AAE 440/590: Spacecraft Attitude Dynamics
Computational Problem Set 5

Prof. Kenshiro Oguri

Due: April 20, 2022 at **9:00** PM ET
(submission at Brightspace>Content>Gradescope)

Your write-up should include snippets of your code that are relevant to solving the problems. Note that, as stated in the Problem Set Format Instruction, your write-up must still **include a full description** of the methods and equations used to solve the problem.

Problem 1

In class, we learned that the gravity force and torque acting on a rigid body satellite are expressed as follows:

$$\begin{aligned}
\boldsymbol{F}_G &= -\frac{\mu}{R_c^3}\left[m\boldsymbol{R}_c + \underbrace{\frac{3}{R_c^2}\bar{\bar{\mathbf{I}}}_c\cdot\boldsymbol{R}_c + \frac{3}{2R_c^2}\mathrm{Tr}[\bar{\bar{\mathbf{I}}}_c]\boldsymbol{R}_c - \frac{15}{2R_c^4}(\boldsymbol{R}_c\cdot\bar{\bar{\mathbf{I}}}_c\cdot\boldsymbol{R}_c)\boldsymbol{R}_c}_{\mathcal{O}(\varepsilon^2)} + \mathcal{O}\left(\varepsilon^3\right)\right] \\
\boldsymbol{L}_G &= \frac{3\mu}{R_c^5}\boldsymbol{R}_c\times\bar{\bar{\mathbf{I}}}_c\cdot\boldsymbol{R}_c + \mathcal{O}\left(\varepsilon^3\right)
\end{aligned} \tag{1}$$

where m and $\bar{\bar{\mathbf{I}}}_c$ are the total mass and the inertia tensor of the satellite; μ is the gravitational parameter of the point-mass gravitational body (usually Earth in this class); $\boldsymbol{R}_c$ is the position of the satellite center of mass (CoM) measured from the gravitational body; and R_c is its magnitude, i.e., $R_c = \|\boldsymbol{R}_c\|_2$. $\varepsilon = \frac{r}{R_c}$ is a small parameter, where r is the distance between the satellite CoM and a satellite differential mass $\mathrm{d}m$.

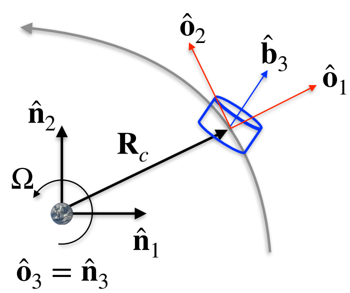

Figure 1: Inertial frame $\{\hat{\boldsymbol{n}}_i\}$, satellite body-fixed frame $\{\hat{\boldsymbol{b}}_i\}$, and orbit frame $\{\hat{\boldsymbol{o}}_i\}$

Consider a satellite in a circular low-Earth orbit (LEO), depicted in Fig. 1. The inertial frame, satellite body-fixed frame, and orbit frame are represented by $\mathcal{N}$-frame, $\mathcal{B}$-frame, and $\mathcal{O}$-frame, where $\{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$, $\{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$, and $\{\hat{\boldsymbol{o}}_1, \hat{\boldsymbol{o}}_2, \hat{\boldsymbol{o}}_3\}$ are right-handed vector bases fixed in $\mathcal{N}$-frame, $\mathcal{B}$-frame, and $\mathcal{O}$-frame, respectively. $\mathcal{N}$-frame and $\mathcal{O}$-frame are taken so that the orbit lies in the plane spanned by $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$, $\hat{\boldsymbol{n}}_3 = \hat{\boldsymbol{o}}_3$, and that the orbit radius vector $\boldsymbol{R}_c$ is aligned with $\hat{\boldsymbol{o}}_1$, i.e., $\boldsymbol{R}_c = R_c\hat{\boldsymbol{o}}_1$. Use $\mu = 3.9860 \times 10^5$ km^3/s^2 for the gravitational parameter of the Earth.

Suppose that the satellite inertia tensor is given by:

$$\bar{\bar{\mathbf{I}}}_c = \frac{5}{12} m l^2 \hat{\boldsymbol{b}}_1 \hat{\boldsymbol{b}}_1 + \frac{5}{6} m l^2 \hat{\boldsymbol{b}}_2 \hat{\boldsymbol{b}}_2 + \frac{13}{12} m l^2 \hat{\boldsymbol{b}}_3 \hat{\boldsymbol{b}}_3 \tag{2}$$

where l represents the characteristic length of the satellite. Also, the orientation of $\mathcal{B}$-frame relative to $\mathcal{O}$-frame in terms of a modified Rodrigues parameter (MRP) vector is measured at $t = 0$ as:

$$\boldsymbol{\sigma}_{B/O}(t = 0) = \frac{1}{3} \hat{\boldsymbol{b}}_1 + \frac{1}{4} \hat{\boldsymbol{b}}_2 + \frac{1}{5} \hat{\boldsymbol{b}}_3,$$

(a): **Discuss** whether the body with the inertia tensor given in Eq. (2) is centrobaric. Why or why not?

(b): When the body is not centrobaric, the center of mass (CoM) and the center of gravity (CoG) do not coincide. **Show** that the position of the CoG measured from the Earth CoM is expressed as in Eq. (3):

$$\boldsymbol{R}_{cg} = -R_{cg} \frac{\boldsymbol{F}_G}{\|\boldsymbol{F}_G\|_2}, \quad R_{cg} = \sqrt{\frac{\mu m}{\|\boldsymbol{F}_G\|_2}} \tag{3}$$

Hint: from the definition of CoG, the gravity force acting on a body can be expressed in terms of $\boldsymbol{R}_{cg}$:

$$\boldsymbol{F}_G = -\frac{\mu m}{R_{cg}^3} \boldsymbol{R}_{cg},$$

where R_{cg} is the magnitude of $\boldsymbol{R}_{cg}$, i.e., $R_{cg} = \|\boldsymbol{R}_{cg}\|_2$.

(c): Suppose that the properties of our satellite are $m = 100$ kg and $l = 70$ cm (i.e., $l = 7.0 \times 10^{-4}$ km) in an LEO of orbit radius $R_c = 6578$ km (~ 200 km altitude). Answer the following problems.

(c.1): Using Eq. (1) and ignoring the terms of the order ε^3 and higher, **determine** the values of $\boldsymbol{F}_G$ and $\boldsymbol{L}_G$ in $\mathcal{B}$-frame at $t = 0$.

(c.2): **Determine** the value of the first-order approximation of $\boldsymbol{F}_G$ (i.e., ignore ε^2 and higher order terms) in $\mathcal{B}$-frame at $t = 0$, and **report** the value of $\boldsymbol{F}_{G,\text{2nd}} - \boldsymbol{F}_{G,\text{1st}}$, where $\boldsymbol{F}_{G,\text{2nd}}$ denotes the second-order approximation of $\boldsymbol{F}_G$ while $\boldsymbol{F}_{G,\text{1st}}$ the first-order approximation.

(c.3): **Determine** the distance between the CoG and CoM of the satellite at $t = 0$, i.e., $\|\boldsymbol{R}_{cg} - \boldsymbol{R}_c\|_2$, where use the second-order approximation of $\boldsymbol{F}_G$ given in Eq. (1) to compute $\boldsymbol{R}_{cg}$.

(d): Next, consider a much larger body with $m = 4.5 \times 10^5$ kg and $l = 33$ m ($= 3.3 \times 10^{-2}$ km)—similar size as the international space station (ISS) but a different shape—in the same orbit as part (c). **Answer** the same questions as (c.1)-(c.3) for this system.

(e): Next, consider the large body assumed in part (d) with a different orbit radius $R_c = 3.58 \times 10^5$ km (close to the geosynchronous orbit). **Answer** the same questions as (c.1)-(c.3) for this system.

(f): **Discuss** the results obtained so far and their implications, addressing the following points:

- comparison of the two systems assumed in (c) and (d), in terms of the (i) gravitational force, (ii), torque, and (iii) distance between the CoM and CoG
- comparison of the two systems assumed in (d) and (e), in terms of the same quantities (i)-(iii)
- based on the numerical results of (c)-(e) as well as the analytical expression given in Eq. (1), what we can infer about the general trend of the three quantities (i)-(iii) for different systems
- whether the second-order term of the total gravity is practically negligible for typical Earth orbiters
- in what kind of systems the second-order term of the total gravity may not be negligible (Hint: what if we consider an orbiter about a small asteroid?)

Problem 2

Let us now investigate the spacecraft attitude motion under the action of the gravity gradient torque.

In class, we used the Euler parameter (EP) as the attitude representation to analyze the stability of the attitude motion in a circular orbit. In this problem, we use the MRP representation.

Thus, our attitude variables consist of the MRP vector that represents the orientation of $\mathcal{B}$-frame relative to $\mathcal{O}$-frame, denoted by $\boldsymbol{\sigma}(=\boldsymbol{\sigma}_{B/O}) = \sigma_1\hat{\boldsymbol{b}}_1 + \sigma_2\hat{\boldsymbol{b}}_2 + \sigma_3\hat{\boldsymbol{b}}_3$, and the angular velocity of $\mathcal{B}$-frame relative to $\mathcal{N}$-frame, $\boldsymbol{\omega}(=\boldsymbol{\omega}_{B/N}) = \omega_1\hat{\boldsymbol{b}}_1 + \omega_2\hat{\boldsymbol{b}}_2 + \omega_3\hat{\boldsymbol{b}}_3$. We use the same $\mathcal{N}$-frame, $\mathcal{B}$-frame, and $\mathcal{O}$-frame as defined in Problem 1. The equations of motion that describe our system are given by:

$$\begin{bmatrix}\dot{\sigma_1}\\ \dot{\sigma_2}\\ \dot{\sigma_3}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1-\sigma^2+2\sigma_1^2 & 2(\sigma_1\sigma_2-\sigma_3) & 2(\sigma_1\sigma_3+\sigma_2)\\ 2(\sigma_2\sigma_1+\sigma_3) & 1-\sigma^2+2\sigma_2^2 & 2(\sigma_2\sigma_3-\sigma_1)\\ 2(\sigma_3\sigma_1-\sigma_2) & 2(\sigma_3\sigma_2+\sigma_1) & 1-\sigma^2+2\sigma_3^2\end{bmatrix}\begin{bmatrix}\omega_1'\\ \omega_2'\\ \omega_3'\end{bmatrix} \quad \text{(Kinematic Diff. Equation)} \tag{4}$$

$$\begin{bmatrix}I_1\dot{\omega}_1\\ I_2\dot{\omega}_2\\ I_3\dot{\omega}_3\end{bmatrix} = \begin{bmatrix}-(I_3-I_2)\omega_2\omega_3\\ -(I_1-I_3)\omega_3\omega_1\\ -(I_2-I_1)\omega_1\omega_2\end{bmatrix} + \begin{bmatrix}L_{G_1}\\ L_{G_2}\\ L_{G_3}\end{bmatrix} \quad \text{(Euler's Rotational Equation)} \tag{5}$$

$$\begin{bmatrix}L_{G_1}\\ L_{G_2}\\ L_{G_3}\end{bmatrix} = \frac{3\mu}{R^5}\begin{bmatrix}(I_3-I_2)R_3R_2\\ (I_1-I_3)R_1R_3\\ (I_2-I_1)R_2R_1\end{bmatrix} \quad \text{(Gravity Gradient)} \tag{6}$$

where $\boldsymbol{\omega}'(=\boldsymbol{\omega}_{B/O}) = \omega_1'\hat{\boldsymbol{b}}_1 + \omega_2'\hat{\boldsymbol{b}}_2 + \omega_3'\hat{\boldsymbol{b}}_3$ is the angular velocity of $\mathcal{B}$-frame relative to $\mathcal{O}$-frame, and $\boldsymbol{R} = R_1\hat{\boldsymbol{b}}_1 + R_2\hat{\boldsymbol{b}}_2 + R_3\hat{\boldsymbol{b}}_3$ represents the satellite position. μ is the gravitational parameter of the central body, and the orbit angular speed is constant in circular orbit and given by $\Omega = \sqrt{\mu/R^3}$.

(a): **Show** that combining Eqs. (5) and (6) leads to the following equations:

$$\begin{bmatrix}\dot{\omega}_1\\ \dot{\omega}_2\\ \dot{\omega}_3\end{bmatrix} = \begin{bmatrix}K_1\omega_2\omega_3\\ K_2\omega_1\omega_3\\ K_3\omega_1\omega_2\end{bmatrix} - \frac{3\Omega^2}{\|\boldsymbol{R}\|_2^2}\begin{bmatrix}K_1R_3R_2\\ K_2R_1R_3\\ K_3R_2R_1\end{bmatrix} \tag{7}$$

where $K_1 = (I_2-I_3)/I_1,\ K_2 = (I_3-I_1)/I_2$, and $K_3 = (I_1-I_2)/I_3$.

(b): In class, we derived a particular solution of our system in terms of EP, given by: $\bar{\boldsymbol{\epsilon}}_{B/O} = [0,\ 0,\ 0,\ 1]^\top$ and $\bar{\boldsymbol{\omega}}_{B/N} = [0,\ 0,\ \Omega]^\top$, where the bar $\bar{(\cdot)}$ indicates the particular solution. **Show** that the particular solution $\{\bar{\boldsymbol{\epsilon}}_{B/O}, \bar{\boldsymbol{\omega}}_{B/N}\}$ is equivalently expressed as in Eq. (8), and that Eq. (8) satisfies Eqs. (4) and (7).

$$\bar{\sigma}_1 = \bar{\sigma}_2 = \bar{\sigma}_3 = 0, \quad \bar{\omega}_1 = \bar{\omega}_2 = 0, \quad \bar{\omega}_3 = \Omega \tag{8}$$

Hint: ω_i' in Eq. (4) need to be first expressed in terms of ω_i.

(c): We use Eq. (8) as the particular solution of our interest. We then linearize Eqs. (4) and (7) to perform the linear stability analysis. To do this, consider small perturbations in $\boldsymbol{\sigma}$ and $\boldsymbol{\omega}$ about the particular solution, i.e., $\sigma_i = \bar{\sigma}_i + \widetilde{\sigma}_i$ and $\omega_i = \bar{\omega}_i + \widetilde{\omega}_i$ with $\widetilde{\sigma}_i \ll 1$ and $\widetilde{\omega}_i \ll 1$, and ignore the second and higher order terms ($\widetilde{\sigma}_i^2 = \widetilde{\omega}_i^2 = 0$). **Show** that the linearized equations are given by Eq. (9).

$$\begin{bmatrix}\dot{\widetilde{\sigma}}_1\\ \dot{\widetilde{\sigma}}_2\\ \dot{\widetilde{\sigma}}_3\end{bmatrix} = \frac{1}{4}\begin{bmatrix}\widetilde{\omega}_1\\ \widetilde{\omega}_2\\ \widetilde{\omega}_3\end{bmatrix} + \Omega\begin{bmatrix}\widetilde{\sigma}_2\\ -\widetilde{\sigma}_1\\ 0\end{bmatrix}, \quad \begin{cases}\dot{\widetilde{\omega}}_1 = K_1\Omega\widetilde{\omega}_2\\ \dot{\widetilde{\omega}}_2 = K_2\Omega\widetilde{\omega}_1 - 12K_2\Omega^2\widetilde{\sigma}_2\\ \dot{\widetilde{\omega}}_3 = 12K_3\Omega^2\widetilde{\sigma}_3\end{cases} \tag{9}$$

(d): As discussed in class, the stability of a linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is determined by the eigenvalues of the system matrix A, which are obtained by solving the characteristic equations of the system. **Show** that the characteristic equations of our linear system Eq. (9) are the same as those for $\boldsymbol{\epsilon}$ derived in class, i.e., given by Eq. (10). Also, **discuss** the linear stability of our system about the particular solution Eq. (8) for different possible combinations of K_1, K_2, and K_3.

$$\lambda^2 - 3K_3\Omega^2 = 0, \quad \lambda^4 + (1 - K_1K_2 + 3K_2)\Omega^2\lambda^2 - 4K_1K_2\Omega^4 = 0. \tag{10}$$

(e): Let us then numerically investigate the nonlinear attitude motion and compare the results against the linear stability analysis. To do this, we numerically integrate the original, nonlinear system (i.e., Eqs. (4) and (7)) simultaneously with a variety of satellite inertia properties for an initial condition. Specifically, we consider a circular orbit around the Earth ($\mu = 3.9860 \times 10^5$ km^3/s^2) with its radius being $R = 6800$ [km] and the following initial condition:

$$\boldsymbol{\sigma}(t=0) = \frac{1}{\sqrt{3}} \tan\frac{\theta_0}{4}(\hat{\boldsymbol{b}}_1 + \hat{\boldsymbol{b}}_2 + \hat{\boldsymbol{b}}_3), \quad \boldsymbol{\omega}(t=0) = \Omega\hat{\boldsymbol{b}}_3, \quad \theta_0 = 3 \text{ [deg] } (= \pi/60 \text{ [rad]}) \tag{11}$$

As for the inertia property, let us first assume the principal inertia moments to be $I_1 = 400$ [kg $\cdot$ m^2], $I_2 = 600$ [kg $\cdot$ m^2], and $I_3 = 800$ [kg $\cdot$ m^2].

(e.1): **Discuss** the kind of perturbation assumed in our initial condition Eq. (11) (e.g., is the perturbation in both/either of the initial orientation and angular velocity? is the perturbation about which axis for what amount?).

(e.2): Numerically integrate Eqs. (4) and (7) *simultaneously* for a time span from $t = 0$ to $t = 8$ **[hours]** with integration tolerance 1.0×10^{-10}, and **show** the plots of $\sigma_i(t)$ and $\omega_i(t)$ over time.
Note: Be careful about the units of I_i and R; numerical simulations must use consistent units.

(e.3): **Compute** the principal rotation angle $\theta(t)$ from $\boldsymbol{\sigma}(t)$, and **show** $\theta(t)$ in degrees as a function of time.

(e.4): **Discuss** the obtained numerical results, addressing the following points:

- which region in Fig. 2 does the current simulation case correspond to?
- how do $\sigma_i(t)$, $\omega_i(t)$, and $\theta(t)$ behave over time? (e.g., are they stay close to the initial values? are they periodic?)
- the maximum value of θ over time; how does it compare against the initial perturbation θ_0?
- does the numerical result agree with the result of the linear stability analysis (to the extent we can tell from the conducted numerical simulations)?

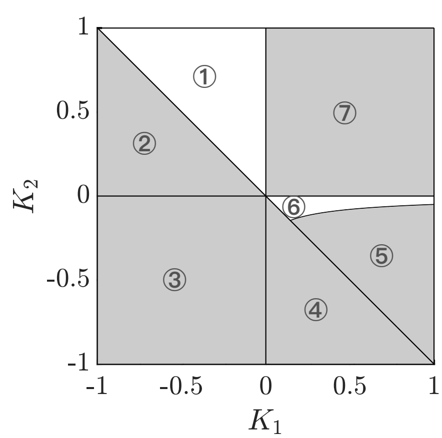

Figure 2: Stability chart

(f): **Repeat** the same investigation (e.2)-(e.4) for the following inertia moments:

(f.1): $I_1 = 600$ [kg $\cdot$ m^2], $I_2 = 400$ [kg $\cdot$ m^2], and $I_3 = 800$ [kg $\cdot$ m^2]

(f.2): $I_1 = 400$ [kg $\cdot$ m^2], $I_2 = 800$ [kg $\cdot$ m^2], and $I_3 = 600$ [kg $\cdot$ m^2]

(g): **Optional** (extra credit for *both* AAE 440 and 590)
Repeat the same investigation (e.2)-(e.4) with a longer time span, from $t = 0$ to $t = 24$ [hours], for the three inertia moments tested so far.

(g.1): $I_1 = 400$ [kg $\cdot$ m^2], $I_2 = 600$ [kg $\cdot$ m^2], and $I_3 = 800$ [kg $\cdot$ m^2]

(g.2): $I_1 = 600$ [kg $\cdot$ m^2], $I_2 = 400$ [kg $\cdot$ m^2], and $I_3 = 800$ [kg $\cdot$ m^2]

(g.3): $I_1 = 400$ [kg $\cdot$ m^2], $I_2 = 800$ [kg $\cdot$ m^2], and $I_3 = 600$ [kg $\cdot$ m^2]

Problem 3

Students in AAE 590 should solve this problem for full score*; students in AAE 440 who complete this problem with correct answers will receive extra credit.*

As we found in Problem 2, the linear stability analysis yields the same result regardless of the choice of attitude representations. In this problem, we confirm this again with another attitude representation: direction cosine matrix (DCM). Thus, our attitude variables consist of $\boldsymbol{\omega}$ (same as Problem 2) and a DCM of $\mathcal{B}$-frame relative to $\mathcal{O}$-frame, denoted by $C \in \mathbb{R}^{3\times 3}$, where

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Note that the column vector format is assumed for the definition of C, i.e., ${}^{\mathcal{B}}\boldsymbol{\omega} = C\ {}^{\mathcal{O}}\boldsymbol{\omega}$.

Considering the same system as assumed in Problem 2 but with the use of DCM C as the attitude representation, answer the following questions.

(a): **Derive** the nonlinear differential equations for our system in terms of C_{ij} and ω_i.

(b): **Express** the same particular solution as in Problem 2, i.e., Eq. (8), in terms of C_{ij} and ω_i. **Confirm** that this particular solution satisfies the differential equations derived in the previous question.

(c): Considering small perturbations in C_{ij} and w_i, denoted by $\widetilde{C}_{ij}$ and $\widetilde{w}_i$, **derive** the linearized equations of motion in terms of $\widetilde{C}_{ij}$ and $\widetilde{w}_i$. **Confirm** that the eigenvalues of the system matrix satisfy the same equations as in Eq. (10).

(d): **Implement** the kinematic differential equations for the DCM representation, and **perform** the same numerical experiments as done in (e) and (f) of Problem 2. **Show** the plots of $C_{ij}(t)$, $\omega_i(t)$, and $\theta(t)$ over time for each experiment case, and confirm the consistency with the numerical results obtained in Problem 2.