# Basic Proofs in Predicate Logic

## Building Arguments Step by Step

Brendan Shea, PhD

Intro to Logic

# What Is a Proof? Building Logical Bridges

- A **proof** is a sequence of logical steps showing a conclusion **must** be true given certain premises.
- Proofs are **deductive**: they preserve truth with 100% certainty from premises to conclusion.
- Each step follows by a **rule of inference**—a logical principle that guarantees truth preservation.
- Unlike inductive reasoning (probable conclusions) or abductive reasoning (best explanations), deductive proofs offer absolute certainty!

## Deductive Certainty

$$\boxed{\text{True Premises}} \xrightarrow{\text{Deductive Steps}} \boxed{\text{True Conclusion}}$$

Truth is guaranteed to flow forward

# Why Proofs Matter: From Ancient Geometry to Modern Cryptography

- **Mathematics**: Proofs ensure that mathematical facts are eternally true, not just probably true.
- **Computer Security**: Your online banking is safe because of mathematical proofs about encryption.
- **Science**: Proofs help us distinguish between correlation and causation in research.
- **Critical Thinking**: Learning to prove things makes you immune to logical fallacies and bad arguments.

### Example

Real-world applications of proofs:

- Proving that a sorting algorithm always works correctly
- Proving that a medical treatment actually causes improvement
- Proving that a voting system is fair and can't be manipulated
- Proving that a bridge design can support its maximum load

# Proofs vs. Persuasion: The Gold Standard of Certainty

- **Deductive proof**: Guarantees truth—if premises are true, conclusion MUST be true.
- **Inductive reasoning**: Suggests probability—"All swans I've seen are white, so probably all swans are white."
- **Abductive reasoning**: Offers best explanation—"The grass is wet, so it probably rained."
- Only deductive proofs provide absolute certainty; the others give us useful but fallible conclusions.

## Types of Reasoning

| **Type** | **Example** | **Certainty** |
|---|---|---|
| Deductive | All birds have wings; X is a bird; so X has wings | 100% |
| Inductive | I've seen 1000 white swans; next one is probably white | High probability |
| Abductive | Patient has symptoms A,B,C; likely has disease D | Best guess |

# The Structure of a Formal Proof: Steps and Justifications

- Every proof consists of numbered lines, each containing a **statement** and its **justification**.
- Justifications can be: premises (given), rules of inference, or references to previous lines.
- Each line follows logically from previous lines according to specific rules.
- The last line of the proof is your conclusion—what you set out to prove!

## Anatomy of a Proof

| **Line** | **Statement** | **Justification** |
|---|---|---|
| 1 | $P \rightarrow Q$ | Premise |
| 2 | $P$ | Premise |
| 3 | $Q$ | Modus Ponens (1,2) |

This proves: From "If $P$ then $Q$" and "$P$", we can conclude "$Q$"

# Valid Arguments: When Conclusions Must Follow

- An argument is **valid** if the conclusion must be true whenever all premises are true.
- Validity is about the structure of the argument, not whether premises are actually true.
- A valid argument with true premises is called **sound**—this guarantees a true conclusion.
- Our proof rules preserve validity: applying them to true statements always yields true statements.

## Example

Valid argument (even with silly premises):

1. All creatures in Wonderland can talk (premise)
2. The Cheshire Cat is in Wonderland (premise)
3. Therefore, the Cheshire Cat can talk (conclusion)

The structure is valid: All A's are B, X is an A, therefore X is B.

# Common Proof Mistakes: Learning from Logical Missteps

- **Circular reasoning**: Using what you're trying to prove as a justification.
- **Unjustified jumps**: Skipping steps that seem "obvious" but need proof.
- **Wrong rule application**: Using a rule incorrectly or in the wrong situation.
- **Assuming the converse**: Thinking "If $P$ then $Q$" means "If $Q$ then $P$".

## Classic Mistakes to Avoid

- "Everyone knows that..." — Not a valid justification!
- Going from "Some cats are black" to "Fluffy is black" — Need more info!
- Using $P \rightarrow Q$ backwards — Can't conclude $P$ from $Q$!
- Forgetting to cite line numbers — Every step needs justification!

# Your First Proof: A Simple Example to Get Started

- Let's prove: "Dorothy has silver shoes and wants to go home, so Dorothy wants to go home."
- Premises: (1) Dorothy has silver shoes AND Dorothy wants to go home
- Conclusion: Dorothy wants to go home
- We'll use our first rule: **conjunction elimination** (taking AND statements apart).

## Example

| Line | Statement | Justification |
|---|---|---|
| 1 | `HasSilverShoes(Dorothy)` $\wedge$ `WantsHome(Dorothy)` | Premise |
| 2 | `WantsHome(Dorothy)` | $\wedge$ Elimination (1) |

We extracted the second part of the AND statement—our first proof is complete!

# The Toolbox: Introduction and Elimination Rules

- For each logical operator, we have two types of rules: **introduction** and **elimination**.
- **Introduction rules** show how to prove statements with that operator.
- **Elimination rules** show how to use statements with that operator.
- Think of it like a toolbox: some tools build things up, others take them apart!

## Rules for Each Operator

| Operator | Introduction | Elimination |
|---|---|---|
| $\wedge$ (and) | Combine statements | Extract parts |
| $\vee$ (or) | Add possibilities | Consider cases |
| $\rightarrow$ (if-then) | Assume and derive | Apply condition |
| $\neg$ (not) | Show contradiction | Cancel double negative |

# Conjunction Introduction: Building "And" Statements

- **Conjunction Introduction** ($\wedge$I): If you've proved $P$ and you've proved $Q$, you can conclude $P \wedge Q$.
- This rule lets us combine separate facts into a single compound statement.
- Symbol: $\frac{P \quad Q}{P \wedge Q}$ (from $P$ and $Q$ separately, infer $P \wedge Q$).
- The most straightforward rule—if you know both parts, you know the whole!

## Example

Rule in action:

| Line | Statement | Justification |
|---|---|---|
| 1 | `InKansas(Dorothy)` | Premise |
| 2 | `HasDog(Dorothy, Toto)` | Premise |
| 3 | `InKansas(Dorothy)` $\wedge$ `HasDog(Dorothy, Toto)` | $\wedge$I (1,2) |

# Proving Dorothy Has Ruby Slippers AND Wants to Go Home: Conjunction Introduction Practice

- Given: (1) Dorothy has ruby slippers, (2) Dorothy wants to go home
- Goal: Prove that Dorothy has ruby slippers AND wants to go home
- Strategy: Use conjunction introduction to combine the two given facts.
- This shows how we build complex statements from simpler ones.

## The Proof

| Line | Statement | Justification |
|---|---|---|
| 1 | `HasRubySlippers(Dorothy)` | Premise |
| 2 | `WantsHome(Dorothy)` | Premise |
| 3 | `HasRubySlippers(Dorothy)` ∧ `WantsHome(Dorothy)` | ∧I (1,2) |

Notice how line 3 references both lines 1 and 2—we need both pieces!

# Conjunction Elimination: Taking "And" Statements Apart

- **Conjunction Elimination** ($\wedge$E): From $P \wedge Q$, you can conclude $P$ or conclude $Q$ (or both).
- This rule lets us extract useful information from compound statements.
- Two forms: $\frac{P \wedge Q}{P}$ (left elimination) and $\frac{P \wedge Q}{Q}$ (right elimination).
- Like unpacking a box—if the box contains both items, you can take out either one!

## Using $\wedge$E

From "`IsWitch(Glinda)` $\wedge$ `IsGood(Glinda)`":

- We can conclude `IsWitch(Glinda)` (left part)
- We can conclude `IsGood(Glinda)` (right part)
- We can use both conclusions in our proof!

# From Complex to Simple: Conjunction Elimination Practice

- Given: The Scarecrow wants a brain AND is made of straw AND can talk
- Goal: Prove that the Scarecrow can talk
- Strategy: Use conjunction elimination to extract the part we need.
- With nested conjunctions, we may need multiple elimination steps!

**Example**

| Line | Statement | Justifi |
|---|---|---|
| 1 | (WantsBrain(Scarecrow) ∧ MadeOfStraw(Scarecrow)) ∧ CanTalk(Scarecrow) | Premis |
| 2 | CanTalk(Scarecrow) | ∧E (1) |

We extracted the rightmost part of the complex conjunction!

# Disjunction Introduction: Creating "Or" Statements

- **Disjunction Introduction** ($\vee$I): If you've proved $P$, you can conclude $P \vee Q$ for any $Q$.
- This might seem strange—we're adding uncertainty where there was none!
- But it's logically valid: if $P$ is true, then "$P$ or $Q$" must be true (regardless of $Q$).
- Useful when you need to match a specific form or work with given disjunctions.

## Two Forms of $\vee$I

- From $P$, conclude $P \vee Q$ (left introduction)
- From $Q$, conclude $P \vee Q$ (right introduction)

Example: From "Alice is in Wonderland", we can conclude "Alice is in Wonderland OR Alice is in Oz"

# Proving "The Cat Is Grinning OR Invisible": Disjunction Introduction Practice

- Given: The Cheshire Cat is grinning
- Goal: Prove that the Cheshire Cat is grinning OR the Cheshire Cat is invisible
- Strategy: Use disjunction introduction to add the alternative possibility.
- Remember: We're not claiming the cat is invisible—just that at least one option is true!

## Example

| Line | Statement | Justification |
|---|---|---|
| 1 | IsGrinning(CheshireCat) | Premise |
| 2 | IsGrinning(CheshireCat) $\vee$ IsInvisible(CheshireCat) | $\vee$I (1) |

Even though we only know the cat is grinning, the "or" statement is certainly true!

# Disjunction Elimination: The Case-by-Case Method

- **Disjunction Elimination** ($\vee$E): The most complex rule so far—reasoning by cases.
- If you know $P \vee Q$, and you can prove $R$ from $P$, and prove $R$ from $Q$, then $R$ must be true.
- Think: "Either way, we get the same result, so that result must hold."
- Requires setting up two sub-proofs, one for each possibility.

### Structure of $\vee$E

To use $P \vee Q$:

1. Assume $P$ temporarily, prove $R$
2. Assume $Q$ temporarily, prove $R$
3. Conclude $R$ (since it follows either way)

Like checking both paths at a fork in the road and finding they lead to the same place!

# Working Through Cases: Disjunction Elimination Practice

- Given: Either the Tin Man needs oil OR he needs a heart
- Given: If he needs oil, he can't move. If he needs a heart, he can't love.
- Goal: Prove that either he can't move OR he can't love
- Strategy: Consider both cases and show each leads to our conclusion.

## Example

| | | |
|---|---|---|
| 1 | `NeedsOil(TinMan)` $\vee$ `NeedsHeart(TinMan)` | Premise |
| 2 | `NeedsOil(TinMan)` $\to$ `CantMove(TinMan)` | Premise |
| 3 | `NeedsHeart(TinMan)` $\to$ `CantLove(TinMan)` | Premise |
| 4 | `NeedsOil(TinMan)` | Assume (for $\vee$E) |
| 5 | `CantMove(TinMan)` | Modus Ponens (2,4) |
| 6 | `CantMove(TinMan)` $\vee$ `CantLove(TinMan)` | $\vee$I (5) |
| 7 | `NeedsHeart(TinMan)` | Assume (for $\vee$E) |
| 8 | `CantLove(TinMan)` | Modus Ponens (3,7) |
| 9 | `CantMove(TinMan)` $\vee$ `CantLove(TinMan)` | $\vee$I (8) |
| 10 | `CantMove(TinMan)` $\vee$ `CantLove(TinMan)` | $\vee$E (1,4-6,7-9) |

# Conditional Introduction: Building "If-Then" Statements

- **Conditional Introduction** ($\rightarrow$I): To prove $P \rightarrow Q$, assume $P$ and derive $Q$.
- This captures how we naturally think about "if-then": "If this were true, what would follow?"
- The assumption is temporary—only valid within the sub-proof.
- Once we derive $Q$ from the assumption $P$, we can conclude $P \rightarrow Q$ outside the sub-proof.

## The Power of Assumption

Assume $P$

... derive ...

Get $Q$

$\longrightarrow P \rightarrow Q$

# Proving "If You Click Your Heels Three Times, Then You'll Go Home": Conditional Introduction

- Given: Anyone who clicks their heels three times while wearing ruby slippers goes home
- Given: Dorothy has ruby slippers
- Goal: If Dorothy clicks her heels three times, then Dorothy goes home
- Strategy: Assume the "if" part and derive the "then" part.

## Example

| | | |
|---|---|---|
| 1 | $\forall x$ ((ClicksHeels3x($x$)$\wedge$ HasRubySlippers($x$)) $\rightarrow$ GoesHome($x$)) | Premise |
| 2 | HasRubySlippers(Dorothy) | Premise |
| 3 | ClicksHeels3x(Dorothy) | Assume (for $\rightarrow$I) |
| 4 | ClicksHeels3x(Dorothy) $\wedge$ HasRubySlippers(Dorothy) | $\wedge$I (3,2) |
| 5 | GoesHome(Dorothy) | $\forall$E, MP (1,4) |
| 6 | ClicksHeels3x(Dorothy) $\rightarrow$ GoesHome(Dorothy) | $\rightarrow$I (3-5) |

# Conditional Elimination (Modus Ponens): Using "If-Then" Statements

- **Modus Ponens** (MP or $\rightarrow$E): From $P \rightarrow Q$ and $P$, conclude $Q$.
- This is perhaps the most famous rule in logic—"the mode of affirming."
- It lets us apply conditional knowledge: if we know "if-then" and the "if" part is true, the "then" part follows.
- Symbol: $\frac{P \rightarrow Q \quad P}{Q}$ (from the conditional and its antecedent, infer the consequent).

## Example

Classic example:

- If you're in Oz, you're not in Kansas anymore
- Dorothy is in Oz
- Therefore, Dorothy is not in Kansas anymore

This pattern of reasoning is used constantly in mathematics, science, and daily life!

# Following the Chain: Modus Ponens Practice

- Given: If you follow the yellow brick road, you reach Emerald City
- Given: If you reach Emerald City, you can see the Wizard
- Given: Dorothy follows the yellow brick road
- Goal: Prove that Dorothy can see the Wizard

## Chain Reasoning with Modus Ponens

| | | |
|---|---|---|
| 1 | `FollowsYBR(Dorothy)` $\rightarrow$ `ReachesEC(Dorothy)` | Premise |
| 2 | `ReachesEC(Dorothy)` $\rightarrow$ `CanSeeWizard(Dorothy)` | Premise |
| 3 | `FollowsYBR(Dorothy)` | Premise |
| 4 | `ReachesEC(Dorothy)` | MP (1,3) |
| 5 | `CanSeeWizard(Dorothy)` | MP (2,4) |

Notice how we chain modus ponens applications to follow a sequence of implications!

# Negation Introduction: Proving Something Is False

- **Negation Introduction** ($\neg$I): To prove $\neg P$, show that assuming $P$ leads to a contradiction.
- A **contradiction** is when we derive both $Q$ and $\neg Q$ for some statement $Q$.
- This rule formalizes proof by contradiction—a powerful technique!
- If assuming something leads to impossibility, that thing must be false.

## Structure of $\neg$I

1. Want to prove: $\neg P$
2. Assume: $P$ (temporarily)
3. Derive: Both $Q$ and $\neg Q$ for some $Q$
4. Conclude: $\neg P$ (since $P$ led to contradiction)

The contradiction shows our assumption was impossible!

# Proof by Contradiction: When Assuming the Opposite Leads to Trouble

- Let's prove: "There is no biggest number"
- Formally: $\neg\exists x \forall y\, (\texttt{Number}(y) \rightarrow y \leq x)$
- Strategy: Assume there IS a biggest number, then derive a contradiction.
- This technique works when direct proof seems impossible.

## Example

Wonderland example: "The Hatter is not both mad and sane"

| | | |
|---|---|---|
| 1 | `IsMad(Hatter)` | Premise |
| 2 | `IsMad(Hatter)` $\wedge$ `IsSane(Hatter)` | Assume (for $\neg$I) |
| 3 | `IsSane(Hatter)` | $\wedge$E (2) |
| 4 | `IsMad(Hatter)` | $\wedge$E (2) |
| 5 | $\neg$`IsMad(Hatter)` | Definition of sane |
| 6 | $\neg$(`IsMad(Hatter)` $\wedge$ `IsSane(Hatter)`) | $\neg$I (2-5) |

# Negation Elimination: The Double Negative Rule

- **Negation Elimination** (¬E): From $\neg\neg P$, you can conclude $P$.
- This is the logical version of canceling a double negative—"not not P" just means "P".
- While it feels obvious, it plays a crucial role in formal proofs—especially when working backwards from contradictions.
- This rule ensures that nothing stays unnecessarily wrapped in extra layers of denial!

## Example

| Line | Statement | Justification |
|---|---|---|
| 1 | `¬¬ IsWizard(Gandalf)` | Premise |
| 2 | `IsWizard(Gandalf)` | ¬E (1) |

We removed the double negation to get the straightforward result.

# Universal Introduction: Proving "All" Statements

- **Universal Introduction** ($\forall$I): To prove $\forall x\ P(x)$, prove $P(a)$ for an arbitrary $a$.
- The key: $a$ must be **arbitrary**—no special properties, could be any individual.
- If we can prove something for a "generic" individual, it holds for all individuals.
- Think: "Take any random thing from the domain. If I can prove the property for it, then all things have that property."

## The Arbitrary Individual Method

1. Introduce a new name $a$ that hasn't been used before
2. Make no assumptions about $a$ except what applies to everything
3. Prove the property holds for $a$
4. Conclude the property holds for all $x$

# The Arbitrary Individual: Why Universal Introduction Works

- An **arbitrary individual** represents "any member of the domain" without being specific.
- We can't use individuals mentioned in premises—they might have special properties!
- The name must be "fresh"—not appearing anywhere else in the proof.
- This ensures our reasoning applies to every possible individual, not just special cases.

## Valid vs Invalid $\forall$I

| **Invalid** | **Valid** |
|---|---|
| IsYellow(BigBird) | Let $a$ be arbitrary |
| Therefore $\forall x$ IsYellow($x$) | Prove HasColor($a$) |
| (BigBird is special!) | Therefore $\forall x$ HasColor($x$) |

# Proving "All Roads in Oz Lead Somewhere": Universal Introduction Practice

- Given: Every road has an endpoint (general principle)
- Given: Everything in Oz is magical
- Goal: Prove that all roads in Oz lead somewhere
- Strategy: Take an arbitrary road in Oz and show it leads somewhere.

## Example

| | | |
|---|---|---|
| 1 | $\forall x\ (\texttt{IsRoad}(x) \rightarrow \exists y\ \texttt{LeadsTo}(x, y))$ | Premise |
| 2 | $\forall x\ (\texttt{InOz}(x) \rightarrow \texttt{IsMagical}(x))$ | Premise |
| 3 | Let $a$ be arbitrary | For $\forall$I |
| 4 | $\texttt{IsRoad}(a) \wedge \texttt{InOz}(a)$ | Assume |
| 5 | $\texttt{IsRoad}(a)$ | $\wedge$E (4) |
| 6 | $\exists y\ \texttt{LeadsTo}(a, y)$ | $\forall$E, MP (1,5) |
| 7 | $\forall x\ ((\texttt{IsRoad}(x) \wedge \texttt{InOz}(x)) \rightarrow \exists y\ \texttt{LeadsTo}(x, y))$ | $\forall$I (3-6) |

# Universal Elimination: Using "All" Statements

- **Universal Elimination** ($\forall$E): From $\forall x\ P(x)$, conclude $P(t)$ for any term $t$.
- If something is true for all individuals, it's true for any specific individual we choose.
- This is the simplest quantifier rule—just plug in the individual you need!
- Symbol: $\frac{\forall x P(x)}{P(t)}$ where $t$ is any individual name.

## Example

Using universal facts:

| | | |
|---|---|---|
| 1 | $\forall x$ (`InEmeraldCity`$(x) \to$ `SeesGreen`$(x)$) | Premise |
| 2 | `InEmeraldCity(Dorothy)` | Premise |
| 3 | `InEmeraldCity(Dorothy)` $\to$ `SeesGreen(Dorothy)` | $\forall$E (1) |
| 4 | `SeesGreen(Dorothy)` | MP (2,3) |

We instantiated the universal rule with the specific individual Dorothy!

# From General to Specific: Universal Elimination Practice

- Given: All witches in Oz can cast spells
- Given: All good witches help travelers
- Given: Glinda is a good witch in Oz
- Goal: Prove that Glinda can cast spells and helps travelers

## Multiple Applications of $\forall$E

| | | |
|---|---|---|
| 1 | $\forall x\ ((\texttt{IsWitch}(x) \wedge \texttt{InOz}(x)) \to \texttt{CanCastSpells}(x))$ | Premise |
| 2 | $\forall x\ (\texttt{IsGoodWitch}(x) \to \texttt{HelpsTravelers}(x))$ | Premise |
| 3 | $\texttt{IsGoodWitch(Glinda)} \wedge \texttt{InOz(Glinda)}$ | Premise |
| 4 | $\texttt{IsGoodWitch(Glinda)}$ | $\wedge$E (3) |
| 5 | $\texttt{IsGoodWitch(Glinda)} \to \texttt{IsWitch(Glinda)}$ | Logic fact |
| 6 | $\texttt{IsWitch(Glinda)}$ | MP (4,5) |
| 7 | $\texttt{InOz(Glinda)}$ | $\wedge$E (3) |
| 8 | $\texttt{IsWitch(Glinda)} \wedge \texttt{InOz(Glinda)}$ | $\wedge$I (6,7) |
| 9 | $\texttt{CanCastSpells(Glinda)}$ | $\forall$E, MP (1,8) |
| 10 | $\texttt{HelpsTravelers(Glinda)}$ | $\forall$E, MP (2,4) |

# Existential Introduction: Proving "Some" Statements

- **Existential Introduction** ($\exists$I): From $P(t)$ for a specific $t$, conclude $\exists x \ P(x)$.
- If we know something is true for at least one individual, then "something" has that property.
- This is straightforward: finding one example proves existence!
- Symbol: $\frac{P(t)}{\exists x P(x)}$ where $t$ is any individual.

## Example

Proving existence:

- We know: The Cowardly Lion lives in Oz and wants courage
- We can conclude: Something in Oz wants courage
- We can conclude: Some lion wants courage
- Both conclusions follow by $\exists$I!

The specific example (Cowardly Lion) proves the existential claim.

# Finding Your Example: Existential Introduction Practice

- Given: The White Rabbit is always late
- Given: The White Rabbit works for the Queen of Hearts
- Goal: Prove that someone who works for the Queen is always late
- Strategy: Use the White Rabbit as our witness for the existential claim.

## From Specific to Existential

| | | |
|---|---|---|
| 1 | `AlwaysLate(WhiteRabbit)` | Premise |
| 2 | `WorksFor(WhiteRabbit, QueenOfHearts)` | Premise |
| 3 | `AlwaysLate(WhiteRabbit)` $\wedge$ `WorksFor(WhiteRabbit, QueenOfHearts)` | $\wedge$I (1,2) |
| 4 | $\exists x$ (`AlwaysLate`($x$)$\wedge$ `WorksFor`($x$, `QueenOfHearts`)) | $\exists$I (3) |

The White Rabbit serves as our proof that such an individual exists!

# Existential Elimination: Using "Some" Statements Carefully

- **Existential Elimination** ($\exists$E): The trickiest quantifier rule!
- From $\exists x\ P(x)$, we know something has property $P$, but we don't know which thing.
- Solution: Give it a temporary name (that hasn't been used before) and reason about it.
- Key restriction: The conclusion cannot mention this temporary name!

## Structure of $\exists$E

1. Have: $\exists x\ P(x)$
2. Say: "Let $c$ be something such that $P(c)$" (where $c$ is fresh)
3. Derive: Some conclusion $Q$ using $P(c)$
4. Conclude: $Q$ (but $Q$ cannot contain $c$!)

We can use the witness temporarily but can't make claims about it specifically.

# The Name Game: Existential Elimination Practice

- Given: Someone stole the tarts
- Given: Anyone who stole the tarts will be punished by the Queen
- Goal: Prove that someone will be punished by the Queen
- Strategy: Name the unknown thief temporarily and reason about them.

## Example

| | | |
|---|---|---|
| 1 | $\exists x$ StoleTarts($x$) | Premise |
| 2 | $\forall x$ (StoleTarts($x$) $\rightarrow$ PunishedByQueen($x$)) | Premise |
| 3 | Let $c$ be such that StoleTarts($c$) | ∃E setup (1) |
| 4 | StoleTarts($c$) $\rightarrow$ PunishedByQueen($c$) | ∀E (2) |
| 5 | PunishedByQueen($c$) | MP (3,4) |
| 6 | $\exists x$ PunishedByQueen($x$) | ∃I (5) |
| 7 | $\exists x$ PunishedByQueen($x$) | ∃E (1,3-6) |

Note: The conclusion (line 7) doesn't mention $c$—it was just a temporary name!

# Proof Strategy: Working Backwards from Your Goal

- **Forward reasoning**: Start with premises and apply rules until you reach the goal.
- **Backward reasoning**: Look at what you want to prove and ask "What would I need?"
- Most proofs combine both approaches—work from both ends until they meet!
- Always keep your goal in mind; every step should bring you closer to it.

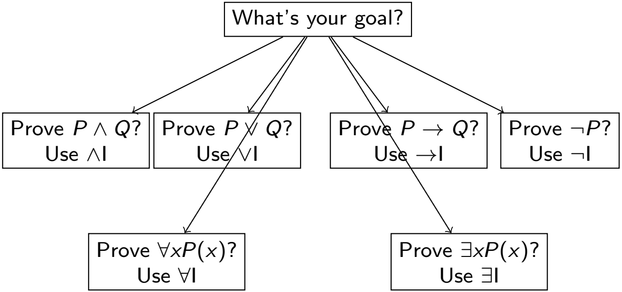
## Strategic Questions

When stuck, ask yourself:

- What form is my goal? (Conditional? Universal? Negation?)
- What rule creates that form? (Need $\rightarrow$? Use $\rightarrow$I!)
- What premises haven't I used yet?
- Can I break complex premises into simpler parts?

# When to Use Which Rule: A Decision Tree

- Look at the **main operator** of what you're trying to prove or use.
- To prove statements: use introduction rules
- To use statements: use elimination rules
- Match the rule to the operator!



# Common Proof Patterns: Recognizing Standard Arguments

- **Chain of implications**: Use repeated modus ponens to follow $P \to Q \to R$.
- **Proof by cases**: When given $P \vee Q$, show your goal follows from each option.
- **Universal to specific**: Apply general rules to particular individuals with $\forall$E.
- **Existence proof**: Find one example and use $\exists$I.

## Example

Classic patterns in Wonderland:

- "All cards are flat; the King is a card; so the King is flat" ($\forall$E + MP)
- "Either eat the cake or drink the potion; both lead to size change" ($\vee$E)
- "If late, then in trouble; late; therefore in trouble" (MP)
- "The Cheshire Cat can disappear; so something can disappear" ($\exists$I)

## Tips for Getting Unstuck: What to Do When You're Lost

- **List what you have**: Write down all premises and what you've proven so far.
- **Clarify your goal**: What exactly are you trying to prove? What's its structure?
- **Work backwards**: What would directly give you the conclusion? What would give you that?
- **Look for unused premises**: Every premise should typically be used in the proof.

### Common Unsticking Techniques

- Can't prove $P \to Q$? Try assuming $P$ and proving $Q$
- Can't prove $\neg P$? Try assuming $P$ and finding a contradiction
- Have $P \vee Q$ but stuck? Try both cases separately
- Can't prove $\forall x P(x)$? Prove it for an arbitrary individual
- Remember: Complex statements can often be broken into simpler parts!

# Checking Your Work: How to Know Your Proof Is Complete

- **Every line is justified**: Each step cites a rule and the exact lines it uses.
- **Rules applied correctly**: Check that you're using each rule properly.
- **Fresh names where required**: For ∀I and ∃E, ensure names are truly new.
- **Conclusion matches goal**: Your last line should be exactly what you set out to prove.

## Proof Checklist

1. ✓ Are all premises listed and numbered?
2. ✓ Does each line follow from previous ones?
3. ✓ Are all subproofs properly closed?
4. ✓ Are variable restrictions respected?
5. ✓ Is the conclusion exactly what was required?

If you can check all boxes, your proof is complete!

# Next Steps: From Basic Proofs to Mathematical Reasoning

- You've learned the fundamental rules of deductive reasoning—the foundation of all mathematical proof!
- These same techniques appear in: calculus proofs, computer science algorithms, philosophical arguments.
- Remember: Proof-writing is a skill that improves with practice—keep working problems!

### Where Proofs Will Take You

- **Mathematics**: Prove theorems about numbers, shapes, and functions
- **Computer Science**: Verify algorithms work correctly every time
- **Philosophy**: Analyze arguments about knowledge, ethics, and reality
- **Science**: Distinguish between correlation and causation