COMP4141 Slides Week 2

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UNSW

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Admin

Tutorial Solutions

- They will be posted once all the tutes have been held for the week

Quizes

(20% in total)

- They start **today**!
- They will be 20 minutes; you will write your answers and I will collect at the end. This will be similar to exam conditions; laptop and phones away. No talking to your fellow students during the quiz.
- Please write down your name, student ID, and the tutorial session on top of the first page.
- Please use a black pen

ϵ-NFAs

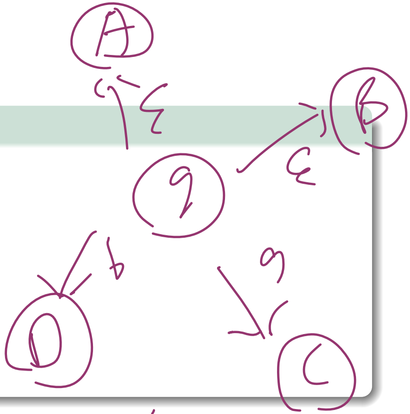
ϵ-Nondeterministic Finite Automata

Definition

An ϵ-NFA is an NFA that allows state changes that do not consume input symbols. The only difference in the formal definition is the transition relation:

$$\delta : Q \times (\Sigma \cup \{\epsilon\}) \to 2^Q.$$

ϵ-Closure



Definition

For $q \in Q$, define $E(q)$ to be the least set satisfying:

- $q \in E(q)$,
- if $s \in E(q)$ and $t \in \delta(s, \epsilon)$, then $t \in E(q)$.

For a set $S \subseteq Q$, define

$$E(S) = \bigcup_{q \in S} E(q).$$

$E(q) = \{q, A, B$

Proof Constructions

Lemma

Let $L \subseteq \Sigma^*$ be a regular language. Then $\Sigma^* \setminus L$ is also a regular language.

Proof idea: Replace F with $Q \setminus F$.

Lemma

Let $L_1, L_2 \subseteq \Sigma^*$ be regular languages. Then $L_1 \cap L_2$ is regular.

Proof idea: Construct a product automaton with $Q = Q_1 \times Q_2$,

$$\delta((q, q'), a) = (\delta_1(q, a), \delta_2(q', a)), \quad F = F_1 \times F_2 = \{(q, q') : q \in F_1 \text{ and } q' \in F_2\}.$$

Regular Languages

Regular Languages

Regular Expressions (Inductive Definition)

The set of regular expressions over an alphabet Σ is defined inductively:

- If $a \in \Sigma$, then a is a regular expression.
- $\emptyset$ is a regular expression.
- ε is a regular expression.
- If R_1 and R_2 are regular expressions, then:
  - $R_1 \cup R_2$ is a regular expression, ← Match R_1 or R_2
  - $R_1 \cdot R_2$ is a regular expression,
  - R_1^* is a regular expression.

$$R_1 \cdot R_2 = \{ w_1 w_2 : w_1 \in R_1, w_2 \in R_2 \}$$

$$R^{*} = \bigcup_{n=0}^{\infty} R^{n}$$

$(L')^2$ have exactly 4 a's

$$R^{n} = \{ w_1 w_2 \ldots w_n : w_i \in R \ \forall i \}$$

$$R^{0} = \varepsilon$$

Equivalence with Automata

Kleene's Theorem

Let $L \subseteq \Sigma^*$ be a language. The following are equivalent:

- L is accepted by a DFA,
- L is accepted by an NFA,
- L is accepted by a GNFA,
- $L = L(R)$ for some regular expression R.

From Regular Expressions to Finite Automata (Union)

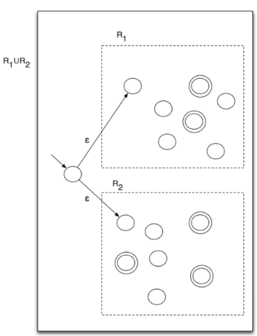


Figure: Proof By Picture (Union)

From Regular Expressions to Finite Automata (Concatenation)

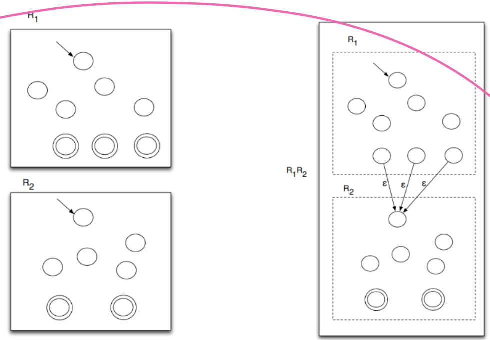


Figure: Proof By Picture (Concatenation)

From Regular Expressions to Finite Automata (Kleene Star)

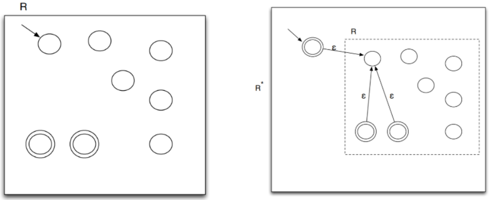


Figure: Proof By Picture (Kleene Star)

Generalised NFAs (GNFAs)

Definition

A **generalised nondeterministic finite automaton (GNFA)** is a 5-tuple $(Q, \Sigma, \delta, q_0, q_F)$ where:

- Q is a finite set of states,
- Σ is the input alphabet,
- $\delta : (Q \setminus \{q_F\}) \times (Q \setminus \{q_0\}) \longrightarrow \mathrm{RE}_\Sigma$ is the transition function,
- $q_0 \in Q$ is the start state,
- $q_F \in Q$ is the accept state.

Pumping Lemma

Pumping Lemma

If $L \subseteq \Sigma^*$ is regular, then there exists $p \in \mathbb{N}$ (the *pumping length*) such that for every $w \in L$ with $|w| \geq p$, we can write $w = xyz$ satisfying:

1. $|xy| \leq p$,
2. $|y| > 0$,
3. $xy^iz \in L$ for all $i \in \mathbb{N}$.

Myhill–Nerode Theorem

Distinguishability

Let $L \subseteq \Sigma^*$.
We say x and y are *distinguishable by L* if there exists $z \in \Sigma^*$ such that:

$$xz \in L \quad \text{and} \quad yz \notin L$$

(or vice versa).
We write $x \equiv_L y$ if x and y are not distinguishable by L.
The *index* of L is the number of $\equiv_L$-equivalence classes.

Myhill–Nerode Theorem

A language $L \subseteq \Sigma^*$ is regular if and only if the index of L is finite.
Moreover, this index equals the minimum number of states of any DFA that accepts L.

Using Myhill–Nerode

Myhill–Nerode proof format

Find an infinite sequence of words (not necessarily in L), $u_1, u_2, u_3, \ldots,$ and a doubly indexed sequence of words $w_{i,j}$ for $i < j \in \mathbb{N}_0$ such that

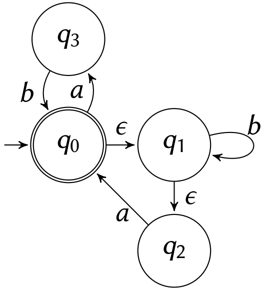
$$u_i w_{i,j} \in L \quad \text{and} \quad u_j w_{i,j} \notin L$$

(or vice versa).

W1 P7 & P8

Problems 7 & 8 (ϵ-Nondeterministic Finite Automata; Week 1)

Assume the alphabet $\Sigma = \{a, b\}$. Consider the following ϵ-NFA.



bab

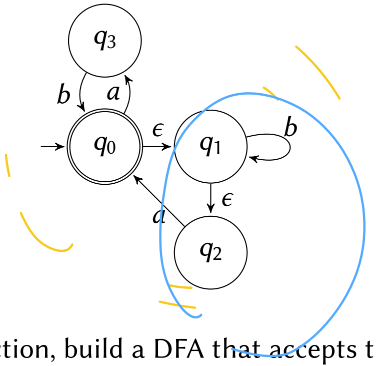
7. List all strings of length ≤ 3 that this automaton accepts.

ε	✓
a	✓
b	✗
ab	✓
bb	✗
ba	✓
aa	✓

aaa	✓
aab	✓
aba	✓
abb	✗
baa	✓
bab	✗
bba	✓
bbb	✗

Problems 7 & 8 (ϵ-Nondeterministic Finite Automata; Week 1)

Assume the alphabet $\Sigma = \{a, b\}$. Consider the following ϵ-NFA.

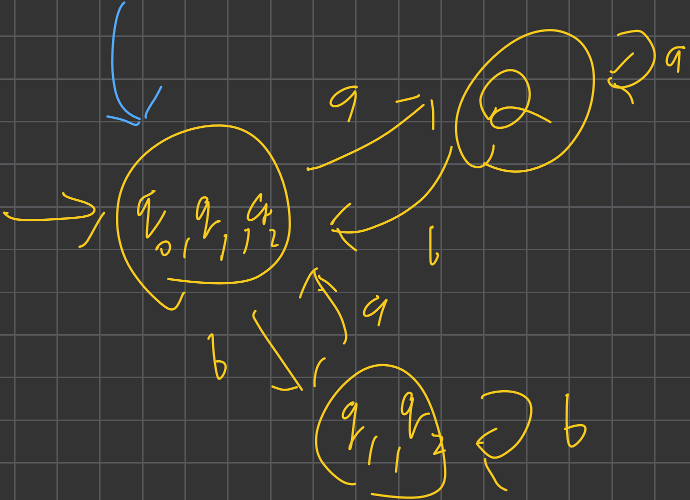


8. Using the subset construction, build a DFA that accepts the same language. Give the graphical representation of this DFA, clearly indicating the set of NFA states associated to each DFA state. Show only the reachable states.

$$E(q_0) = \{q_0, q_1, q_2\}$$

$$E(q_0, q_3) = \emptyset$$

$$E(q_1) = q_1, q_2$$



W2 P1

Problem 1 (Week 2)

Give regular expressions for each of the following subsets of $\{a, b\}^*$.

(a) $\{x : x$ contains an even number of a's$\}$

$\supseteq$

L

$L' := \{x : x$ contains 2 a's$\}$

$\underbrace{\quad\quad}_{b\text{'s}}\ a\ \underbrace{\quad\quad}_{b\text{'s}}\ a\ \underbrace{\quad\quad}_{b\text{'s}}$

Regular expression for L'

$b^* a b^* a b^*$

Try for L

$$(b^*ab^*ab^*)^*$$

Counterexamples are
b
bb
⋮

Correct is
$$(b^*ab^*ab^*)^* \cup b^*$$

Another way is $(b^*ab^*ab^*)^* \cdot b^*$

Problem 1 (Week 2)

Give regular expressions for each of the following subsets of $\{a, b\}^*$.

(b) $\{x : x$ contains an odd number of b's$\}$

Problem 1 (Week 2)

Give regular expressions for each of the following subsets of $\{a, b\}^*$.

(c) $\{x : x$ contains an even number of a's or an odd number of b's$\}$

W2 P2

Problem 2 (Week 2)

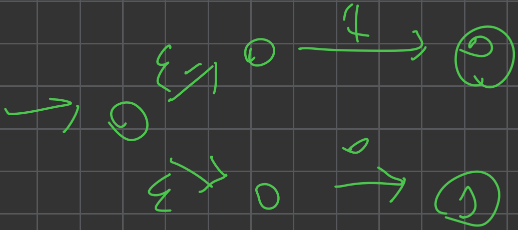
Convert the following regular expressions into DFAs:

(a) $(0 \cup 1) \cdot (0 \cup 1) \cdot (0 \cup 1)$

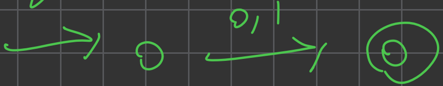
$R = (0 \cup 1)$

union of ε-NFA for 0:

with the ε-NFA for 1



Equivalent to



(For $(0 \vee 1)$)

To get $(0 \cup 1) \cdot (0 \cup 1) \cdot (0 \cup 1)$

$$\to \circ \xrightarrow{0,1} \circ \xrightarrow{\varepsilon} \circ \xrightarrow{0,1} \circ \xrightarrow{\varepsilon} \circ \xrightarrow{0,1} \circledcirc$$

To get DFA:

$$\begin{tikzcd}
\arrow[r] & \circ \arrow[r, "{0,1}"] & \circ \arrow[r, "{0,1}"] & \circ \arrow[r, "{0,1}"] & \circledcirc \arrow[r, "{0,1}"] & \circ \arrow[loop right, "{0,1}"]
\end{tikzcd}$$

Problem 2 (Week 2)

Convert the following regular expressions into DFAs:

(b) $0^* \cdot 1^*$

Problem 2 (Week 2)

Convert the following regular expressions into DFAs:

(c) $(0 \cup 1^*) \cdot (0^* \cup 1)$

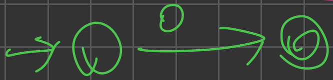
$$(0 \cup 1^*) \cdot (1 \cup 0^*)$$

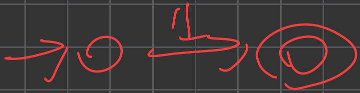
$$R_1 = 0 \cup 1^*$$

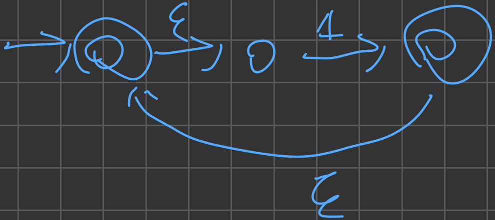
$$R_2 = 1 \cup 0^*$$

Want $R_1 \cdot R_2$

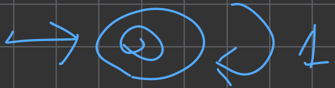
$R_1 = 0 \cup 1^*$



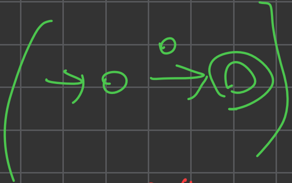


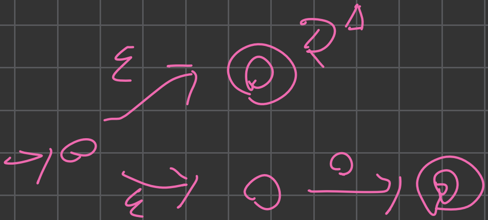


For 1^*, simplify to



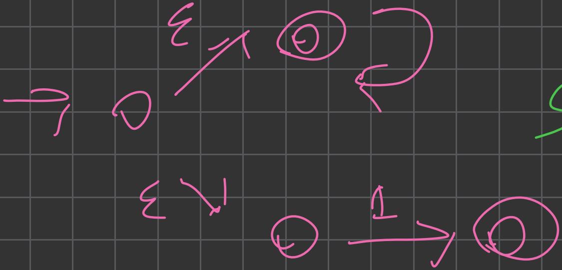
Combine with 0



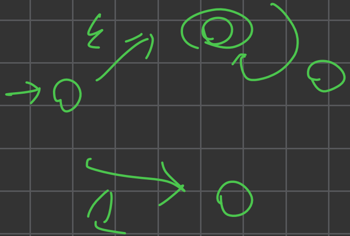


Simplify to

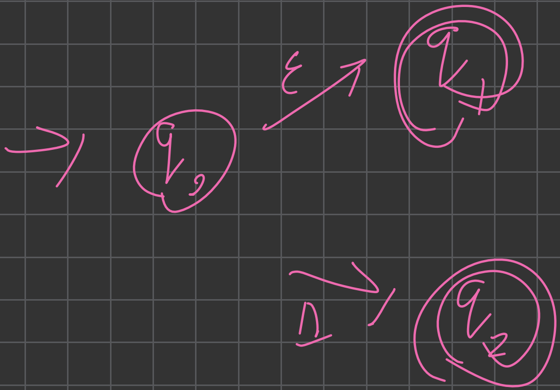
$R_2 = 1 \cup \emptyset^*$



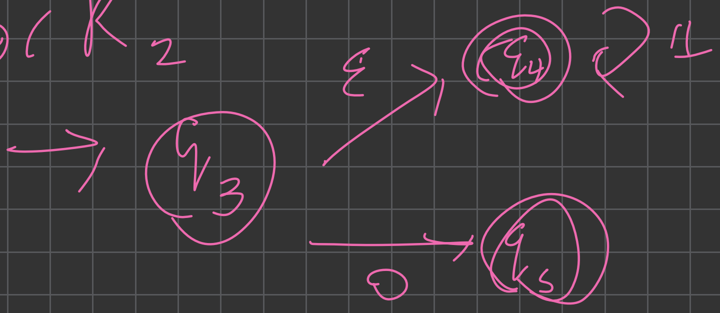
Simplify to

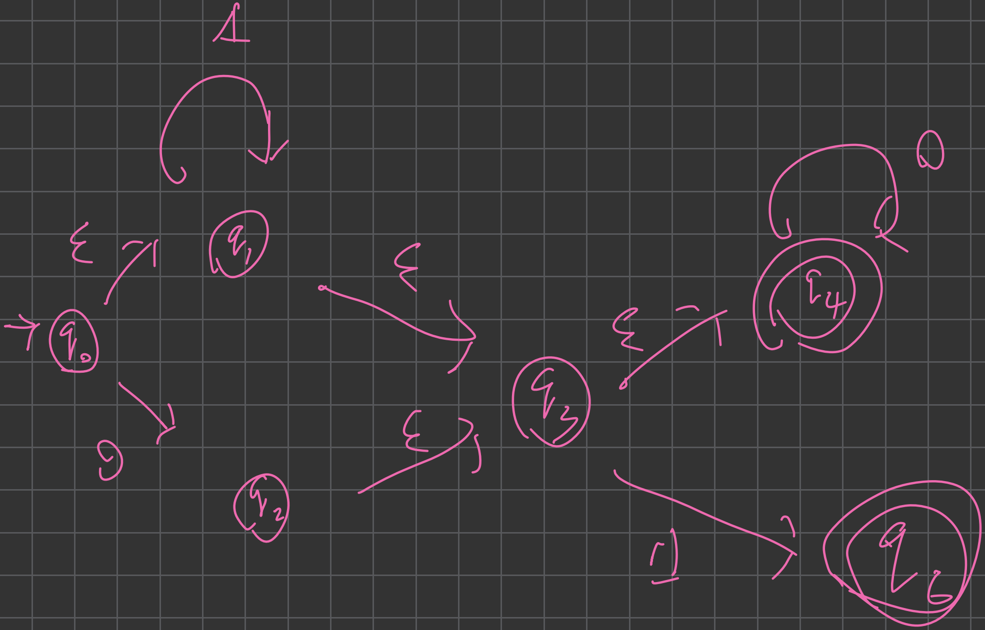


For R,



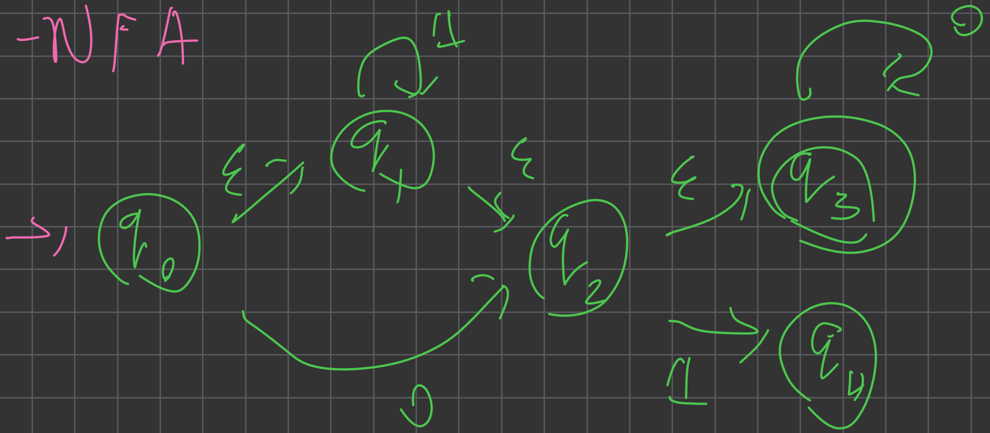
Fork 2



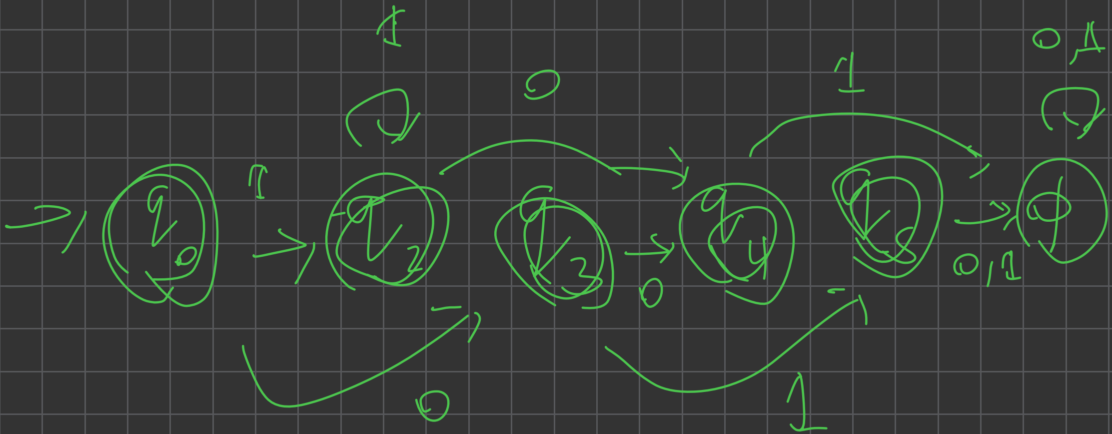


1
ε
q1
q0
0
q2
ε
ε
q3
ε
q4
0
1
q6

ε-NFA



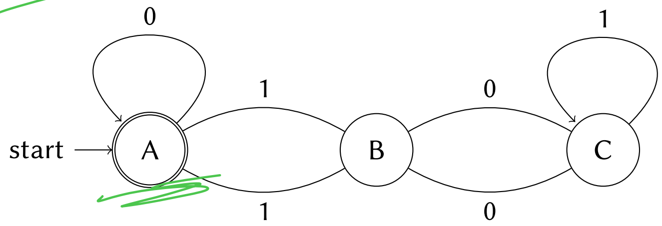
After word: DFA



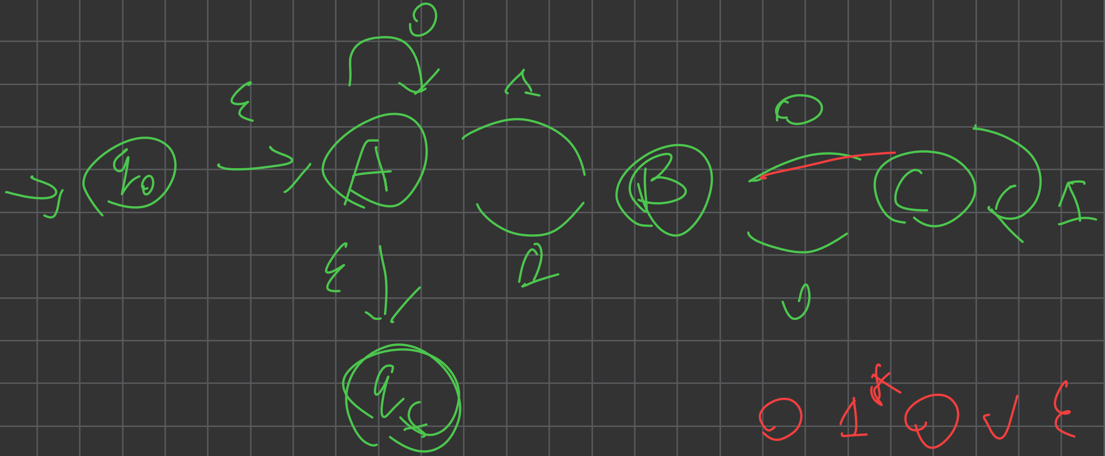
W2 P3

Problem 3 (Week 2)

Give a regular expression that matches the language of the following DFA:

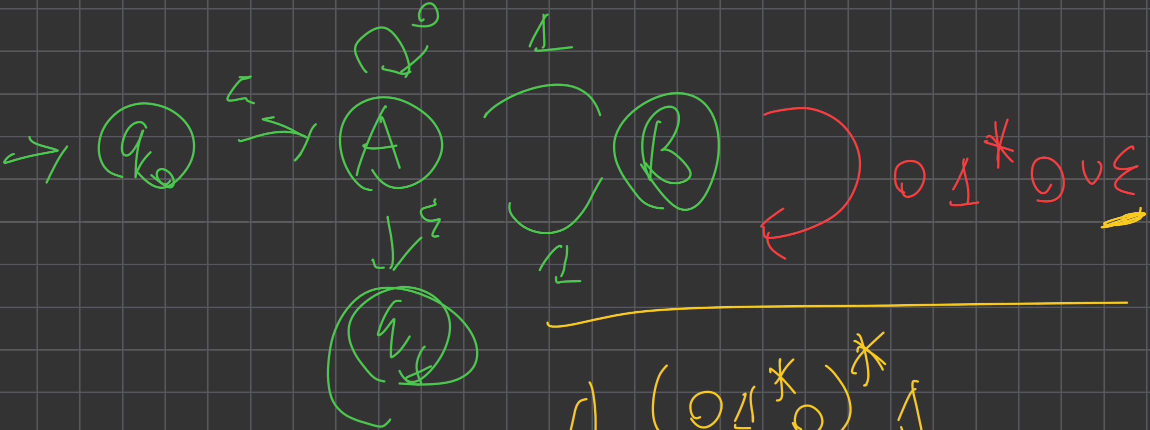


Build GNFA



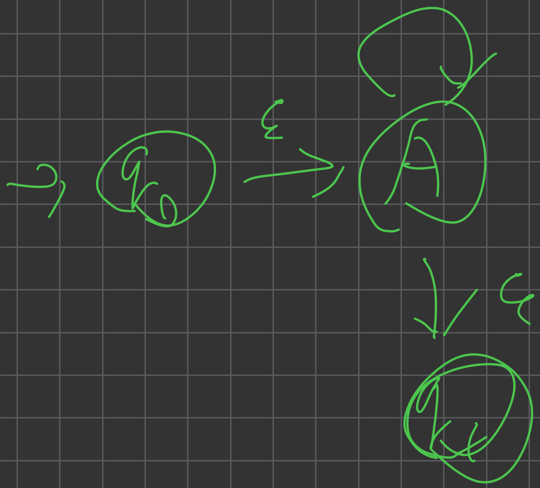
$0 1^* 0 \cup \varepsilon$

Get rid of C



$1(01^*0)^*1$

$\emptyset \vee 1(01^*0)^*1$



$$\to q_0 \xrightarrow{\varepsilon \cup (0 \cup 1(01^*0)^*1))^*} q_f$$

$$\boxed{\varepsilon \cup (0 \cup 1(01^*0)^*1)^*}$$ | Regular expression

W2 P4

Problem 4 (Week 2)

Show that the following languages are not regular:

(a) $\{0^i 1^j : 0 \leq i \leq j\}$

Problem 4 (Week 2)

Show that the following languages are not regular:

(b) $\{w \in \{0,1\}^* :$ the number of 0's in w is equal to the number of 1's in $w\}$

Problem 4 (Week 2)

Show that the following languages are not regular:

(c) $\{ww : w \in \{0,1\}^*\}$

Bonus

Discussion

(a) Given an NFA A, how can you determine if $L(A) = \emptyset$?

(b) Using the above process, outline an algorithm that takes two regular expressions E_1 and E_2 and determines if $L(E_1) \subseteq L(E_2)$.

(c) If $|E_1| = n$ and $|E_2| = m$, what is an upper bound (using big-O notation) for the number of states in the DFA that is used in the previous algorithm.