SOME SELECTED ELEMENTS OF FLUID MECHANICS

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This appendix is designed to give a primer on some basic concepts in fluid mechanics that are employed in this thesis. As is the case with the companion document on tensor calculus, this appendix is meant more to give an introduction to those with no background in fluid mechanics, as well as establish a consistent notation used in this thesis. Thus many concepts will not be rigorously defined, but rather introduced in as practical a way as possible.

1. The Navier-Stokes Equation

The fundamental equation of fluid mechanics is the *Navier-Stokes equation*. Much as Maxwell's equations define electromagnetics, and the Schrödinger equation defines basic quantum mechanics, Navier-Stokes is the defining equation of viscous fluid flow. Although a derivation of Navier-Stokes (NS) is out of the scope of this appendix, several sources give excellent derivations of the equation from first principles, including an excellent Wikipedia page [1]. That having been said, in the course of describing the elements of NS, I hope to elucidate some of the physical assumptions that go into it.

1.1. Physical Interpretation of Navier-Stokes.

The incompressible NS equation describes how the vector flow field $\mathbf{u}(x, y, z, t)$ as a function of location (x, y, z) and time t depends on a number of different factors:

$$\overbrace{\underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{\text{temporal acceleration}} + \underbrace{(\mathbf{u} \cdot \nabla)\mathbf{u}}_{\text{convective acceleration}}}^{\text{inertia per unit volume}} = \underbrace{\mathbf{f}}_{\text{body forces}} - \underbrace{\frac{1}{\rho}\nabla p}_{\text{pressure}} + \underbrace{\frac{\mu}{\rho}\nabla^2 \mathbf{u}}_{\text{diffusion of momentum}} \tag{1}$$

where f are body forces (in units of acceleration), ρ is the density of the fluid, μ is the viscosity of the fluid, and p is the pressure.

Although the Navier-Stokes equation is somewhat long, by breaking it down into its constituent pieces, we can gain an inutitive understanding of the underlying physics. From a very basic perspective, it can be thought of as a restatement of Newton's second equation, $\frac{d\mathbf{v}}{dt} = \frac{\mathbf{F}}{m}$. The left hand equation in general describes the changes in inertia per unit volume, and is sometimes written compactly as the material derivative $\frac{D}{Dt}\mathbf{u}$. The right hand side is a statment of the internal and external forces affecting each parcel of fluid that are causing those changes.

More specifically, the first term on the left hand side $\frac{\partial \mathbf{u}}{\partial t}$ is the change in the velocity flow field at a given location over time. The second term on the left hand side describes not how inertia changes with time, but rather how it changes as a particle moves spatially, or *convective acceleration.* As we will see, in low Reynold's number flow, inertia is quickly dissipated as a particle moves in space, and this term can be neglected. On the right hand side of the equation, we see the effects of different forces on each parcel of fluid. $\mathbf{f}$ is a statement of the external or body forces. The other two terms are internal forces, referring to forces directed normal to each fluid parcel, or pressure p, and those directed tangentially to each parcel, or shearing forces, given by $\frac{\mu}{\rho}\nabla^2\mathbf{u}$.

The general setup of Navier-Stokes is that one solves for the flow field $\mathbf{u}(x, y, z; t)$ and pressure $p(x, y, z; t)$ given a specific set of forces $\mathbf{f}$; boundary conditions $\mathbf{u}(x_0, y_0, z_0; t)$; and initial conditions $\mathbf{u}(x, y, z; t_0)$[1] [2].

1.2. **Vectorial Bookkeeping with Navier Stokes.** As with other vector equations, the incompressible NS equation compactly summarizes multiple equations. As an exercise in vector notation, we will look at each term in Navier-Stokes and make sure that it contains the correct number of components.

On the left hand side, the first term is the product of a scalar $\frac{\partial}{\partial t}$, with a vector $\mathbf{u}$, and thus yields a vector. The second term can be thought of as the dot product of two vectors $\mathbf{u} \cdot \nabla$, which yields a scalar, multiplied by a vecotr $\mathbf{u}$.

On the right hand side, the body force term is itself a vector $\mathbf{f}$. The pressure term consists of a scalar $1/\rho$ multiplied by the gradient of a scalar ∇p, which is a vector. Finally, the last term consists of a scalar μ/ρ, multiplied by the dot product of two vectors $\nabla \cdot \nabla$, multiplied by a vector $\mathbf{u}$, also yielding a vector. In Cartesian coordinates, then, we can expand NS as three separate equations:

$$u_t + (uu_x + vu_y + wu_z) = f - \frac{1}{\rho}p_x + \frac{\mu}{\rho}(u_{xx} + u_{yy} + u_{zz}) \tag{2}$$

$$v_t + (uv_x + vv_y + wv_z) = g - \frac{1}{\rho}p_y + \frac{\mu}{\rho}(v_{xx} + v_{yy} + v_{zz}) \tag{3}$$

$$w_t + (uw_x + vw_y + ww_z) = h - \frac{1}{\rho}p_z + \frac{\mu}{\rho}(w_{xx} + w_{yy} + w_{zz}) \tag{4}$$

[1]Although as in other differential equations, boundary conditions can either be Dirichlet, specifying the value of the function itself; Neumann, specifying the value of the first derivative of the function; mixed, a combination of Neumann and Dirichlet that varies spatially; or Robin, where boundaries at a given location are described as a linear combination of Neumann and Dirichlet.

where $\mathbf{u} = (u, v, w)$ and $\mathbf{f} = (f, g, h)$, and where for compactness we have now defined $\frac{\partial u}{\partial x} \equiv u_x$, $\frac{\partial u}{\partial y} = u_y$, etc. (That is, subscripts denote partial derivatives as opposed to vector components in Eqs. 2 - 4.

2. Non-dimensionalization

One common procedure in fluid mechanics is to rewrite an equation that has units into one that has no units. This procedure is known as *non-dimensionalization.* Similar to the bookkeeping in vector notation, dimensional analysis allows us to ensure that all our quantities have the correct units after being multiplied together. In doing so, we are also left with non-dimensional parameters that have meaningful physical interpretations.

The basic idea of non-dimensionalization is to (1) determine the units of each quantity, and (2) divide through by a characteristic quantity that is related to the geometry of the actual problem. For example, the left-most term in of Eq. 2 has the units of u, [m/s], divided by t, [s]. The entire equation, then, has units of [m/s^2]. We begin by redefining $u^* = u/U$, where U is now a characteristic velocity that describes our problem. U could be, for example, the speed of a moving boundary in the setup of the problem. The time component can also be non-dimensionalized as $t^* = t/T$. In that case, the first term becomes:

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial (U\mathbf{u}^*)}{\partial (t^* T)}$$
$$\frac{\partial \mathbf{u}}{\partial t} = \frac{U}{T}\frac{\partial \mathbf{u}^*}{\partial t^*}$$

Parameter	Definition	Units
u^*	u/U	[m/s]
t^*	t/T	[s]
∇^*	$L\nabla$	[1/m]
p^*	$pL/\mu U$	[kg/m s2]
f^*	f/f_0	[m/s2]

Table 1. Non-dimensionalized parameters.

where now $\frac{\partial \mathbf{u}^*}{\partial t^*}$ is unitless and the coefficient U/T gives us a characteristic acceleration. As shown in table 1, each of the terms in Navier-Stokes can be rewritten. $x^* = x/L$ terms appear due to ∇. L might be the diameter of a pipe in Poiseuille flow, or the size of a sphere moving at terminal velocity under the influence of gravity. Finally, the pressure term be non-dimensionalized in multiple ways. The specific choice we took is in anticipation of the fact that viscous forces will dominate inertia.

The Navier Stokes equation then becomes:

$$\frac{U}{T}\frac{\partial \mathbf{u}^*}{\partial t^*} + \frac{U^2}{L}\mathbf{u}^* \cdot \nabla \mathbf{u}^* = -\frac{\mu U}{L^2 \rho}\nabla p^* + \frac{\mu U}{L^2 \rho}\nabla^2 \mathbf{u}^* + f_0 \mathbf{f}^*$$

and dividing through by $\mu/(LU\rho)$, we get

$$\frac{LU^2\rho}{\mu T}\frac{\partial \mathbf{u}^*}{\partial t^*} + \frac{LU\rho}{\mu}\mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\nabla^* p^* + \nabla^{*2}\mathbf{u}^* + \frac{LU\rho}{\mu}\frac{f_0 L}{U^2}\mathbf{f}^* \tag{5}$$

$$\beta \frac{\partial \mathbf{u}^*}{\partial t^*} + \mathrm{Re}\, \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\nabla^* p^* + \nabla^{*2}\mathbf{u}^* + \frac{\mathrm{Re}}{\mathrm{Fr}^2}\mathbf{f}^* \tag{6}$$

where $\beta = LU^2\rho/\mu T$, $Re = \rho U L/\mu$ and $Fr = U/\sqrt{f_0 L}$. These three quantities, the acceleration parameter, the Reynolds number, and the Froude number, number respectively, arise naturally from the non-dimensionalization process. From here, we now drop the * subscript.

3. Stokes equation

In the types of flows we will be considering, we can usually make several important approximations from Navier-Stokes. First, we typically consider flows with no body forces, i.e. $\mathbf{f} = 0$. Additionally, we will consider steady-state flows, meaning that that $\frac{\partial \mathbf{u}}{\partial t} = 0$. Finally, and most importantly, we consider flows that are in the low Reynolds number regime, $Re \ll 1$. The low Re regime means that, according to Eq 6, the convective term $\mathbf{u} \cdot \nabla \mathbf{u}$ can be neglected. Making these approximations leaves us with the *Stokes equation*, here written in its dimensional form:

$$\mu \nabla^2 \mathbf{u} - \nabla p = 0 \tag{7}$$

To justify briefly low Reynolds number in the case of ciliary flow, we consider some typical values of ciliary flow. Assuming $\rho = 1[g/cm^3]$, $\mu = 1[cP]$, $U = 1[mm/s]$, and $L = 10[\mu m]$, where L is the length of a cilium, we get a $Re = 0.01$.

In comparison to Navier-Stokes, the Stokes equation has several important interpretations. First, because there is no time dependence, the Stokes equation is said to be instantaneous. That is, once a solution has been found, it holds over the entire domain at that time, instaneously. The Stokes equation can be thought of as an dynamic equilibrium equation [2]. In the Stokes equation, it is assumed that the system comes to equilibrium much faster than any relevant flow dynamics are occuring.[2]. What is left in the Stokes equation is a balance of forces. There are no external forces or time dependent acceleration, and thus the Stokes equation expresses the

[2]It is often said that viscous dissipation dominates inertial forces

balance of viscous shearing and pressure that is needed to maintain a steady flow in a given geometry with specified boundary conditions.

A second important property of the Stokes equation is linearity. That is, given two solutions to the equation, $\mathbf{u}_1$, and $\mathbf{u}_2$, we can add the solutions together to get a valid solution $\mathbf{u}_3 = \mathbf{u}_2 + \mathbf{u}_1$. This property, known as the superposition principle, is not true about the more general incompressible Navier-Stokes equation due to the convective inertial term $\mathbf{u} \cdot \nabla \mathbf{u}$. The propery of linearity is important because it opens up a number of numerical and theoretical methods to solve the Stokes equation, making analytic and computational simulations much simpler than the full blown Navier-Stokes equation.

As a simple example in Matlab, we consider the solution to Poiseuille flow. We consider a single cross-section between two parallel plates, separated by a height h in the y direction. The pressure is held at either end of the tube (of length ℓ such that there is a pressure gradient $-\Delta p/\ell$ set up in the x direction. We know that resulting parabolic flow profile is directed purely along the x axis but varies parabolically in y:

$$u(x,y,z) = \frac{4u_{\max}}{h^2}(h-y)y$$
$$v(x,y,z) = w(x,y,z) = 0$$

To see this in Matlab, we first generate this flow field:

```
h=1; l=5; umax=1; npts=20; x=NaN(npts+2,npts+2); y=x;
dx=l/(npts-1); dy=h/(npts-1); visc=1;
[x(2:end-1,2:end-1),y(2:end-1,2:end-1)]= ...
       meshgrid(linspace(0,l,npts),linspace(0,h,npts));
u=4*umax/h^2*(h-y).*y;v=x*0;
quiver(x,y,u,v);
```

we can verify the pressure gradient generated by Stokes equation by taking the Laplacian of our vector field:

```
dpx=visc*del2(u,dx,dy);
dpy=visc*del2(v,dx,dy);
quiver(x,y,dpx,dpy);
```

Where we see now that the pressure gradient required to sustain flow is equal to the one we imposed at the boundaries.

4. Continuity Equation and Incompressibility

One of the elements the Navier-Stokes equation is the incompressibility equation. The incompressibility equation comes from the more general *continuity equation*, which is in general a statement of the conservation of some physical quantity. To understand the continuity equation, we consider a small parcel of fluid, typically drawn in two-dimensions as a small square,.

The basic idea of the continuity equation is that if we consider all the mass (for example) in a small parcel, we can account for that mass by two processes: (1) the mass q can be created or destroyed within the parcel $\dot{q}_{\text{source}}$, $\dot{q}_{\text{sink}}$ (where $\dot{q}$ refers to the rate of change over time), or (2) the mass can leave the parcel, $\dot{q}_{\text{flux}}$.

$$\dot{q} = -\dot{q}_{\text{flux}} + \dot{q}_{\text{source}} - \dot{q}_{\text{sink}} \tag{8}$$

If we consider the flux term specifically, the total amount of mass leaving a volume is the net flux through a surface.

$$\dot{q}_{\text{flux}} = \oint_S \mathbf{j} \cdot d\mathbf{S}$$

where $\mathbf{j}$ is the flux of mass, and $\mathbf{S}$ is the surface. The cyclic integral indicates that the surface of the integral is closed. For a fluid of constant density, the flux is just given by the density, multiplied by the velocity vector at that location

$$\mathbf{j} = \rho \mathbf{u}$$

Now typically, we will not have any sources or sinks, so $\dot{q}_{\text{source}} = \dot{q}_{\text{sink}} = 0$. Moreover, if we apply the divergence theorem to Eq 9, and consider the definition of the density in a small unit volume:

$$\oint_S \mathbf{j} \cdot d\mathbf{S} = \int\int\int_V \nabla \cdot \mathbf{j} dV \tag{9}$$

$$\int\int\int \rho dV = q \tag{10}$$

$$\tag{11}$$

and plugging in Eq 8,

$$\frac{\partial \rho}{\partial t}\rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

and if the fluid is incompressible, i.e. ρ is constant, then the final equation simplifies down to:

$$\nabla \cdot \mathbf{u} = 0 \tag{12}$$

And from Eq 12 we see that the criteria of having an incompressible fluid is equivalent to having a divergence-free (sometimes called solenoidal) vector field.

To verify that our previous Poiseuille flow profile is indeed divergence-free, we can simply compute the divergence in Matlab:

```
[ux,uy]=gradient(u,dx,dy);
[vx,vy]=gradient(v,dx,dy);
divergence = ux+vy;
imagesc(divergence);
```

5. The Streamfunction

One important vector identity states that the divergence of the curl of a function is zero, that is:

$$\nabla \cdot (\nabla \times \mathbf{\Psi}) = 0$$

Knowing that our fluid is incompressible, and thus that the divergence is equal to zero $\nabla \cdot \mathbf{u} = 0$, this motivates us to write our vector field $\mathbf{u}$ as a function of a more fundamental field.

$$\mathbf{u} = \nabla \times \mathbf{\Psi}$$

where $\mathbf{\Psi}$, also known as the streamfunction, is a different vector field that determines $\mathbf{u}$ by the curl operation. The reason to write our vector field as the curl of another function is several fold. First, it guarantees that our vector field is incompressible. Secondly, it turns out that while we need to specify 2-3 functions for $\mathbf{u}$ in 2 and 3 dimensions, we need to specify one less dimensions for the stream function. Finally, in order to solve Stokes equation, we need to solve for $\mathbf{u}$ and p simultaneously. It also ends up that the equivalent equation for the streamfunction is significantly simpler to solve in many cases.

For example, in two-dimensions, the streamfunction simplifies to $\Psi = \langle 0, 0, \psi \rangle$, and the vector flow field is given by

$$u = \frac{\partial \psi}{\partial y}$$
$$v = -\frac{\partial \psi}{\partial x}$$

Moreover, the Stokes equation for the streamfunction is just:

$$\nabla^4 \psi = 0$$

which is known as the biharmonic equation. It is often easier to solve this biharmonic equation than the coupled differential equations that can directly be written from the Stokes equation.

To continue our example in Matlab, we write down the streamfunction for Poiseuille flow:

```
psi=4*umax/h^2*(h*y.^2/2-y.^3/3);
```

We first verify qualitatively that our flow profile resembles Poiseuille flow:

```
[psix,psiy]=gradient(psi,dx,dy);
u1=psiy;
v1=-psix;
quiver(x,y,u1,v1);
```

We can verify more explicitly that the streamfunction satisfies the Stokes equation by calculating $\nabla^4\psi$:

```
d4=del2(del2(psi,dx,dy),dx,dy);
imagesc(d4)
```

Where this quantity is zero within machine tolerance.

Finally, the contours of the streamfunction give the paths of particles seeded in the flow field:

```
contour(x,y,psi)
```

6. Derivative Quantities of the Flow Field

Once a flow field has been determined, either theoretically by solving the Stokes equation, or experimentally by measuring the flow field, several important quantities can be calculated. Here, we will focus on just two: one that quantifies shearing forces in the fluid (viscous stress tensor), one that quantifies viscous dissipation of energy (dissipation function).

6.1. **Newtonian Fluids and Viscous Stress Tensor.** One important classification of a fluid is how it resists flow. In general, the frictional forces arising within a fluid can be thought of as due to the fact that areas of the fluid are flowing at different speeds with respect to one another. For example, in the limit that a fluid reaches plug flow, for example, where the whole fluid moves as a block, there is no internal frictional dissipation. The fluid appears more like a rigid object, with frictional forces only at the interface with the boundary.

The definition of a Newtonian fluid is that the frictional shearing forces τ are proportional to the strain rate, i.e. difference in velocity between different fluid layers. The coefficient of proportionality, or "frictional coefficient," is the viscosity of the fluid. More formally, this is typically written in one dimension as:

$$\tau = \mu\frac{du}{dz}$$

where μ is the viscosity, and $\frac{du}{dz}$ is the velocity gradient in the z-direction. Fluid flow is described by three vector components, however, and each of these components can change in three spatial dimensions. Thus, to properly describe frictional shearing forces in a fluid, we need an object that describes all these possible combinations. That construct is known as the *viscous stress tensor*, a 3x3 tensor that describes the shearing forces in each of three dimensions with respect to each of three velocity components.

More specifically, as we described in the accompanying tensor tutorial, a tensor can be formed by the dyadic product of two vectors. Here we consider the dyadic product of the del operator with the velocity field:

$$\nabla \mathbf{u} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix}$$

For an incompressible fluid, the *rate-of-strain tensor* $\underline{\underline{S}}$, which describes the local rate of deformation, is then given by

$$\underline{\underline{S}} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \tag{13}$$

and because the fluid is Newtonian, the viscous stress tensor τ is given by

$$\underline{\underline{\tau}} = \frac{\mu}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \tag{14}$$

As an example, we again consider Poiseuille flow:

```
[ux,uy]=gradient(u,dx,dy);
[vx,vy]=gradient(v,dx,dy);
ros = zeros([size(ux),2,2]);
ros(:,:,1,1)=ux;
ros(:,:,2,1)=1/2*(uy+vx);
ros(:,:,1,2)=1/2*(uy+vx);
ros(:,:,2,2)=uy;
figure(1);imagesc(squeeze(ros(:,:,2,1)));
figure(2);imagesc(squeeze(ros(:,:,1,1)));
```

We see that the rate of strain tensor vanishes along the diagonals, but off diagonal elements are present. We note that the rate of strain tensor is actually a tensor field, with a 2x2 tensor specified at every location in space. As such, the Matlab array size needed to contain the information is 4-dimensional.

6.2. **Dissipation Function.** As with classical mechanics, once we have tallied up the frictional forces in a system, we can then calculate the energy dissipated due to friction. To start, we note that the *traction* $\mathbf{f}$, or force per unit area, is given by

$$\mathbf{f} = \hat{\mathbf{n}} \cdot \underline{\underline{\sigma}}$$

where $\hat{\mathbf{n}}$ is the normal vector on the surface in the parcel, and $\underline{\underline{\sigma}}$ is the Cauchy stress tensor, which in an incompressible fluid is simly equal to:

$$\underline{\underline{\sigma}} = -p\mathbb{1} + \underline{\underline{\tau}}$$

where $\mathbb{1}$ is the identity matrix, and $\underline{\underline{\tau}}$ is defined in Eq. 14. To obtain the rate of energy dissipation, we consider that in classical mechanics, the power is given by $P = \mathbf{F} \cdot \mathbf{u}$ where $\mathbf{F}$ is the force acting on an object. To obtain the power of dissipation in a fluid parcel, then, we integrate the traction over the surface of parcel, such that

$$P = \oint_S \mathbf{f} \cdot \mathbf{u} \, dS \tag{15}$$

$$P = \oint_S \hat{\mathbf{n}} \cdot \underline{\underline{\sigma}} \cdot \mathbf{u} \, dS \tag{16}$$

applying the divergence theorem, Eq. 16 can be written as:

$$P = \int_V \nabla \cdot \underline{\underline{\sigma}} \cdot \mathbf{u} \, dV$$

and using a vector / tensor identity:

$$P = \int_V \underline{\underline{\sigma}} : \nabla \mathbf{u} \, dV + \int_V \mathbf{u} \cdot \nabla \cdot \underline{\underline{\sigma}} \, dV$$

It turns out that the interpretation of the right term is the energy due to potential energy changes as well as convective flow, and thus these are conserved if one considers the entirety of a closed system. On the other hand, the left term, which is often denoted with ϵ, is a measure of the viscous dissipation within the system. The argument of the integral is often termed the *dissipation function*, denoted as ϕ, and is a measure of viscous dissipation per unit volume.

$$\epsilon \equiv \int_V \underline{\underline{\sigma}} : \nabla \mathbf{u} \, dV$$

$$\phi \equiv \underline{\underline{\sigma}} : \nabla \mathbf{u}$$

and it can be shown that, given an incompressible fluid with $\nabla \cdot \mathbf{u} = 0$, the definition of ϕ is equivalent to:

$$\phi = \mu \underline{\underline{S}} : \underline{\underline{S}}$$

where $\underline{\underline{S}}$ is defined in 13. To return once again to our example of Poiseuille flow, we can now calculate the dissipation function from our rate of strain tensor[3].

```
phi=2*visc*sum(dot(ros,ros,4),3);
imagesc(phi);
```

And we can integrate over the entire area to get our energy dissipation rate per unit length:

[3]Of note, for experimental data, our flow fields may not actually be incompressible. As such, the full expression for the dissipation function is slightly more complicated, but can still be calculated in full from the vectorial flow field

```
Energy=nansum(phi(:))
```

References

[1] Derivation of the navier–stokes equations.
[2] C. Pozrikidis. *Stresses, Equation of Motion, and Vorticity Transport Equation*, pages x, 675 p. Oxford University Press, New York, 1997.