AAE 440/590: Spacecraft Attitude Dynamics
Computational Problem Set 4

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Due: April 13, 2022 at **7:00** PM ET
(submission at Brightspace>Content>Gradescope)

Your write-up should include snippets of your code that are relevant to solving the problems. Note that, as stated in the Problem Set Format Instruction, your write-up must still **include a full description** of the methods and equations used to solve the problem.

Problem 1

Consider a satellite moving in a force-free, torque-free environment. The inertial frame and satellite body-fixed frame are represented by $\mathcal{N}$-frame and $\mathcal{B}$-frame, where $\{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$ and $\{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$ are right-handed vector bases fixed in $\mathcal{N}$-frame and $\mathcal{B}$-frame, respectively. The inertia tensor of the body is given by:

$$\bar{\bar{\mathbf{I}}}_c = \frac{3}{4}\hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_1 + \hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_2 + \frac{3}{2}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_3 \quad \text{(dimensionless)} \tag{1}$$

We consider the same initial satellite angular velocity as used in the Problem 2 of CPS3, i.e.,:

$$\boldsymbol{\omega}(t = 0) = -0.1\ \hat{\boldsymbol{b}}_1 + 0.05\ \hat{\boldsymbol{b}}_2 + 0.1\ \hat{\boldsymbol{b}}_3 \quad \text{(dimensionless)} \tag{2}$$

(a): Using the function `dwdt_torqueFree` implemented in CPS3, **compute** the time history of $\boldsymbol{\omega}$ over a time span from $t = 0$ to $t = 200$ with integration tolerance 1×10^{-10}. **Show** the plots of $\omega_i(t)$ over time.

(b): To compare the obtained numerical result against an axisymmetric case, numerically integrate the same system as in the Problem 2 of CPS3 from $t = 0$ to $t = 200$ with integration tolerance 1×10^{-10}. Denote by $\bar{\bar{\mathbf{I}}}_{c,\text{axisym}}$ the inertia tensor assumed in the Problem 2 of CPS3, that is,

$$\bar{\bar{\mathbf{I}}}_{c,\text{axisym}} = \hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_1 + \hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_2 + \frac{3}{2}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_3 \quad \text{(dimensionless)} \tag{3}$$

Show the plots of $\omega_i(t)$ for the system with $\bar{\bar{\mathbf{I}}}_{c,\text{axisym}}$ over time.

(c): **Discuss** the qualitative differences and similarities of $\omega_i(t)$ obtained in (a) and (b) above, addressing the following points: overall behaviors (periodic? constant?), maximum/minimum values, and periods (if periodic).

(d): The two constant integrals of motion, namely the angular momentum magnitude, $\|\boldsymbol{H}\|_2$, and the rotational Kinetic energy, T_{rot}, are still useful tools to check the validity of numerical simulations for general inertia bodies. Regarding the result of (a), **compute** $\|\boldsymbol{H}(t)\|_2$ and $T_{\text{rot}}(t)$. **Show** the plots of $\|\boldsymbol{H}(t)\|_2$ and $T_{\text{rot}}(t)$, and **confirm** that they are indeed constant over time.

(e): We learned in class that there are three additional constant integrals of motion that can be found from the analogy to undamped Duffing equations. Repeating the equations in the lecture slides, the constants are given by:

$$C_i = \dot{\omega}_i^2 + A_i\omega_i^2 + \frac{B_i}{2}\omega_i^4 \quad \text{for } i = 1, 2, 3, \tag{4}$$

where A_i and B_i are constants that depend on the initial condition and $\bar{\bar{\mathbf{I}}}_c$, given by:

$$A_1 = \frac{(I_1 - I_2)(2I_3 T_{\mathrm{rot}} - \|\boldsymbol{H}\|_2^2) + (I_3 - I_1)(\|\boldsymbol{H}\|_2^2 - 2I_2 T_{\mathrm{rot}})}{I_1 I_2 I_3} \quad B_1 = \frac{2(I_1 - I_2)(I_1 - I_3)}{I_2 I_3}$$
$$A_2 = \frac{(I_2 - I_3)(2I_1 T_{\mathrm{rot}} - \|\boldsymbol{H}\|_2^2) + (I_1 - I_2)(\|\boldsymbol{H}\|_2^2 - 2I_3 T_{\mathrm{rot}})}{I_1 I_2 I_3} \quad B_2 = -\frac{2(I_1 - I_2)(I_2 - I_3)}{I_1 I_3}$$
$$A_3 = \frac{(I_3 - I_1)(2I_2 T_{\mathrm{rot}} - \|\boldsymbol{H}\|_2^2) + (I_2 - I_3)(\|\boldsymbol{H}\|_2^2 - 2I_1 T_{\mathrm{rot}})}{I_1 I_2 I_3} \quad B_3 = \frac{2(I_1 - I_3)(I_2 - I_3)}{I_1 I_2}$$

Compute the constants C_i in Eq. (4) over time for the result of (a), **show** the plots of C_i as functions of t, and **confirm** that they are indeed constant over time.
(Hint: $\dot{\omega}_i$ in Eq. (4) can be computed by using the Euler's equations of motion given $\omega_i(t)$)

(f): Another way to look at the torque-free attitude motion of general inertia bodies is the Poinsot construction. According to the Poinsot construction, the angular velocity vector $\boldsymbol{\omega}(t)$ moves in a plane perpendicular to the angular momentum vector $\boldsymbol{H}$ in the inertial frame (called *invariable plane*).
Focusing on the case with the unsymmetric inertia tensor given in Eq. (1), let us numerically verify this argument by visualizing $\boldsymbol{\omega}(t)$ and $\boldsymbol{H}$ in the inertial frame, as follows:

(f.1): In order to visualize these quantities in the inertial frame, we also need information of the satellite orientation over time to map $\boldsymbol{\omega}(t)$ and $\boldsymbol{H}$ from $\mathcal{B}$-frame to $\mathcal{N}$-frame. Here we use the modified Rodrigues parameter (MRP) to represent the orientation of $\mathcal{B}$-frame relative to $\mathcal{N}$-frame, and suppose that the MRP vector at $t = 0$ is evaluated as:

$$\boldsymbol{\sigma}(t = 0) = \frac{1}{3}\hat{\boldsymbol{b}}_1 + \frac{1}{3}\hat{\boldsymbol{b}}_2 + \frac{1}{3}\hat{\boldsymbol{b}}_3 \tag{5}$$

Compute $\boldsymbol{\sigma}(t)$ over a time span from $t = 0$ to $t = 200$ by numerically integrating `KDE_MRP` and `dwdt_torqueFree` simultaneously, with integration tolerance 1×10^{-10}, where the shadow-set switching is not necessary (you may perform the switching at $\|\boldsymbol{\sigma}\|_2 = 1$ if you want too). **Show** the plots of $\sigma_i(t)$ over time.

(f.2): **Compute** ${}^{\mathcal{N}}\boldsymbol{H}(t)$, i.e., time history of $\boldsymbol{H}$ in $\mathcal{N}$-frame, and **show** the plots of ${}^{\mathcal{N}}H_i(t)$ to confirm that ${}^{\mathcal{N}}H_i(t)$ are indeed constant over time.

(f.3): **Compute** ${}^{\mathcal{N}}\boldsymbol{\omega}(t)$, and **show** the three-dimensional plot of $\boldsymbol{\omega}(t)$ and $\boldsymbol{H}$ in $\mathcal{N}$-frame, where represent $\boldsymbol{H}$ by a line that connects the origin and the point given by ${}^{\mathcal{N}}\boldsymbol{H}(t = 0)$ (showing $\boldsymbol{H}$ only at $t = 0$ is sufficient since ${}^{\mathcal{N}}\boldsymbol{H}(t)$ is time-invariant). Visually **confirm** that $\boldsymbol{\omega}(t)$ is indeed confined in a plane perpendicular to $\boldsymbol{H}$ in $\mathcal{N}$-frame, where include at least two different views of the three-dimensional plot: a view that illustrates the perpendicularity of the invariable plane to $\boldsymbol{H}$ and a view that projects the locus of $\boldsymbol{\omega}(t)$ on the invariable plane (called *herpolhode*).

(g): Let us compare the herpolhode curve obtained in (f) against the one for an axisymmetric inertia case. To do this, **repeat** the procedure in (f.1)-(f.3) using $\bar{\bar{\mathbf{I}}}_{c,\mathrm{axisym}}$ given in Eq. (3) with the same initial conditions Eqs. (2) and (5). **Discuss** the comparison of the two herpolhode curves obtained in (f) and in this question (e.g., are the curves circular? closed?).

Problem 2

This problem is concerned with numerically verifying the instability of the rotation about the intermediate moment axis. In class, we learned that the Euler's rotational equations of motion without torque (Eq. (7)) have the following three particular solutions:

$$\begin{cases} \omega_2(t) = \omega_3(t) = 0, \quad \omega_1(t) = \omega_{1_0}(\neq 0) & \cdots \text{Solution 1} \\ \omega_3(t) = \omega_1(t) = 0, \quad \omega_2(t) = \omega_{2_0}(\neq 0) & \cdots \text{Solution 2} \\ \omega_1(t) = \omega_2(t) = 0, \quad \omega_3(t) = \omega_{3_0}(\neq 0) & \cdots \text{Solution 3} \end{cases} \tag{6}$$

Recall that the Euler's rotational equations of motion in a torque-free environment is given as follows:

$$\begin{cases} \dot{\omega}_1 = -\frac{I_3-I_2}{I_1}\omega_2\omega_3 \\ \dot{\omega}_2 = -\frac{I_1-I_3}{I_2}\omega_3\omega_1 \\ \dot{\omega}_3 = -\frac{I_2-I_1}{I_3}\omega_1\omega_2 \end{cases} \tag{7}$$

Answer the following questions assuming the inertia tensor given by Eq. (1).

(a): **Show** that each of the particular solutions in Eq. (6) indeed satisfies the differential equations Eq. (7).

(b): **Discuss** which of the particular solutions in Eq. (6) corresponds to the maximum-energy, minimum-energy, and intermediate-energy rotational states, respectively.

(c): Let us also numerically confirm that Eq. (6) are the particular solutions of Eq. (7). Consider the following three cases for the initial angular velocity vectors:

$$\begin{cases} \boldsymbol{\omega}(t=0) = 0.1\ \hat{\boldsymbol{b}}_1 & \cdots \text{Case 1} \\ \boldsymbol{\omega}(t=0) = 0.1\ \hat{\boldsymbol{b}}_2 & \cdots \text{Case 2} \\ \boldsymbol{\omega}(t=0) = 0.1\ \hat{\boldsymbol{b}}_3 & \cdots \text{Case 3} \end{cases} \quad \text{(dimensionless)} \tag{8}$$

For each case in Eq. (8), **compute** $\boldsymbol{\omega}(t)$ by numerically integrating Eq. (7) from $t = 0$ to $t = 200$ with integration tolerance 1×10^{-10}. **Show** the plots of $\omega_i(t)$ for each of the three cases, and **confirm** that the results are consistent with Eq. (6).

(d): Now, let us consider a small perturbation in $\boldsymbol{\omega}(t = 0)$ for each of the three cases in Eq. (8). Specifically, we consider the following perturbed angular velocity at $t = 0$:

$$\begin{cases} \boldsymbol{\omega}(t=0) = 0.1\ \hat{\boldsymbol{b}}_1 + 0.001\ \hat{\boldsymbol{b}}_2 + 0.001\ \hat{\boldsymbol{b}}_3 & \cdots \text{Case 1}' \\ \boldsymbol{\omega}(t=0) = 0.001\ \hat{\boldsymbol{b}}_1 + 0.1\ \hat{\boldsymbol{b}}_2 + 0.001\ \hat{\boldsymbol{b}}_3 & \cdots \text{Case 2}' \\ \boldsymbol{\omega}(t=0) = 0.001\ \hat{\boldsymbol{b}}_1 + 0.001\ \hat{\boldsymbol{b}}_2 + 0.1\ \hat{\boldsymbol{b}}_3 & \cdots \text{Case 3}' \end{cases} \quad \text{(dimensionless)} \tag{9}$$

(d.1): For each case in Eq. (9), **compute** $\boldsymbol{\omega}(t)$ over a time span from $t = 0$ to $t = 200$ with integration tolerance 1×10^{-10}. **Show** the plots of $\omega_i(t)$ over time.

(d.2): **Discuss** the numerical results obtained in the previous question, addressing the following points:

- how $\omega_i(t)$ behaves over time for each case
- how $\omega_i(t)$ compares against the unperturbed results (i.e., results obtained in (b)) for each case; does the perturbation grow over time or stay around the same level? why?
- which of the maximum-energy, minimum-energy, and intermediate-energy rotational states corresponds to the unstable case

(d.3): **Optional** (extra credit for *both* AAE 440 and 590 sections):
Show the three-dimensional herpolhode plots for each of the perturbed cases in Eq. (9), and **discuss** the results (assume the same initial condition as Eq. (5)).

Problem 3

Students in AAE 590 should solve this problem for full score*; students in AAE 440 who complete this problem with correct answers will receive extra credit.*

In class, we also learned that the angular momentum vector of the torque-free attitude motion of a general inertia body must always lie on an intersection of the angular momentum ellipsoid and energy ellipsoid.

(a): Answer the following questions to visually verify the argument, where consider a system with the inertia tensor and the initial angular velocity given by Eqs. (1) and (2), respectively.

(a.1): Recall from class that the analysis is simplified by using the angular momenta coordinates, ${}^{\mathcal{B}}H_i$, which makes the angular momentum ellipsoid a sphere. **Determine** the radius of the angular momentum sphere and the semi-axes of the energy ellipsoid in the angular momenta coordinates.

(a.2): **Show** the angular momentum sphere and the energy ellipsoid in a three-dimensional space with the angular momenta coordinates, overlaid with ${}^{\mathcal{B}}\boldsymbol{H}(t)$. Use different colors for the two surfaces for better illustration. Include at least three different views of the plot, each showing the projection of the plot on the ${}^{\mathcal{B}}H_1$-${}^{\mathcal{B}}H_2$ plane, ${}^{\mathcal{B}}H_1$-${}^{\mathcal{B}}H_3$ plane, and ${}^{\mathcal{B}}H_2$-${}^{\mathcal{B}}H_3$ plane. **Confirm** that ${}^{\mathcal{B}}\boldsymbol{H}(t)$ is indeed on a intersection of the two surfaces (called *polhode* curve).
Hint: if you are using Matlab, a surface can be visualized by using the `surf` function (for python users, `plot_surface` function in the `matplotlib` library). Data points for spheres and ellipsoids can be generated by using `sphere` and `ellipsoid` functions in Matlab.

(b): **Optional** (extra credit for *both* AAE 440 and 590 sections):

(b.1): **Repeat** the procedure in (a) for an axisymmetric inertia case, where use $\bar{\bar{\mathbf{I}}}_{c,\text{axisym}}$ given in Eq. (3) with the same initial condition Eq. (2). Again, include at least three different views of the plot, each showing the projection onto the three planes.

(b.2): **Discuss** the comparison of the two polhode curves obtained in (a) and in the previous question.