AAE 440/590: Spacecraft Attitude Dynamics
Computational Problem Set 3

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Due: April 6, 2022 at **9:00** PM ET
(submission at Brightspace>Content>Gradescope)

Your write-up should include snippets of your code that are relevant to solving the problems. Note that, as stated in the Problem Set Format Instruction, your write-up must still **include a full description** of the methods and equations used to solve the problem.

Problem 1

In CPS2, we implemented kinematic differential equations (KDEs) for two attitude representations: Euler parameters and Euler angle sequences. Here, we implement KDEs for other attitude representations, namely classical Rodrigues parameter (CRP) vectors and modified Rodrigues parameter (MRP) vectors, and simulate the spacecraft attitude motion using these attitude representations.

We assume the same system as considered in CPS2, which is repeated in the following. The orientation of the body-fixed frame ($\mathcal{B}$ frame, with vector bases $\{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$) relative to the inertial frame ($\mathcal{N}$ frame, $\{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$) at time $t = 0$ is given by a body (3-2-1) sequence of Euler angles $(-\pi/4, \pi/8, \pi/5)$. The time history of the angular velocity $\boldsymbol{\omega}(t)$ is measured as:

$$\boldsymbol{\omega}(t) = \omega_1(t)\hat{\boldsymbol{b}}_1 + \omega_2(t)\hat{\boldsymbol{b}}_2 + \omega_3(t)\hat{\boldsymbol{b}}_3, \quad \begin{cases} \omega_1(t) = \omega_{1_0}\cos(0.2t) + \omega_{2_0}\sin(0.2t) \\ \omega_2(t) = \omega_{2_0}\cos(0.2t) - \omega_{1_0}\sin(0.2t) \\ \omega_3(t) = \omega_{3_0} \end{cases} \tag{1}$$

where

$$\omega_{1_0} = 0.1, \quad \omega_{2_0} = 0.15, \quad \omega_{3_0} = 0.3, \qquad [\text{rad/sec}], \tag{2}$$

(a): Answer the following questions concerned with the simulation of attitude motions via CRP vectors.

(a.1): Define a CRP vector $\boldsymbol{\rho} = [\rho_1, \rho_2, \rho_3]^\top$ that represents the orientation of $\mathcal{B}$ frame relative to $\mathcal{N}$ frame. **Determine** the value of $\boldsymbol{\rho}$ at time $t = 0$.

(a.2): **Implement** the CRP's KDE as a function named `KDE_CRP`, which computes $\dot{\boldsymbol{\rho}}$ given $\boldsymbol{\rho}$ and $\boldsymbol{\omega}$. **Discuss** which vector bases are used to express the elements of $\boldsymbol{\rho}$, $\boldsymbol{\omega}$, and $\dot{\boldsymbol{\rho}}$ that appear in the function `KDE_CRP`.

(a.3): **Compute** the time history of $\boldsymbol{\rho}$ by numerically integrating `KDE_CRP` over a time span from $t = 0$ to $t = 9.5$ seconds with integration tolerance 1×10^{-10}. **Show** the plots of $\rho_i(t)$ and the magnitude $\|\boldsymbol{\rho}(t)\|_2$ over time.

(a.4): To understand the attitude motion in terms of the principal axis rotation, **compute** the principal rotation angle and axis (i.e., θ and $\hat{\boldsymbol{\lambda}}$) by converting from $\boldsymbol{\rho}(t)$ obtained in (a.3), and **show** the plots of $\theta(t)$ and $\hat{\lambda}_i(t)$ over time. Use degrees for the $\theta(t)$ plot.

(a.5): **Show** that the result obtained in (a.3) is consistent with the result of CPS2 by converting $\boldsymbol{\rho}(t)$ obtained in (a.3) to a history of Euler parameter vectors $\boldsymbol{\epsilon}(t)$. To compare the obtained $\boldsymbol{\epsilon}(t)$ against the result of CPS2, **include** the plots of $\epsilon_i(t)$ from this problem and from CPS2 (i.e., $\epsilon_i(t)$ obtained by numerically integrating the EP's KDE) over time from $t = 0$ to $t = 9.5$ seconds, where choose $\epsilon_4 \geq 0$ for both plots.

(a.6): Try to numerically integrate the same system with the same initial conditions over a time span from $t = 0$ to $t = 100$ seconds with the same integration tolerance. **Discuss** if the numerical integration is successful, and if not, why this is the case.

(b): Now let us turn our attention to MRP. Answer the following questions.

(b.1): Define a MRP vector $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3]^\top$ that represents the orientation of $\mathcal{B}$ frame relative to $\mathcal{N}$ frame. **Determine** the value of $\boldsymbol{\sigma}$ at time $t = 0$.

(b.2): **Implement** the MRP's KDE as a function named `KDE_MRP`, which computes $\dot{\boldsymbol{\sigma}}$ given $\boldsymbol{\sigma}$ and $\boldsymbol{\omega}$. **Discuss** which vector bases are used to express the elements of $\boldsymbol{\sigma}$, $\boldsymbol{\omega}$, and $\dot{\boldsymbol{\sigma}}$ that appear in the function `KDE_MRP`.

(b.3): **Compute** the time history of $\boldsymbol{\sigma}$ by numerically integrating `KDE_MRP` over a time span from $t = 0$ to $t = 100$ seconds with integration tolerance 1×10^{-10}. **Show** the plots of $\sigma_i(t)$ and the magnitude $\|\boldsymbol{\sigma}(t)\|_2$ over time.

(b.4): Like we did for CRP, **compute** $\theta(t)$ and $\hat{\lambda}_i(t)$ from $\boldsymbol{\sigma}(t)$ obtained in (b.3), and **show** their plots over time from $t = 0$ to $t = 100$, where allow $\theta(t)$ to take values greater than 180 degrees.

(b.5): Also, **show** the consistency of the results obtained in (b.3) and of CPS2 by converting $\boldsymbol{\sigma}(t)$ to $\boldsymbol{\epsilon}(t)$ by **including** the plots of $\epsilon_i(t)$ from this problem and from CPS2 over a time span from $t = 0$ to $t = 100$ seconds, where choose $\epsilon_4 \geq 0$ for both plots.

(b.6): **Discuss** the time history of the magnitude of the MRP vectors (e.g., does the magnitude exceeds unity? what does that imply?). **Discuss** also the advantage of MRP over CRP in terms of singularity based on the simulation results in this Problem.

Problem 2

Consider an axisymmetric satellite moving in a force-free, torque-free environment. The inertial frame and satellite body-fixed frame are represented by $\mathcal{N}$-frame and $\mathcal{B}$-frame, where $\{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$ and $\{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$ are right-handed vector bases fixed in $\mathcal{N}$-frame and $\mathcal{B}$-frame, respectively. The inertia tensor of the body and angular velocity about the center of mass (CoM) at $t = 0$ are measured by:

$$\bar{\bar{\mathbf{I}}}_c = \hat{\boldsymbol{b}}_1\hat{\boldsymbol{b}}_1 + \hat{\boldsymbol{b}}_2\hat{\boldsymbol{b}}_2 + \frac{3}{2}\hat{\boldsymbol{b}}_3\hat{\boldsymbol{b}}_3, \quad \boldsymbol{\omega}_0 = -0.1\ \hat{\boldsymbol{b}}_1 + 0.05\ \hat{\boldsymbol{b}}_2 + 0.1\ \hat{\boldsymbol{b}}_3 \quad \text{(dimensionless)}$$

The orientation of $\mathcal{B}$-frame relative to $\mathcal{N}$-frame at $t = 0$ is given in terms of an EP vector as:

$$\boldsymbol{\epsilon} = \frac{1}{2}\begin{bmatrix} -\sqrt{3}\hat{\boldsymbol{n}}_1 \\ 1 \end{bmatrix} \tag{3}$$

Answer the following questions to numerically study the attitude behavior of the system described above.

(a): Recall from class that the Euler's rotational equations of motion in a torque-free environment is derived as follows:

$$\begin{cases} \dot{\omega}_1 = -\frac{I_3 - I_2}{I_1}\omega_2\omega_3 \\ \dot{\omega}_2 = -\frac{I_1 - I_3}{I_2}\omega_3\omega_1 \\ \dot{\omega}_3 = -\frac{I_2 - I_1}{I_3}\omega_1\omega_2 \end{cases} \tag{4}$$

Implement Eq. (4) as a function named `dwdt_torqueFree`, which computes $\dot{\boldsymbol{\omega}}$ given $\boldsymbol{\omega}$ and I_1, I_2, I_3. **Discuss** which vector bases are used to express the elements of $\boldsymbol{\omega}$ and $\bar{\bar{\mathbf{I}}}_c$ that appear in the function `dwdt_torqueFree`.

(b): **Compute** the time history of $\boldsymbol{\omega}$ by numerically integrating `dwdt_torqueFree` over a time span from $t = 0$ to $t = 100$ with integration tolerance 1×10^{-10}. **Show** the plots of $\omega_i(t)$ over time.

(c): Some useful tools to check the validity of the attitude simulation code include the two constant integrals of motion, namely the angular momentum magnitude, $\|\boldsymbol{H}\|_2$, and the rotational Kinetic energy, T_{rot}. **Compute** the values of these values over time using the result of the previous question. **Show** the plots of $\|\boldsymbol{H}(t)\|_2$ and $T_{\text{rot}}(t)$ over time, and **confirm** that they are indeed constant over time.

(d): Also, **show** the plots of the angular momentum vector in the body-fixed frame over time, i.e., ${}^{\mathcal{B}}H_i(t)$, from $t = 0$ to $t = 100$. **Discuss** the results by addressing the following points:

- the time variation of each element of ${}^{\mathcal{B}}\boldsymbol{H}(t)$
- the third element of ${}^{\mathcal{B}}\boldsymbol{H}(t)$ should be constant over time; what does this imply about the angle formed by the precession axis and the spin axis (i.e., the cone angle)?

(e): The angular momentum vector in the inertial frame, ${}^{\mathcal{N}}\boldsymbol{H}$, is also constant over time. **Compute** the time history of ${}^{\mathcal{N}}\boldsymbol{H}(t)$ by performing the coordinate transformation of ${}^{\mathcal{B}}\boldsymbol{H}(t)$, **show** the plots of ${}^{\mathcal{N}}H_i(t)$ over time from $t = 0$ to $t = 100$, and **confirm** that these values are indeed constant over time. Hint: to perform the coordinate transformation of the angular momentum vector from $\mathcal{B}$-frame to $\mathcal{N}$-frame, a set of KDEs needs to be numerically integrated given the initial attitude Eq. (3).

(f): We also learned in class that, for axisymmetric bodies, the solution of the differential equations Eq. (4) is analytically expressed via trigonometric functions. **Compute** $\boldsymbol{\omega}(t)$ from $t = 0$ to $t = 100$ by using the analytical solution. **Show** the plots of $\omega_i(t)$ over time obtained by this analytical solution, and **confirm** that these are indeed consistent with those obtained by numerical integration.

Problem 3

Students in AAE 590 should solve this problem for full score*; students in AAE 440 who complete this problem with correct answers will receive extra credit.*

This Problem is concerned with the numerical integration of the MRP's KDE performed in part (b) of Problem 1. Consider the same system assumed in Problem 1.

(a): As we learned in class, MRP has a singularity, and numerical integration of the MRP's KDE may encounter the singularity depending on the initial conditions. Such initial conditions are, however, fairly rare, implying the robustness of MRP as an attitude representation (you can try different initial conditions in Problem 1 to find initial conditions that lead to the MRP singularity).

Here, let us intentionally use an initial condition that leads to the singularity: initial attitude via a body (3-2-1) Euler angle sequence $(-4/\pi, 0, 0)$ and the initial angular velocity $\{\omega_{1_0}, \omega_{2_0}, \omega_{3_0}\} = \{0.0, 0.0, 0.3\}$. Using this initial condition, **compute** $\boldsymbol{\sigma}(t)$ by numerically integrating `KDE_MRP` from $t = 0$ to $t = 25$ seconds with integration tolerance 1×10^{-10}. **Show** the plots of $\sigma_i(t)$, $\|\boldsymbol{\sigma}(t)\|_2$, and $\theta(t)$ over time. Also, **discuss** what happens if we try to numerically integrate it from $t = 0$ to $t = 100$ seconds.

(b): We also learned in class that we can avoid the singularity by switching our MRP to its shadow set, $\boldsymbol{\sigma}^s$. **Implement** the MRP shadow set switching at $\|\boldsymbol{\sigma}\|_2 = 1$ in your simulation code, and **show** the plots of $\sigma_i(t)$ from $t = 0$ to $t = 100$ seconds for the initial condition given in (a). Note the following hints:

- If you are using Matlab, you can detect the event $\|\boldsymbol{\sigma}\|_2 = 1$ within your `ode45` by using *event* function (see `https://www.mathworks.com/help/matlab/math/ode-event-location.html`; for python users, see `https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.solve_ivp.html`). Every time the event $\|\boldsymbol{\sigma}\|_2 = 1$ is detected, you may terminate `ode45`, switch the MRP set, and re-run `ode45` with the updated initial conditions, until reaching $t = 100$ seconds.
- If you are using a language that does not have ode solvers that can detect events, you may manually split your time span to shorter arcs (say, 20 of 5-second arcs), check $\|\boldsymbol{\sigma}\|_2$ after every integration, and switch the MRP set as necessary. Recall that the switching does not need to occur exactly at $\|\boldsymbol{\sigma}\|_2 = 1$ because MRP-based attitude representation remains valid when $\|\boldsymbol{\sigma}\|_2 > 1$.

(c): **Show** the plot of $\|\boldsymbol{\sigma}(t)\|_2$ over time from $t = 0$ to $t = 100$ seconds, and **report** the number of the MRP switching occurrences between $t = 0$ and $t = 100$ seconds.