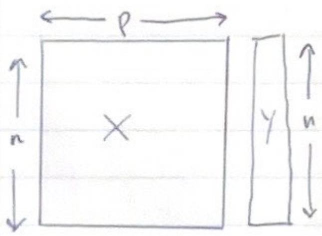
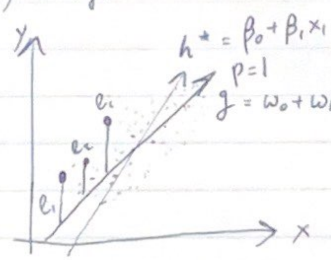
M342W 2/17

So far, the responses space was $\{0,1\}$ and the models were "binary classification" models. What if $\mathcal{Y} = \mathbb{R}$ or $\mathcal{Y} \subset \mathbb{R}$? This means the response is continuous and our predictions will be continuous. These models are called "regression" models. The word "regression" is used because of historical circumstances only (see lab).

What is the null model $g_0$? $g_0 = \bar{y}$ (avg)




$$\mathcal{H} = \{ \underset{w_0 + w_1 x_1 + \dots w_p x_p}{\vec{w} \cdot \vec{x}} : \vec{w} \in \mathbb{R}^{p+1} \}$$

Like before, this candidate set requires a "1" appended to each of the original p-length x-vectors.

$$h^*(\vec{x}) = w_0^* + w_1^* x_1 + \dots + w_p^* x_p = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p$$

$$\Downarrow$$

$$y = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p + \varepsilon$$

Standard notation for the best/"true" values of the linear coefficients

We have training data and the candidate set of linear models. We need an algorithm that will compute $w_0$ and $w_1$ for us. We first need an "objective function" or "error function" or "loss function" which gauges the degree of our model mistakes. Let $e_i = y_i - \hat{y}_i$. Consider the loss function:

$$\sum_{i=1}^{n} e_i^2 = \text{SSE (Sum of squared error)} = \sum (y_i - \hat{y}_i)^2 = \sum (y_i - w_0 - w_1 x_i)^2$$

Our algorithm will seek to $\text{argmin}\{SSE\}$ over all possible $w_0, w_1$ values. To do this, we take the partial derivative with respect to $w_0$ and set equal to zero and solve for $b_0$ then take the partial deriv. wrt $w_1$ and set equal to zero and solve for $b_1$. We will call $g(x) = b_0 + b_1 x$ the "least squares" regression model or "ordinary least squares" (OLS).

$$\sum y_i^2 + w_0^2 + w_1^2 x_i^2 - 2y_i w_0 - 2y_i w_1 x_i + 2w_0 w_1 x_i$$
$$= \sum y_i^2 + n w_0^2 + w_1^2 \sum x_i^2 - 2w_0 n\bar{y} - 2w_1 \sum x_i y_i + 2 w_0 w_1 n \bar{x}$$

$$\frac{\partial}{\partial w_0}[SSE] = \frac{\partial}{\partial w_0}[\quad] = \cancel{2} n w_0 - \cancel{2} n\bar{y} + \cancel{2} w_1 n \bar{x} \overset{set}{=} 0 \Rightarrow b_0 = \frac{n\bar{y} - w_1 n \bar{x}}{n}$$
$$= \bar{y} - w_1 \bar{x}$$
$$= \bar{y} - b_1 \bar{x}$$

$$\frac{\partial}{\partial w_1}[SSE] = \frac{\partial}{\partial w_1}[\quad] = \cancel{2} w_1 \sum x_i^2 - \cancel{2} \sum x_i y_i + \cancel{2} w_0 n \bar{x} \overset{set}{=} 0$$
$$\Rightarrow \overset{(b_1)}{w_1} \sum x_i^2 = \sum x_i y_i - \overset{(b_0)}{w_0} n \bar{x} =$$
$$= \sum x_i y_i - (\bar{y} - b_1 \bar{x}) n x$$
$$\Rightarrow b_1 \sum x_i^2 = \sum x_i y_i - n\bar{x}\bar{y} + n\bar{x}^2 b_1 \Rightarrow b_1 \sum x_i^2 - b_1 n \bar{x}^2 = \sum x_i y_i - n\bar{x}\bar{y}$$
$$\Rightarrow b_1 = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum x_i^2 - n\bar{x}^2}$$ this is the answer and now we simplify... using M241-like notation.

$$S_x^2 = \frac{1}{n-1}\sum (x_i - \bar{x})^2 = \frac{1}{n-1}\left(\sum x_i^2 - 2\bar{x}\overbrace{\sum x_i}^{n\bar{x}} - n\bar{x}^2\right)$$
$$= \frac{1}{n-1}\left(\sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2\right) = \frac{1}{n-1}\left(\sum x_i^2 - n\bar{x}^2\right)$$

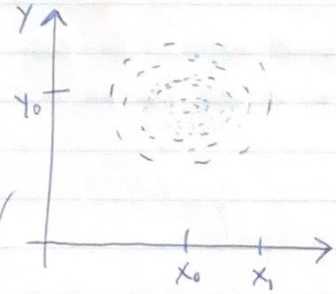
$$r := \frac{S_{xy}}{S_x S_y} \Downarrow S_{xy} = r S_x S_y$$

$$\rho := Corr[X,Y] := \frac{Cov[X,Y]}{SE[X]\,SE[Y]} = \frac{E[(X-\mu_x)(Y-\mu_y)]}{\sqrt{Var[Y]\,Var[X]}}$$

Covariance is estimated with

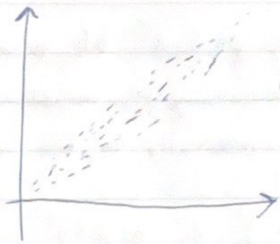
$$S_{xy} = \frac{1}{n}\sum (x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n-1}\left(\sum x_i y_i - \bar{y}\overbrace{\sum x_i}^{n\bar{x}} - \bar{x}\overbrace{\sum y_i}^{n\bar{y}} + n\bar{x}\bar{y}\right)$$
$$= \frac{1}{n-1}\left(\sum x_i y_i - n\bar{x}\bar{y} - n\bar{x}\bar{y} + n\bar{x}\bar{y}\right) = \frac{1}{n-1}\left(\sum x_i y_i - n\bar{x}\bar{y}\right)$$

$$b_1 = \frac{(n-1)S_{xy}}{(n-1)S_x^2} = \frac{S_{xy}}{S_x^2} = \frac{r S_x S_y}{S_x^2} = r\frac{S_y}{S_x} \Rightarrow b_0 = \bar{y} - r\frac{S_y}{S_x}\bar{x}$$



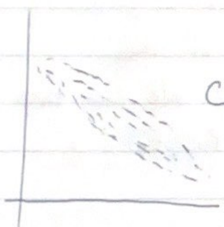

doesn't covary

$Cov[X,Y]=0$



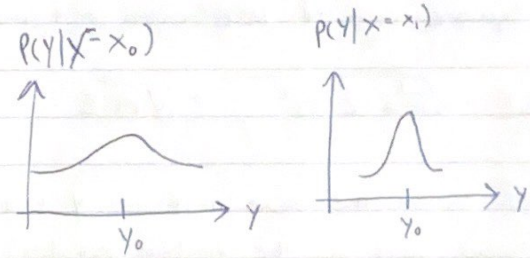
$Cov[X,Y] > 0$

Covariance measures change in expected value of the 2nd rv if the 1st changes.



$Cov[X,Y] < 0$

are $X, Y$ independent? Yes




The word "association" just means "dependence". Correlation means linear dependence. (and covariance means linear dependence). Correlation is a type of association (it is linear association)

Let's examine a special case of OLS where $p=1$. Let the only feature be a binary feature e.g. $x_i$ is either "red" or "green". Lets create a new $x$ which is a dummy/binary variable which is 0 if red, 1 if green. What is a good model for prediction?

$g(x) =$

$$\left.\begin{array}{l} g(\text{red}) = \bar{Y}_{red} \\ g(\text{green})\ \bar{Y}_{green} \end{array}\right\}\ \text{OLS model}$$