General Relativity and Cosmology: Assignment 1

Due date: Midnight Monday 13th September

Assignments are to be submitted by the due date; penalties apply for late submission. All questions on the assignment should be posted to the discussion forum. I will not be available for consultation on the assignment questions on the due date or the day before. You may use standard integrals or packages such as *Mathematica* to solve any integrals you encounter; sources must be listed. Students are encouraged to discuss the questions with each other, and on the on-line discussion forum. However, submitted work must reflect an individual's effort. Assignments may be handwritten or typed but must be legible; no marks will be awarded if mathematical derivations cannot be followed.

Question 1:

In the lectures we considered a rocket undergoing a constant acceleration in the *x*-direction in flat Minkowski spacetime. The motion is governed by the relationships

$$
\begin{aligned}
\mathbf{u}\cdot\mathbf{u} &= \eta_{\alpha\beta}\; u^\alpha u^\beta &= -1\\
\mathbf{u}\cdot\mathbf{a} &= \eta_{\alpha\beta}\; u^\alpha a^\beta &= 0\\
\mathbf{a}\cdot\mathbf{a} &= \eta_{\alpha\beta}\; a^\alpha a^\beta &= a^2
\end{aligned}
$$

where a is a constant. In a coordinate system that remains at rest, the rocket begins from the coordinates $(t,x) = (0,0)$ at the time of $\tau = 0$ on the clock of the rocketeer (its *proper time*). In the following, consider spatial motion in the *x*-direction only.

a: Explicitly solve for the motion of the rocket, showing that the components of the position, 4-velocity and 4-acceleration are given by:

$$
\begin{aligned}
x^\alpha(\tau) &= a^{-1} & &(sinh(a\tau), cosh(a\tau)-1)\\
u^\alpha(\tau) &= & &(cosh(a\tau), sinh(a\tau))\\
a^\alpha(\tau) &= a & &(sinh(a\tau), cosh(a\tau))
\end{aligned}
$$

b: As the rocket travels, an observer at rest at the origin $(x=0)$ fires photons in the positive *x*-direction which are detected on the rocket. Show that the relationship between the time the photon is emitted from the origin, t_e, and the proper time on the rocket when the photon is received, τ_r, is:

$$
\tau_r = -\frac{1}{a}\ln\left(1 - at_e\right)
$$

c: Show that the energy of an exchanged photon detected by the observer on the rocket, E_r, compared to that emitted by observer at rest at the origin, E_e, is given by:

$$
\frac{E_r}{E_e} = \exp(-a\tau_r) = 1 - at_e
$$

d: With the use of sketches, briefly describe the view of observer at rest at the origin as seen by those on the rocket. Comment how this reveals the existence of the *Rindler Horizon.*

Question 2:

A spherically symmetric spacetime can be described by the Schwarzschild metric:

$$
ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)
$$

where m is the mass of the spherical object curving the spacetime. The non-zero values of the Christoffel symbols for this spacetime are given in the lecture notes and Hartle's textbook.

a: Using the Lagrangian approach outlined in the lectures, derive the equations of motions in each of the coordinates (t,r,θ,ϕ) for objects moving in this spacetime (use either L or K formulations).

b: Using the results derived in **a:**, determine the non-zero Christoffel symbols for the Schwarzschild spacetime.

The Christoffel symbols can also be determined directly from the metric through:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(g_{\beta\delta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta} \right)$$

where the comma refers to a partial derivative.

c: Noting that the Schwarzschild metric is diagonal, explicitly determine the non-zero values of the Christoffel symbols using the above expression.

Question 3:
Now consider motion in the Schwarzschild metric described in **Question 2:** Using the results for the equations of motion derived above (and given in the lecture notes and textbook), and considering motion only in the equatorial plane, so $\theta=\pi/2$ and $u^{\theta}=0$, show that:

a: For an object (either massive or a photon) at an initial location of

$$x^{\alpha}_{o}(\tau = 0) = (t_o, r_o, \theta_o, \phi_o) = \left(0, R, \frac{\pi}{2}, 0\right)$$

with initial spatial motion only in the ϕ-direction, with $u^{\phi}(\tau=0) = C$. Show that the initial 4-velocity is given by:

$$u^{\alpha}_{o}(\tau = 0) = \left(\sqrt{\frac{(R^2C^2 - \mathbf{u}\cdot\mathbf{u})R}{(R-2m)}}, 0, 0, C \right)$$

Here **u.u** is -1 for a massive object and 0 for a photon. In the following, you will be required to numerically integrate the equations of motion in the Schwarzschild metric. You may use any numerical approach (*matlab, python, Mathematica, bespoke integrator)* but your code must be included as part of your solution.

Consider a Schwarzschild spacetime with $m=1$, and an initial radius of $R=10$.

c: Integrate the path of a massive object from $\tau=0$ to $\tau=1000$ for $C=(10\sqrt{7})^{-1}$. Plot the resultant t, r and ϕ as a function of the proper time. Demonstrate that the result is a circular orbit (hint: plot $x=r\cos(\phi)$ and $y=r\sin(\phi)$). Also show that the normalization of the 4-velocity holds along the world-line of the massive object. Repeat the integration for $C=1.1\,(10\sqrt{7})^{-1}$. Briefly comment on the form of the orbit and how it differs from orbits in Newtonian gravity.

d: Repeat **c:** for a massless particle, but for the affine parameter from $\lambda=0$ to $\lambda=10$, and $C=0.5$, 1 and 2. Briefly comment on the relationship between the initial components of the 4-velocity and affine parameter (hint: consider the photon motion in cartesian coordinates).