**PHY407: Lab 10**

**Date: November 26th. 2021**

**Lab Partners: Brendan Halliday and Nikolaos Rizos**

**Contributions:**

- Q1. Brendan Halliday
- Q2. Brendan and Nikolaos Rizos
- Q3. Nikolaos

## Q1.a.

Q1. (a)

$$0 \leq \theta \leq \pi$$

$$0 \leq \phi \leq 2\pi$$

$$\int_0^{\pi} P_1(\theta)\, d\theta = \int_0^{\pi} \frac{\sin(\theta)}{2}\, d\theta$$

$$= -\frac{1}{2}\cos\theta \Big|_0^{\pi} = 1 \quad \checkmark$$

$$\int_0^{2\pi} P_2(\phi)\, d\phi = \int_0^{2\pi} \frac{1}{2\pi}\, d\phi$$

$$= \frac{1}{2\pi}(2\pi - 0) = 1 \quad \checkmark$$

Formulas for generating angles $\theta$ and $\phi$ can be computed in the following way

$$z = \int_0^{x(z)} P_2(x')\, dx'$$

$$z = \frac{1}{2\pi}\int_0^{x(z)} dx'$$

$$\Rightarrow \boxed{2\pi z = x(z) = \phi(z)}$$

s.t. $0 < z < 1$

where $z$ is a random number

Likewise for $\theta$

$$z = \int_0^{\theta(z)} \frac{\sin\theta'}{2}\, d\theta'$$

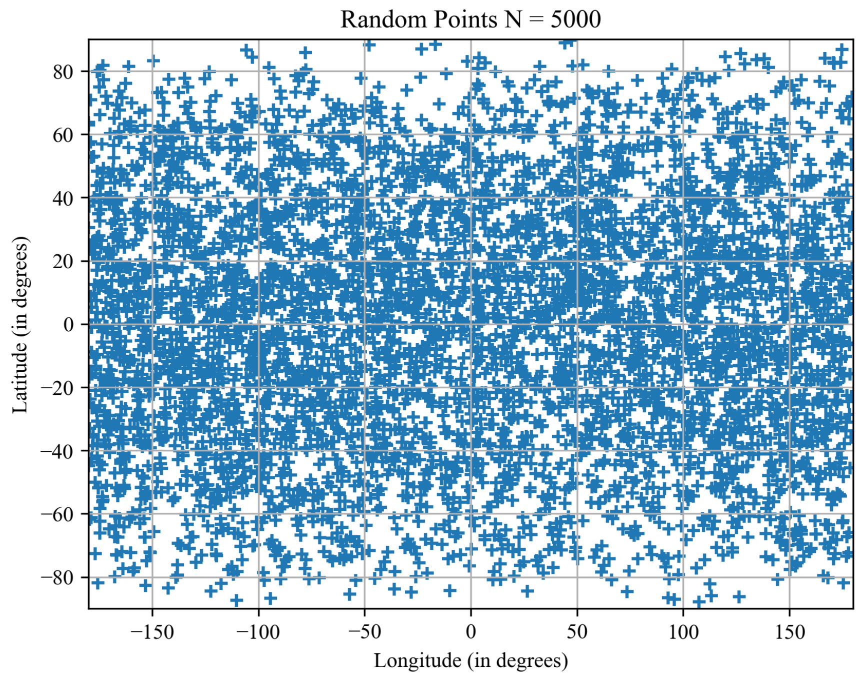
$$2z = -\cos\theta' \Big|_0^{\theta(z)}$$

$$-2z = \cos\theta(z) - 1$$

$$\boxed{\theta(z) = \arccos(1 - 2z)}$$

**Q1.b.**

Below is a plot of the random points plotted on a latitude vs. longitude plot:




Here we see that as latitude increases in magnitude, the random points become more and more sparse. This was expected as when you project these points to their corresponding coordinates on a sphere, the result is a uniform distribution over the surface.

**Q1.c.**

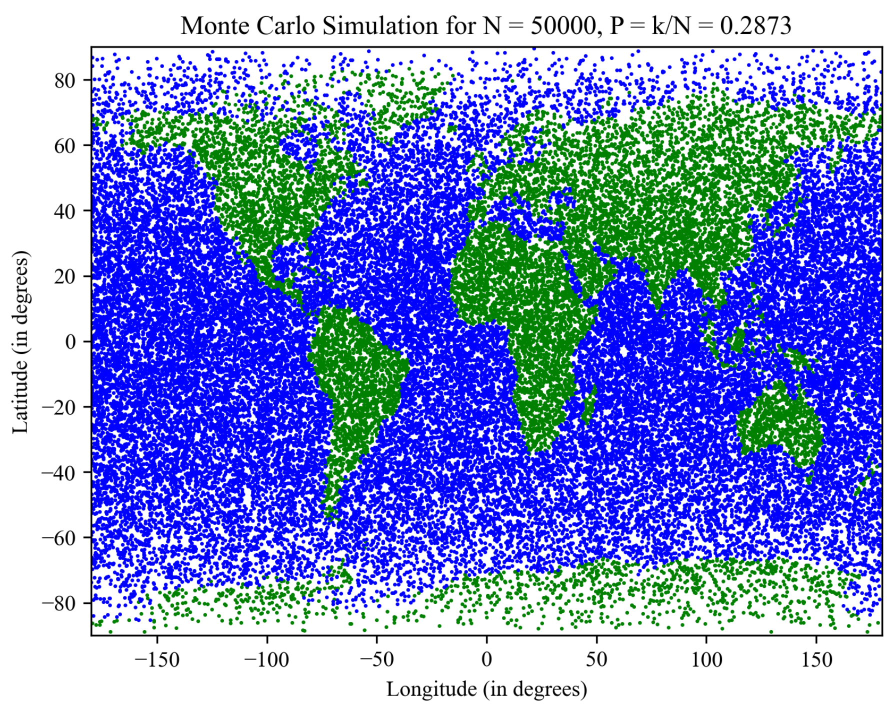
Numerical integration was done in a very simple way by summing all the points in the array data from Earth.npz. SInce all land points are represented by ones, the sum is the total number of land points in the set. The probability of choosing a land point is the sum divided by the total number of elements in the data array. This probability is roughly 0.3319. This value multiplied by the surface area of a sphere of radius one meter is 4.1703 $m^2$ (this value is the area of land).

**Q1.d.**

______Below is a table that lists the calculated values for land fractions for N = 50, 500, 5000, and 50000. These calculations were done by sampling N random points and then determining the number of points that correspond to a land value. The ratio between these numbers gives an approximation of the land fraction of the planet.

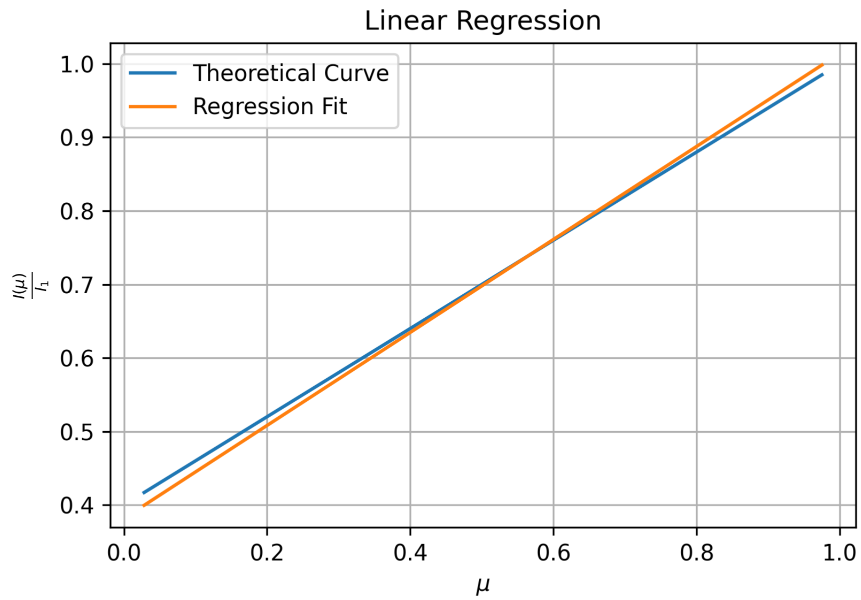
| Values of N | Land Fraction |
|---|---|
| 50 | 0.3800 |
| 500 | 0.2760 |
| 5000 | 0.2882 |
| 50000 | 0.2873 |

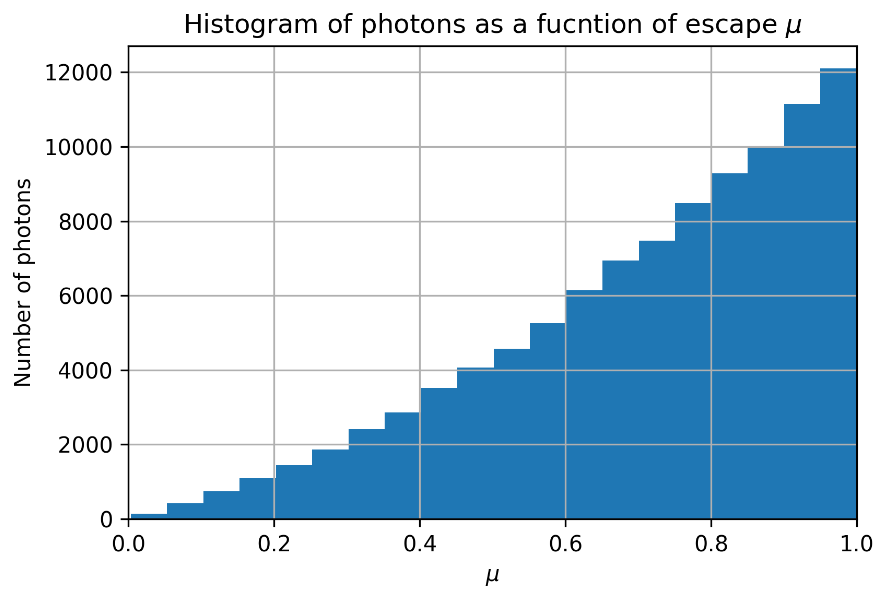
Below is a visualization of the N = 50000 case. The green points are land values and the blue are non-land values.




**Q2.b.**

Using N = 100000 photons and maximum optical depth of 10, we fitted a linear regression model for the specific intensity as a function of average $\mu$. We compare the resulting line, with the line corresponding to the analytical solution of the equation and provide a plot with both lines on it below. We also provide a histogram of the number of photons given the angle $\mu$.



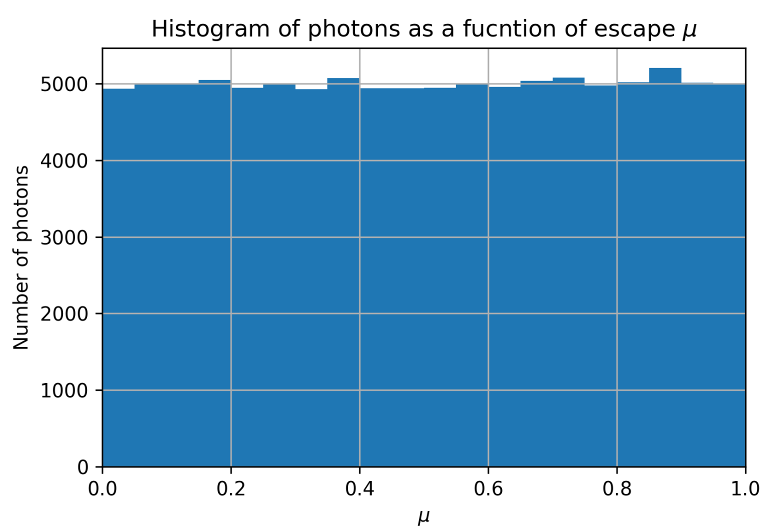

For the regression model, the slope coefficient estimate was: 0.654, while the intercept estimate was: 0.375, both rounded to 3 decimal points.

The first plot suggests that when observing the sun in the sky, it seems that the highest intensity one can record corresponds to its center ($\mu = 1$ corresponds to light being emitted perpendicular to the surface of the sun, and thus directly towards the observer who is looking at the center). As the angle of emission becomes smaller and smaller, it seems that the intensities become smaller. Looking at the bar-plot of the number of photons given angle of emission, we again observe an exponential increase in the emitted photons as the angles of emission etch closer to be perpendicular to the surface.

These two plots thus suggest that an observer looking at the sun in the sky will observe stronger light on its center, and weaker light on the edges. The reason for that is that at the observed center of the sun when looking at it in the sky, the light reaching the observer is the one emitted perpendicular to the surface (and thus with the highest intensity) while at the edges, the light finally reaching the observer is the one being emitted at an angle not perpendicular to the surface (the perpendicular emitted light on what the observer defines as "edges" from their point of view, does not move towards them). Thus, the observer would note that the light as they look towards the edges of the sun becomes dimmer and dimmer, which is a direct result of limb-darkening.

**<u>Q2.c.</u>**

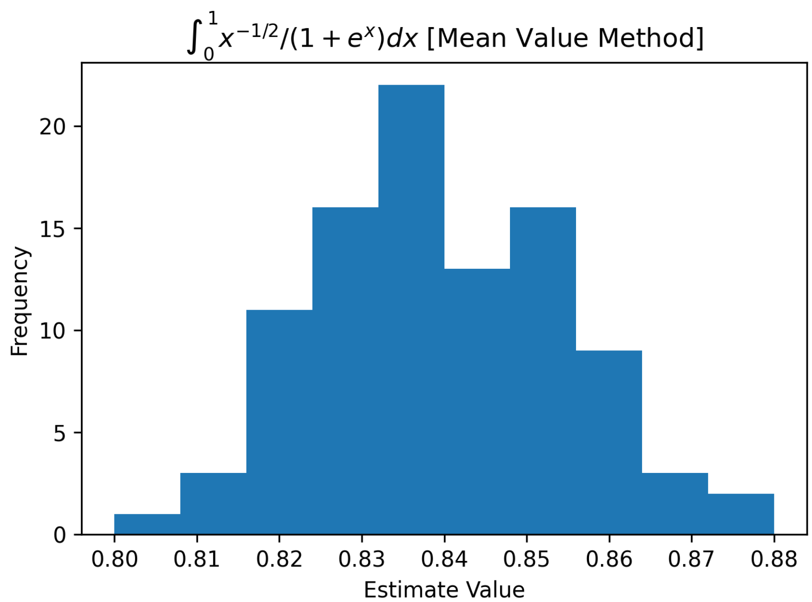
Decreasing the maximum optical depth to a value of 0.0001 while maintaining the same number of photons, we would expect a lot less scattering of the light, and a reduction in limb darkening. This is because the probability that the light travels the new optical depth without interaction is substantially higher than before. We would thus expect that the light is emitted uniformly towards all directions with approximately the same intensity, regardless of whether it was emitted perpendicular to the surface of the sun or not. The observer would thus notice no limb-darkening and would instead observe constant intensity of light being emitted from the center compared to the edges of the sun, from their point of view. We provide the bar plot of the number of photons corresponding to the angle of emission below.

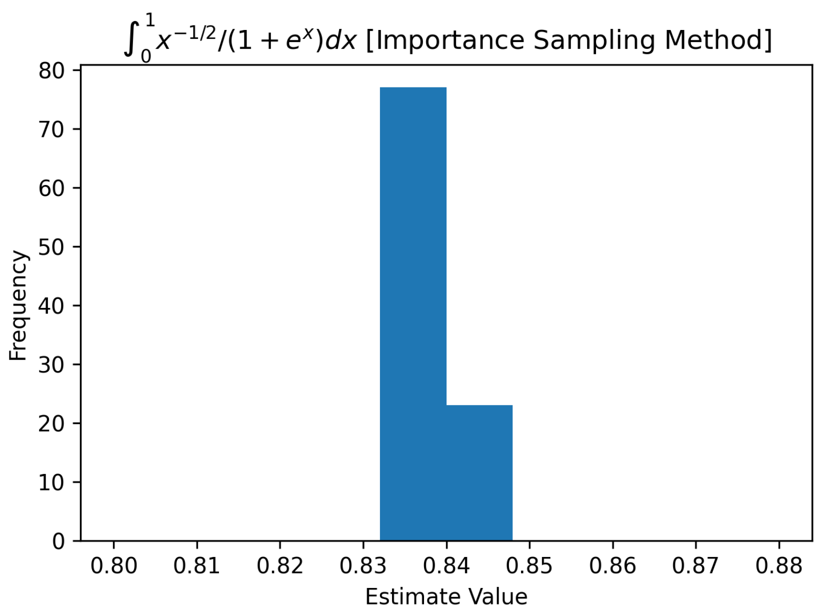



This bar plot confirms our expectation(uniform distribution), as we observe approximately the same number of photons being emitted (intensity) regardless of angle of emission in reference to the surface of the sun.

**Q3.a.**

Evaluating the given integral using the Mean Value, and Importance Sampling methods 100 times for each method, using N = 10000 sample points for each of the 100 estimations of each integral, we created 2 bar-plots consisting of the 100 estimates for the two methods. These appear below.

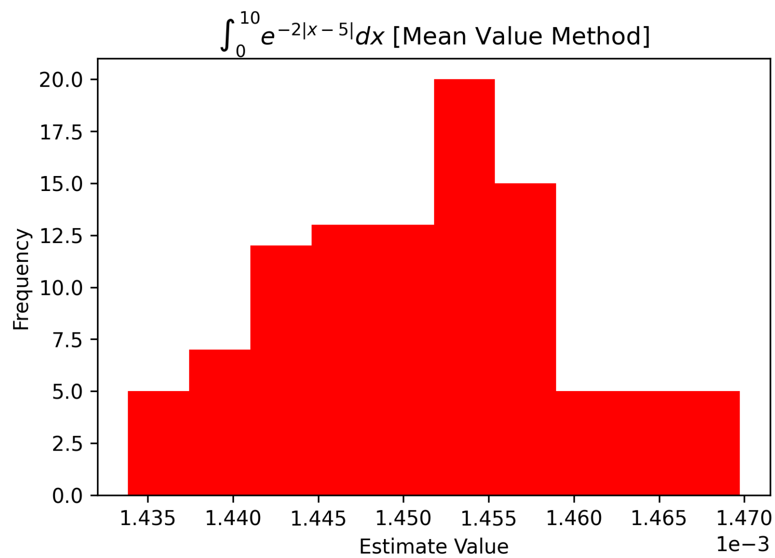


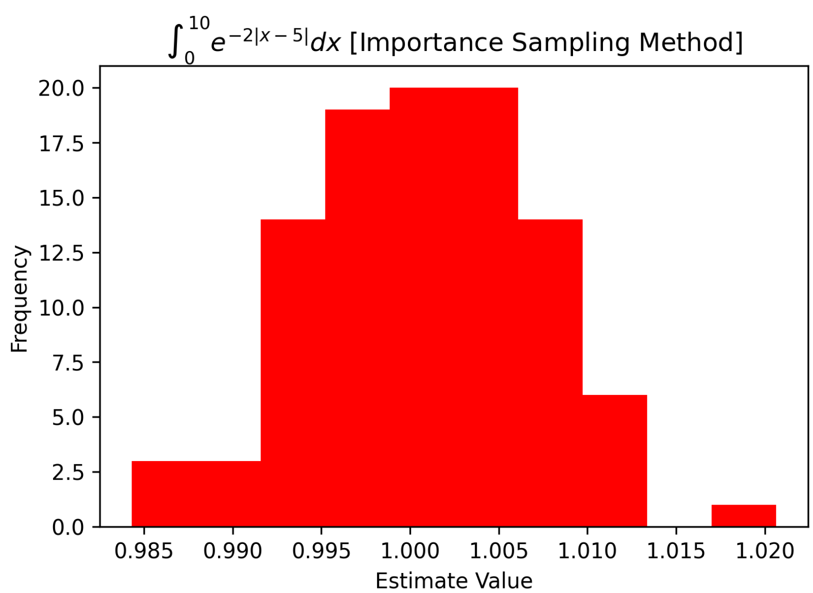



For the first integral, we see that both methods appear to result in estimates centered around the 0.84 value. The Importance Sampling method though yielded substantially more accurate results, as the 100 estimates had much smaller spread than the Mean Value method.

**Q3.b.**

We followed the exact same procedure, this time evaluating the second provided integral with the Importance sampling, and mean value methods 100 times using 10000 sample points for each of the 100 estimates. We again provide the bar-plots of these estimates for the two methods below.



This time, we observe that both methods result in the same spread amongst the estimated values, but the Mean Value Method appears to give estimates closer to 0, while the Importance Sampling Method appears to give results closer to 1. The integral in reality results in a value very close to 1. We thus observe that in the Importance Sampling Method in both cases appears to be superior to the Mean Value Method, as in the first case it provided much higher accuracy in the estimates of the integral, while in the second case it captured approximately the correct result, while the Mean Value Method's estimate of the integral was not close to the actual value.