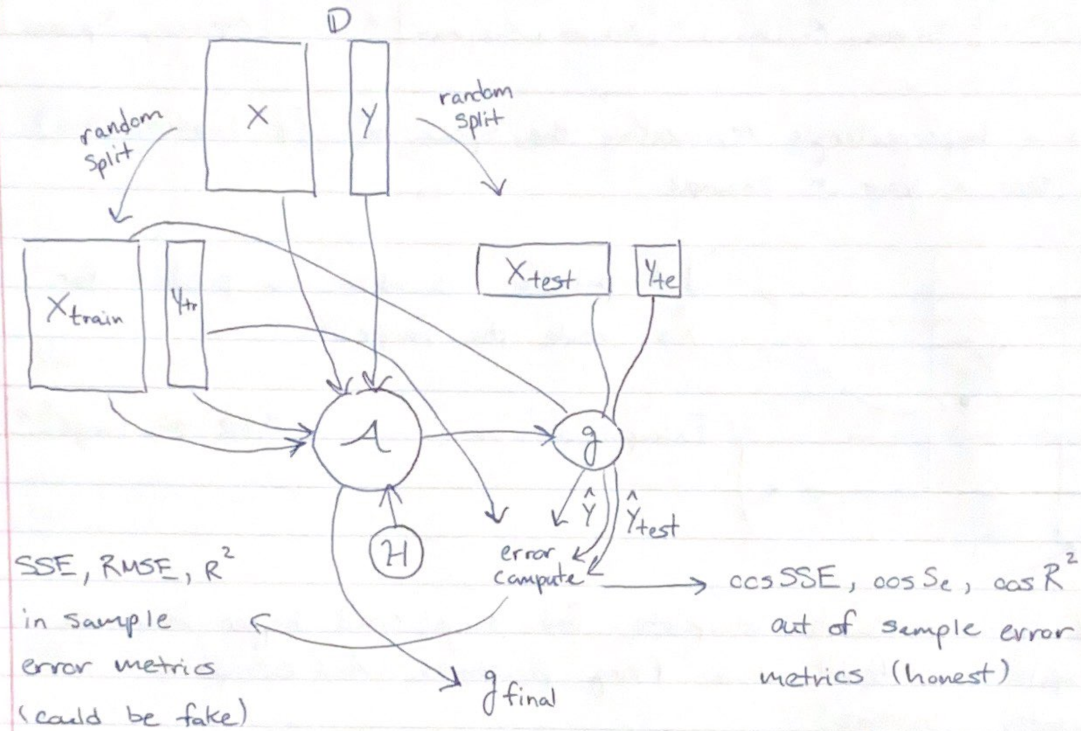
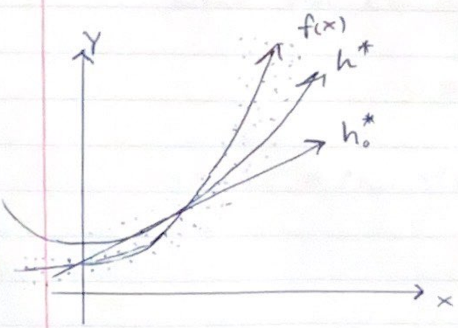
$K = 10 \Rightarrow$ test set is 10% n.



The g_{final} is the function used for future prediction. Its performance is at least as good as the oos metrics since you're running the same model fitting procedure but now n is slightly higher.

~

let $p = 1$ feature, $y = g(x) + \underbrace{h^*(x) - g(x)}_{\text{high if } n \text{ is not much} > p} + \underbrace{f(x) - h^*(x)}_{\text{misspecification error}} + \underbrace{t(z) - f(x)}_{\delta}$



$\mathcal{H}_0 = \{ w_0 + w_1 x : w_0, w_1 \in \mathbb{R} \}$

$\mathcal{H} = \{ w_0 + w_1 x + w_2 x^2 : w_0, w_1, w_2 \in \mathbb{R} \}$

$f(x)$ is not linear and therefore even the best possible linear model h_0^* will perform poorly. So why not allow for a more expressive candidate set?

We can do that by expanding the basis/complexity in $\mathcal{H}$. For example, we now allow for a quadratic term so we can fit parabolic-shaped curves. This allows us to get closer to the real f (which may be very complex and nonlinear), reducing misspecification error. We now have $p = 2$ which is greater than $p_{raw} = 1$. We call this x^2 a "derived feature" in contrast to a "raw feature" (original). E.g. $x_2 = g(x_1) = x_1^2$. It's a transformation of a raw feature.

You're at liberty to use any transformed features you want. If they're useless, they appear as random noise and you overfit.

Using squares and cubes is a well-known modeling procedure called "polynomial regression".

Is polynomial regression "linear"? Yes and no. "Yes" in the sense that you create a design matrix and use OLS and thus linear in the transformed features but "no" ~~to the~~ because the g model is not linear in the raw features.

Advanced math note: polynomial regression is a principled approach because of the Weierstrauss Approximation Thm (1885) which says that any continuous function f who domain is $x \in [a,b]$ can be approximated by a polynomial function p_d with arbitrary precision by picking d, its degree:

$$\forall_\varepsilon > 0 \quad \forall_{x \in [a,b]} \quad \exists_d \quad |f(x) - p_d(x)| < \varepsilon.$$

The Stone-Weierstrauss Thm (1937) generalizes the above. One implication of this thm is that a multivariate polynom func can approx any cont func $f(x_1, \ldots, x_p)$

How do we do a polynomial regression of degree d. E.g. $d=2$

$$X_{raw} = \begin{array}{c} \\ \vec{X}_{\cdot 1} \end{array} \hspace{-2em} \begin{bmatrix} 1 & x_{11} \\ 1 & x_{12} \\ \vdots & \vdots \\ 1 & x_{1n} \end{bmatrix} \xrightarrow{\text{transform}} X = \begin{bmatrix} 1 & x_{11} & x_{11}^2 \\ 1 & x_{12} & x_{12}^2 \\ \vdots & \vdots & \vdots \\ 1 & x_{1n} & x_{1n}^2 \end{bmatrix}$$

(columns of X: $\vec{X}_{\cdot 1}$, $\vec{X}_{\cdot 2}$)

$p=2$

The transformed matrix X is still full rank since a polynomial function cannot be expressed with finite linear terms.

$$\vec{b} = (X^T X)^{-1} X^T \vec{y} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$$

$$g(x) = \hat{y} = b_0 + b_1 x + b_2 x^2 = b_0 + (b_1 + b_2 x) x$$

Can you do polynomial regression of degree $d=3$? Yes. Same way! Just make a new feature and cube x_1. How far can you go in OLS? $p = n-1$ i.e. $d = n-1$. That would yield a perfect fit. Any higher d, and you can't invert $X^T X$. E.g. $n=5$

$$X = \begin{bmatrix} \underset{=x_{11}^0}{1} & x_{11} & x_{11}^2 & x_{11}^3 & x_{11}^4 \\ 1 & x_{12} & \vdots & \vdots & \vdots \\ 1 & x_{13} & & & \\ 1 & x_{14} & & & \\ 1 & x_{15} & & & \end{bmatrix}$$

is this full rank? This is a special matrix called a Vandermonde Matrix and it's proven to be full rank if:

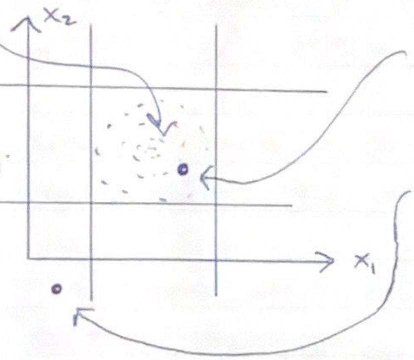
$$\det[X] = \prod_{i=1}^{n} \prod_{j=1}^{n} x_j - x_i \neq 0$$

Consider p raw features given by the columns of X. Define:

$$\text{Range}[X] = [x_{\cdot 1, min}, x_{\cdot 1, max}] \times [x_{\cdot 2, min}, x_{\cdot 2, max}] \times \cdots \times [x_{\cdot p, min}, x_{\cdot p, max}]$$

This is a hyperrectangle representing the space of $\vec{x}$s (observations) you've seen in your n example.

e.g.



"Interpolation" is when you predict for $\vec{x}$s inside the Range[X]

"Extrapolation" is . . . outside the Range[X]

We build models to interpolate. Bad things could happen when you extrapolate. Different model fitting procedures ($\mathcal{A}$) extrapolate differently beware!

We expanded ~~our~~ the complexity of our candidate set $\mathcal{H}$ using polynomials. But we found that high degree polynomials had unintended consequences (Runge's Phenomenon). Is there another transformation of raw features that we can employ to expand $\mathcal{H}$? Of course... there are tons of functions! Exponentials, logs, sines, etc. Let's examine logs because they are very popular and very useful:

$$\ln(x+1) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \approx x \quad \text{if} \quad x \approx 0$$

$$\Rightarrow \ln(x) = \ln((x+1)-1) \approx x - 1 \quad \text{e.g.} \quad \ln(1.02) = .019 \approx 1.02 - 1$$

Consider the following linear model:

$y = b_0 + b_1 \ln(x)$ e.g.

$\Delta x = x_f - x_0 = 1.07 - 1.00$

$$\Delta y = (b_0 + b_1 \ln(x_f)) - (b_0 + b_1 \ln(x_0)) = b_1 \ln\left(\frac{x_f}{x_0}\right) \approx b_1 \underbrace{\left(\frac{x_f}{x_0} - 1\right)}^{\%\text{ change in } X}$$

This simple log model can be approx interpreted as proportional change in x yields b_1 change in y (in y's units). i.e if x increases 100%, y goes up by b_1.

Likewise you can do $\ln(y) = b_0 + b_1 x$ and this approx interpreted as unit change in x yields b_1 proportion change in y and $\ln(y) = b_0 + b_1 \ln(x)$ is approx interpreted as proportion change in x yields b_1 proportion change in y.