Introduction to Propositional Logic

Brendan Shea, PhD

March 20, 2025

What is Propositional Logic?

- **Propositional logic** is a branch of logic that studies how simple statements combine to form complex statements.
- Propositional logic helps us determine whether arguments are valid based on their structure alone.
- We use symbols to represent statements and show how they connect to each other.
- Propositional logic is the foundation for mathematical proofs, computer programming, and clear reasoning.

Why Study Logic?

Logic gives us tools to evaluate arguments, avoid fallacies, and communicate precisely in mathematics, computer science, law, and everyday reasoning.

Statements vs. Non-Statements

- A **statement** (or proposition) is a declarative sentence that is either true or false, but not both.
- Statements make claims about the world that can be evaluated as true or false.
- Questions, commands, exclamations, and opinions are not statements in propositional logic.
- We use letters like p, q, and r to represent simple statements.

Examples

Statements	**Non-Statements**
Paris is in France.	Is Paris in France?
$2 + 2 = 5$.	Go to Paris!
All squares are rectangles.	I wish I were in Paris.

Introducing Logical Symbols

- **Logical symbols** allow us to write complex statements in a precise, mathematical way.
- Each symbol represents a specific logical operation or connection between statements.
- Using symbols helps us avoid the ambiguity that can occur in everyday language.
- We combine symbols to translate complex English sentences into logical form.

$\neg$ or $\sim$	Negation ("not")
$\wedge$	Conjunction ("and")
$\vee$	Disjunction ("or")
$\rightarrow$	Conditional ("if...then")
$\leftrightarrow$	Biconditional ("if and only if")

The Negation Symbol ($\sim$): Saying "Not"

- **Negation** is the simplest logical operation, which simply reverses the truth value of a statement.
- We write the negation of statement p as $\sim p$ (read as "not p").
- If p is true, then $\sim p$ is false; if p is false, then $\sim p$ is true.
- Double negation cancels out: $\sim (\sim p)$ is equivalent to p.

Example

Let $p =$ "It is raining."
$\sim p =$ "It is not raining."
Let $q =$ "The earth is flat."
$\sim q =$ "The earth is not flat."

The Conjunction Symbol ($\wedge$): Saying "And"

- **Conjunction** connects two statements with "and," requiring both to be true.
- We write the conjunction of statements p and q as $p \wedge q$ (read as "p and q").
- $p \wedge q$ is true only when both p and q are true.
- $p \wedge q$ is false if either p is false, q is false, or both are false.

Real-World Example

To get an A in this class, you need to pass the midterm exam AND complete the final project.
Let p = "You pass the midterm exam."
Let q = "You complete the final project."
Getting an A = $p \wedge q$

The Disjunction Symbol ($\vee$): Saying "Or"

- **Disjunction** connects two statements with "or," requiring at least one to be true.
- We write the disjunction of statements p and q as $p \vee q$ (read as "p or q").
- $p \vee q$ is true when p is true, q is true, or both are true.
- $p \vee q$ is false only when both p and q are false.

Important Note

In logic, we use the **inclusive or** by default, which means "p or q or both." This is different from the **exclusive or** (sometimes called "either/or"), which means "p or q but not both."

The Conditional Symbol ($\rightarrow$): "If...Then"

- A **conditional statement** expresses that one thing depends on another.
- We write "if p then q" as $p \rightarrow q$ (read as "p implies q").
- The statement p is called the **antecedent**, and q is called the **consequent**.
- $p \rightarrow q$ is false only when p is true and q is false; it is true in all other cases.

If you study hard, then you will pass the test.

p: You study hard (antecedent)
q: You will pass the test (consequent)
$p \rightarrow q$: If you study hard, then you will pass the test.

The conditional is only false when you study hard but do not pass.

The Biconditional Symbol ($\leftrightarrow$): "If and Only If"

- A **biconditional statement** expresses that two statements have the same truth value.
- We write "p if and only if q" as $p \leftrightarrow q$ (read as "p if and only if q").
- $p \leftrightarrow q$ is true when both p and q are true, or both are false.
- $p \leftrightarrow q$ is false when one statement is true and the other is false.

Understanding Biconditionals

$p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$
It means "if p then q, AND if q then p"
Example: "A triangle is equilateral if and only if all its angles are equal."

Translating English to Symbols: Simple Statements

- Translating English sentences into logical symbols requires identifying the logical structure.
- First, identify simple statements and assign letters like p, q, r to them.
- Then identify logical connectives (and, or, not, if-then, if and only if).
- Finally, combine the symbols according to the structure of the original sentence.

Example

"It is not raining, but it is cold."
Let $p =$ "It is raining." Let $q =$ "It is cold."
Translation: $\sim p \wedge q$

"Either I will go to the movie or I will stay home."
Let $r =$ "I will go to the movie." Let $s =$ "I will stay home."
Translation: $r \vee s$

Translating English to Symbols: Complex Statements

- Complex statements may contain multiple logical connectives.
- **Parentheses** help clarify the order of operations in complex statements.
- Work step by step, breaking down complicated sentences into simpler parts.
- Be careful with negations, as they can apply to individual statements or entire expressions.

Example Translation

"If it's sunny and warm, then I'll go swimming or hiking."

Let p = "It's sunny."
Let q = "It's warm."
Let r = "I'll go swimming."
Let s = "I'll go hiking."

Translation: $(p \wedge q) \rightarrow (r \vee s)$

Common Phrases for Conditionals

- Conditionals ($p \to q$) can be expressed in many different ways in English.
- The "if-then" form is just one way to express a conditional relationship.
- Understanding these different forms helps translate real-world statements into logic.
- The logical meaning remains the same regardless of the phrasing.

Different Ways to Say $p \to q$

- If p, then q.
- p implies q.
- p only if q.
- q if p.
- p is sufficient for q.
- q is necessary for p.
- Unless not-q, p.

The Biconditional Symbol ($\leftrightarrow$): "If and Only If"

- A **biconditional statement** expresses that two statements have the same truth value.
- We write "p if and only if q" as $p \leftrightarrow q$ (read as "p if and only if q").
- $p \leftrightarrow q$ is true when both p and q are true, or both are false.
- $p \leftrightarrow q$ is false when one statement is true and the other is false.

Understanding Biconditionals

$p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$
It means "if p then q, AND if q then p"
Example: "A triangle is equilateral if and only if all its angles are equal."

Translating English to Symbols: Simple Statements

- Translating English sentences into logical symbols requires identifying the logical structure.
- First, identify simple statements and assign letters like p, q, r to them.
- Then identify logical connectives (and, or, not, if-then, if and only if).
- Finally, combine the symbols according to the structure of the original sentence.

Example

"It is not raining, but it is cold."
Let $p =$ "It is raining." Let $q =$ "It is cold."
Translation: $\sim p \wedge q$

"Either I will go to the movie or I will stay home."
Let $r =$ "I will go to the movie." Let $s =$ "I will stay home."
Translation: $r \vee s$

Translating English to Symbols: Complex Statements

- Complex statements may contain multiple logical connectives.
- **Parentheses** help clarify the order of operations in complex statements.
- Work step by step, breaking down complicated sentences into simpler parts.
- Be careful with negations, as they can apply to individual statements or entire expressions.

Example Translation

"If it's sunny and warm, then I'll go swimming or hiking."

Let p = "It's sunny."
Let q = "It's warm."
Let r = "I'll go swimming."
Let s = "I'll go hiking."

Translation: $(p \wedge q) \rightarrow (r \vee s)$

Common Phrases for Conditionals

- Conditionals ($p \rightarrow q$) can be expressed in many different ways in English.
- The "if-then" form is just one way to express a conditional relationship.
- Understanding these different forms helps translate real-world statements into logic.
- The logical meaning remains the same regardless of the phrasing.

Different Ways to Say $p \rightarrow q$

- If p, then q.
- p implies q.
- p only if q.
- q if p.
- p is sufficient for q.
- q is necessary for p.
- Unless not-q, p.

Spotting Hidden Conditionals in Everyday Language

- Many English sentences contain **hidden conditionals** that are not expressed in the "if-then" form.
- Signs, warnings, and rules often contain implicit conditional statements.
- Recognizing these allows us to translate a wider range of statements into logical form.
- Translating these statements correctly is crucial for analyzing arguments.

Examples of Hidden Conditionals

"No pets allowed" → "If you have a pet, then you are not allowed."
"Students must complete homework to pass" → "If a student does not complete homework, then the student will not pass."
"Only adults may enter" → "If a person enters, then that person is an adult."

Truth Values: True or False

- In propositional logic, every statement has exactly one **truth value**: True (T) or False (F).
- The truth value of a complex statement depends on the truth values of its simple components.
- Logical operators ($\sim$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$) have specific rules for determining truth values.
- These rules are consistent and do not depend on the content of the statements.

Statement: "Paris is the capital of France."
Truth value: TRUE

Statement: "New York is the capital of the United States."
Truth value: FALSE

Statement: "Paris is the capital of France AND New York is the capital of the United States."
Truth value: FALSE (since one component is false)

Introduction to Truth Tables

- A **truth table** is a systematic way to determine the truth value of a compound statement.
- Truth tables list all possible combinations of truth values for the simple statements.
- Each row represents one possible scenario of truth values.
- For n simple statements, there are 2^n possible combinations of truth values.

Structure of a Truth Table

- Left columns show truth values of simple statements (p, q, etc.)
- Right columns show truth values of increasingly complex expressions
- Final column shows truth value of the complete expression
- Each row represents one possible "world" or scenario

Truth Table for Negation

- The truth table for negation ($\sim p$) shows how negation flips truth values.
- It is the simplest truth table, with only two rows (since there is only one statement, p).
- Understanding negation is fundamental to building more complex truth tables.
- We can read the table as: "When p is true, $\sim p$ is false; when p is false, $\sim p$ is true."

p	$\sim p$
T	F
F	T

Example

Let $p =$ "The sun is shining."
If it's true that the sun is shining, then "The sun is not shining" ($\sim p$) is false.
If it's false that the sun is shining, then "The sun is not shining" ($\sim p$) is true.

Truth Table for Conjunction

- The truth table for conjunction ($p \wedge q$) shows when two statements joined by "and" are true.
- A conjunction is true only when both of its components (called **conjuncts**) are true.
- If either component is false, the entire conjunction is false.
- This aligns with our everyday understanding of "and" in statements.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example

"It is raining and it is cold."
This statement is true only when both "It is raining" AND "It is cold" are true.

Truth Table for Disjunction

- The truth table for disjunction ($p \vee q$) shows when two statements joined by "or" are true.
- A disjunction is true when at least one of its components (called **disjuncts**) is true.
- A disjunction is false only when both of its components are false.
- Remember that we use the inclusive "or" in logic, meaning "either or both."

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Real-World Example

"To apply for this job, you need a college degree or five years of experience."
You qualify if you have: a degree only, experience only, or both.

Truth Table for Conditional

- The truth table for the conditional ($p \rightarrow q$) often seems counterintuitive at first.
- A conditional is false only when the antecedent (p) is true and the consequent (q) is false.
- A conditional with a false antecedent is always true, regardless of the consequent.
- Think of a conditional as making a promise: it's broken only when the promise isn't fulfilled.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Why Are The Bottom Two Rows True?

If the antecedent (p) is false, the conditional is automatically true. This is called a **vacuously true** conditional. It's like saying, "If I'm the Queen of England, then I'll give you a million dollars." Since I'm not the Queen, the promise can't be broken!

Truth Table for Biconditional

- The truth table for the biconditional ($p \leftrightarrow q$) shows when two statements have the same truth value.
- A biconditional is true when both components are true or both components are false.
- A biconditional is false when one component is true and the other is false.
- The biconditional can be thought of as a "double conditional" or "two-way implication."

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example

"A number is even if and only if it is divisible by 2."
When a number is even AND divisible by 2: true
When a number is not even AND not divisible by 2: true
When a number is even BUT not divisible by 2: false (impossible)
When a number is not even BUT divisible by 2: false (impossible)

Building Complex Truth Tables: Step by Step

- Complex logical expressions require building truth tables in systematic steps.
- Work from the inside out, calculating intermediate expressions before the final result.
- Add columns for each sub-expression to track your work clearly.
- The order of operations in logic: parentheses, negation, conjunction/disjunction, conditional, biconditional.

Example

Truth table for $(p \vee q) \rightarrow \sim r$

p	q	r	$p \vee q$	$\sim r$	$(p \vee q) \rightarrow \sim r$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	T

Logical Equivalence: When Statements Mean the Same Thing

- Two statements are **logically equivalent** if they have identical truth values in all possible scenarios.
- We write "p is logically equivalent to q" as $p \equiv q$ (note this is not a logical operator).
- Truth tables can be used to determine if two statements are logically equivalent.
- Recognizing logical equivalences helps simplify complex expressions.

Important Logical Equivalences

- Double Negation: $\sim(\sim p) \equiv p$
- Commutative Laws: $p \wedge q \equiv q \wedge p$; $p \vee q \equiv q \vee p$
- De Morgan's Laws: $\sim(p \wedge q) \equiv \sim p \vee \sim q$; $\sim(p \vee q) \equiv \sim p \wedge \sim q$
- Conditional Equivalence: $p \to q \equiv \sim p \vee q$

Contradictions: Statements That Are Always False

- A **contradiction** is a compound statement that is false in every possible scenario.
- Contradictions have "F" in every row of their truth table.
- Contradictions are logically impossible statements.
- Identifying contradictions helps us avoid logical errors in reasoning.

Example of a Contradiction

Example of a contradiction: $p \wedge \sim p$

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

"It is raining and it is not raining" is always false.

Tautologies: Statements That Are Always True

- A **tautology** is a compound statement that is true in every possible scenario.
- Tautologies have "T" in every row of their truth table.
- Tautologies represent logical truths that hold regardless of the truth of their components.
- Tautologies are the foundation of valid logical arguments.

Example

Example of a tautology: $p \vee \sim p$ (Law of Excluded Middle)

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Another tautology: $p \rightarrow p$ (Law of Identity)

p	$p \rightarrow p$
T	T
F	T

Contingent Statements: Sometimes True, Sometimes False

- A **contingent statement** is neither a tautology nor a contradiction.
- Contingent statements are true in some scenarios and false in others.
- Most statements we encounter in everyday life are contingent.
- In a truth table, contingent statements have at least one T and at least one F.

Examples of Contingent Statements

- "It is raining today." (True on some days, false on others)
- $p \vee q$ (True in 3 cases, false in 1 case)
- $p \rightarrow q$ (True in 3 cases, false in 1 case)
- $(p \wedge q) \vee r$ (Truth depends on the truth values of p, q, and r)

Valid Arguments: When Conclusions Must Follow

- An argument consists of premises and a conclusion.
- A **valid argument** is one where the conclusion necessarily follows from the premises.
- If all premises are true, the conclusion must be true in a valid argument.
- The structure of an argument determines its validity, not the actual truth of its statements.

Example of a valid argument:

Premise 1: If it rains, the game will be canceled.
Premise 2: It is raining.
Conclusion: Therefore, the game will be canceled.

If we let $p =$ "It rains" and $q =$ "The game will be canceled"
This argument has the form: $(p \to q) \wedge p \therefore q$

Modus Ponens: Affirming the Antecedent

- **Modus ponens** (Latin for "method of affirming") is a fundamental rule of inference.
- Structure: If $p \to q$ is true, and p is true, then q must be true.
- This rule allows us to derive new true statements from established ones.
- Modus ponens is one of the most commonly used inference rules in logic.

Modus Ponens Format

1. $p \to q$	(If p, then q)
2. p	(p is true)
$\therefore q$	(Therefore, q is true)

Example:
1. If it's raining, then the ground is wet.
2. It is raining.

$\therefore$ The ground is wet.

Modus Tollens: Denying the Consequent

- **Modus tollens** (Latin for "method of denying") is another key rule of inference.
- Structure: If $p \rightarrow q$ is true, and q is false, then p must be false.
- This rule works by elimination: if the consequent doesn't occur, the antecedent couldn't have occurred.
- Modus tollens involves reasoning with negation and a conditional statement.

Modus Tollens Format

1. $p \rightarrow q$	(If p, then q)
2. $\sim q$	(q is false)
$\therefore \sim p$	(Therefore, p is false)

Example:
1. If it's raining, then the ground is wet.
2. The ground is not wet.

$\therefore$ It is not raining.

Fallacies to Avoid: Affirming the Consequent

- **Affirming the consequent** is a common logical fallacy that looks similar to modus ponens.
- Structure: From $p \to q$ and q, wrongly concluding p.
- This reasoning is invalid because there could be other causes for q besides p.
- The conditional only promises that if p occurs, then q will follow (not the reverse).

Affirming the Consequent (Invalid!)

1. $p \to q$	(If p, then q)
2. q	(q is true)
$\therefore p$	(INVALID conclusion!)

Example:
1. If it's raining, then the ground is wet.
2. The ground is wet.

$\therefore$ It is raining. (INVALID! The ground could be wet for many reasons.)

Fallacies to Avoid: Denying the Antecedent

- **Denying the antecedent** is another common fallacy that resembles modus tollens.
- Structure: From $p \rightarrow q$ and $\sim p$, wrongly concluding $\sim q$.
- This reasoning is invalid because q might still be true for reasons other than p.
- The conditional only tells us what happens if p is true, not what happens if p is false.

Denying the Antecedent (Invalid!)

1. $p \rightarrow q$	(If p, then q)
2. $\sim p$	(p is false)
$\therefore \sim q$	(INVALID conclusion!)

Example:
1. If you study hard, then you will pass the test.
2. You did not study hard.

$\therefore$ You will not pass the test. (INVALID! You might pass for other reasons.)

Using Multiple Inference Rules Together

- Complex arguments often require using multiple inference rules in sequence.
- Each step in the argument must be a valid inference from previous statements.
- We can derive new conclusions by applying inference rules to both premises and derived statements.
- This process is the foundation of logical proofs and deductive reasoning.

Example

Deductive chain of reasoning:

1.	$p \to q$	Premise
2.	$q \to r$	Premise
3.	$\sim r$	Premise
4.	$\sim q$	Modus Tollens: from 2 and 3
5.	$\sim p$	Modus Tollens: from 1 and 4

Example in words:
1. If it rains, the streets will be wet.
2. If the streets are wet, traffic will slow down.
3. Traffic did not slow down.
4. Therefore, the streets were not wet.
5. Therefore, it did not rain.

Analyzing Arguments from Daily Life

- Propositional logic helps us analyze and evaluate everyday arguments.
- By translating arguments into logical form, we can identify valid and invalid reasoning.
- Many arguments in advertising, politics, and daily discussions contain logical fallacies.
- Critical thinking requires recognizing the logical structure beneath the rhetoric.

Analyzing a real-world argument:

"If the economy is strong, unemployment will be low.
Unemployment is low.
Therefore, the economy is strong."

Logical form: $(p \to q) \land q \therefore p$
This is the fallacy of affirming the consequent.
Low unemployment could have other causes.

Conditional Statements in Math Problems

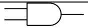
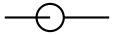
- Mathematics frequently uses conditional statements in definitions, theorems, and problems.
- Understanding the logical structure of these statements is crucial for solving math problems.
- **If-then statements** in math often establish necessary or sufficient conditions.
- Being able to recognize contrapositive statements is especially useful in mathematical reasoning.

Mathematical Examples

- "If a triangle is equilateral, then all its angles are equal."
  Contrapositive: "If not all angles of a triangle are equal, then the triangle is not equilateral."
- "If n is even, then n^2 is even."
  Contrapositive: "If n^2 is not even, then n is not even."
- "If $x > 5$, then $x^2 > 25$."
  Contrapositive: "If $x^2 \leq 25$, then $x \leq 5$."

Propositional Logic in Computer Science

- Propositional logic is fundamental to computer science and programming.
- Boolean operators in programming languages (AND, OR, NOT) directly correspond to logical operators.
- Conditional statements in code (if-then-else) rely on the principles of propositional logic.
- Circuit design uses logic gates that implement basic logical operations.

Logic	**Python**	**Circuit Symbol**
$p \wedge q$	p and q	
$p \vee q$	p or q	
$\sim p$	not p	

Review: Putting It All Together

- Propositional logic provides tools for analyzing the structure of arguments.
- We use symbols ($\sim$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$) to represent logical operations.
- Truth tables help us determine when complex statements are true or false.
- Valid argument forms like modus ponens and modus tollens allow us to draw correct conclusions.

Key Takeaways

- Learn to recognize logical patterns in everyday language.
- Be wary of common fallacies like affirming the consequent and denying the antecedent.
- Use truth tables to analyze complex statements systematically.
- Remember that logical validity is about the form of an argument, not its content.
- Propositional logic is the foundation for more advanced logical systems.