). **Answer** the same questions as (c.1)-(c.3) for this system.

(f): **Discuss** the results obtained so far and their implications, addressing the following points:

- comparison of the two systems assumed in (c) and (d), in terms of the (i) gravitational force, (ii), torque, and (iii) distance between the CoM and CoG
- comparison of the two systems assumed in (d) and (e), in terms of the same quantities (i)-(iii)
- based on the numerical results of (c)-(e) as well as the analytical expression given in Eq. (1), what we can infer about the general trend of the three quantities (i)-(iii) for different systems
- whether the second-order term of the total gravity is practically negligible for typical Earth orbiters
- in what kind of systems the second-order term of the total gravity may not be negligible (Hint: what if we consider an orbiter about a small asteroid?)

Problem 01: Problem Solution

Part (a):

We know that a body is centrobaric if it has the same moment of inertia about every line that passes through the CoM; $I_1 = I_2 = I_3$ for any body frame. Because this spacecraft has three unique principal moments of inertia we can confidently say that this satellite is **not** centrobaric.

Part (b):

Starting with the definition of CoG, we know that the gravity force on a body can be expressed in terms of $\boldsymbol{R_{cg}}$. We can then take the norm of the gravity force and solve for $\boldsymbol{R_{cg}}$

$$\boldsymbol{F_G} = -\frac{\mu m}{R_{cg}^3}\boldsymbol{R_{cg}}$$

$$\|\boldsymbol{F_G}\|_2 = \frac{\mu m}{R_{cg}^3} R_{cg} = \frac{\mu m}{R_{cg}^2}$$

$$R_{cg} = \sqrt{\frac{\mu m}{\|\boldsymbol{F_G}\|_2}}$$

$$\frac{\boldsymbol{F_G}}{\|\boldsymbol{F_G}\|_2} = -\frac{\frac{\mu m}{R_{cg}^3}\boldsymbol{R_{cg}}}{\frac{\mu m}{R_{cg}^2}} = -\frac{\boldsymbol{R_{cg}}}{R_{cg}}$$

$$\boldsymbol{R_{cg}} = -\frac{\boldsymbol{F_G}}{\|\boldsymbol{F_G}\|_2} \cdot R_{cg}$$

Part (c):

Part (c.1):

Before we can compute Eq. (1) we must first determine ${}^{\mathcal{B}}\boldsymbol{R}_c$. This can be done using $[\mathcal{BO}]$.

$$[\mathcal{BO}] = \mathbf{MRPtoDCM}\left(\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{O}}\right) = I_3 + \frac{8\widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\sigma}} - 4(1 - \boldsymbol{\sigma}\cdot\boldsymbol{\sigma})\widetilde{\boldsymbol{\sigma}}}{(1 + \boldsymbol{\sigma}\cdot\boldsymbol{\sigma})^2}$$

$${}^{\mathcal{B}}\boldsymbol{R}_c = [\mathcal{BO}]\,{}^{\mathcal{O}}\boldsymbol{R}_c = [\mathcal{BO}]\begin{bmatrix} R_c \\ 0 \\ 0 \end{bmatrix}$$

Next we can compute the following eequations:

$$\boldsymbol{F}_G = -\frac{\mu}{R_C^3}\left[m\boldsymbol{R_c} + \frac{3}{R_c^2}\bar{\bar{\mathbf{I}}}_{\boldsymbol{c}} \cdot \boldsymbol{R_c} + \frac{3}{2R_c^2}\mathrm{Tr}\left[\bar{\bar{\mathbf{I}}}_{\boldsymbol{c}}\right]\boldsymbol{R_c} - \frac{15}{2R_c^4}\left(\boldsymbol{R_c}\cdot\bar{\bar{\mathbf{I}}}_{\boldsymbol{c}}\cdot\boldsymbol{R_c}\right) + \boldsymbol{\mathcal{O}}(\epsilon^3)\right]$$

$$\boldsymbol{L}_G = \frac{3\mu}{R_c^5}\boldsymbol{R_c} \times \bar{\bar{\mathbf{I}}}_{\boldsymbol{c}} \cdot \boldsymbol{R_c} + \boldsymbol{\mathcal{O}}(\epsilon^3)$$

$$\boldsymbol{F}_G = \begin{bmatrix} -408.6 \\ -23.5 \\ -826.0 \end{bmatrix} N, \qquad \boldsymbol{L}_G = \begin{bmatrix} 1.177E-06 \\ -5.455E-05 \\ 9.702E-07 \end{bmatrix} N \cdot m$$

```python
# Part C
mu      = 3.989e14 # m/s^2
m       = 100   # [kg]
l       = 0.7   # [m]
I       = m*l**2 * np.array([[5/12, 0, 0],
                             [0, 5/6, 0],
                             [0, 0, 13/12]])

Rc      = 6578e3  # [m] 200km altitude
Rc_O    = np.array([Rc, 0, 0])

MRP_BO  = np.array([1/3, 1/4, 1/5])
DCM_BO  = MRPtoDCM(MRP_BO)
Rc_B    = np.dot(DCM_BO, Rc_O)

Fg      = -mu/Rc**3*(m*Rc_B+3/Rc**2*np.dot(I, Rc_B)+3/(2*Rc**2)*np.trace(I)*Rc_B
                  - 15/(2*Rc**4)*np.dot(Rc_B, np.dot(I, Rc_B))*Rc_B)

Lg      = 3*mu/(Rc**5)*np.cross(Rc_B, np.dot(I, Rc_B))
```

<u>Part (c.2):</u>

We can compare the first order and second order approximations for the gravitation force and as shown below, confirm that the difference is extremely small.

$$\boldsymbol{F}_{G,1st} = \begin{bmatrix} -408.6 \\ -23.5 \\ -826.0 \end{bmatrix} N, \qquad \boldsymbol{F}_{G,2nd} - \boldsymbol{F}_{G,1st} = \begin{bmatrix} 1.108E-11 \\ 3.055E-13 \\ 3.752E-12 \end{bmatrix} N$$

<u>Part (c.3):</u>

Using the equations determined in Part (a) we can apply the following relation using the second order approximation for gravitational force. Note all vectors we expressed in the B frame for these calculations.

$$R_{cg} = \sqrt{\frac{\mu m}{\|\boldsymbol{F}_G\|_2}}, \qquad \boldsymbol{R}_{cg} = -\frac{\boldsymbol{F}_G}{\|\boldsymbol{F}_G\|_2} \cdot R_{cg}$$

$$\left\|\boldsymbol{R}_{cg} - \boldsymbol{R}_c\right\| = 6.622E-08\ m$$

<u>Part (d):</u>

Using the same calculations and code as in Part (c), we find that:

$$\boldsymbol{F}_G = \begin{bmatrix} -1.839 \\ -0.106 \\ -3.717 \end{bmatrix} MN, \qquad \boldsymbol{L}_G = \begin{bmatrix} 11.769 \\ -545.547 \\ 9.703 \end{bmatrix} N \cdot m$$

$$\boldsymbol{F}_{G,2nd} - \boldsymbol{F}_{G,1st} = \begin{bmatrix} 1.107E-5 \\ 3.039E-7 \\ 3.662E-6 \end{bmatrix} N$$

$$\left\|\boldsymbol{R}_{cg} - \boldsymbol{R}_c\right\| = 1.467E-04\ m$$

Part (e):

Using the same calculations and code as in Part (c), we find that:

$$\boldsymbol{F}_G = \begin{bmatrix} -62.082 \\ -3.571 \\ -125.497 \end{bmatrix} kN, \qquad \boldsymbol{L}_G = \begin{bmatrix} 0.0733 \\ -3.384 \\ 0.060 \end{bmatrix} N \cdot m$$

$$\boldsymbol{F}_{G,2nd} - \boldsymbol{F}_{G,1st} = \begin{bmatrix} 1.262E-7 \\ 3.463E-9 \\ 4.172E-8 \end{bmatrix} N$$

$$\left\|\boldsymbol{R}_{cg} - \boldsymbol{R}_c\right\| = 2.696E-5\ m$$

Part (f):

- Comparing systems in (c) to (d) we can see that those in (d) have roughly 4500x larger gravitational force, 1 million times larger torque, and 2200x larger distance between CoM and CoG.
- Comparing systems in (d) to (e) we can see that those in (d) have roughly 30x larger gravitational force, 160x larger torque, and 5x larger distance between CoM and CoG.
- Broadly speaking, we can notice that increasing the length of the object increases the torque and distance between CoM and CoG. Additionally increasing mass increases the gravitational force. Finally, increasing radius will reduce all 3 terms (with the largest impact on torque). These relations align with the structure of Eq. (1).
- How negligible the term is may depend on the specific application. However, in these 3 examples we see that the second order terms contribution is billions of times smaller than the first order term. For the vast majority of use cases this means the second order term is effectively negligible for Earth orbit.
- One case where the second order term may become significant is when $\epsilon = r/R_c$ becomes large. This can occur when the spacecraft is very close to the center of mass of a body such as a satellite orbiting an asteroid, where the relative size of the satellite compared to its distance from the center of the asteroid is quite large.

Problem 02: Problem Statement

Let us now investigate the spacecraft attitude motion under the action of the gravity gradient torque.

In class, we used the Euler parameter (EP) as the attitude representation to analyze the stability of the attitude motion in a circular orbit. In this problem, we use the MRP representation.

Thus, our attitude variables consist of the MRP vector that represents the orientation of $\mathcal{B}$-frame relative to $\mathcal{O}$-frame, denoted by $\boldsymbol{\sigma}(=\boldsymbol{\sigma}_{B/O}) = \sigma_1\hat{\boldsymbol{b}}_1 + \sigma_2\hat{\boldsymbol{b}}_2 + \sigma_3\hat{\boldsymbol{b}}_3$, and the angular velocity of $\mathcal{B}$-frame relative to $\mathcal{N}$-frame, $\boldsymbol{\omega}(=\boldsymbol{\omega}_{B/N}) = \omega_1\hat{\boldsymbol{b}}_1 + \omega_2\hat{\boldsymbol{b}}_2 + \omega_3\hat{\boldsymbol{b}}_3$. We use the same $\mathcal{N}$-frame, $\mathcal{B}$-frame, and $\mathcal{O}$-frame as defined in Problem 1. The equations of motion that describe our system are given by:

$$\begin{bmatrix}\dot{\sigma_1}\\ \dot{\sigma_2}\\ \dot{\sigma_3}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1-\sigma^2+2\sigma_1^2 & 2(\sigma_1\sigma_2-\sigma_3) & 2(\sigma_1\sigma_3+\sigma_2)\\ 2(\sigma_2\sigma_1+\sigma_3) & 1-\sigma^2+2\sigma_2^2 & 2(\sigma_2\sigma_3-\sigma_1)\\ 2(\sigma_3\sigma_1-\sigma_2) & 2(\sigma_3\sigma_2+\sigma_1) & 1-\sigma^2+2\sigma_3^2\end{bmatrix}\begin{bmatrix}\omega_1'\\ \omega_2'\\ \omega_3'\end{bmatrix} \quad \text{(Kinematic Diff. Equation)} \tag{4}$$

$$\begin{bmatrix}I_1\dot{\omega}_1\\ I_2\dot{\omega}_2\\ I_3\dot{\omega}_3\end{bmatrix} = \begin{bmatrix}-(I_3-I_2)\omega_2\omega_3\\ -(I_1-I_3)\omega_3\omega_1\\ -(I_2-I_1)\omega_1\omega_2\end{bmatrix} + \begin{bmatrix}L_{G_1}\\ L_{G_2}\\ L_{G_3}\end{bmatrix} \quad \text{(Euler's Rotational Equation)} \tag{5}$$

$$\begin{bmatrix}L_{G_1}\\ L_{G_2}\\ L_{G_3}\end{bmatrix} = \frac{3\mu}{R^5}\begin{bmatrix}(I_3-I_2)R_3R_2\\ (I_1-I_3)R_1R_3\\ (I_2-I_1)R_2R_1\end{bmatrix} \quad \text{(Gravity Gradient)} \tag{6}$$

where $\boldsymbol{\omega}'(=\boldsymbol{\omega}_{B/O}) = \omega_1'\hat{\boldsymbol{b}}_1 + \omega_2'\hat{\boldsymbol{b}}_2 + \omega_3'\hat{\boldsymbol{b}}_3$ is the angular velocity of $\mathcal{B}$-frame relative to $\mathcal{O}$-frame, and $\boldsymbol{R} = R_1\hat{\boldsymbol{b}}_1 + R_2\hat{\boldsymbol{b}}_2 + R_3\hat{\boldsymbol{b}}_3$ represents the satellite position. μ is the gravitational parameter of the central body, and the orbit angular speed is constant in circular orbit and given by $\Omega = \sqrt{\mu/R^3}$.

(a): **Show** that combining Eqs. (5) and (6) leads to the following equations:

$$\begin{bmatrix}\dot{\omega}_1\\ \dot{\omega}_2\\ \dot{\omega}_3\end{bmatrix} = \begin{bmatrix}K_1\omega_2\omega_3\\ K_2\omega_1\omega_3\\ K_3\omega_1\omega_2\end{bmatrix} - \frac{3\Omega^2}{\|\boldsymbol{R}\|_2^2}\begin{bmatrix}K_1R_3R_2\\ K_2R_1R_3\\ K_3R_2R_1\end{bmatrix} \tag{7}$$

where $K_1 = (I_2-I_3)/I_1,\ K_2 = (I_3-I_1)/I_2$, and $K_3 = (I_1-I_2)/I_3$.

(b): In class, we derived a particular solution of our system in terms of EP, given by: $\bar{\boldsymbol{\epsilon}}_{B/O} = [0,\ 0,\ 0,\ 1]^\top$ and $\bar{\boldsymbol{\omega}}_{B/N} = [0,\ 0,\ \Omega]^\top$, where the bar $\bar{(\cdot)}$ indicates the particular solution. **Show** that the particular solution $\{\bar{\boldsymbol{\epsilon}}_{B/O}, \bar{\boldsymbol{\omega}}_{B/N}\}$ is equivalently expressed as in Eq. (8), and that Eq. (8) satisfies Eqs. (4) and (7).

$$\bar{\sigma}_1 = \bar{\sigma}_2 = \bar{\sigma}_3 = 0, \quad \bar{\omega}_1 = \bar{\omega}_2 = 0, \quad \bar{\omega}_3 = \Omega \tag{8}$$

Hint: ω_i' in Eq. (4) need to be first expressed in terms of ω_i.

(c): We use Eq. (8) as the particular solution of our interest. We then linearize Eqs. (4) and (7) to perform the linear stability analysis. To do this, consider small perturbations in $\boldsymbol{\sigma}$ and $\boldsymbol{\omega}$ about the particular solution, i.e., $\sigma_i = \bar{\sigma}_i + \widetilde{\sigma}_i$ and $\omega_i = \bar{\omega}_i + \widetilde{\omega}_i$ with $\widetilde{\sigma}_i \ll 1$ and $\widetilde{\omega}_i \ll 1$, and ignore the second and higher order terms ($\widetilde{\sigma}_i^2 = \widetilde{\omega}_i^2 = 0$). **Show** that the linearized equations are given by Eq. (9).

$$\begin{bmatrix}\dot{\widetilde{\sigma}}_1\\ \dot{\widetilde{\sigma}}_2\\ \dot{\widetilde{\sigma}}_3\end{bmatrix} = \frac{1}{4}\begin{bmatrix}\widetilde{\omega}_1\\ \widetilde{\omega}_2\\ \widetilde{\omega}_3\end{bmatrix} + \Omega\begin{bmatrix}\widetilde{\sigma}_2\\ -\widetilde{\sigma}_1\\ 0\end{bmatrix}, \quad \begin{cases}\dot{\widetilde{\omega}}_1 = K_1\Omega\widetilde{\omega}_2\\ \dot{\widetilde{\omega}}_2 = K_2\Omega\widetilde{\omega}_1 - 12K_2\Omega^2\widetilde{\sigma}_2\\ \dot{\widetilde{\omega}}_3 = 12K_3\Omega^2\widetilde{\sigma}_3\end{cases} \tag{9}$$

(d): As discussed in class, the stability of a linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is determined by the eigenvalues of the system matrix A, which are obtained by solving the characteristic equations of the system. **Show** that the characteristic equations of our linear system Eq. (9) are the same as those for $\boldsymbol{\epsilon}$ derived in class, i.e., given by Eq. (10). Also, **discuss** the linear stability of our system about the particular solution Eq. (8) for different possible combinations of K_1, K_2, and K_3.

$$\lambda^2 - 3K_3\Omega^2 = 0, \quad \lambda^4 + (1 - K_1K_2 + 3K_2)\Omega^2\lambda^2 - 4K_1K_2\Omega^4 = 0. \tag{10}$$

(e): Let us then numerically investigate the nonlinear attitude motion and compare the results against the linear stability analysis. To do this, we numerically integrate the original, nonlinear system (i.e., Eqs. (4) and (7)) simultaneously with a variety of satellite inertia properties for an initial condition. Specifically, we consider a circular orbit around the Earth ($\mu = 3.9860 \times 10^5$ km^3/s^2) with its radius being $R = 6800$ [km] and the following initial condition:

$$\boldsymbol{\sigma}(t=0) = \frac{1}{\sqrt{3}} \tan\frac{\theta_0}{4}(\hat{\boldsymbol{b}}_1 + \hat{\boldsymbol{b}}_2 + \hat{\boldsymbol{b}}_3), \quad \boldsymbol{\omega}(t=0) = \Omega\hat{\boldsymbol{b}}_3, \quad \theta_0 = 3 \text{ [deg] } (= \pi/60 \text{ [rad]}) \tag{11}$$

As for the inertia property, let us first assume the principal inertia moments to be $I_1 = 400$ [kg $\cdot$ m^2], $I_2 = 600$ [kg $\cdot$ m^2], and $I_3 = 800$ [kg $\cdot$ m^2].

(e.1): **Discuss** the kind of perturbation assumed in our initial condition Eq. (11) (e.g., is the perturbation in both/either of the initial orientation and angular velocity? is the perturbation about which axis for what amount?).

(e.2): Numerically integrate Eqs. (4) and (7) *simultaneously* for a time span from $t = 0$ to $t = 8$ **[hours]** with integration tolerance 1.0×10^{-10}, and **show** the plots of $\sigma_i(t)$ and $\omega_i(t)$ over time.
Note: Be careful about the units of I_i and R; numerical simulations must use consistent units.

(e.3): **Compute** the principal rotation angle $\theta(t)$ from $\boldsymbol{\sigma}(t)$, and **show** $\theta(t)$ in degrees as a function of time.

(e.4): **Discuss** the obtained numerical results, addressing the following points:

- which region in Fig. 2 does the current simulation case correspond to?
- how do $\sigma_i(t)$, $\omega_i(t)$, and $\theta(t)$ behave over time? (e.g., are they stay close to the initial values? are they periodic?)
- the maximum value of θ over time; how does it compare against the initial perturbation θ_0?
- does the numerical result agree with the result of the linear stability analysis (to the extent we can tell from the conducted numerical simulations)?

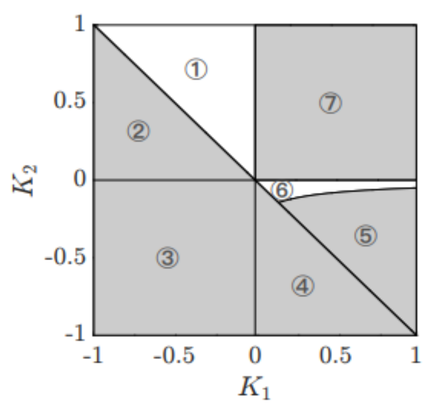


Figure 2: Stability chart

(f): **Repeat** the same investigation (e.2)-(e.4) for the following inertia moments:

(f.1): $I_1 = 600$ [kg $\cdot$ m^2], $I_2 = 400$ [kg $\cdot$ m^2], and $I_3 = 800$ [kg $\cdot$ m^2]

(f.2): $I_1 = 400$ [kg $\cdot$ m^2], $I_2 = 800$ [kg $\cdot$ m^2], and $I_3 = 600$ [kg $\cdot$ m^2]

(g): **Optional** (extra credit for *both* AAE 440 and 590)
Repeat the same investigation (e.2)-(e.4) with a longer time span, from $t = 0$ to $t = 24$ [hours], for the three inertia moments tested so far.

(g.1): $I_1 = 400$ [kg $\cdot$ m^2], $I_2 = 600$ [kg $\cdot$ m^2], and $I_3 = 800$ [kg $\cdot$ m^2]

(g.2): $I_1 = 600$ [kg $\cdot$ m^2], $I_2 = 400$ [kg $\cdot$ m^2], and $I_3 = 800$ [kg $\cdot$ m^2]

(g.3): $I_1 = 400$ [kg $\cdot$ m^2], $I_2 = 800$ [kg $\cdot$ m^2], and $I_3 = 600$ [kg $\cdot$ m^2]

Problem 02: Problem Solution

Part (a):

Starting with Eq. (5), we can expand the gravity gradient term using Eq. (6) then elementwise divide by the principal moments of inertia and simplify using the K_i values

$$\begin{bmatrix} I_1\dot{\omega}_1 \\ I_2\dot{\omega}_2 \\ I_3\dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} -(I_3 - I_2)\omega_2\omega_3 \\ -(I_3 - I_2)\omega_1\omega_3 \\ -(I_3 - I_2)\omega_1\omega_2 \end{bmatrix} + \begin{bmatrix} L_{G_1} \\ L_{G_2} \\ L_{G_3} \end{bmatrix}, \qquad \boldsymbol{L}_G = \frac{3\mu}{R_c^5}\begin{bmatrix} (I_3 - I_2)R_3R_2 \\ (I_1 - I_3)R_1R_3 \\ (I_2 - I_1)R_2R_1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} \frac{-(I_3 - I_2)}{I_1}\omega_2\omega_3 \\ \frac{-(I_1 - I_3)}{I_2}\omega_1\omega_3 \\ \frac{-(I_2 - I_1)}{I_3}\omega_1\omega_2 \end{bmatrix} + \frac{3\mu}{R_c^5}\begin{bmatrix} \frac{(I_3 - I_2)}{I_1}R_3R_2 \\ \frac{(I_1 - I_3)}{I_2}R_1R_3 \\ \frac{(I_2 - I_1)}{I_3}R_2R_1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} K_1\omega_2\omega_3 \\ K_2\omega_1\omega_3 \\ K_3\omega_1\omega_2 \end{bmatrix} - \frac{3\mu}{R_c^5}\begin{bmatrix} K_1R_3R_1 \\ K_2R_1R_3 \\ K_3R_2R_1 \end{bmatrix} = \begin{bmatrix} K_1\omega_2\omega_3 \\ K_2\omega_1\omega_3 \\ K_3\omega_1\omega_2 \end{bmatrix} - \frac{3\Omega^2}{\|\boldsymbol{R_c}\|_2^2}\begin{bmatrix} K_1R_3R_2 \\ K_2R_1R_3 \\ K_3R_2R_1 \end{bmatrix}$$

Part (b):

First, we can show that the solution given in MRP is equivalent to the one derived in class. Then we can plug the solution into Eq. (4) and Eq. (7) and demonstrate their validity.

$$\boldsymbol{\sigma} = \frac{\boldsymbol{\epsilon}_{1:3}}{1 + \epsilon_4} = \frac{1}{1+1}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{\omega}' = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} = \Omega\hat{\boldsymbol{b}}_3 - \Omega\hat{\boldsymbol{b}}_3 = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_2 \\ \dot{\sigma}_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 - \sigma^2 + 2\sigma_1^2 & 2(\sigma_1\sigma_2 - \sigma_3) & 2(\sigma_1\sigma_2 + \sigma_2) \\ 2(\sigma_2\sigma_1 + \sigma_3) & 1 - \sigma^2 + 2\sigma_2^2 & 2(\sigma_2\sigma_3 - \sigma_1) \\ 2(\sigma_3\sigma_1 - \sigma_2) & 2(\sigma_3\sigma_2 + \sigma_1) & 1 - \sigma^2 + 2\sigma_3^2 \end{bmatrix}\begin{bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Because $\boldsymbol{\sigma} = 0$, we know that $\boldsymbol{R} = -R_c\hat{\boldsymbol{o}}_{\boldsymbol{1}} = -R_c\hat{\boldsymbol{b}}_{\boldsymbol{1}},\ R_1 = R_c, R_2 = R_3 = 0$

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} K_1\omega_2\omega_3 \\ K_2\omega_1\omega_3 \\ K_3\omega_1\omega_2 \end{bmatrix} - \frac{3\Omega^2}{\|\boldsymbol{R_c}\|_2^2}\begin{bmatrix} K_1R_3R_2 \\ K_2R_1R_3 \\ K_3R_2R_1 \end{bmatrix} = \begin{bmatrix} K_1 0\bar{\omega}_3 \\ K_2 0\bar{\omega}_3 \\ K_3 0 \end{bmatrix} - \frac{3\Omega^2}{\|\boldsymbol{R_c}\|_2^2}\begin{bmatrix} K_1 0 \\ K_2 R_1 0 \\ K_3 0 R_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Part (c):

To confirm our linearized equations, we can write our $\boldsymbol{\sigma}$ and $\boldsymbol{\omega}$ in terms of the particular solution and perturbation, then substitute into Eq. (4) and Eq. (7), and neglect 2nd or higher order terms.

$$\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}} + \tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}, \quad \boldsymbol{\omega} = \bar{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}} = \begin{bmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \\ \Omega + \tilde{\omega}_3 \end{bmatrix}, \quad \boldsymbol{\omega}' = \boldsymbol{\omega} - \boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}}$$

$$\begin{bmatrix} \dot{\tilde{\sigma}}_1 \\ \dot{\tilde{\sigma}}_2 \\ \dot{\tilde{\sigma}}_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 - \tilde{\sigma}^2 + 2\tilde{\sigma}_1^2 & 2(\tilde{\sigma}_1\tilde{\sigma}_2 - \tilde{\sigma}_3) & 2(\tilde{\sigma}_1\tilde{\sigma}_2 + \tilde{\sigma}_2) \\ 2(\tilde{\sigma}_2\tilde{\sigma}_1 + \tilde{\sigma}_3) & 1 - \tilde{\sigma}^2 + 2\tilde{\sigma}_2^2 & 2(\tilde{\sigma}_2\tilde{\sigma}_3 - \tilde{\sigma}_1) \\ 2(\tilde{\sigma}_3\tilde{\sigma}_1 - \tilde{\sigma}_2) & 2(\tilde{\sigma}_3\tilde{\sigma}_2 + \tilde{\sigma}_1) & 1 - \tilde{\sigma}^2 + 2\tilde{\sigma}_3^2 \end{bmatrix}\begin{bmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \\ \tilde{\omega}_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & -\tilde{\sigma}_3 & \tilde{\sigma}_2 \\ \tilde{\sigma}_3 & 1 & -\tilde{\sigma}_1 \\ -\tilde{\sigma}_2 & \tilde{\sigma}_1 & 1 \end{bmatrix}\begin{bmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \\ \Omega + \tilde{\omega}_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \tilde{\omega}_1' \\ \tilde{\omega}_2' \\ \tilde{\omega}_3' \end{bmatrix} + \Omega\begin{bmatrix} \tilde{\sigma}_2 \\ -\tilde{\sigma}_1 \\ 0 \end{bmatrix}$$

Next we determine our ${}^{\mathcal{B}}\boldsymbol{R}$ after the perturbation by calculating the DCM using our perturbed state:

$$C = \frac{1}{(1+\sigma^2)^2}\begin{bmatrix} 4(\sigma_1^2-\sigma_2^2-\sigma_3^2)+(1-\sigma^2)^2 & 8\sigma_1\sigma_2+4\sigma_3(1-\sigma^2) & 8\sigma_1\sigma_3-4\sigma_2(1-\sigma^2) \\ 8\sigma_2\sigma_1-4\sigma_3(1-\sigma^2) & 4(-\sigma_1^2+\sigma_2^2-\sigma_3^2)+(1-\sigma^2)^2 & 8\sigma_2\sigma_3+4\sigma_1(1-\sigma^2) \\ 8\sigma_3\sigma_1+4\sigma_2(1-\sigma^2) & 8\sigma_3\sigma_2-4\sigma_1(1-\sigma^2) & 4(-\sigma_1^2-\sigma_2^2+\sigma_3^2)+(1-\sigma^2)^2 \end{bmatrix}$$

$$\sigma^2 = \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}$$

$${}^{\mathcal{B}}\boldsymbol{R} = [\mathcal{BN}]^{\mathcal{N}}\boldsymbol{R} = \begin{bmatrix} 1 & 4\tilde{\sigma}_3 & -4\tilde{\sigma}_2 \\ -4\tilde{\sigma}_3 & 1 & 4\tilde{\sigma}_1 \\ 4\tilde{\sigma}_2 & -4\tilde{\sigma}_1 & 1 \end{bmatrix}\begin{bmatrix} -R_c \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -R_c \\ 4\tilde{\sigma}_3 R_c \\ -4\tilde{\sigma}_2 R_c \end{bmatrix}$$

$$\begin{bmatrix} \dot{\tilde{\omega}}_1 \\ \dot{\tilde{\omega}}_2 \\ \dot{\tilde{\omega}}_3 \end{bmatrix} = \begin{bmatrix} K_1\tilde{\omega}_2(\Omega+\tilde{\omega}_3) \\ K_2\tilde{\omega}_1(\Omega+\tilde{\omega}_3) \\ K_3\tilde{\omega}_1\tilde{\omega}_2 \end{bmatrix} - \frac{3\Omega^2}{\|\boldsymbol{R}_c\|_2^2}\begin{bmatrix} K_1R_3R_2 \\ K_2R_1R_3 \\ K_3R_2R_1 \end{bmatrix} = \begin{bmatrix} K_1\tilde{\omega}_2\Omega \\ K_2\tilde{\omega}_1\Omega \\ 0 \end{bmatrix} - \frac{3\Omega^2}{\|\boldsymbol{R}_c\|_2^2}\begin{bmatrix} K_1\cdot 0 \\ K_2\cdot 4R_c^2\cdot\tilde{\sigma}_2 \\ K_3\cdot -4R_c^2\cdot\tilde{\sigma}_3 \end{bmatrix} = \begin{bmatrix} K_1\Omega\tilde{\omega}_2 \\ K_2\Omega\tilde{\omega}_1 - 12K_2\Omega^2\tilde{\sigma}_2 \\ 12K_3\Omega^2\tilde{\sigma}_3 \end{bmatrix}$$

Part (d):

To find the characteristic equations for this system we can first identify that $\tilde{\sigma}_3$ and $\tilde{\omega}_3$ are decoupled from the rest of the system. We can separate and combine those. For the other 4 state variables we can write the system of ODEs in the form $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ and solve for its eigenvalues.

$$\begin{bmatrix} \dot{\tilde{\sigma}}_1 \\ \dot{\tilde{\sigma}}_2 \\ \dot{\tilde{\sigma}}_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \\ \tilde{\omega}_3 \end{bmatrix} + \Omega\begin{bmatrix} \tilde{\sigma}_2 \\ -\tilde{\sigma}_1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \dot{\tilde{\omega}}_1 \\ \dot{\tilde{\omega}}_2 \\ \dot{\tilde{\omega}}_3 \end{bmatrix} = \begin{bmatrix} K_1\Omega\tilde{\omega}_2 \\ K_2\Omega\tilde{\omega}_1 - 12K_2\Omega^2\tilde{\sigma}_2 \\ 12K_3\Omega^2\tilde{\sigma}_3 \end{bmatrix}$$

$$\dot{\tilde{\omega}}_3 = 12K_3\Omega^2\tilde{\sigma}_3, \qquad \dot{\tilde{\sigma}}_3 = \frac{1}{4}\tilde{\omega}_3, \qquad \ddot{\tilde{\sigma}}_3 = 3K_3\Omega^2\tilde{\sigma}_3, \qquad \lambda^2 - 3K_3\Omega^2 = 0$$

$$\begin{bmatrix} \dot{\tilde{\sigma}}_1 \\ \dot{\tilde{\sigma}}_2 \\ \dot{\tilde{\omega}}_1 \\ \dot{\tilde{\omega}}_2 \end{bmatrix} = \begin{bmatrix} 0 & \Omega & 1/4 & 0 \\ -\Omega & 0 & 0 & 1/4 \\ 0 & 0 & 0 & K_1\Omega \\ 0 & -12K_2\Omega^2 & K_2\Omega & 0 \end{bmatrix}\begin{bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tilde{\omega}_1 \\ \tilde{\omega}_2 \end{bmatrix}$$

$$\det(A - \lambda I_{4\times4}) = \lambda^4 + (1 - K_1K_2 + 3K_2)\Omega^2\lambda^2 - 4K_1K_2\Omega^4 = 0$$

For clarity's sake, let us define:

$$b = (1 - K_1K_2 + 3K_2)\Omega^2, \qquad c = -4K_1K_2\Omega^4$$

To find where the linearized system is unstable, we must determine the K values such that the real component of λ_i is > 0

$$\lambda^2 - 3K_3\Omega^2 = 0, \qquad \lambda^4 + b\lambda^2 + c = 0$$

We find that the system is unstable if any of the following holds:

$$\begin{cases} K_3 > 0 \\ b < 0 \\ c < 0 \\ b^2 - 4c < 0 \end{cases} \leftrightarrow \begin{cases} K_1 + K_2 < 0 \\ 1 - K_1K_2 + 3K_2 < 0 \\ K_1K_2 > 0 \\ (1 - K_1K_2 + 3K_2)^2 + 16K_1K_2 < 0 \end{cases}$$

Part (e):

Part (e.1):

For this part, the initial conditions given represent a perturbation in initial orientation of the spacecraft. Specifically, by a 3-degree rotation about the axis $\hat{\boldsymbol{\lambda}} = \frac{1}{\sqrt{3}}(\hat{\boldsymbol{b}}_1 + \hat{\boldsymbol{b}}_2 + \hat{\boldsymbol{b}}_3)$

Part (e.2):

Several steps must be taken to numerically integrate the system. First, I calculate the radius vector in the B frame ${}^{\mathcal{B}}\boldsymbol{R}$. Next, I calculate $\boldsymbol{\omega}'$. Finally, we can construct a state variable and integrate the orientation and angular velocity using Eq. (4) and Eq. (7).

$$ {}^{\mathcal{B}}\boldsymbol{R} = [\mathcal{BO}(\boldsymbol{\sigma})]{}^{\mathcal{O}}\boldsymbol{R} = [\mathcal{BO}(\boldsymbol{\sigma})][-R \quad 0 \quad 0]^T $$

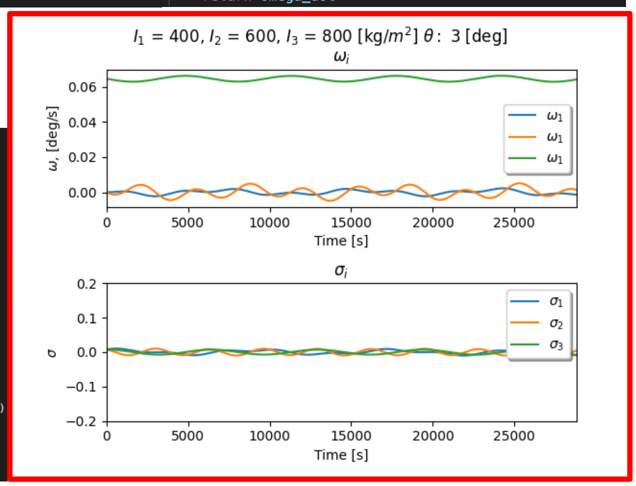
$$ {}^{\mathcal{B}}\boldsymbol{\omega}' = {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} - [\mathcal{BO}(\boldsymbol{\sigma})]{}^{\mathcal{O}}\boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} = {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} - [\mathcal{BO}(\boldsymbol{\sigma})][0 \quad 0 \quad \Omega]^T $$

$$ \begin{bmatrix} {}^{\mathcal{B}}\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{O}} \\ \dot{\boldsymbol{\sigma}} \end{bmatrix} = \begin{bmatrix} f(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}}, \bar{\bar{I}}, {}^{\mathcal{B}}\boldsymbol{R}) \\ g(\boldsymbol{\sigma}, \boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}}) \end{bmatrix} = \begin{bmatrix} \text{dwdt\_gravTorque()} \\ \text{KDE\_MRP()} \end{bmatrix} $$

```python
def dwdt_torqueFree(omega, I):
    """
    Compute the derivative of the angular velocity
    given the angular velocity and the inertia matrix
    """
    omega_dot = np.zeros(3)
    omega_dot[0] = -(I[2]-I[1])/I[0]*omega[1]*omega[2]
    omega_dot[1] = -(I[0]-I[2])/I[1]*omega[2]*omega[0]
    omega_dot[2] = -(I[1]-I[0])/I[2]*omega[0]*omega[1]
    return omega_dot
```

```python
def dwdt_gravTorque(omega, I, Omega, R):
    """
    Compute the derivative of the angular velocity
    given the angular velocity and the inertia matrix
    Include the torque from gravity gradient
    """
    K1 = (I[1]-I[2])/I[0]
    K2 = (I[2]-I[0])/I[1]
    K3 = (I[0]-I[1])/I[2]

    B = 3*Omega**2/LA.norm(R)**2
    omega_dot = np.zeros(3)
    omega_dot[0] = K1*omega[1]*omega[2]-B*K1*R[2]*R[1]
    omega_dot[1] = K2*omega[2]*omega[0]-B*K2*R[0]*R[2]
    omega_dot[2] = K3*omega[0]*omega[1]-B*K3*R[1]*R[0]
    return omega_dot
```

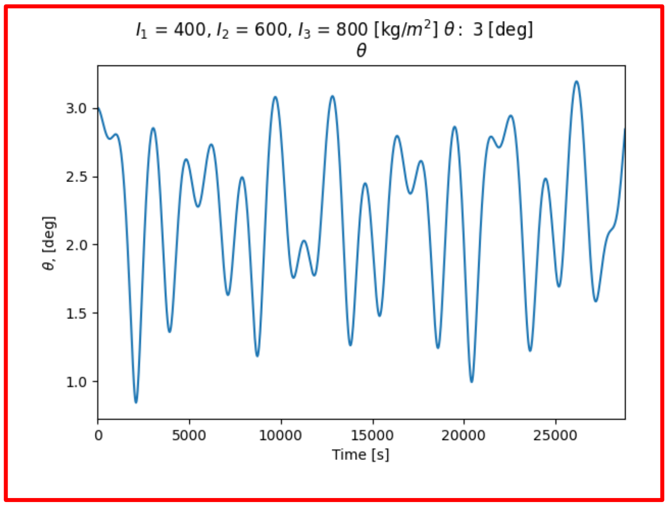
```python
def sattelite_orientation_MRP(t, state, I1, I2, I3, Omega, R_O):
    """
    Return the derivative of omega and
    MRP for sattelite orientation
    Account for gravity gradient torque
    **note state is an array of [omega, MRP]
    R_O is the orbit radius in Orbit frame
    """
    state_dot   = np.zeros_like(state)
    omega       = state[0:3]
    MRP         = state[3:6]
    omega_ON    = np.array([0, 0, Omega])

    BO          = MRPtoDCM(MRP)
    R_B         = np.dot(BO, R_O)
    omega_BO    = omega - np.dot(BO, omega_ON)

    omega_dot   = dwdt_gravTorque(omega, [I1, I2, I3], Omega, R_B)
    MRP_dot          = KDE_MRP(MRP, omega_BO)
    state_dot[0:3]   = omega_dot
    state_dot[3:6]   = MRP_dot
    return state_dot
```



Part (e.3):

Here in order to compute the angle associated with the principle axis of rotation I calculate the DCM, then the PRP from the DCM.

$$ [\hat{\boldsymbol{\lambda}}, \theta] = \text{DCMtoPRP}(\text{MRPtoDCM}(\boldsymbol{\sigma})) $$



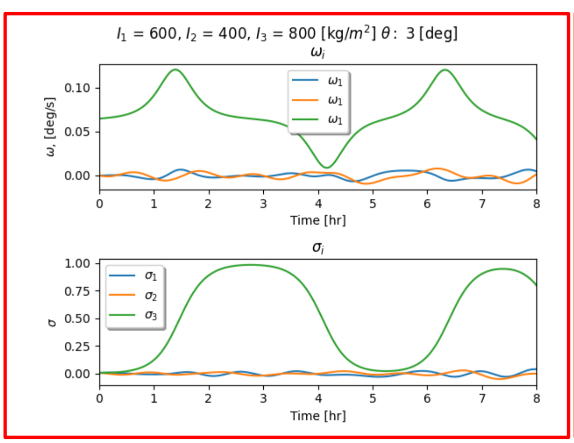
Part (e.4):

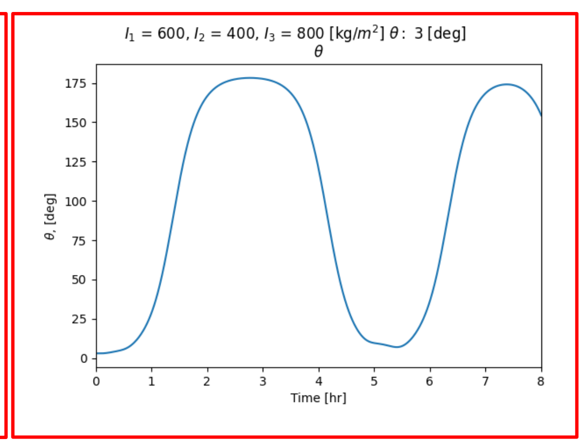
- This simulation corresponds to the stable Region 1, $I_1 < I_2 < I_3$
- $\boldsymbol{\omega}, \boldsymbol{\sigma}, \theta$ all remain very close to their initial values. Additionally both $\boldsymbol{\omega}, \boldsymbol{\sigma}$ appear periodic however do not follow a simple sinusoid.
- The maximum value of θ is 6% higher than the initial perturbation.
- From the numerical simulation, we can say that this system agrees with our linear stability analysis which predicted that the system would be stable.

Part (f):

Part (f.1):

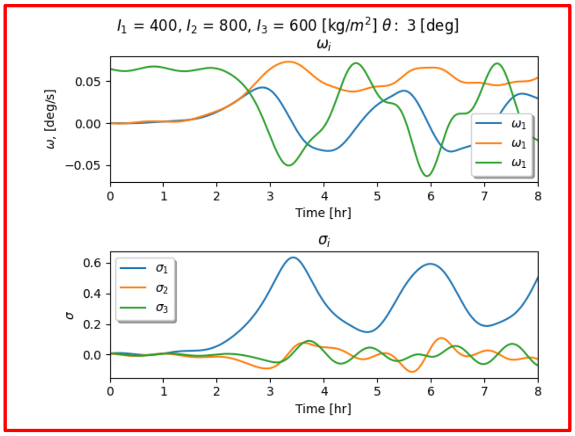
- This simulation corresponds to the unstable Region 2
- ω_3, σ_3, and θ all vary significantly from their initial values and generally appear periodic. $\boldsymbol{\omega}_{1:2}, \boldsymbol{\sigma}_{1:2}$ remain very close to their initial values but do not show clear periodicity
- The maximum value of θ is 5900% higher than the initial perturbation.
- From the numerical simulation, we can say that this system agrees with our linear stability analysis which predicted that the system would be unstable.

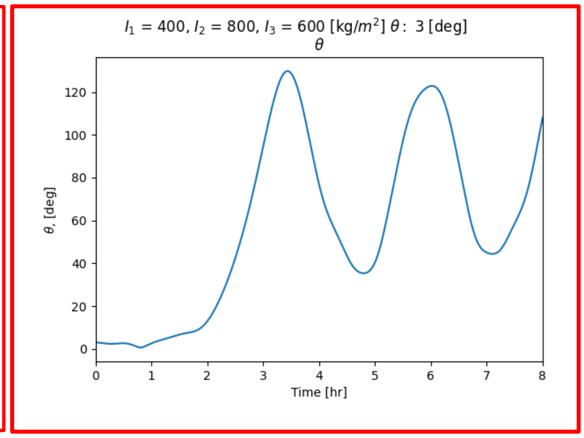




Part (f.2):

- This simulation corresponds to the unstable Region 3
- $\boldsymbol{\omega}, \boldsymbol{\sigma}, \theta$ all vary significantly from their initial perturbation, and none demonstrate periodicity.
- The maximum value of θ is 4300% higher than the initial perturbation.
- From the numerical simulation, we can say that this system agrees with our linear stability analysis which predicted that the system would be unstable and chaotic.

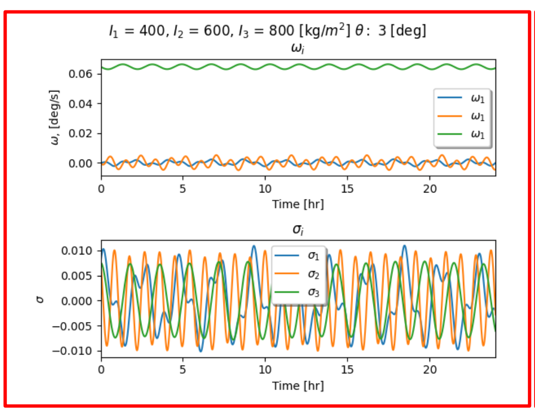


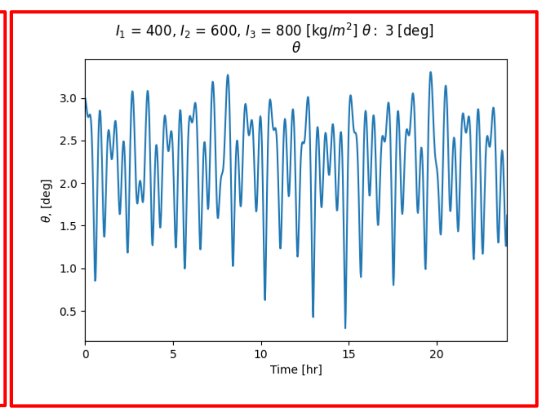


Part (g):

Part (g.1):

- This simulation corresponds to the stable Region 1, $I_1 < I_2 < I_3$
- $\boldsymbol{\omega}, \boldsymbol{\sigma}, \theta$ all remain very close to their initial values. Additionally both $\boldsymbol{\omega}, \boldsymbol{\sigma}$ appear oscillatory however do not follow a simple sinusoid.
- The maximum value of θ is 10% higher than the initial perturbation.
- From the numerical simulation, we can say that this system agrees with our linear stability analysis which predicted that the system would be stable.

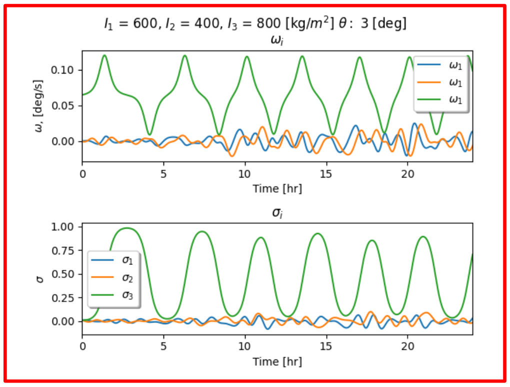


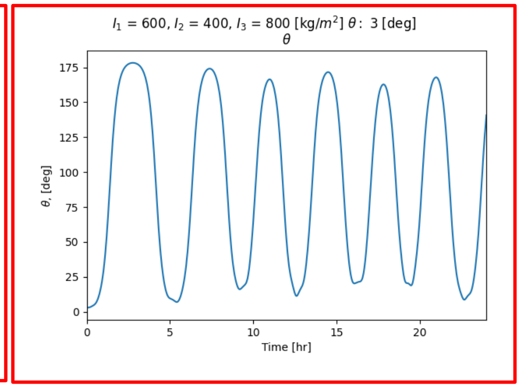


Part (g.2):

- This simulation corresponds to the unstable Region 2

- $\boldsymbol{\omega_3}$, σ_3, and θ all vary significantly from their initial values and generally appear oscillatory. $\boldsymbol{\omega_{1:2}}$, $\boldsymbol{\sigma_{1:2}}$ remain very close to their initial values but do not show clear periodicity
- The maximum value of θ is 5940% higher than the initial perturbation.
- From the numerical simulation, we can say that this system agrees with our linear stability analysis which predicted that the system would be unstable.





Part (g.3):

- This simulation corresponds to the unstable Region 3
- $\boldsymbol{\omega}, \boldsymbol{\sigma}, \theta$ all vary significantly from their initial perturbation, and none demonstrate periodicity.
- The maximum value of θ is 4742% higher than the initial perturbation.
- From the numerical simulation, we can say that this system agrees with our linear stability analysis which predicted that the system would be unstable and chaotic.