<u>PHY407: Lab 9</u>

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Contributions:

- Q1. Nikolaos Rizos
- Q2. Brendan Halliday
- Q3. Nikolaos

Q1.a.

1) a) Our equation is: $\dfrac{\partial^2\phi}{\partial t^2} = u^2 \cdot \dfrac{\partial^2\phi}{\partial x^2}$ (1),

$\forall t>0, \quad \phi|_{x=0} = \phi|_{x=L} = 0$ and

$\forall x \in [0,L]: \phi_0(x) = \phi|_{t=0}, \ \psi_0(x) = \psi|_{t=0} = \dfrac{\partial\phi}{\partial t}\Big|_{t=0}$

We are given a trial solution:

$$\phi(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi x}{L}\right)\left(\tilde{\phi}_{0,k}\cos(\omega_k t) + \frac{\tilde{\psi}_{0,k}}{\omega_k}\sin(\omega_k t)\right)$$

$$\cdot \ \frac{\partial^2\phi}{\partial x^2} = \sum_{k=1}^{\infty} -\left(\frac{k\pi}{L}\right)^2 \sin\left(\frac{k\pi x}{L}\right)\left(\tilde{\phi}_{0,k}\cos(\omega_k t) + \frac{\tilde{\psi}_{0,k}}{\omega_k}\sin(\omega_k t)\right)$$

$$\cdot \ \frac{\partial^2\phi}{\partial t^2} = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi x}{L}\right)\left(-\tilde{\phi}_{0,k}\,\omega_k^2 \cos(\omega_k t) - \tilde{\psi}_{0,k}\cdot\omega_k \sin(\omega_k t)\right)$$

$$(1) \Rightarrow \sum_{k=1}^{\infty}\left[\sin\left(\frac{k\pi x}{L}\right)\left(-\tilde{\phi}_{0,k}\cdot\omega_k^2\cos(\omega_k t) - \tilde{\psi}_{0,k}\,\omega_k\sin(\omega_k t)\right)\right] =$$

$$= \sum_{k=1}^{\infty}\left[-\left(\frac{k \cdot u \cdot \pi}{L}\right)^2 \sin\left(\frac{k\pi x}{L}\right)\left(\tilde{\phi}_{0,k}\cos(\omega_k t) + \frac{\tilde{\psi}_{0,k}}{\omega_k}\sin(\omega_k t)\right)\right] \Rightarrow$$

$$\Rightarrow \sum_{k=1}^{\infty}\left[-\sin\left(\frac{k\pi x}{L}\right)\tilde{\phi}_{0,k}\,\omega_k^2\cos(\omega_k t) - \sin\left(\frac{k\pi x}{L}\right)\tilde{\psi}_{0,k}\,\omega_k\sin(\omega_k t)\right] =$$

$$= \sum_{k=1}^{\infty}\left[-\left(\frac{ku\pi}{L}\right)^2\sin\left(\frac{k\pi x}{L}\right)\tilde{\phi}_{0,k}\cos(\omega_k t) - \left(\frac{ku\pi}{L}\right)^2\sin\left(\frac{k\pi x}{L}\right)\cdot\frac{\tilde{\psi}_{0,k}}{\omega_k}\sin(\omega_k t)\right]$$

$$\Rightarrow \sum_{k=1}^{\infty}\left[\sin\left(\frac{k\pi x}{L}\right)\cdot\tilde{\phi}_{0,k}\cos(\omega_k t)\left(-\omega_k^2 + \left(\frac{ku\pi}{L}\right)^2\right) + \sin\left(\frac{k\pi x}{L}\right)\cdot\tilde{\psi}_{0,k}\sin(\omega_k t)\left(-\omega_k + \frac{1}{\omega_k}\left(\frac{ku\pi}{L}\right)^2\right)\right]$$

$$= 0 \Rightarrow$$

$$\Rightarrow \sum_{k=1}^{\infty}\left[\sin\left(\frac{k\pi x}{L}\right)\tilde{\phi}_{0,k}\cos(\omega_k t)\left(-\omega_k^2 + \left(\frac{ku\pi}{L}\right)^2\right) + \sin\left(\frac{k\pi x}{L}\right)\cdot\tilde{\psi}_{0,k}\sin(\omega_k t)\left(\frac{1}{\omega_k}\right)\cdot\left(-\omega_k^2 + \left(\frac{ku\pi}{L}\right)^2\right)\right]$$

$$= 0 \Rightarrow$$

$$\Rightarrow \sum_{k=1}^{\infty}\left[\left(-\omega_k^2+\left(\frac{k u \pi}{L}\right)^2\right)\cdot\left(\sin\left(\frac{k\pi x}{L}\right)\cdot\tilde{\phi}_{q k}\cos(\omega_k t) + \sin\left(\frac{k\pi x}{L}\right)\tilde{\phi}_{q k}\sin(\omega_k t)\cdot\left(\frac{1}{\omega_k}\right)\right)\right] = 0$$

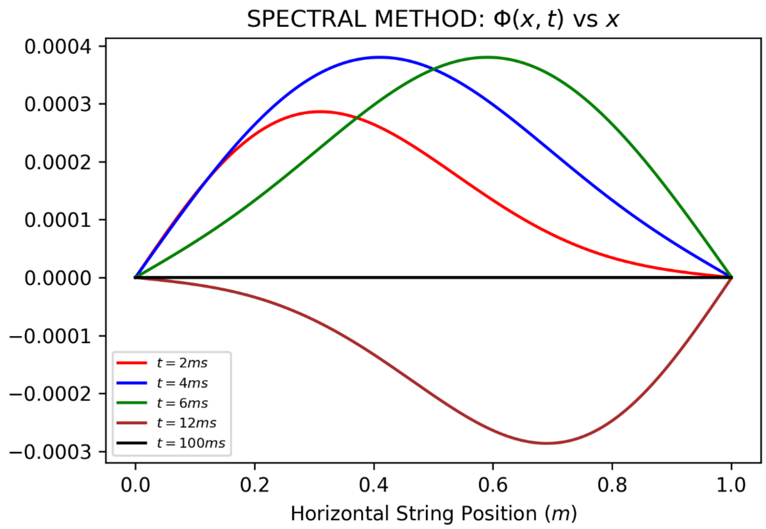
In order for the trial solution to be a valid solution for ①, it needs to satisfy ①, or, the above expression needs to be true.

That happens when: $-\omega_k^2+\left(\frac{k u \pi}{L}\right)^2=0 \Rightarrow$

$\Rightarrow \boxed{\omega_k = \frac{k u \pi}{L}}$ For this value, the trial solution is a valid solution of our equation.

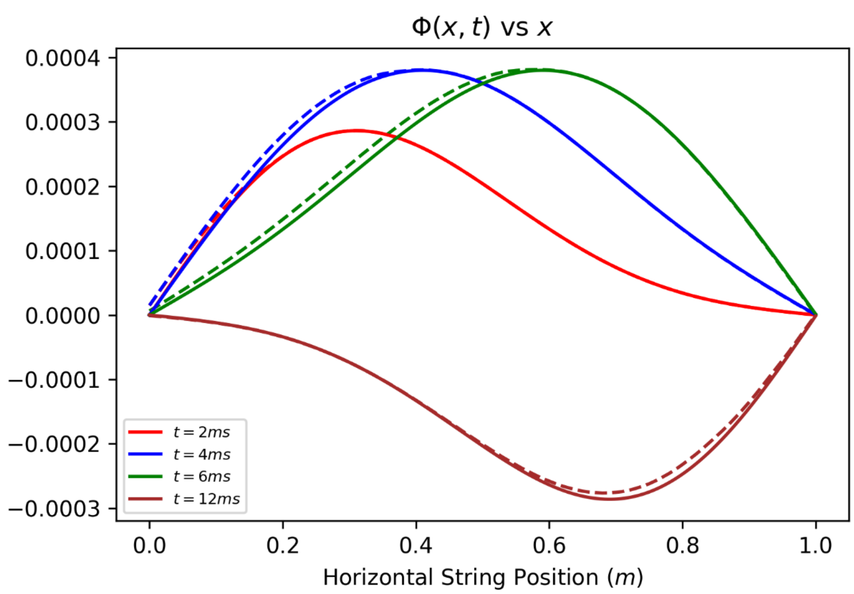
Q1.b.

Using the series solution derived above with a truncated value of N = 1000, in order to estimate the actual solution of the wave equation using the spectral method, we plot below the solution corresponding to the times t = 2, 4, 6, 12, 100ms, in one plot.

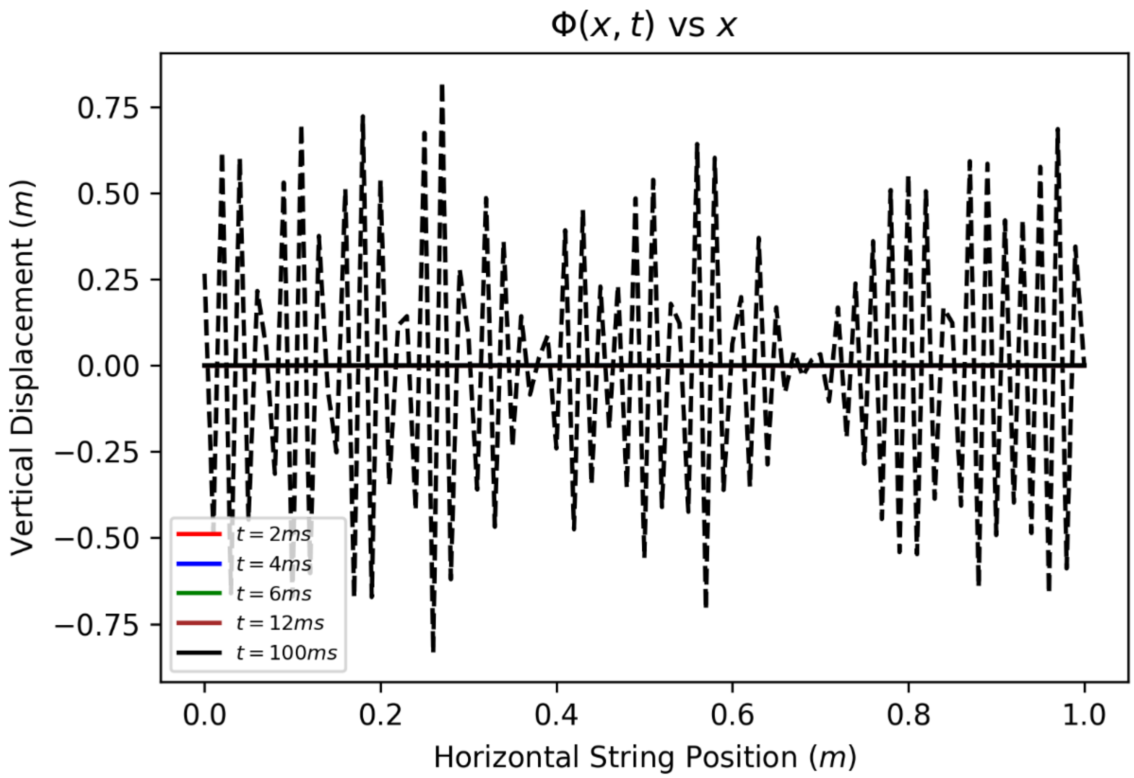


Q1.c.

We graph the above solutions along with the ones resulting from the FTCS method on the same plot, but only for small times (i.e., t = 2, 4, 6, 12ms. Dotted line is the FTCS method), again in one plot provided below.



We provide the above plot but with all the required times included (i.e., t = 2, 4, 6, 12, 100ms. Dotted curve is the FTCS method).

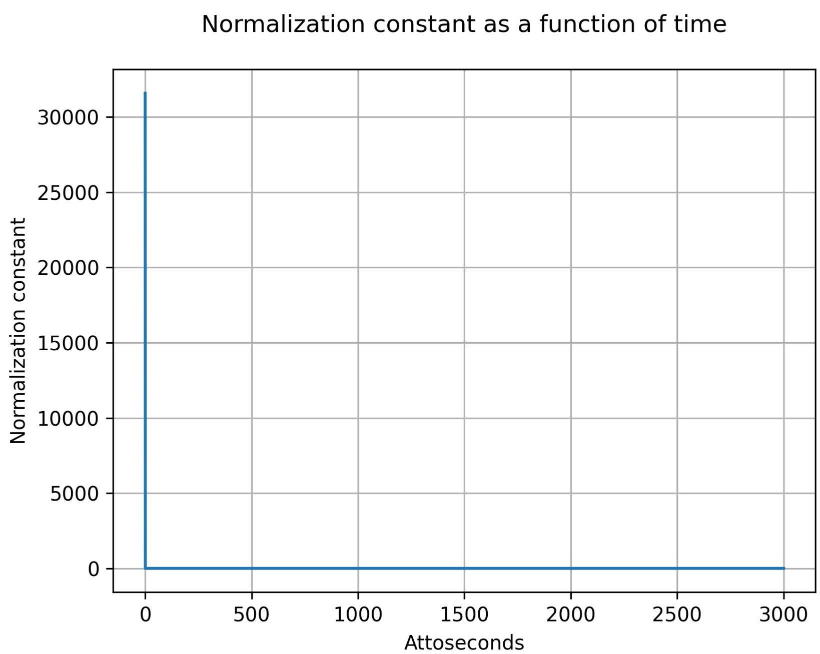


We can make 2 observations when comparing the two methods in this way. The first observation is that the Spectral method is much faster than the FTCS method. The Spectral method took about 0.008s to estimate the required solutions, while the FTCS method took about 30 seconds. The Spectral method is about 3000 times faster.

For small times, the two methods provide almost identical results (although it seems that only for t = 2ms the two solutions coincide almost perfectly on the plot). The biggest difference is shown in the last plot, where for t = 100ms the FTCS method breaks down and outputs highly inaccurate results. In contrast, in the first plot we can see that the Spectral method yielded correct results for t = 100ms. Thus, the Spectral method is not only several thousand times faster than the FTCS method, but it is more versatile as it works even for large times where the FTCS method does not.

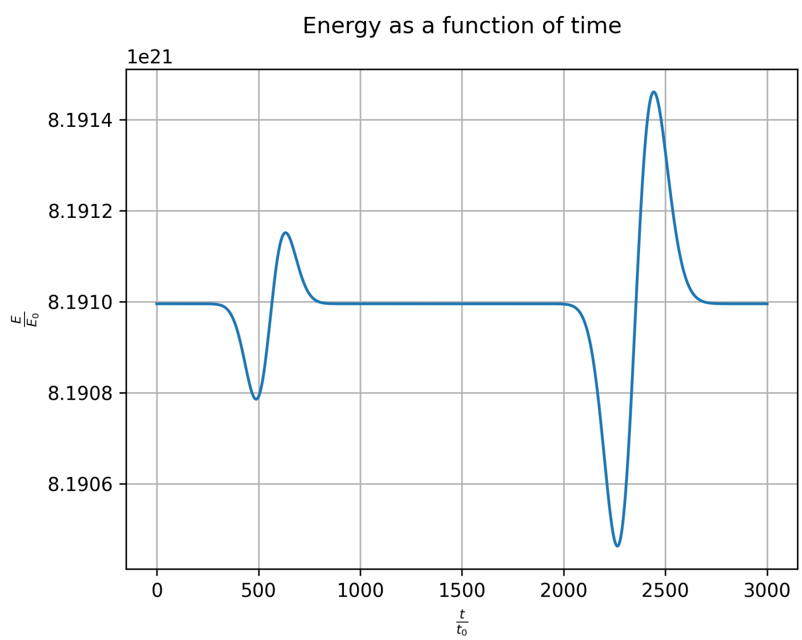
Q2.a.

Below is a graph of the normalization constant.



The C-N is shown to be stable since after normalizing the first time,the normalization constant never changes.

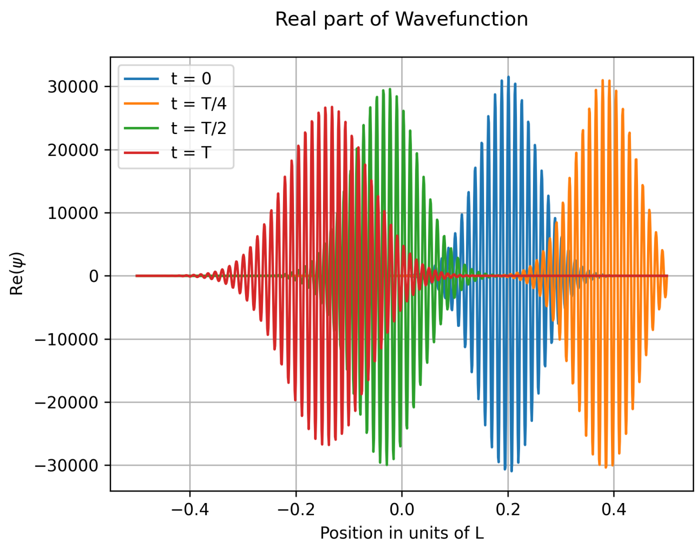
Below is a plot of the energy as a function of time:

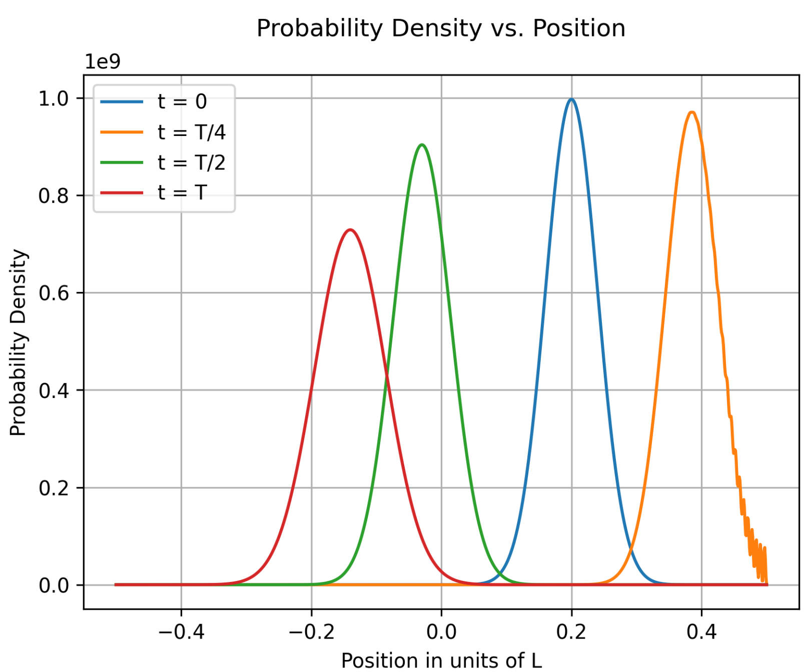


The energy is meant to be constant however it is not here. The reason for this anomaly is due to machine precision. The units should have been scaled before simulation to give a more accurate representation of energy.

Q2.b.

Below is a plot of the real part of psi at the 4 different times and the probability density function at 4 different times respectively:

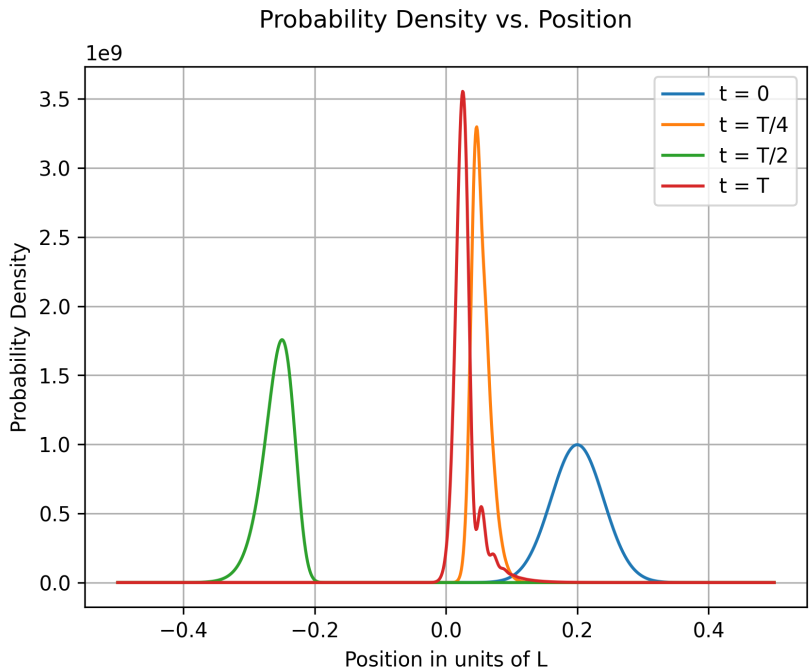




Over time, the probability density function spreads out.

<u>Q2.c.</u>

Below is the plot for the QHO for the 4 different times:



As time evolves, the wave functions become more localized in the center of the well.

Q3.a.

3) a) we wish to estimate the value of $u(x,t)$, at time $t+\Delta t$. We Taylor expand $u(x,t+\Delta t)$ about t:

- $u(x,t+\Delta t) = u(x,t) + \frac{\partial u}{\partial t}(t+\Delta t - t) + \frac{\partial^2 u}{\partial t^2}\frac{(t+\Delta t - t)^2}{2} \Rightarrow$

$$\Rightarrow \boxed{u(x,t+\Delta t) = u(x,t) + \frac{\partial u}{\partial t}(\Delta t) + \frac{\partial^2 u}{\partial t^2}\frac{(\Delta t)^2}{2}} \quad (22)$$

- the equation is: $\frac{\partial u}{\partial t} + \varepsilon \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \Rightarrow$

$$\Rightarrow \frac{\partial u}{\partial t} = -\varepsilon \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) \quad (*)$$

$$\frac{\partial^2 u}{\partial t^2} = -\varepsilon \cdot \frac{\partial^2}{\partial x \partial t}\left(\frac{u^2}{2}\right) \Rightarrow$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = -\varepsilon \cdot \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right)\right) \Rightarrow$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = -\varepsilon \cdot \frac{\partial}{\partial x}\left(u \cdot \frac{\partial u}{\partial t}\right) \Rightarrow$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = -\varepsilon \cdot \frac{\partial}{\partial x}\left(u \cdot \left(-\varepsilon \cdot \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\right)\right) \Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = \varepsilon^2 \frac{\partial}{\partial x}\left[u \cdot \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\right]} \quad (23)$$

• (2), (22), (23) $\Rightarrow$

$$\Rightarrow u(x, t+\Delta t) = u(x,t) - \Delta t \cdot \varepsilon \cdot \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) + \frac{(\Delta t)^2}{2}\varepsilon^2 \frac{\partial}{\partial x}\left[u \cdot \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\right] \quad (3)$$

• we can approximate:

$$\frac{\partial}{\partial x}(u^2) = \frac{\left(u_{i+1}^j\right)^2 - \left(u_{i-1}^j\right)^2}{2\Delta x} \quad (4)$$

and

$$\frac{\partial}{\partial x}\left(u \cdot \frac{\partial u^2}{\partial x}\right) = \frac{1}{\Delta x}\left[u\left(x+\frac{\Delta x}{2}\right) \cdot \frac{\partial}{\partial x}\left[u^2\left(x+\frac{\Delta x}{2}, t\right)\right] - u\left(x-\frac{\Delta x}{2}\right) \cdot \frac{\partial}{\partial x}\left[u^2\left(x-\frac{\Delta x}{2}, t\right)\right]\right] \quad (5)$$

(3),(4),(5) $\Rightarrow u(x, t+\Delta t) =$

$$= u(x,t) - \Delta t \cdot \varepsilon \left[\frac{\left(u_{i+1}^j\right)^2 - \left(u_{i-1}^j\right)^2}{4\Delta x}\right] + \frac{\Delta t^2}{2}\varepsilon^2 \frac{1}{2\Delta x}\left[u\left(x+\frac{\Delta x}{2}, t\right) \cdot \frac{\partial}{\partial x}\left[u^2\left(x+\frac{\Delta x}{2}, t\right)\right]\right.$$

$$\left. - u\left(x-\frac{\Delta x}{2}, t\right) \cdot \frac{\partial}{\partial x}\left[u^2\left(x-\frac{\Delta x}{2}, t\right)\right]\right] \quad (6)$$

we can approximate:

• $u\left(x+\frac{\Delta x}{2}, t\right) = \dfrac{u(x,t) + u(x+\Delta x, t)}{2} = \dfrac{u_i^j + u_{i+1}^j}{2}$

• $u\left(x-\frac{\Delta x}{2}, t\right) = \dfrac{u(x,t) + u(x-\Delta x, t)}{2} = \dfrac{u_i^j + u_{i-1}^j}{2}$

• $\dfrac{\partial}{\partial x} u^2\left(x+\frac{\Delta x}{2}, t\right) = \dfrac{\left(u_{i+1}^j\right)^2 - \left(u_i^j\right)^2}{\Delta x}$

• $\dfrac{\partial}{\partial x} u^2\left(x-\frac{\Delta x}{2}, t\right) = \dfrac{\left(u_{i-1}^j\right)^2 - \left(u_i^j\right)^2}{-\Delta x}$

substituting these approximations in ①:

$$u(x, t+\Delta t) = u(x,t) - \Delta t \cdot \varepsilon \left[\frac{(u_{i+1}^j)^2 - (u_{i-1}^j)^2}{4\Delta x} \right]$$

$$+ \frac{\Delta t^2}{2} \varepsilon^2 \cdot \frac{1}{2\Delta x} \left[\left(\frac{u_i^j + u_{i+1}^j}{2} \right) \cdot \left(\frac{(u_{i+1}^j)^2 - (u_i^j)^2}{\Delta x} \right) - \left(\frac{u_i^j + u_{i-1}^j}{2} \right) \cdot \left(\frac{(u_{i-1}^j)^2 - (u_i^j)^2}{-\Delta x} \right) \right] \Rightarrow$$

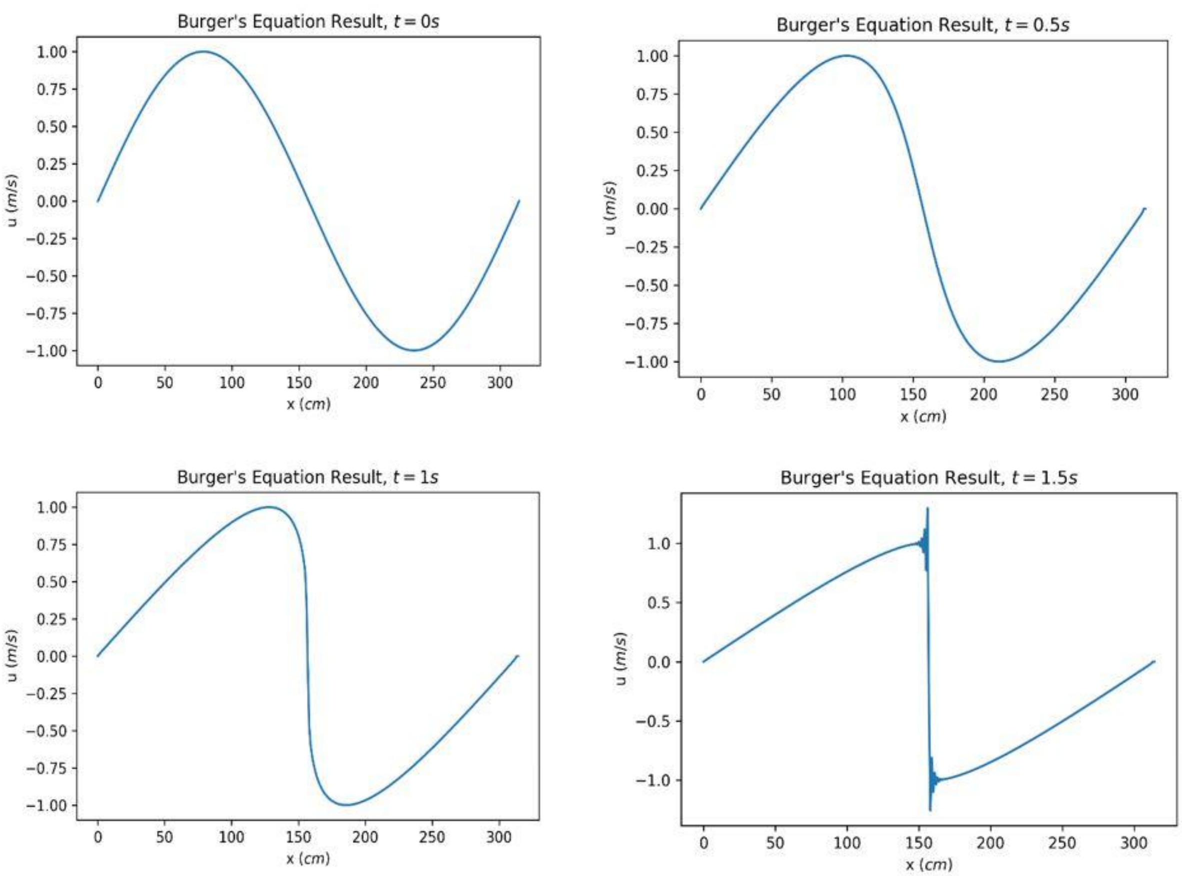
$$\Rightarrow u_i^{j+1} = u(x,t) - \frac{1}{4} \cdot \frac{\Delta t}{\Delta x} \cdot \varepsilon \left[(u_{i+1}^j)^2 - (u_{i-1}^j)^2 \right]$$

$$+ \frac{\Delta t^2}{2} \cdot \varepsilon^2 \cdot \frac{1}{4\Delta x^2} \left[(u_i^j + u_{i+1}^j)\left((u_{i+1}^j)^2 - (u_i^j)^2 \right) + (u_i^j + u_{i-1}^j)\left((u_{i-1}^j)^2 - (u_i^j)^2 \right) \right] \Rightarrow$$

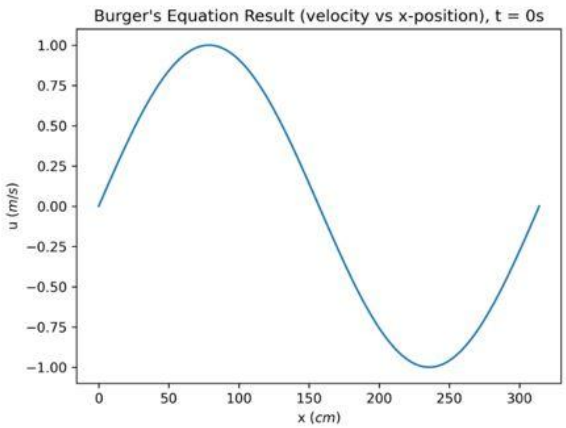
$$\Rightarrow u_i^{j+1} = u_i^j - \frac{\varepsilon}{4} \left((u_{i+1}^j)^2 - (u_{i-1}^j)^2 \right) + \frac{\varepsilon^2}{8} \left[(u_i^j + u_{i+1}^j)\left((u_{i+1}^j)^2 - (u_i^j)^2 \right) + (u_i^j + u_{i-1}^j)\left((u_{i-1}^j)^2 - (u_i^j)^2 \right) \right]$$

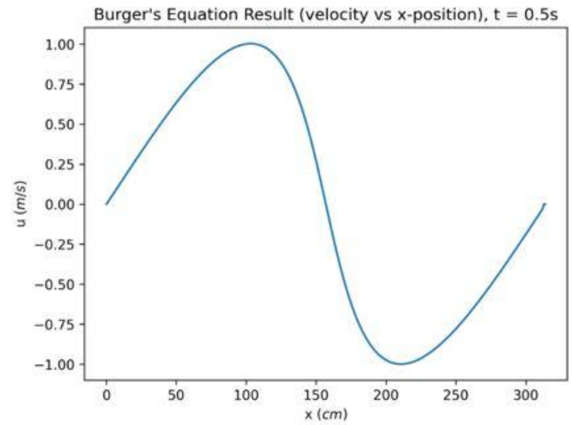
Q3.b.

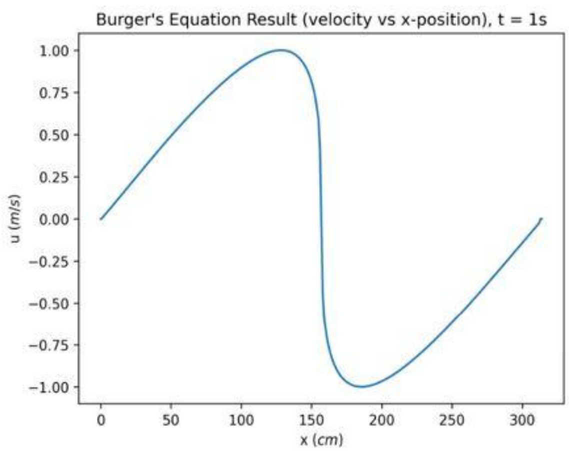
Using the above final formula in order to estimate the solution to the Burger's equation using the Lax-Wendroff method, we plot the solutions corresponding to t = 0, 0.5, 1, 1.5s below

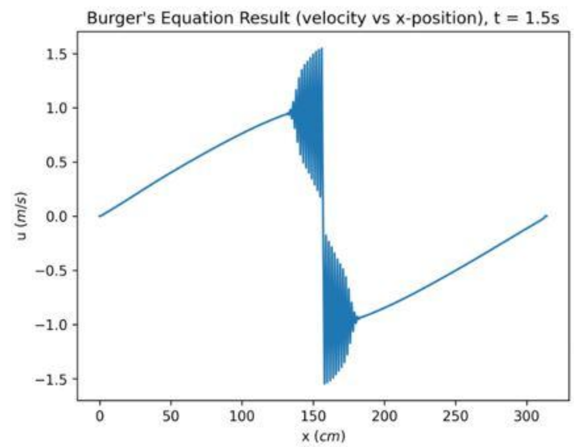


Below appear the plots of the solutions of Burger's equation using the leapfrog method corresponding to t = 0, 0.5, 1, 1.5s.









When looking at the 2 sets of plots, we can observe that the two methods yield almost identical results for t = 0, 0.5, 1s, and it seems that the solution appears to approach something resembling a sawtooth function as time increases. The main difference between the two plots is that the Lax-Wendroff method's solution for t = 1.5s does not result in as dramatic aberrations near the "discontinuities", compared to the Leapfrog method. Thus, the Lax-Wendroff method is more stable than the Leapfrog method, and also provides more accurate estimates.