General Relativity and Cosmology: Assignment 1
Due date: Midnight Monday 13th September

Assignments are to be submitted by the due date; penalties apply for late submission. All questions on the assignment should be posted to the discussion forum. I will not be available for consultation on the assignment questions on the due date or the day before. You may use standard integrals or packages such as *Mathematica* to solve any integrals you encounter; sources must be listed. Students are encouraged to discuss the questions with each other, and on the on-line discussion forum. However, submitted work must reflect an individual's effort. Assignments may be handwritten or typed but must be legible; no marks will be awarded if mathematical derivations cannot be followed.

Question 1:
In the lectures we considered a rocket undergoing a constant acceleration in the *x*-direction in flat Minkowski spacetime. The motion is governed by the relationships

$$\begin{aligned} \mathbf{u}\cdot\mathbf{u} &= \eta_{\alpha\beta}\ u^\alpha u^\beta &= -1 \\ \mathbf{u}\cdot\mathbf{a} &= \eta_{\alpha\beta}\ u^\alpha a^\beta &= 0 \\ \mathbf{a}\cdot\mathbf{a} &= \eta_{\alpha\beta}\ a^\alpha a^\beta &= a^2 \end{aligned}$$

where a is a constant. In a coordinate system that remains at rest, the rocket begins from the coordinates $(t,x) = (0,0)$ at the time of $\tau = 0$ on the clock of the rocketeer (its *proper time*). In the following, consider spatial motion in the *x*-direction only.

a: Explicitly solve for the motion of the rocket, showing that the components of the position, 4-velocity and 4-acceleration are given by:

$$\begin{aligned} x^\alpha(\tau) &= a^{-1} & (sinh(a\tau), cosh(a\tau) - 1) \\ u^\alpha(\tau) &= & (cosh(a\tau), sinh(a\tau)) \\ a^\alpha(\tau) &= a & (sinh(a\tau), cosh(a\tau)) \end{aligned}$$

b: As the rocket travels, an observer at rest at the origin $(x=0)$ fires photons in the positive *x*-direction which are detected on the rocket. Show that the relationship between the time the photon is emitted from the origin, t_e, and the proper time on the rocket when the photon is received, τ_r, is:

$$\tau_r = -\frac{1}{a}\ln(1 - at_e)$$

c: Show that the energy of an exchanged photon detected by the observer on the rocket, E_r, compared to that emitted by observer at rest at the origin, E_e, is given by:

$$\frac{E_r}{E_e} = \exp(-a\tau_r) = 1 - at_e$$

d: With the use of sketches, briefly describe the view of observer at rest at the origin as seen by those on the rocket. Comment how this reveals the existence of the *Rindler Horizon.*

Question 2:
A spherically symmetric spacetime can be described by the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)$$

where m is the mass of the spherical object curving the spacetime. The non-zero values of the Christoffel symbols for this spacetime are given in the lecture notes and Hartle's textbook.

a: Using the Lagrangian approach outlined in the lectures, derive the equations of motions in each of the coordinates (t,r,θ,ϕ) for objects moving in this spacetime (use either L or K formulations).

b: Using the results derived in **a:**, determine the non-zero Christoffel symbols for the Schwarzschild spacetime.

The Christoffel symbols can also be determined directly from the metric through:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(g_{\beta\delta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta} \right)$$

where the comma refers to a partial derivative.

c: Noting that the Schwarzschild metric is diagonal, explicitly determine the non-zero values of the Christoffel symbols using the above expression.

Question 3:

Now consider motion in the Schwarzschild metric described in **Question 2:** Using the results for the equations of motion derived above (and given in the lecture notes and textbook), and considering motion only in the equatorial plane, so $\theta=\pi/2$ and $u^{\theta}=0$, show that:

a: For an object (either massive or a photon) at an initial location of

$$x^{\alpha}_{o}(\tau = 0) = (t_o, r_o, \theta_o, \phi_o) = \left(0, R, \frac{\pi}{2}, 0\right)$$

with initial spatial motion only in the ϕ-direction, with $u^{\phi}(\tau=0) = C$. Show that the initial 4-velocity is given by:

$$u^{\alpha}_{o}(\tau = 0) = \left(\sqrt{\frac{(R^2C^2 - \mathbf{u}\cdot\mathbf{u})R}{(R-2m)}}, 0, 0, C \right)$$

Here **u.u** is -1 for a massive object and 0 for a photon. In the following, you will be required to numerically integrate the equations of motion in the Schwarzschild metric. You may use any numerical approach (*matlab, python, Mathematica, bespoke integrator)* but your code must be included as part of your solution.

Consider a Schwarzschild spacetime with $m=1$, and an initial radius of $R=10$.

c: Integrate the path of a massive object from $\tau=0$ to $\tau=1000$ for $C=(10\sqrt{7})^{-1}$. Plot the resultant t, r and ϕ as a function of the proper time. Demonstrate that the result is a circular orbit (hint: plot $x=r\cos(\phi)$ and $y=r\sin(\phi)$). Also show that the normalization of the 4-velocity holds along the world-line of the massive object. Repeat the integration for $C=1.1\,(10\sqrt{7})^{-1}$. Briefly comment on the form of the orbit and how it differs from orbits in Newtonian gravity.

d: Repeat **c:** for a massless particle, but for the affine parameter from $\lambda=0$ to $\lambda=10$, and $C=0.5$, 1 and 2. Briefly comment on the relationship between the initial components of the 4-velocity and affine parameter (hint: consider the photon motion in cartesian coordinates).

PHYS4123 GR Assignment 1

SID: 480344342

September 13, 2021

1

1.a

$$
\begin{aligned}
-1 &= \eta_{\alpha\beta} u^\alpha u^\beta \\
0 &= \eta_{\alpha\beta} u^\alpha a^\beta \\
a^2 &= \eta_{\alpha\beta} a^\alpha a^\beta
\end{aligned}
$$

Assuming spatial motion only occurs in x^1 gives:

$$
\begin{aligned}
-1 &= -u^0 u^0 + u^1 u^1 \\
0 &= -u_0 a_0 + u_1 a_1 \\
a^2 &= -a^0 a^0 + a^1 a^1
\end{aligned}
$$

Then:

$$u^0 u^0 = u^1 u^1 + 1 \tag{1}$$

$$a_0 = \frac{u_1}{u_0} a_1 \tag{2}$$

$$a^1 a^1 = a^0 a^0 + a^2 \tag{3}$$

Substituting (2) into (3):

$$
\begin{aligned}
a^1 a^1 &= a^2 \frac{1}{1 - u^1 u^1 / u^0 u^0} \\
a^0 a^0 &= a^2 \frac{1}{u^0 u^0 / u^1 u^1 - 1}
\end{aligned}
$$

Then using (1):

$$
\begin{aligned}
a^1 a^1 &= a^2 (1 + u^1 u^1) = \left(\frac{du^1}{d\tau}\right)^2 \\
a^0 a^0 &= a^2 (u^0 u^0 - 1) = \left(\frac{du^0}{d\tau}\right)^2
\end{aligned}
$$

Integrating:

$$
\begin{aligned}
\tau &= \pm \int \frac{1}{|a|\sqrt{1 + u^1 u^1}} du^1 = \pm \frac{1}{|a|} \sinh^{-1}(u^1) + A \\
\tau &= \pm \int \frac{1}{|a|\sqrt{u^0 u^0 - 1}} du^1 = \pm \frac{1}{|a|} \cosh^{-1}(u^0) + B
\end{aligned}
$$

Assuming $u^0 > 0$. Since the rocket begins at rest, $(u^0, u^1) = (1, 0)$ at $\tau = 0$ and$C = 0$ and $B = 0$. Also, letting $\mp|a| = a$ (such that the sign of a is the direction of a^1):

$$(u^0, u^1) = (\cosh(a\tau), \sinh(a\tau))$$

Differentiating gives:

$$(a^0, a^1) = a(\sinh(a\tau), \cosh(a\tau))$$

Whereas integrating gives:

$$(x^0, x^1) = \frac{1}{a}(\sinh(a\tau) + C, \cosh(a\tau) + D)$$

Where the initial conditions $(x^0, x^1) = (0, 0)$ at $\tau = 0$ constrain $C = 0$ and $D = -1$.

1.b

As shown in figure 1, a photon launched at t_e by a stationary observer will intersect with the world line of the rocket (as long as t_e is less than some critical value). The photon's world line is a straight line with gradient 1 (a speed of c) intersecting $x = 0$ at t_e. The rocket's world line is the curve $(x, t) = x^\alpha(\tau)$, derived in part a. The intersection of these two world lines occurs when:

$$\begin{aligned}
x &= t - t_e \\
\implies \frac{1}{a}\cosh(a\tau_r) - \frac{1}{a} &= \frac{1}{a}\sinh(a\tau_r) - t_e \\
\implies 1 - at_e &= \cosh(a\tau_r) - \sinh(a\tau_r) \\
&= e^{-a\tau_r} \\
\implies -a\tau_r &= \ln(1 - at_e) \\
\implies \tau_r &= -\frac{1}{a}\ln(1 - at_e)
\end{aligned}$$

1.c

The relativistic Doppler shift with an angle of 0 between a photon and a target is:

$$E_r = E_e\sqrt{\frac{1-v}{1+v}},$$

where E_e is the energy of the photon in the rest frame of the source moving at $-v$ relative to the rocket, which receives a photon of energy E_r. The rocket's velocity at the time of receiving the photon is:

$$v(\tau_r) = \frac{dx}{dt} = \frac{dx}{d\tau}\cdot\frac{d\tau}{dt} = \frac{u^x(\tau_r)}{u^t(\tau_r)} = \tanh(a\tau_r)$$

Therefore:

$$\begin{aligned}
\frac{E_r}{E_e} &= \sqrt{\frac{1 - \tanh(a\tau_r)}{1 + \tanh(a\tau_r)}} \\
&= \sqrt{\frac{1 - \frac{\exp(2a\tau_r)-1}{\exp(2a\tau_r)+1}}{1 + \frac{\exp(2a\tau_r)-1}{\exp(2a\tau_r)+1}}} \\
&= \sqrt{\frac{\exp(2a\tau_r) + 1 - \exp(2a\tau_r) + 1}{\exp(2a\tau_r) + 1 + \exp(2a\tau_r) - 1}} \\
&= e^{-a\tau_r}.
\end{aligned}$$

From the previous part, this also means $\frac{E_r}{E_e} = 1 - at_e$.

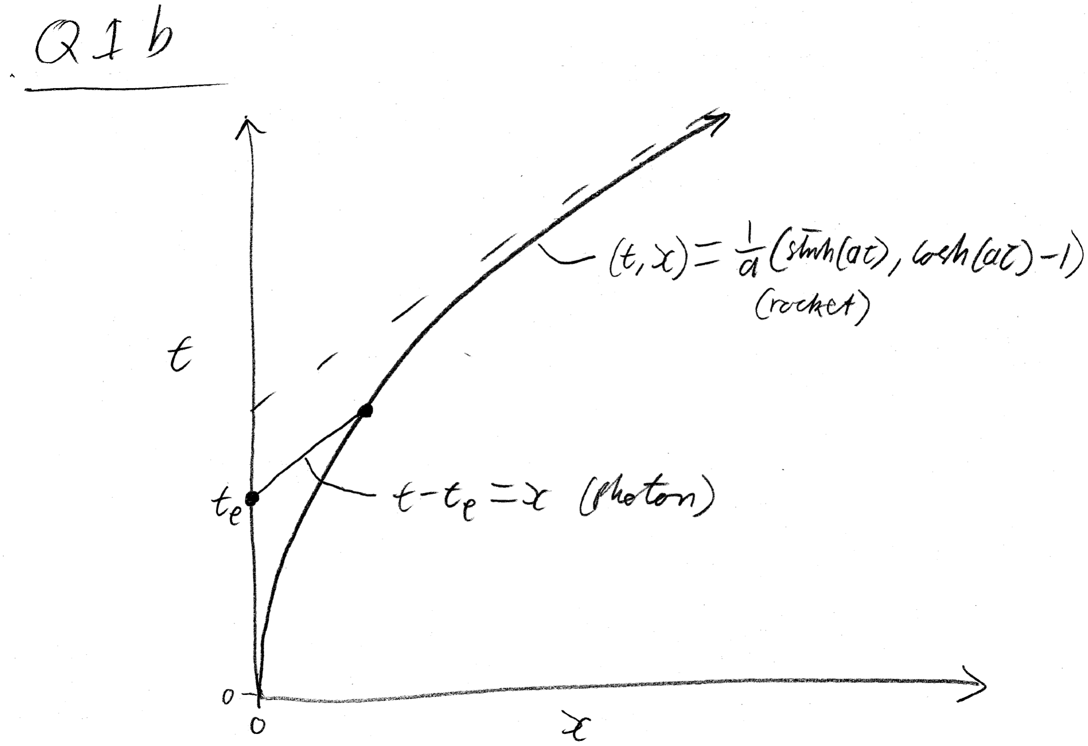


Figure 1: **A photon launched at an accelerating rocket.** The time at which the rocket receives the photon can be calculated from the intersection of their world lines.

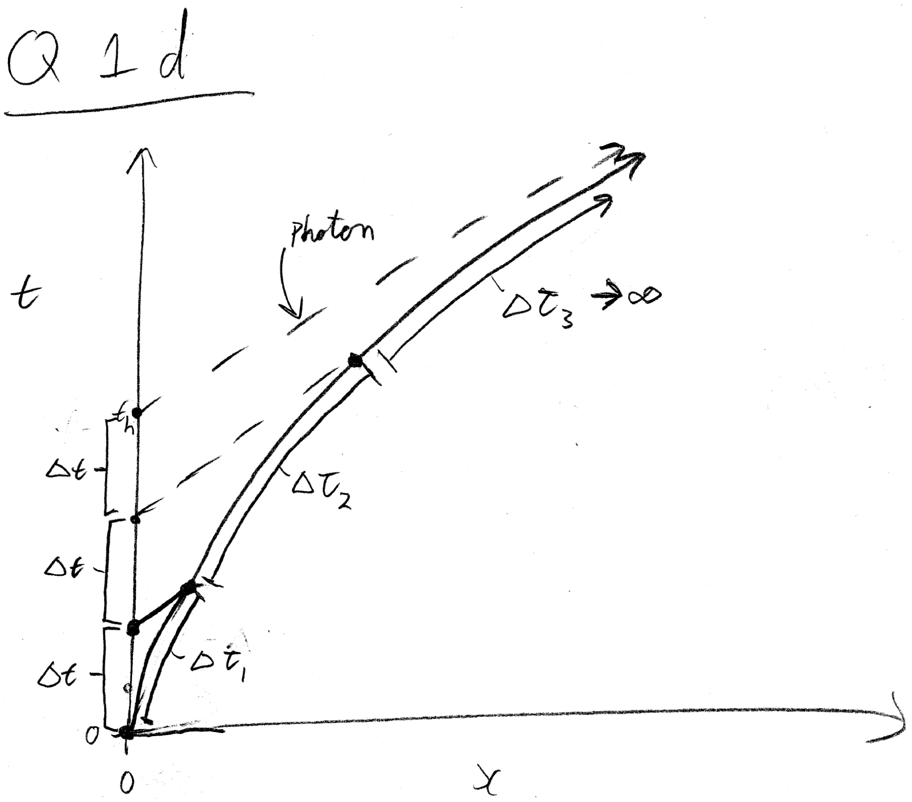


Figure 2: **Successive photons launched at an accelerating rocket**. The length of intervals between received photons along the rocket's world line increases to infinity for photons launched at equally spaced intervals by a non-accelerating observer.

1.d

The velocity of the rocket, measured by the stationary observer, asymptotically approaches the speed of light, as indicated by the equations derived in part a. This means the gradient of the rocket's world line approaches unity, shown in figure 1. If the rocket observes the stationary observer, what it sees will slow down increasingly as it accelerates. For example, figure 2 shows the world-lines of a scenario in which the stationary observer sends a photon to the moving rocket at regular intervals of Δt (in the stationary observers own time). While they have a small acceleration, the proper time of the moving rocket at which it receives the photon (related to the length of its world line) will be slightly larger than the time at which the stationary observer emits the photon. This discrepancy between $\Delta\tau_1$ incorporates both the travel time of the photon and the acceleration of the rocket, since and this proper time is also larger than the time at which the photon arrives at the rocket. At the second increment, $2\Delta t$, another photon is emitted by the stationary observer. The proper time at which it arrives at the rocket is again greater than the coordinate time, and moreso. Hence the time between equally spaced events in the stationary observers frame is increased when viewed by the rocket, so the rocket's view of the origin is slowed-down as it accelerates. Finally, the third photon emitted by the stationary observer has a world-line that is asymptotic to world-line of the rocket. The rocket never receives this photon; the rockets view of the stationary observer becomes 'frozen' as it approaches this critical time, t_h (the third interval of proper time, $\Delta\tau_3$, is infinite, assuming the rocket never ceases accelerating). This represents the Rindler horizon; no signals emitted by the observer later t_h can be received by the rocket. The Rindler horizon is clear from the equation in part c, which shows the proper time at which emitted photons are received asymptotes to ∞ as $t_e \to \frac{1}{a} = t_h$. There is also a point behind the origin (the intersection of the light line passing through the original at t_h) that the rocket cannot view past, although the rocket will see points closer than this spatial horizon (such as the origin) approach it asymptotically. The image of the origin will also appear red-shifted as the rocket gains velocity, and fainter as it gains distance.

2

2.a

The Lagrangian equivalent K for the Schwarzschild metric is:

$$\begin{aligned}K &= g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \\ &= -\left(1-\frac{2m}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2m}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2(\theta)\left(\frac{d\phi}{d\tau}\right)^2.\end{aligned}$$

Then:

$$\frac{\partial K}{\partial\left(dx^\alpha/d\tau\right)} = 2\left[-\left(1-\frac{2m}{r}\right)\frac{dt}{d\tau}, \left(1-\frac{2m}{r}\right)^{-1}\frac{dr}{d\tau}, r^2\frac{d\theta}{d\tau}, r^2\sin^2(\theta)\frac{d\phi}{d\tau}\right],$$

and:

$$\frac{\partial K}{\partial x^\alpha} = 2\left[0, \frac{1}{2}\frac{\partial K}{\partial r}, r^2\sin(\theta)\cos(\theta)\left(\frac{d\phi}{d\tau}\right)^2, 0\right],$$

where:

$$\frac{1}{2}\frac{\partial K}{\partial r} = -\frac{m}{r^2}\left(\frac{dt}{d\tau}\right)^2 - \frac{m}{(r-2m)^2}\left(\frac{dr}{d\tau}\right)^2 + r\left(\frac{d\theta}{d\tau}\right)^2 + r\sin^2(\theta)\left(\frac{d\phi}{d\tau}\right)^2.$$

Using the Euler-Lagrange equation:

$$\frac{d}{d\tau}\frac{\partial K}{\partial(dx^\alpha/d\tau)} = \frac{\partial K}{\partial x^\alpha},$$

gives:

$$
\begin{aligned}
\frac{d}{d\tau}\left[\left(1-\frac{2m}{r}\right)\frac{dt}{d\tau}\right] &= 0\\
\frac{d}{d\tau}\left[\left(1-\frac{2m}{r}\right)^{-1}\frac{dr}{d\tau}\right] &= -\frac{m}{r^2}\left(\frac{dt}{d\tau}\right)^2-\frac{m}{(r-2m)^2}\left(\frac{dr}{d\tau}\right)^2+r\left(\frac{d\theta}{d\tau}\right)^2+r\sin^2(\theta)\left(\frac{d\phi}{d\tau}\right)^2\\
\frac{d}{d\tau}\left[r^2\frac{d\theta}{d\tau}\right] &= r^2\sin(\theta)\cos(\theta)\left(\frac{d\phi}{d\tau}\right)^2\\
\frac{d}{d\tau}\left[r^2 sin^2(\theta)\frac{d\phi}{d\tau}\right] &= 0
\end{aligned}
$$

2.b

Expanding and rearranging the equations of motion above:

$$
\begin{aligned}
0 &= \left(1-\frac{2m}{r}\right)\frac{d^2t}{d\tau^2}+\frac{dt}{d\tau}\frac{2m}{r^2}\frac{dr}{d\tau}\\
\frac{d^2r}{d\tau^2}\frac{r}{r-2m}-\left(\frac{dr}{dt}\right)^2\frac{2m}{(r-2m)^2} &= -\frac{m}{r^2}\left(\frac{dt}{d\tau}\right)^2-\frac{m}{(r-2m)^2}\left(\frac{dr}{d\tau}\right)^2+r\left(\frac{d\theta}{d\tau}\right)^2+r\sin^2(\theta)\left(\frac{d\phi}{d\tau}\right)^2\\
r^2\frac{d^2\theta}{d\tau^2}+2r\frac{dr}{d\tau}\frac{d\theta}{d\tau} &= r^2\sin(\theta)\cos(\theta)\left(\frac{d\phi}{d\tau}\right)^2\\
0 &= r^2\sin^2(\theta)\frac{d^2\phi}{d\tau^2}+2r\sin^2(\theta)\frac{d\phi}{d\tau}\frac{dr}{d\tau}+2r^2\cos(\theta)\sin(\theta)\frac{d\theta}{d\tau}\frac{d\phi}{d\tau}
\end{aligned}
$$

Then:

$$
\begin{aligned}
\frac{d^2t}{d\tau^2} &= -\frac{2m}{r^2}\left(1-\frac{2m}{r}\right)^{-1}\frac{dt}{d\tau}\frac{dr}{d\tau}\\
\frac{d^2r}{d\tau^2} &= -\frac{m}{r^2}\left(1-\frac{2m}{r}\right)\left(\frac{dt}{d\tau}\right)^2+\frac{m}{r^2\left(1-\frac{2m}{r}\right)}\left(\frac{dr}{d\tau}\right)^2+(r-2m)\left(\frac{d\theta}{d\tau}\right)^2+(r-2m)\sin^2(\theta)\left(\frac{d\phi}{d\tau}\right)^2\\
\frac{d^2\theta}{d\tau^2} &= \sin(\theta)\cos(\theta)\left(\frac{d\phi}{d\tau}\right)^2-\frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau}\\
\frac{d^2\phi}{d\tau^2} &= -\frac{2}{r}\frac{d\phi}{d\tau}\frac{dr}{d\tau}-2\frac{\cos(\theta)}{\sin(\theta)}\frac{d\theta}{d\tau}\frac{d\phi}{d\tau}
\end{aligned}
$$

The general geodesic equation is:

$$
\frac{d^2x^\alpha}{d\tau^2}=-\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}
$$

Comparing the rearranged equations of motion to the general geodesic equation gives:

$$
\begin{aligned}
\Gamma^t_{rt} &= \Gamma^t_{tr} = \left(1-\frac{2m}{r}\right)^{-1}\frac{m}{r^2}\\
\Gamma^r_{tt} &= \left(1-\frac{2m}{r}\right)\frac{m}{r^2}\\
\Gamma^r_{rr} &= -\frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1}\\
\Gamma^r_{\theta\theta} &= -(r-2m)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^r_{\phi\phi} &= -(r-2m)\sin^2(\theta)\\
\Gamma^\theta_{\theta r} &= \Gamma^\theta_{r\theta} = \frac{1}{r}\\
\Gamma^\theta_{\phi\phi} &= -\sin(\theta)\cos(\theta)\\
\Gamma^\phi_{\phi r} &= \Gamma^\phi_{r\phi} = \frac{1}{r}\\
\Gamma^\phi_{\theta\phi} &= \Gamma^\phi_{\phi\theta} = \frac{\cos(\theta)}{\sin(\theta)}
\end{aligned}
$$

Note that for symbols with two unique lower indices the coefficients of the equations of motion are halved (one half to each ordering of the indices/derivatives, to ensure the symbols are symmetric).

2.c

The metric is:

$$g_{\alpha\beta} = \begin{bmatrix} -\left(1-\frac{2m}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2m}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{bmatrix}$$

Since the metric is diagonal ($g^{\alpha\delta} = 0$ when $\alpha \neq \delta$), the relationship between the metric and the Christoffel symbols can be simplified to:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\alpha}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}),$$

since all other terms are zero. Also, the lower indices can be interchanged since Christoffel symbols are symmetric.

I $\boldsymbol{\Gamma^{t}_{\beta\gamma}}$

Then for $\alpha = t$, $g_{\beta\gamma,t} = 0$ and:

$$\begin{aligned} \Gamma^{t}_{\beta\gamma} &= \frac{1}{2} g^{tt}(g_{t\beta,\gamma} + g_{t\gamma,\beta}) \\ &= \frac{1}{2} g^{tt} \begin{bmatrix} 0 & -\frac{2m}{r^2} & 0 & 0 \\ -\frac{2m}{r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \implies \Gamma^{t}_{rt} = \Gamma^{t}_{tr} &= \left(1 - \frac{2m}{r}\right)^{-1} \frac{m}{r^2} \end{aligned}$$

where all other symbols are 0.

II $\boldsymbol{\Gamma^{r}_{\beta\gamma}}$

If $\alpha = r$:

$$g_{\beta\gamma,r} = \begin{bmatrix} -\frac{2m}{r^2} & 0 & 0 & 0 \\ 0 & -\frac{2m}{(r-2m)^2} & 0 & 0 \\ 0 & 0 & 2r & 0 \\ 0 & 0 & 0 & 2r\sin^2(\theta) \end{bmatrix}$$

And:

$$g_{r\beta,\gamma} + g_{r\gamma,\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{4m}{(r-2m)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\implies g_{r\beta,\gamma} + g_{r\gamma,\beta} - g_{\beta\gamma,r} = \begin{bmatrix} \frac{2m}{r^2} & 0 & 0 & 0 \\ 0 & -\frac{2m}{(r-2m)^2} & 0 & 0 \\ 0 & 0 & -2r & 0 \\ 0 & 0 & 0 & -2r\sin^2(\theta) \end{bmatrix}$$

$$\implies \Gamma^r_{tt} = \left(1 - \frac{2m}{r}\right)\frac{m}{r^2}$$

$$\Gamma^r_{rr} = -\left(1 - \frac{2m}{r}\right)\frac{m}{(r-2m)^2} = -\frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1}$$

$$\Gamma^r_{\theta\theta} = -\left(1 - \frac{2m}{r}\right) r = -(r-2m)$$

$$\Gamma^r_{\phi\phi} = -\left(1 - \frac{2m}{r}\right) - 2r\sin^2(\theta) = -(r-2m)\sin^2(\theta)$$

III $\Gamma^{\theta}_{\beta\gamma}$

If $\alpha = \theta$:

$$g_{\beta\gamma,\theta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2r^2\sin(\theta)\cos(\theta) \end{bmatrix}$$

And:

$$g_{\theta\beta,\gamma} + g_{\theta\gamma,\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2r & 0 \\ 0 & 2r & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\implies g_{\theta\beta,\gamma} + g_{\theta\gamma,\beta} - g_{\beta\gamma,\theta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2r & 0 \\ 0 & 2r & 0 & 0 \\ 0 & 0 & 0 & -2r^2\sin(\theta)\cos(\theta) \end{bmatrix}$$

$$\implies \Gamma^{\theta}_{\theta r} = \Gamma^{\theta}_{r\theta} = \frac{1}{2}r^{-2}2r = \frac{1}{r}$$

$$\Gamma^{\theta}_{\phi\phi} = -\frac{1}{2}r^{-2}2r^2\sin(\theta)\cos(\theta) = -\sin(\theta)\cos(\theta)$$

IV $\Gamma^{\theta}_{\beta\gamma}$

Finally, if $\alpha = \phi$ then $g_{\beta\gamma,\phi} = 0$ and:

$$g_{\phi\beta,\gamma} + g_{\phi\gamma,\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2r\sin^2(\theta) \\ 0 & 0 & 0 & 2r^2\sin(\theta)\cos(\theta) \\ 0 & 2r\sin^2(\theta) & 2r^2\sin(\theta)\cos(\theta) & 0 \end{bmatrix}$$

$$\implies \Gamma^{\phi}_{\phi r} = \Gamma^{\phi}_{r\phi} = \frac{1}{2}r^{-2}\sin^{-2}(\theta)\cdot 2r\sin^2(\theta) = \frac{1}{r}$$

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{1}{2}r^{-2}\sin^{-2}(\theta)\cdot 2r^2\sin(\theta)\cos(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

```julia
begin
    import Pkg
    Pkg.activate(mktempdir())
    Pkg.add(url="https://github.com/brendanjohnharris/NonstationaryProcesses.jl")

    Pkg.add("DifferentialEquations")
    using NonstationaryProcesses, DifferentialEquations, Plots
    plotlyjs(); fourseas!(); # Plots initialisation
end
```

Click here to view in a web browser

Question 3

The equations of motion in the Schwarzschild metric (question 2) are:

$$a^t = -\frac{2m}{r}\left(1-\frac{2m}{r}\right)^{-1} u^t u^r,$$

$$a^r = -\frac{m}{r^2}\left(1-\frac{2m}{r}\right)\left(u^t\right)^2 + \frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1}\left(u^r\right)^2 + (r-2m)\left(u^\phi\right)^2,$$

$$a^\phi = -\frac{2}{r}u^\phi u^r,$$

where θ has been set to $\frac{\pi}{2}$ for motion in the equatorial plane.

a)

Given:

$$x_0^\alpha = (0, R, \frac{\pi}{2}, 0),$$

and that:

$$u_0^\alpha = (u_0^t, 0, 0, C),$$

the initial condition for u^t can be calculated from the normalisation of the four-velocity. .

The metric is:

$$g_{\alpha\beta} = \begin{bmatrix} -\left(1-\frac{2m}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2m}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{bmatrix},$$

and the norm of the four-velocity is:

$$\mathbf{u}\cdot\mathbf{u} = g_{\alpha\beta}u^\alpha u^\beta,$$

Substituting the initial conditions:

$$\mathbf{u}\cdot\mathbf{u} = -\left(1-\frac{2m}{R}\right)(u_0^t)^2 + R^2(u_0^\phi)^2$$

$$\implies \mathbf{u}\cdot\mathbf{u} = -\left(1-\frac{2m}{R}\right)(u_0^t)^2 + R^2C^2$$

$$\implies \left(1-\frac{2m}{R}\right)(u_0^t)^2 = R^2C^2 - \mathbf{u}\cdot\mathbf{u}$$

$$\implies (u_0^t)^2 = \frac{R^2C^2 - \mathbf{u}\cdot\mathbf{u}}{1-\frac{2m}{R}}$$

$$\implies (u_0^t)^2 = \frac{(R^2C^2 - \mathbf{u}\cdot\mathbf{u})R}{R-2m}$$

$$\implies u_0^t = \sqrt{\frac{(R^2C^2 - \mathbf{u}\cdot\mathbf{u})R}{R-2m}}$$

Taking the positive solution so that the coordinate time and the proper time pass in the same direction.

c)

First set the parameters and initial conditions:

```
m, R = 1.0, 10.0;
```

```
uᵗ₀ (generic function with 1 method)
```

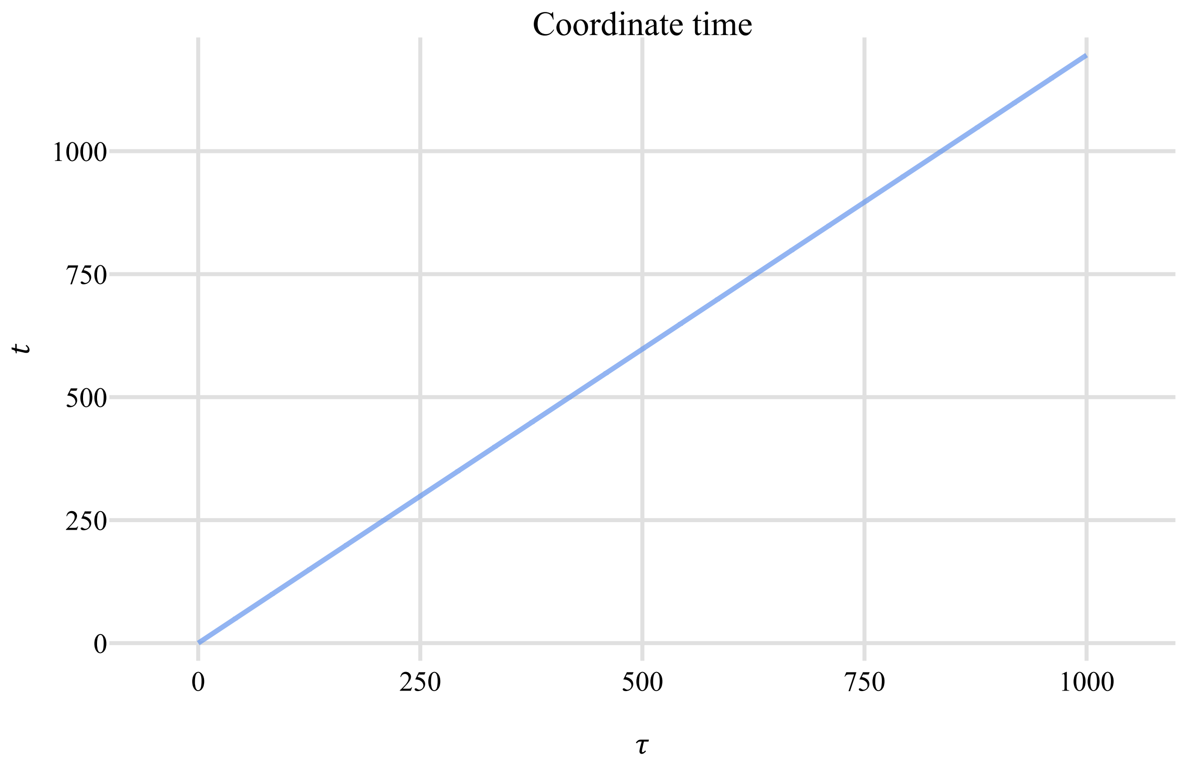
```
uᵗ₀(C, uu) = sqrt(R*(R^2*C^2 - uu)/(R - 2*m))
```

Then the equations of motion, in a function for convenience::

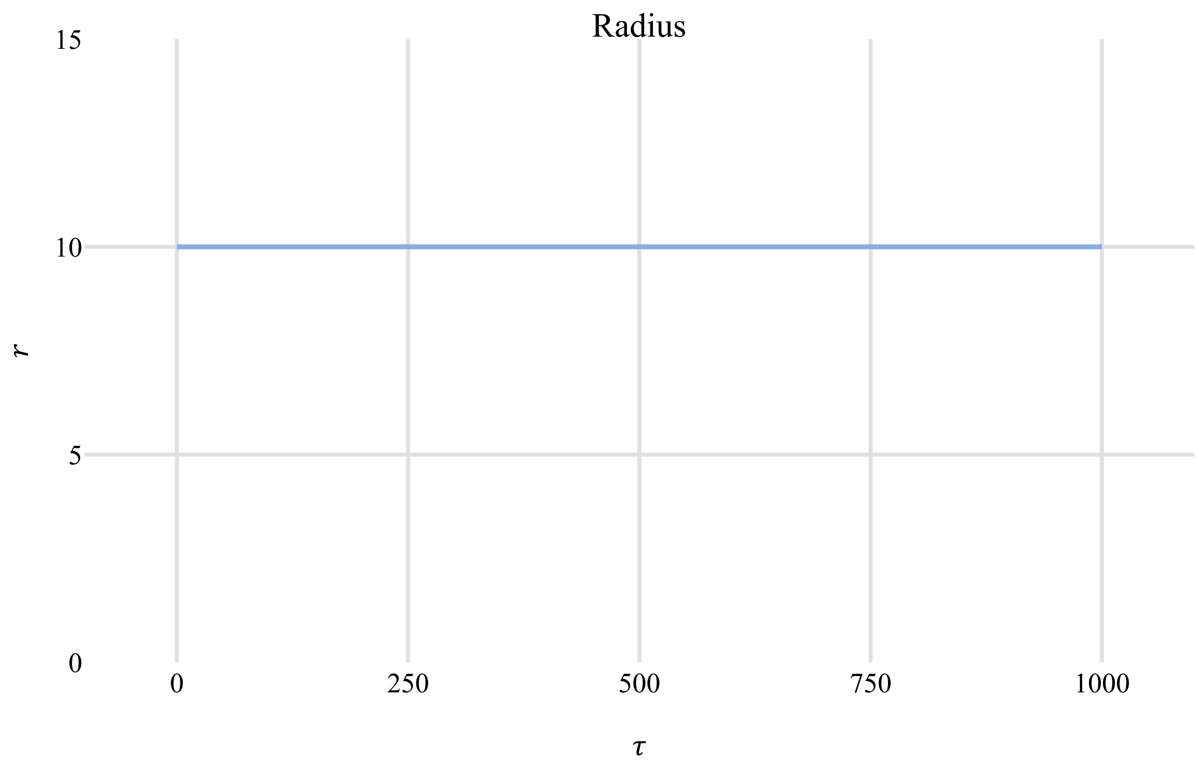
```
function f(C, uu, tmax)
    dXdτ((t, r, ϕ, uᵗ, uʳ, uᵠ), (m, C, R), τ) = [uᵗ, uʳ, uᵠ,
            (-2m/r^2)*(1-2m/r)^(-1)*uᵗ*uʳ,                                              # aᵗ
            (-m/r^2)*(1-2m/r)*uᵗ^2 + (m/r^2)*(1-2m/r)^(-1)*uʳ^2 + (r - 2*m)*uᵠ^2, # aʳ
            (-2/r)*uʳ*uᵠ ];                                                          # aᵠ

    🚀 = Process(
      process = dXdτ(P) = process2solution(P),
      parameter_profile = [m, C, R],
      varnames = [:t, :r, :ϕ, :uᵗ, :uʳ, :uᵠ],
      X0       = [0.0, R, 0.0, uᵗ₀(C, uu), 0, C],
      t0 = 0.0,
      tmax = tmax,
      alg = Vern9(),
      solver_opts = Dict(:adaptive => true, :reltol => 1e-10, :abstol => 1e-10));

    return [times(🚀), (eachcol∘timeseries)(🚀)...]
end;
```

i) $C = (10\sqrt{7})^{-1}$

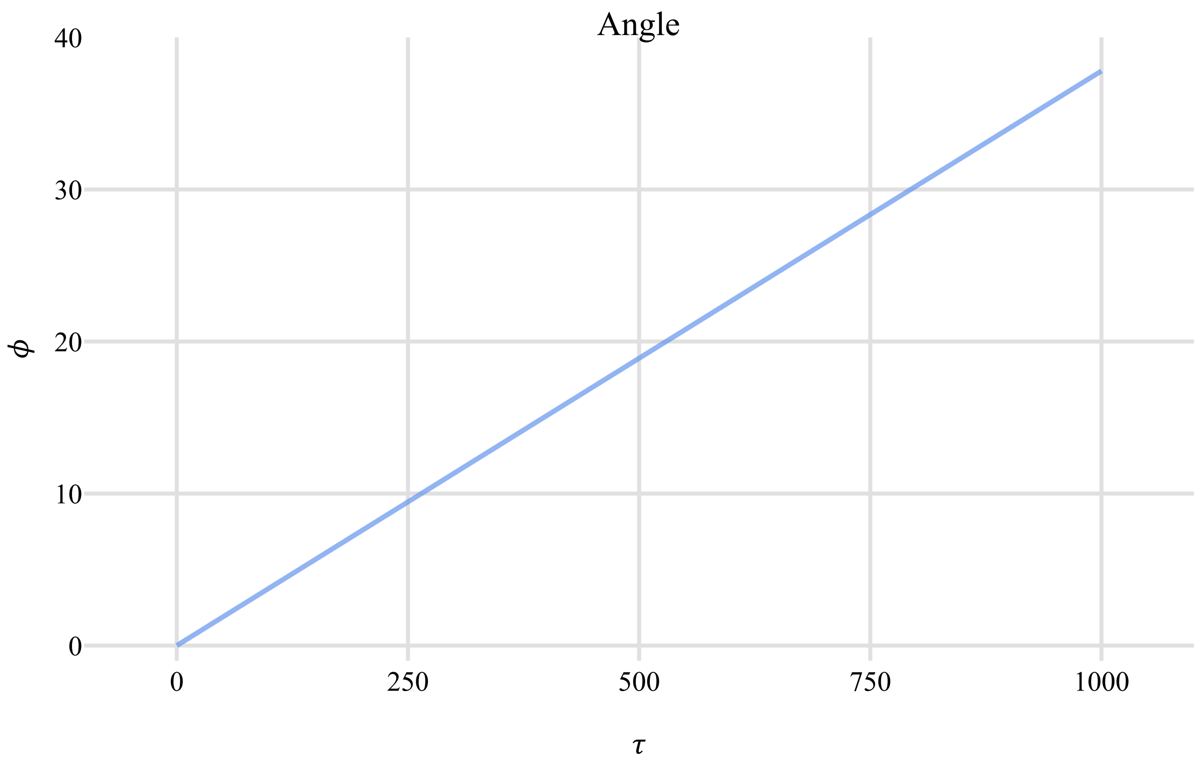
```
τ₁, t₁, r₁, ϕ₁, uᵗ₁, uʳ₁, uᵠ₁ = f((10*sqrt(7))^-1, -1, 1000);
```



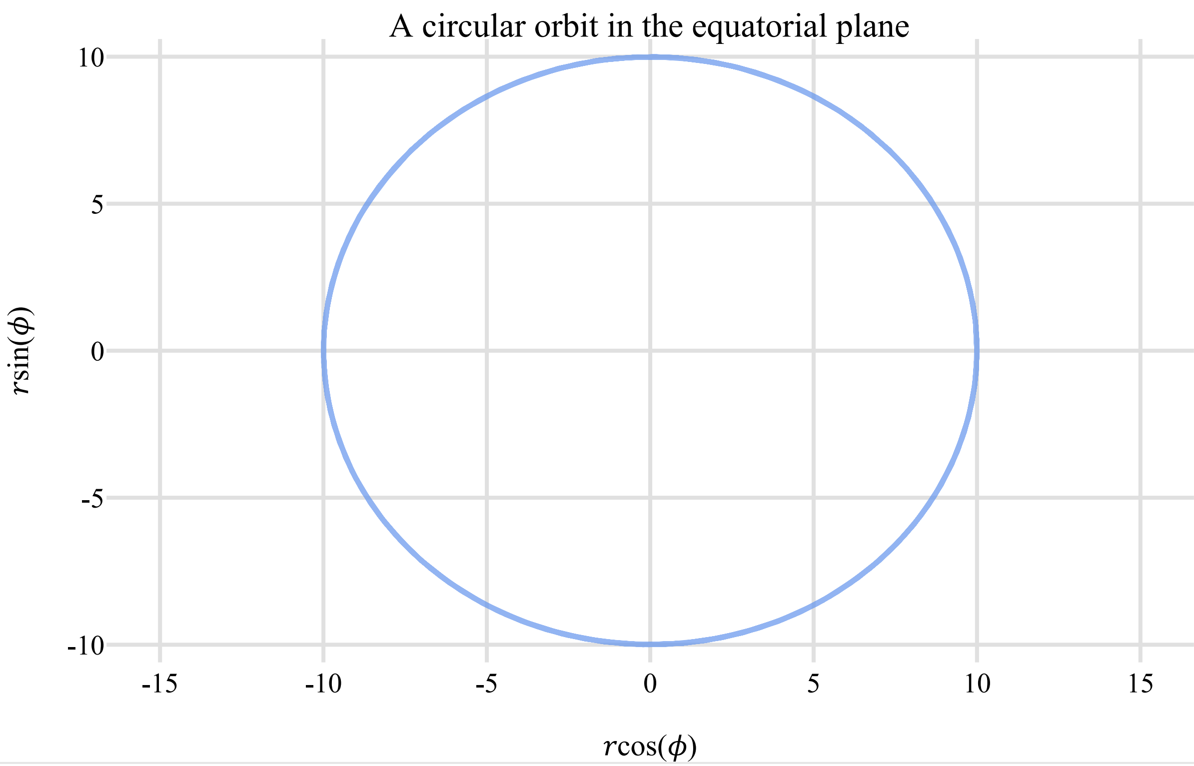

```
plot(τ₁, t₁, xguide="τ", yguide="t", xlims=(-100, 1100),
    title="Coordinate time")
```




```
plot(τ₁, r₁, xguide="τ", yguide="r", xlims=(-100, 1100), ylims=(0, 15),
    title="Radius")
```



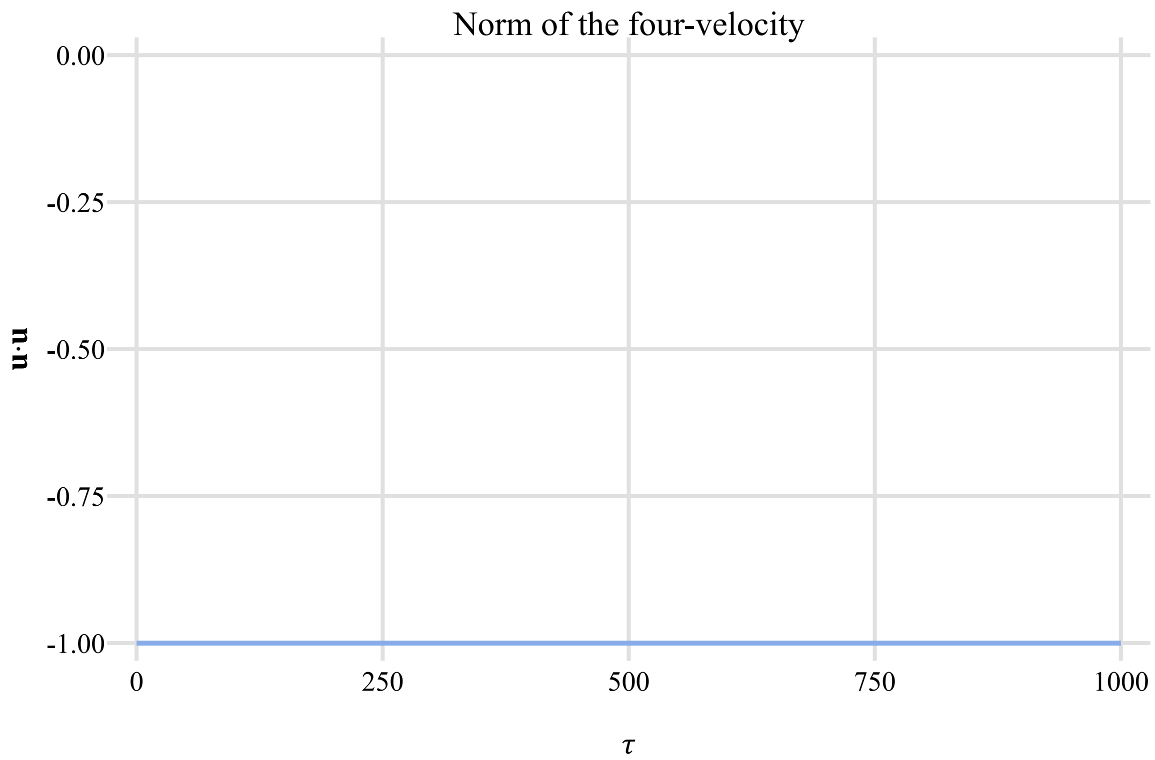

```
plot(τ₁, ϕ₁, xguide="τ", yguide="ϕ", xlims=(-100, 1100), ylims=(-1, 40),
    title="Angle")
```




```
plot(r₁.*cos.(ϕ₁), r₁.*sin.(ϕ₁), xguide="rcos(ϕ)", yguide="rsin(ϕ)",

            aspect_ratio=:equal, title="A circular orbit in the equatorial plane")
```

Using the metric and equation for the norm of the four-velocity written in part a):

```
uu(r, uᵗ, uʳ, uᵠ) = -(1-2m/r)*uᵗ^2 + (1-2m/r)^(-1)*uʳ^2 + r^2*uᵠ^2;
```

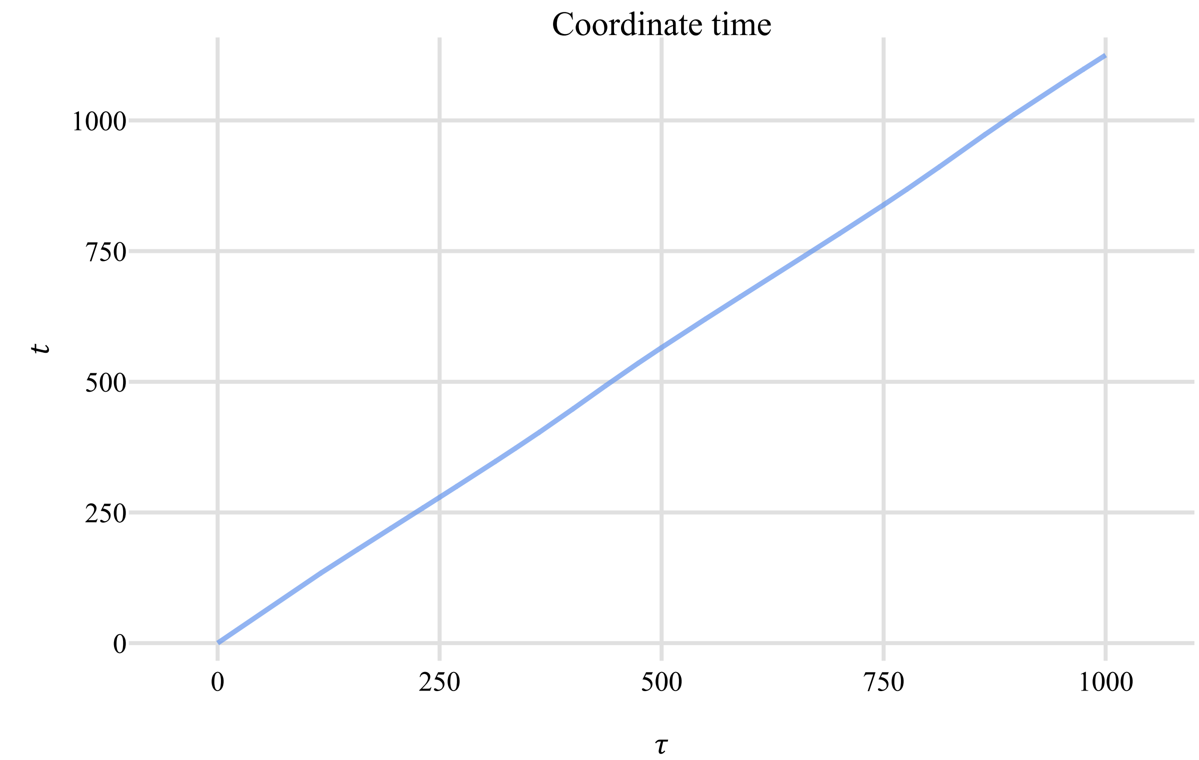



```julia
plot(τ₁, uu.(r₁, uᵗ₁, uʳ₁, uᵠ₁), xguide="τ", yguide="u·u", title="Norm of the four-velocity")
```

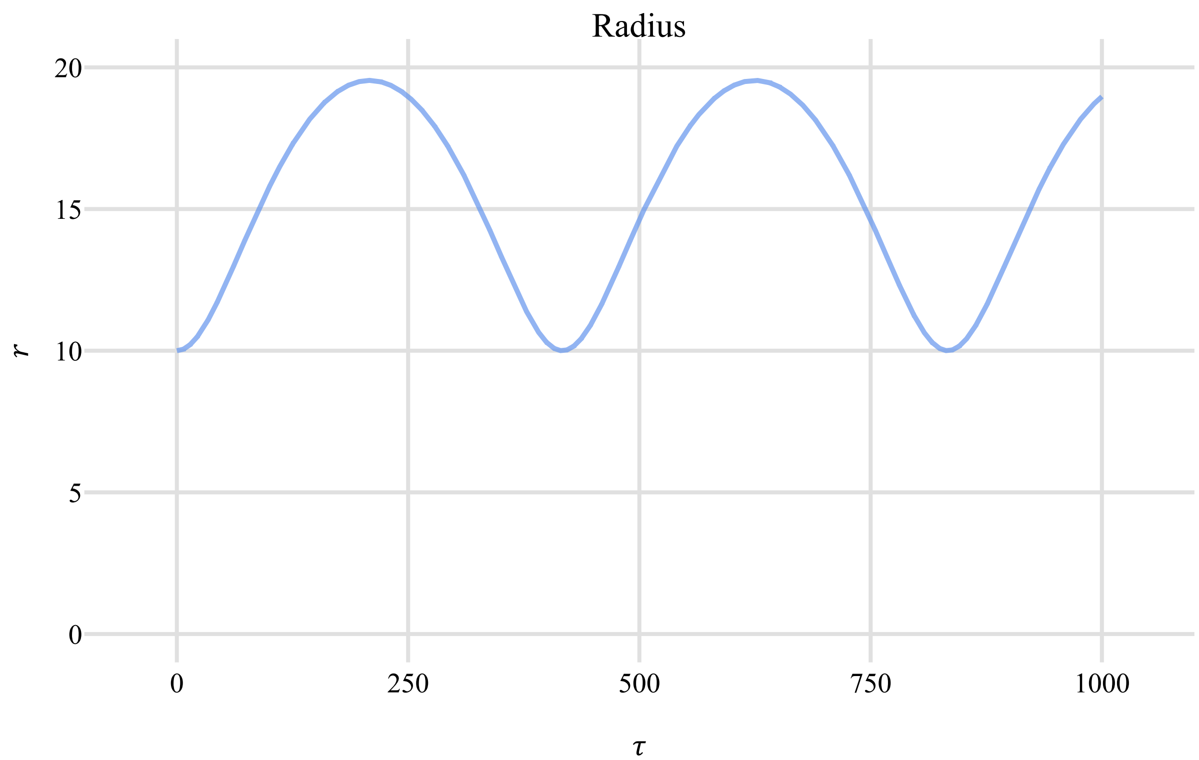
Note that with `Vern9()` the error in the normof the four-velocity is to small to be appear in float values. For a poorer integrator, such as the midpoint method, it is on the order of 10^{-16}.

ii) $C = 1.1(10\sqrt{7})^{-1}$

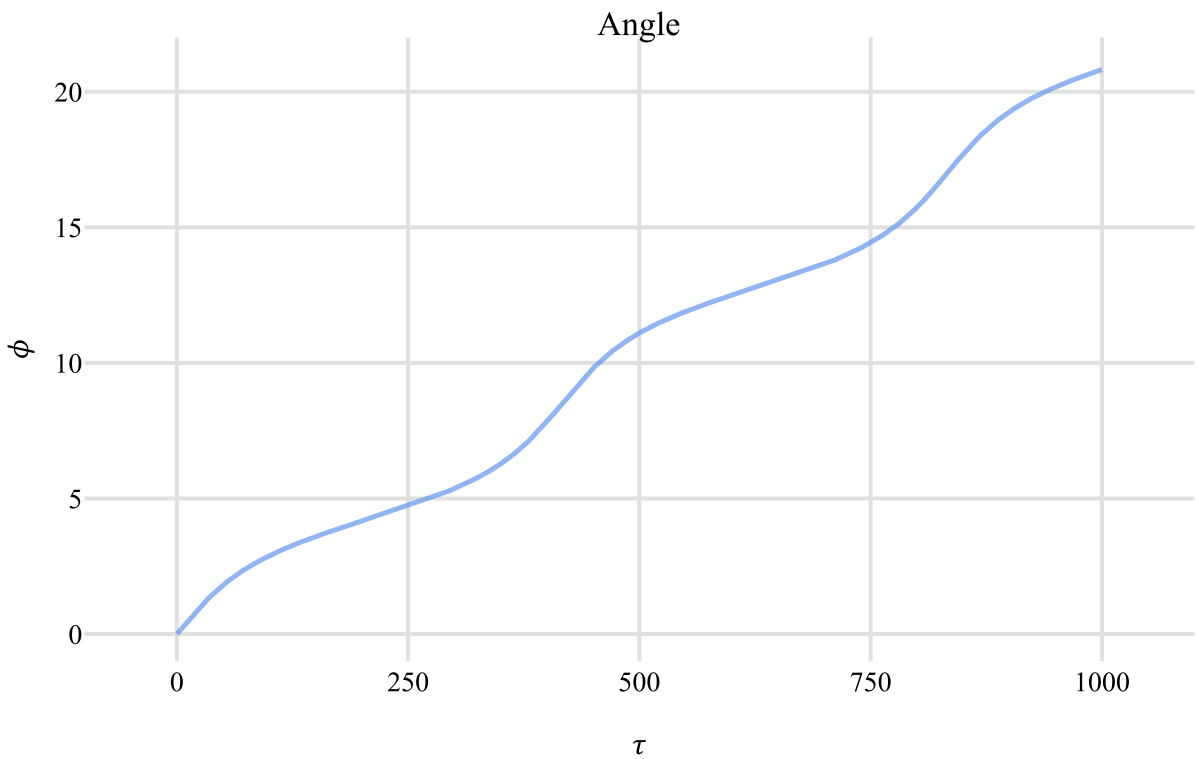
```julia
τ₂, t₂, r₂, ϕ₂, uᵗ₂, uʳ₂, uᵠ₂ = f(1.1*(10*sqrt(7))^-1, -1, 1000);
```



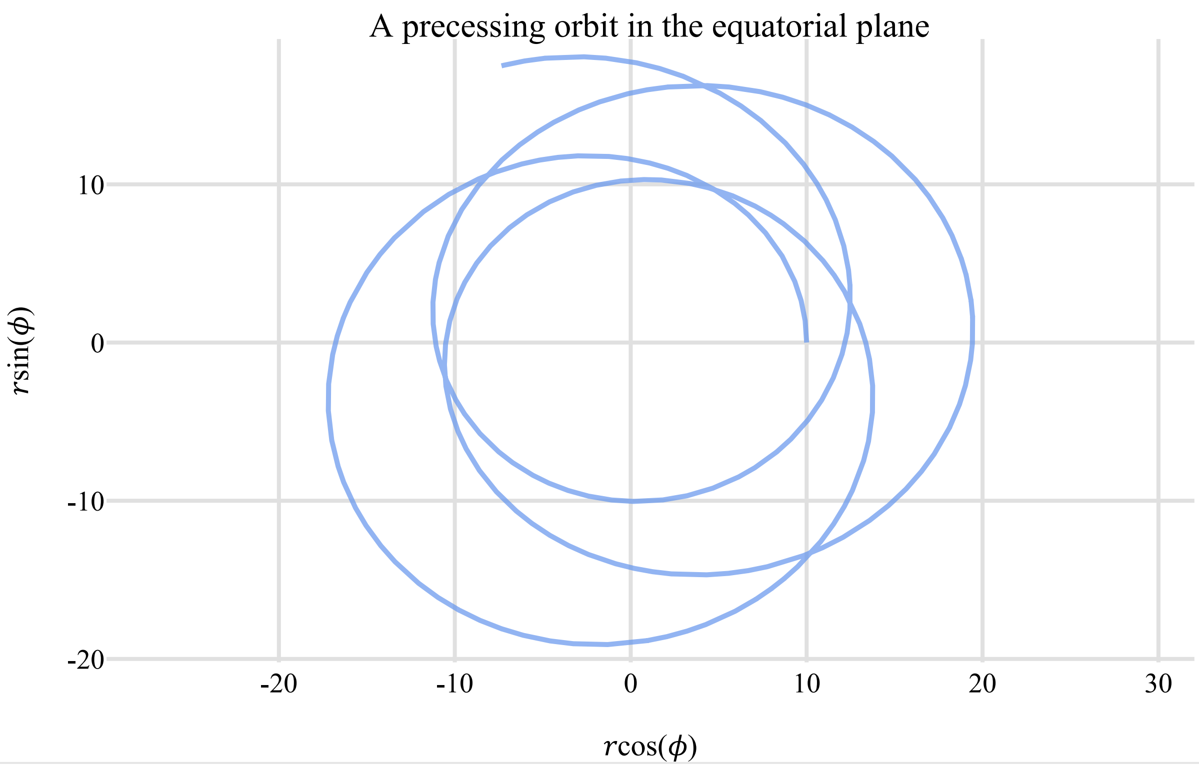

```
plot(τ₂, t₂, xguide="τ", yguide="t", xlims=(-100, 1100),
    title="Coordinate time")
```



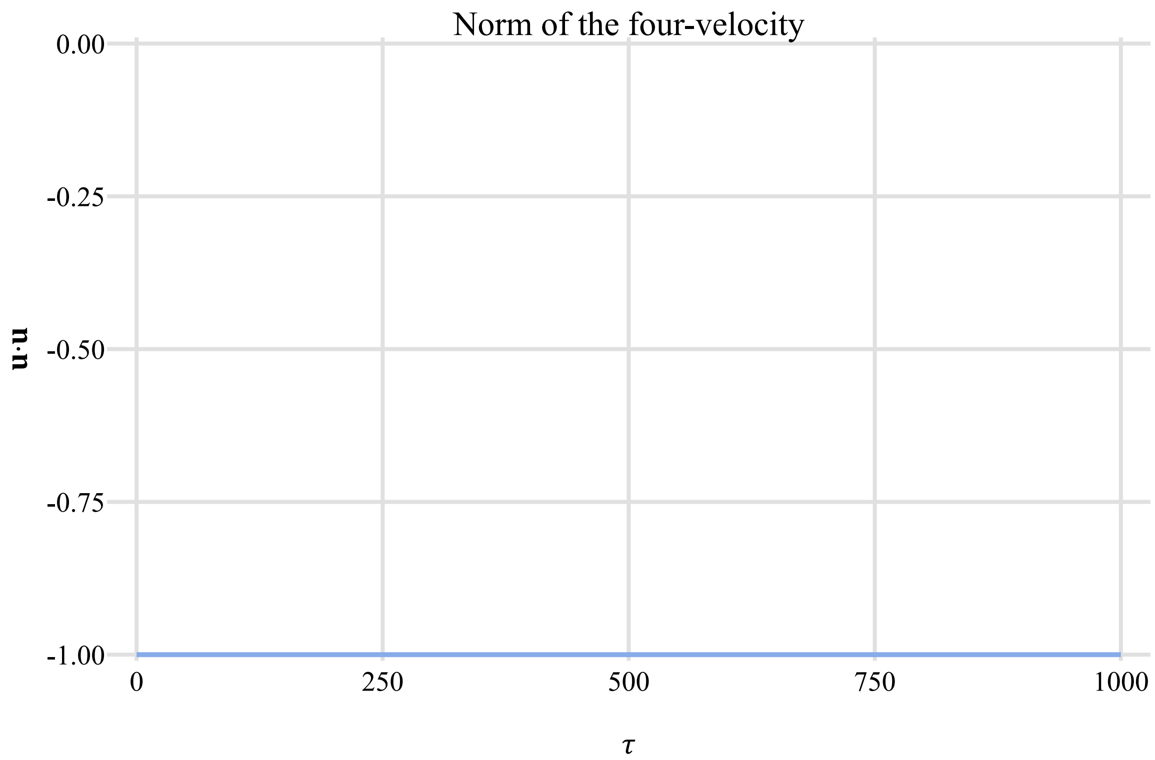

```
plot(τ₂, r₂, xguide="τ", yguide="r", xlims=(-100, 1100), ylims=(-1, 21),
    title="Radius")
```




```
plot(τ₂, ϕ₂, xguide="τ", yguide="ϕ", xlims=(-100, 1100), ylims=(-1, 22),
    title="Angle")
```

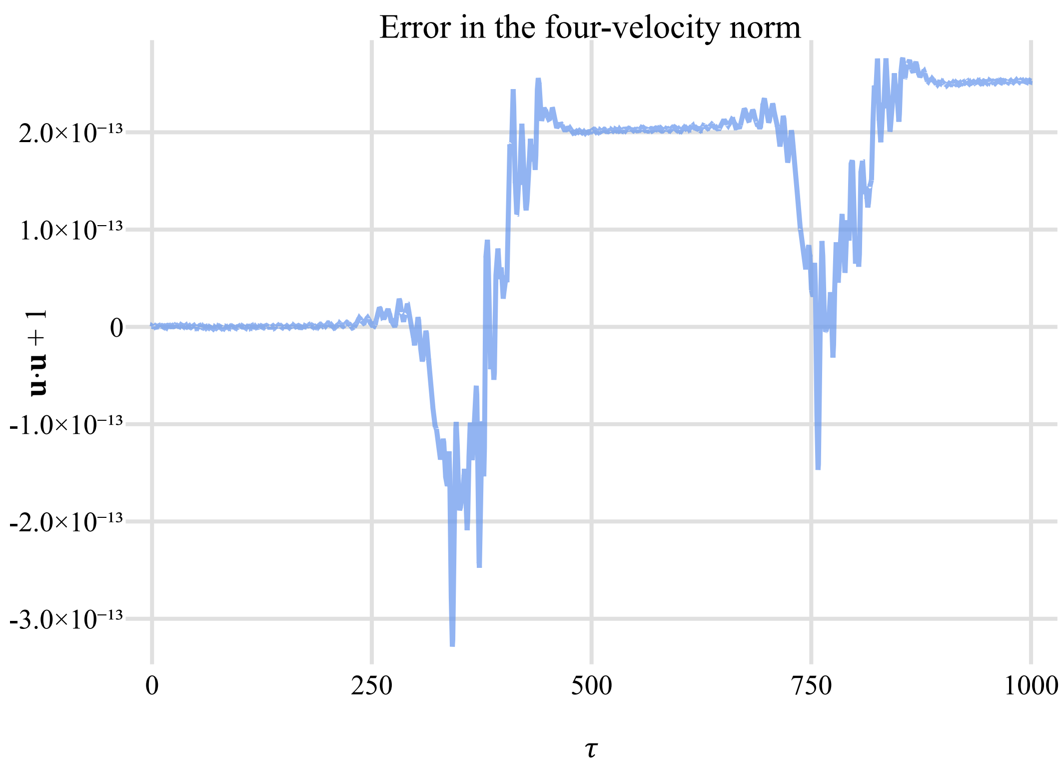



```
plot(r₂.*cos.(ϕ₂), r₂.*sin.(ϕ₂), xguide="rcos(ϕ)", yguide="rsin(ϕ)",
            aspect_ratio=:equal, title="A precessing orbit in the equatorial
plane")
```




```
plot(τ₂, uu.(r₂, uᵗ₂, uʳ₂, uᵠ₂), xguide="τ", yguide="u·u", title="Norm of the four-velocity", ylims=(-1.01, 0.01))
```

This time the error is slightly greater, since the orbit is no longer a perfect circle:




```
plot(τ₂, uu.(r₂, uᵗ₂, uʳ₂, uᵠ₂) .+ 1, xguide="τ", yguide="u·u + 1", title="Error in the four-velocity norm")
```

This orbit precesses, unlike orbits under Netwonian gravity which are strictly conic sections. This is because general relativity, or the procedure to derive the path of an object from geodesic equaitons and the structure of spacetime, introduces an additional term to the effective potential of Newtonian gravity. Both descriptions of gravity have stable minima at someradius that is determined by the mass of the central body and the angular momentum of the orbiting object. Objects with apsides at this stable radius will follow a circular path, as for the first orbit shown in this question; their efective energy is minimised. Objects with apsides displaced from this stable minimum will precess; their effective energy is greather than the potential at the stable minimum. Their radius will oscillate between a point with an r smaller than the stable minimum, and one with an r greater (i.e. an elliptical orbit). This occurs in both Newtonian gravity and general relativity. However, Newtonian gravity has an effective potential with two terms: a radial force and an angular momentum. Hence elliptical orbits under Newtonian gravity are angularly stable and do not precess. General relativity introduces a third term that is cubic in the radius and also depends on angular momentum; this term breaks the (angular) stability of elliptical orbits and causes them to precess (the angle ϕ of the apsides is not constant).

(The effective potential described by general relativity also approaches negative infinity near the origin, so there are radially unstable states that do not appear in this assignment.)

d)

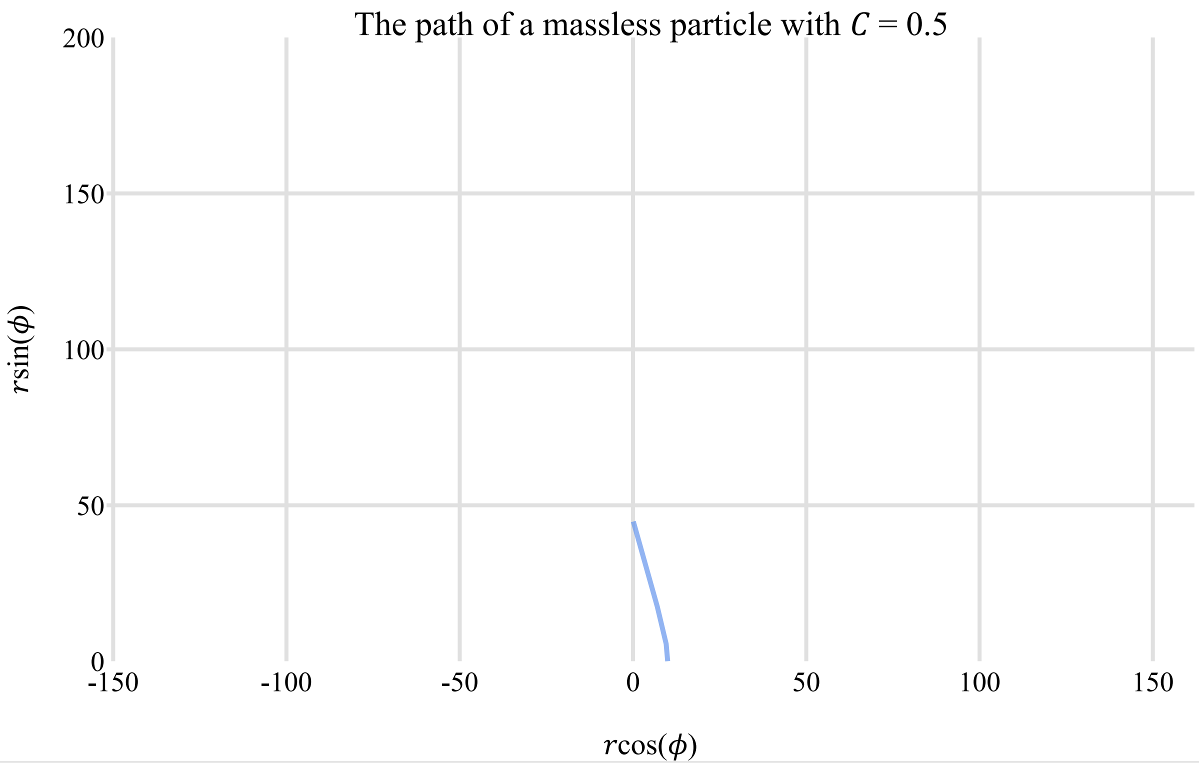
For a massless particle, $\mathbf{u} \cdot \mathbf{u}$ is 0. Additionally, the variable τ is the equations of motion no longer represents proper time but an affine parameter ($0 \leq \lambda \leq 10$).

i) $C = 0.5$

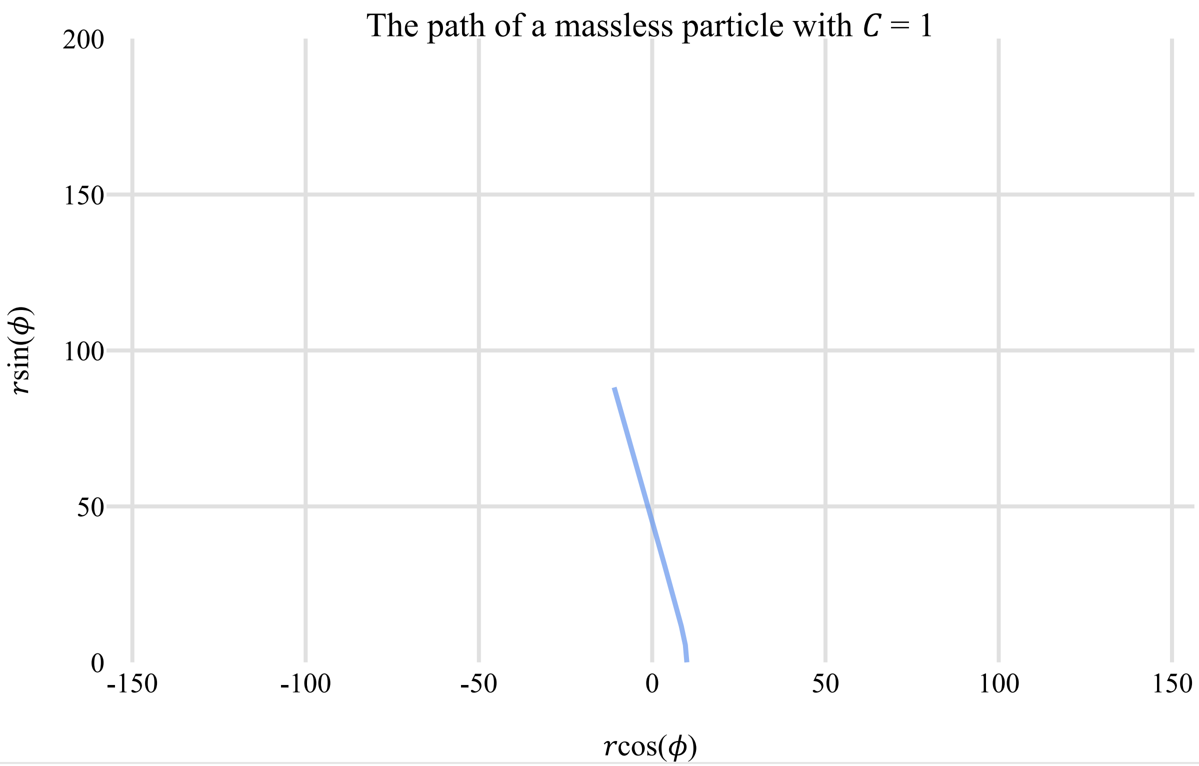
```
τ₃, t₃, r₃, ϕ₃, uᵗ₃, uʳ₃, uᵠ₃ = f(0.5, 0, 10);
```

```
τ₄, t₄, r₄, ϕ₄, uᵗ₄, uʳ₄, uᵠ₄ = f(1, 0, 10);
```

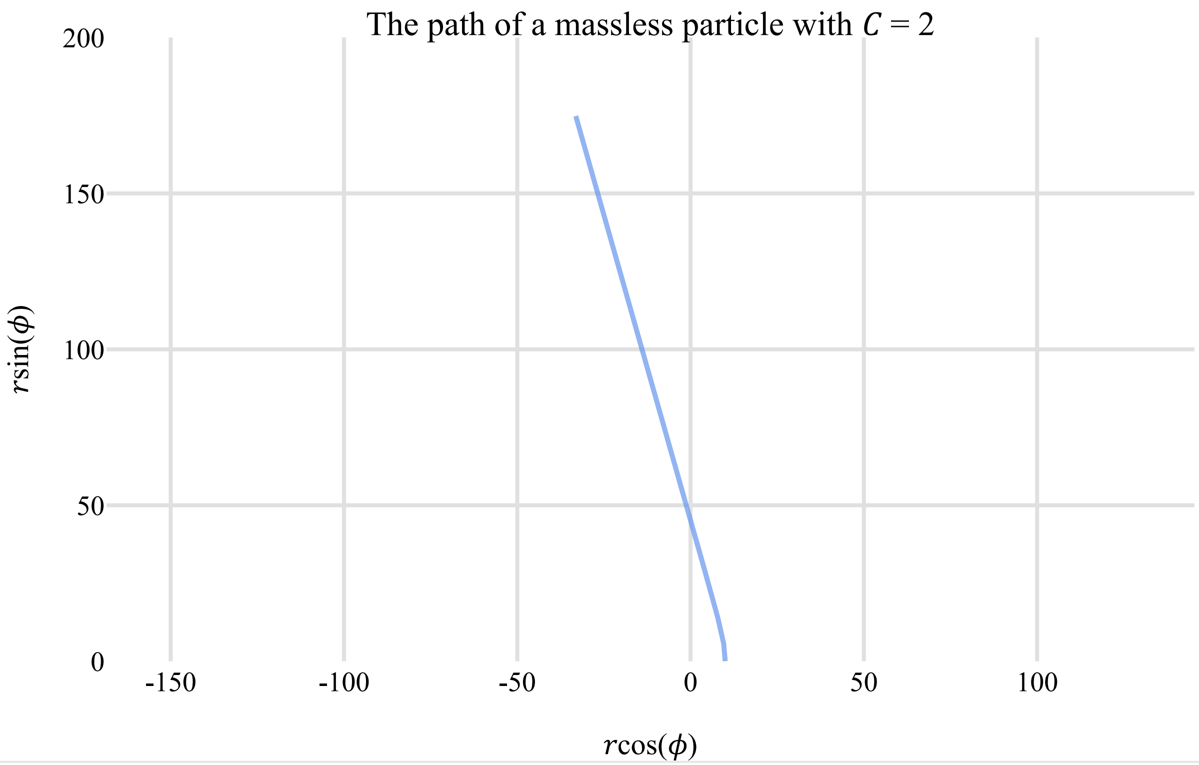
```
τ₅, t₅, r₅, ϕ₅, uᵗ₅, uʳ₅, uᵠ₅ = f(2, 0, 10);
```




```
plot(r₃.*cos.(ϕ₃), r₃.*sin.(ϕ₃), xguide="rcos(ϕ)", yguide="rsin(ϕ)", ylims=(0,
200),
         aspect_ratio=:equal, title="The path of a massless particle with C =
0.5")
```




```
plot(r₄.*cos.(ϕ₄), r₄.*sin.(ϕ₄), xguide="rcos(ϕ)", yguide="rsin(ϕ)", ylims=(0,
200),
         aspect_ratio=:equal, title="The path of a massless particle with C =
1")
```




```
plot(r₅.*cos.(ϕ₅), r₅.*sin.(ϕ₅), xguide="rcos(ϕ)", yguide="rsin(ϕ)", ylims=(0, 200),
          aspect_ratio=:equal, title="The path of a massless particle with C = 2")
```

In this question, the massless particle passes by a massive body and its trajectory (in Cartesian coordinates) is bent. This makes clear the existence of gravity as curvature in space-time, since it can influence even the paths of massless particles. Increasing the intial component of the four-velocity, C, causes the total spatial length of the massless particle's path to increase. However, the trajectory it ultimatley follows is the same; this is because the affine parameter is defined such that it can be substituted in place of (or, to generalise) τ in the equations of motion for a massive particle This gives a geodesic equation valid for a massless (0 four-velocity norm) particles. As such, the affine parameter can be linearly rescaled to give the same trajectory in spacetime for a massless photon; the affine parameter does not have an absolute correspondence to a proper time. Then C has no physical meaning as a velocity over a proper time for a massless partical, but does act to rescale the affine parameter. The result is that increasing C produces a longer path (in space, or, Cartesian coordinates) for the same interval of affine parameters, as demonstrated above.