ENPM662 Homework 2

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1 Introduction

The purpose of this assignment is to practice working with rotation, translation, and transformation matrices as well as develop a better understanding of rigid body transformations.

2 Problem 1.1: Composition of Transforms

2.1 Steps Involved and Explanation

In order to produce a 4x4 homogeneous transformation matrix for a series of rotations and translations, I have to multiply the corresponding matrices in the correct order. The rule I am following in order to multiply these matrices in the correct order goes as follows:

- When rotating or translating with respect to the <u>current axis</u>, I post-multiply (multiply at the end of the expression) the matrices.
- When rotating or translating with respect to the <u>world axis</u>, I pre-multiply (multiply at the beginning of the expression) the matrices.

In order, the expression is developed as follows:

1. $$H = R_{z,\phi}$$
2. $$H = R_{z,\phi} * T_{y,y}$$
3. $$H = R_{z,\phi} * T_{y,y} * R_{z,\theta}$$
4. $$H = R_{x,\psi} * R_{z,\phi} * T_{y,y} * R_{z,\theta}$$

Thus, the raw expression for the given set of transformations is given by:

$$H = R_{x,\psi} * R_{z,\phi} * T_{y,y} * R_{z,\theta}$$

Finally, the correct matrix multiplication expression is given by:

$$\text{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c(\psi) & -s(\psi) & 0 \\ 0 & s(\psi) & c(\psi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} c(\phi) & -s(\phi) & 0 & 0 \\ s(\phi) & c(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} *$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} c(\theta) & -s(\theta) & 0 & 0 \\ s(\theta) & c(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.2 Final Answer

I used SymPy to perform the matrix multiplication outlined above and yielded the final answer:

$$\mathrm{R} = \begin{bmatrix} c(\phi)c(\theta) - s(\phi)s(\phi) & -c(\phi)s(\theta) - s(\phi)c(\theta) & 0 & -y * s(\phi) \\ c(\psi)s(\phi)c(\theta) + c(\psi)c(\psi)s(\theta) & c(\psi)c(\phi)c(\theta) - c(\psi)s(\psi)s(\theta) & -s(\psi) & y * c(\psi)c(\phi) \\ s(\psi)s(\phi)c(\theta) + s(\psi)c(\psi)s(\theta) & s(\psi)c(\phi)c(\theta) - s(\psi)s(\psi)s(\theta) & c(\psi) & y * s(\psi)c(\phi) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3 Problem 1.2: Modeling Beyond Rigid Transformations

3.1 Steps Involved and Explanation

To start this problem, I first had to gather some information from the internet. From a ScienceDirect article, I found that the minimum distance from the Earth to the Sun is 147 million km and the maximum distance from the Earth to the Sun is 152 million km. I will be referring to these variables moving forward as:

$$D_{min} = 147,000,000km$$

$$D_{max} = 152,000,000km$$

Additionally, I had to redraw the situation introducing a new frame: the sun frame. Below in Figure 1 is my new drawing of the situation with D_{min}, D_{max}, and the sun frame added:

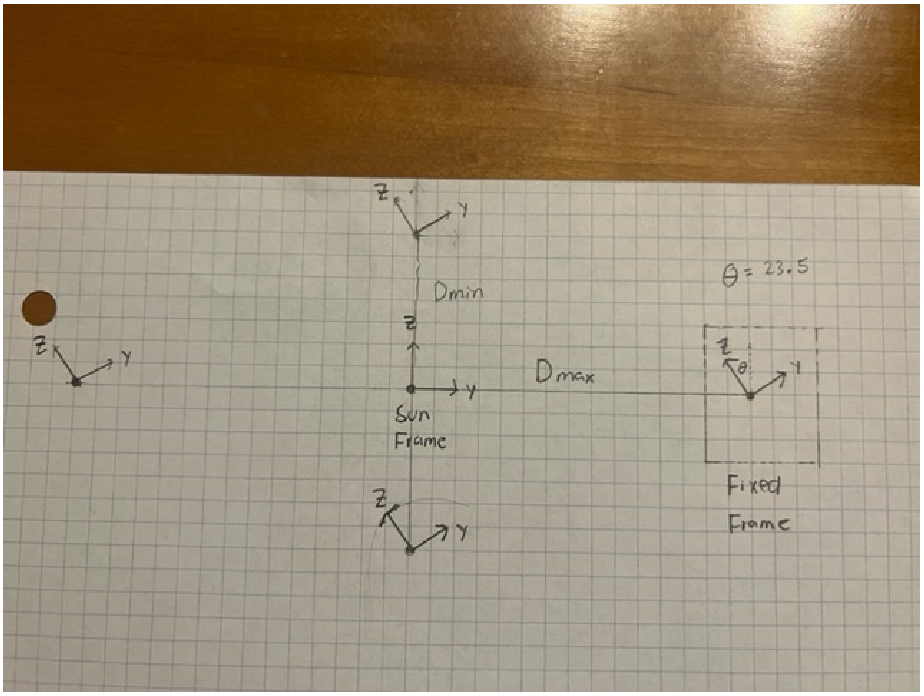

Figure 1: Problem Statement

To solve this problem, I broke it up into multiple steps. I defined my time variable, t, as hours. First, I started by representing the rotation of the Earth around its z axis as a function of time. To represent this, I used a rotation matrix. Since we know that the Earth rotates around its Z axis once every 24 hours, I edited the trigonometric functions located within the rotation matrix to have a period of 24 hours. The sine equations become $sin(\frac{2*\pi*t}{24})$ and the cosine equations become $cos(\frac{2*\pi*t}{24})$. Thus, the transformation matrix for the Earth's axial rotation in the fixed frame is as follows:

$$H_{Axial,Fixed}(t) = \begin{bmatrix} cos(\frac{2*\pi*t}{24}) & -sin(\frac{2*\pi*t}{24}) & 0 & 0 \\ sin(\frac{2*\pi*t}{24}) & cos(\frac{2*\pi*t}{24}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The next part of the problem stems from the revolution of the earth around the sun in an elliptical form. Even though this is called a revolution, this actually constitutes the translation of the earth around the sun. My approach to this part of the problem was to define the translation of the earth around the sun in the sun's frame, then transform that translation from the sun's frame to the fixed frame.

To define the translation of the earth in the sun's frame, I followed a similar process as before. Below is the trigonometric form of an ellipse as a function of time:

$$X(t) = \frac{A_{Major}}{2} * cos(t)$$

$$Y(t) = \frac{A_{Minor}}{2} * sin(t)$$

Since I have defined my frames in Figure 1, the applied equations would be Y and Z substituted for X and Y respectively. Because I have defined this translation as a function of time using trigonometric equations, I can follow the same process I used earlier to accurately capture the period. The Earth translates around the sun once every 365 days. To match my time variable from before, this equals 8760 hours. Thus, the period of these trigonometric equations is 8760 hours. The equations then become:

$$Y(t) = D_{max} * cos(\frac{2 * \pi * t}{8760})$$

$$Z(t) = D_{min} * sin(\frac{2 * \pi * t}{8760})$$

Thus, I build the translation matrix for the revolution around the sun in the sun's frame as follows:

$$H_{Rev,Sun}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & D_{max} * cos(\frac{2*\pi*t}{8760}) \\ 0 & 0 & 1 & D_{min} * sin(\frac{2*\pi*t}{8760}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The next step in this problem is to transform the previous result into the fixed frame. The fixed frame is at a distance of $+D_{max}$ in the y direction and rotated 23.5 degrees around the x axis relative to the sun's frame. I form the transformation matrix as follows:

$$H_{SunToFixed} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(23.5) & -sin(23.5) & D_{max} \\ 0 & sin(23.5) & cos(23.5) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Because I am performing these operations with respect to the current frame (the sun frame) I post-multiply these matrices. Thus the expression looks like this:

$$H_{Rev,Fixed}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & D_{max} * cos(\frac{2*\pi*t}{8760}) \\ 0 & 0 & 1 & D_{min} * sin(\frac{2*\pi*t}{8760}) \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(23.5) & -sin(23.5) & D_{max} \\ 0 & sin(23.5) & cos(23.5) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This yields:

$$H_{Rev,Fixed}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(23.5) & -sin(23.5) & D_{max} + D_{max} * cos(\frac{2*\pi*t}{8760}) \\ 0 & sin(23.5) & cos(23.5) & D_{min} * sin(\frac{2*\pi*t}{8760}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now that I have both Earth's rotation around the Z axis and the translation as functions of time in the fixed frame, I multiply the these matrices together ($H_{Rev,Fixed} * H_{Axial,Fixed}$) to get a singular transformation matrix that represents the Earth's position and orientation as a function of time.

3.2 Final Answer

Using SymPy to perform the matrix multiplication outlined before, I form the transformation matrix that fully defines the Earth's motion with respect to the fixed frame as a function of time:

$$H_{Earth}(t) = \begin{bmatrix} cos(\frac{2*\pi*t}{24}) & -sin(\frac{2*\pi*t}{24}) & 0 & 0 \\ cos(23.5)sin(\frac{2*\pi*t}{24}) & cos(23.5)cos(\frac{2*\pi*t}{24}) & -sin(23.5) & D_{max} + D_{max} * cos(\frac{2*\pi*t}{8760}) \\ sin(23.5)sin(\frac{2*\pi*t}{24}) & sin(23.5)cos(\frac{2*\pi*t}{24}) & cos(23.5) & D_{min} * sin(\frac{2*\pi*t}{8760}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4 Problem 1.3: Transform Estimation

4.1 Steps Involved and Explanation

I found the series of rotations and translations through physical trial and error. I got a cardboard box in my apartment, marked what corners are what with a sharpie, and started a trial-and-error process to solve the problem. My process first started with aligning the EFGH face with the X-Z plane. Since the box has dimensions and the rotations and translations are defined with respect to the given coordinate frame, I had to be very careful to correctly order my transformations. To align the EFGH face, I first rotated the cube around the Y axis by -180 degrees. This results in the ABCD plane to be in line with the X-Y plane with the EFGH plane parallel to the X-Y plane, but 1m below. Next, I translated the cube by +1m in z. This results in the EFGH plane to be in line with the X-Y plane with the ABCD plane parallel to the X-Y plane, but 1m above. Next, I rotated the cube by 270 degrees around the X axis. This results in the EFGH plane to be in line with the X-Z plane with the ABCD plane parallel to the X-Z plane, but shifted +1m in the Y direction. Naturally, this position/orientation results in point G to be found on the X axis. To align point F with the Z axis, I have to translate The cube +1m in the X direction. Thus, EFGH is in line with the X-Z plane, G is aligned with the X axis, and F is aligned with the Z axis.

Below are the Rotations and Translations (in order) in order to accomplish the task:

1. Rotate around Y-Axis by $\theta = -180$
2. Translate along Z-Axis by +1m
3. Translate along Y-Axis by -1m
4. Rotate around X-Axis by $\psi = 270$
5. Translate along X-Axis by +1m

The raw expression for the given set of transformations is given by (pre-multiply because rotating w.r.t. a fixed frame):

$$H = T_x(1m) * R_x(270) * T_y(-1m) * T_z(1m) * R_y(-180)$$

The correct matrix multiplication expression is given by (pre-multiply because rotating w.r.t. a fixed frame):

$$\text{H} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c(270) & -s(270) & 0 \\ 0 & s(270) & c(270) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} *$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} c(-180) & 0 & -s(-180) & 0 \\ 0 & 1 & 0 & 0 \\ -s(-180) & 0 & c(-180) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

I used SymPy to carry out the matrix multiplication above and yielded the final transformation matrix:

$$\text{H} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, I need to pull the Euler angles and translation from the rotation matrix (top right 3x3 matrix). These Euler angles are given by:

$$\psi = atan2(r_{32}, r_{33}) = atan2(-1, 0)$$

$$\theta = atan2(-r_{31}, \sqrt{(r_{32})^2 + (r_{33})^2}) = atan2(0, \sqrt{(-1)^2 + (0)^2})$$

$$\phi = atan2(r_{21}, r_{11}) = atan2(0, -1)$$

4.2 Final Answer: Euler Angles, Translation, and Sketch

$$\psi = -90$$
$$\theta = 0$$
$$\phi = 180$$
$$T_x = 1$$
$$T_y = 1$$
$$T_z = 1$$

Below in Figure 2 is a sketch of the final orientation of the cube after this series of operations.

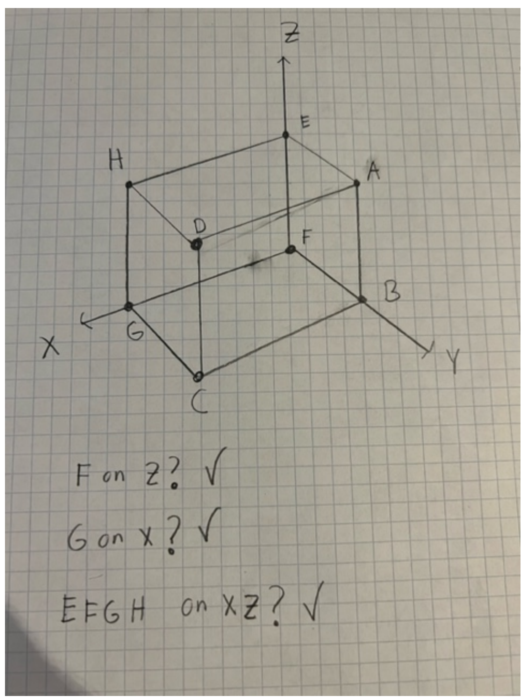

Figure 2: Final Cube Position and Orientation

5 Problem 2.1: Trajectory Optimization

5.1 Steps Involved and Explanation

The first step to completing this problem is to generate the rotation matrix based on the given Euler angles. The raw expression for this is as follows:

$$R = R_{z,\phi} * R_{y,\theta} * R_{x,\psi}$$

Simplifying this expression into a single matrix gives the following:

$$\mathrm{R} = \begin{bmatrix} c(\phi) * c(\theta) & c(\phi) * s(\theta) * s(\psi) - s(\phi) * c(\psi) & s(\phi) * s(\psi) + c(\phi) * s(\theta) * c(\psi) \\ s(\phi) * c(\theta) & c(\phi) * c(\psi) + s(\phi) * s(\theta) * s(\psi) & s(\phi) * s(\theta) * c(\psi) - c(\phi) * s(\psi) \\ -s(\theta) & c(\theta) * s(\psi) & c(\theta) * c(\psi) \end{bmatrix}$$

Substituting the Euler angle values in radians gives:

$$\mathrm{R} = \begin{bmatrix} 0.871 & 0.092 & 0.482 \\ 0.250 & 0.762 & -0.597 \\ -0.422 & 0.641 & 0.641 \end{bmatrix}$$

Next, I converted the rotation matrix to axis-angle representation. Using the following formulas:

$$\theta = cos^{-1}(\frac{r_{11} + r_{22} + r_{33} - 1}{2})$$

$$k = \frac{1}{2sin(\theta)} * \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

This yields:

$$\theta = 0.880rad$$

$$\begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} 0.803 \\ 0.587 \\ 0.102 \end{bmatrix}$$

Since k_x is the largest, I will set $\omega_x = \omega_{max} = 2deg/s = 0.0349rad/s$.

To solve for ω_y and ω_z, I first have to solve for $\dot{\theta}$:

$$\omega_x = \dot{\theta} * k_x$$

$$\dot{\theta} = \frac{\omega_x}{k_x} = 0.0435rad/s$$

To solve for ω_y:

$$\omega_x = \dot{\theta} * k_y = 0.0255rad/s$$

To solve for ω_z:

$$\omega_z = \dot{\theta} * k_z = 0.0045 rad/s$$

Notice how ω_y and ω_z are both less than ω_{max}. Additionally, I have selected a constant-angular-velocity profile. To calculate the shortest time to reach orientation, I perform the following calculations:

$$\dot{\theta} = \frac{\theta}{T}$$

$$T = \frac{0.880 rad}{0.0435 rad/s} = 20.25s$$

To convert back to a rotation matrix from axis-angle representation, I use the formula:

$$R_k = \begin{bmatrix} (k_x)^2 * V_\theta + cos(\theta) & k_x * k_y * V_\theta - k_z * sin(\theta) & k_x * k_z * V_\theta + k_y * sin(\theta) \\ k_x * k_y * V_\theta + k_z * sin(\theta) & (k_y)^2 * V_\theta + cos(\theta) & k_y * k_z * V_\theta - k_x * sin(\theta) \\ k_x * k_z * V_\theta - k_y * sin(\theta) & k_y * k_z * V_\theta + k_x * sin(\theta) & (k_z)^2 * V_\theta + cos(\theta) \end{bmatrix}$$

Where:

$$V_\theta = V * cos(\theta) = 1 - cos(\theta)$$

Additionally, the Euler angles can be calculated through the following formulas:

$$\psi = atan2(r_{32}, r_{33})$$

$$\theta = atan2(-r_{31}, \sqrt{(r_{32})^2 + (r_{33})^2})$$

$$\phi = atan2(r_{21}, r_{11})$$

Finally I used a Python script to plot the Euler angles as well as ω_x, ω_y, and ω_z with respect to time by substituting $\theta = \dot{\theta} * t$ where $\dot{\theta} = 0.0435 rad/s$ and t is the time step (0.1s) into R_k. See the next section for the resulting plots.

5.2 Final Answer: Angular Plots, ω Plots, and Minimum Completed Rotation Time

The minimum time it takes to reach the desired orientation is: 20.251 seconds. Pictured below in Figure 3 are the plots containing information for the Euler angles and angular velocities as a function of time.

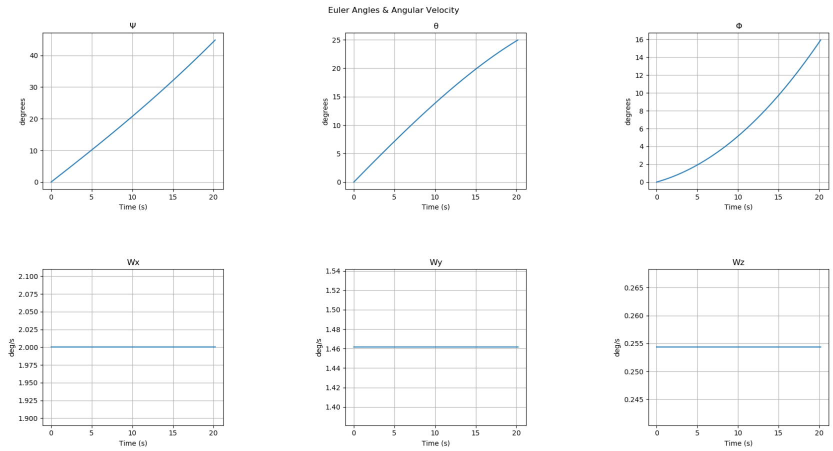

Figure 3: Euler Angles and Angular Velocities