Introduction to Inductive Logic and Bayes Theorem

Understanding Reasoning Under Uncertainty

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Introduction to Logic

Welcome to Inductive Logic: Reasoning with Uncertainty

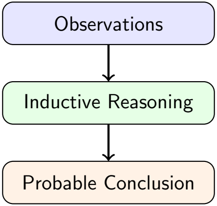
- In everyday life, we often must make decisions without complete information.
- Inductive logic provides tools to reason effectively when certainty is impossible.
- This course will show how to quantify and update beliefs as new evidence emerges.
- Learning these skills will improve your critical thinking in school and daily life.

Course Goals

- Understand the foundations of inductive reasoning
- Master basic probability concepts
- Learn to apply Bayes Theorem to real-world situations
- Develop better decision-making skills

What is Inductive Logic? Making Educated Guesses

- **Inductive logic** is reasoning that provides probable but not certain conclusions.
- Unlike deductive logic, inductive conclusions go beyond what's contained in the premises.
- We use induction whenever we learn from experience and apply it to new situations.
- Induction allows us to form generalizations and make predictions about the future.



Deductive vs. Inductive Reasoning: What's the Difference?

- **Deductive reasoning** provides conclusions that must be true if the premises are true.
- **Inductive reasoning** provides conclusions that are probably true, but not guaranteed.
- In deduction, the conclusion is contained implicitly in the premises.
- In induction, the conclusion goes beyond what is strictly contained in the premises.

Deductive Example

All mammals have lungs.
Whales are mammals.
Therefore, whales have lungs.

Inductive Example

Every swan I've seen is white.
Therefore, probably all swans are white.
(Actually false: black swans exist!)

The Role of Probability in Inductive Reasoning

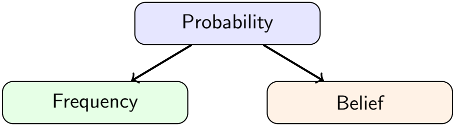
- **Probability** is the mathematical language we use to quantify uncertainty.
- Inductive reasoning relies on probability to express how likely conclusions are.
- Probability allows us to update our beliefs as new evidence emerges.
- We can use probability to compare competing explanations based on available evidence.

Key Question

Instead of asking "Is this true or false?", inductive reasoning asks "How likely is this to be true given what we know?"

Thinking About Probability: Frequency vs. Belief

- There are two main ways to interpret what probability means.
- Each interpretation is useful in different contexts and for different problems.
- These interpretations lead to different approaches to statistical reasoning.
- Understanding both views helps us apply probability appropriately.



Frequency-Type Probability: Counting Outcomes

- **Frequency probability** views probability as the long-run frequency of events.
- It answers: "If we repeat this experiment many times, how often will this outcome occur?"
- This interpretation works well for events that can be repeated, like coin flips or dice rolls.
- Frequency probability is used in many scientific disciplines to analyze data from repeated trials.

Example: Coin Flip

When we say a fair coin has a 50% probability of landing heads, we mean that if we flip the coin many times, approximately half of the outcomes will be heads.

$P(heads) = \frac{Number\ \ of\ \ heads}{Total\ \ number\ \ of\ \ flips} \approx 0.5$

Belief-Type Probability: Measuring Confidence

- **Belief probability** (also called **Bayesian probability**) represents a degree of confidence.
- It answers: "How strongly do I believe this statement based on available evidence?"
- This interpretation works for one-time events that cannot be repeated, like election outcomes.
- Belief probability can be updated as new information becomes available.

Example: Weather Forecast

When a meteorologist says "70% chance of rain tomorrow," they're expressing a degree of belief based on current evidence (weather models, atmospheric conditions, etc.).
$P(rain) = 0.7$ means "Based on current evidence, our confidence level in rain occurring is 70%."

Basic Probability Rules: The Foundation

- Probability is always measured between 0 (impossible) and 1 (certain).
- The probability of all possible outcomes for an event must sum to 1.
- For independent events A and B, the probability of both occurring is $P(A) \times P(B)$.
- For mutually exclusive events A and B, the probability of either occurring is $P(A) + P(B)$.

Formula Reference

- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- $P(A \text{ and } B) = P(A) \times P(B)$ (if independent)
- $P(\text{not } A) = 1 - P(A)$

Conditional Probability: When Events Affect Each Other

- **Conditional probability** measures the likelihood of an event given another has occurred.
- We write this as $P(A|B)$, read as "probability of A given B."
- Conditional probability captures how new information changes our assessment of likelihood.
- This concept is fundamental to understanding Bayes Theorem.

Example: Test Scores

Suppose 80% of students who study pass the test, while only 30% of students who don't study pass.
$P(\text{pass}|\text{studied}) = 0.8$
$P(\text{pass}|\text{didn't study}) = 0.3$
The vertical bar — means "given that."

Introduction to Bayes Theorem: Updating What We Know

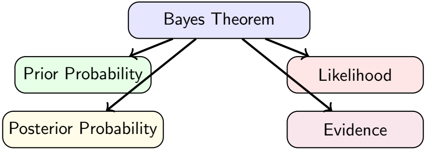
- **Bayes Theorem** allows us to update probability estimates when new evidence emerges.
- It helps us move from what we knew before (prior) to what we know now (posterior).
- The theorem was developed by Reverend Thomas Bayes in the 18th century.
- Bayes Theorem has become one of the most important formulas in statistics and AI.

Why Bayes Theorem Matters

Bayes Theorem formalizes how rational people should change their minds when they encounter new information. It's the mathematical foundation for learning from experience.

The Components of Bayes Theorem: Breaking It Down

- Bayes Theorem involves four key components that we need to understand.
- These components represent different aspects of our knowledge and evidence.
- Understanding each component helps us apply the theorem correctly.
- We'll explore each component in detail over the next few slides.



Prior Probability: What We Believe Before Evidence

- **Prior probability** represents our belief about an event before considering new evidence.
- It's written as $P(H)$, where H stands for our hypothesis or claim of interest.
- Prior probabilities can come from previous studies, logical reasoning, or background knowledge.
- Even when priors are subjective, Bayes Theorem ensures that with enough evidence, different people will converge to similar conclusions.

Example: Disease Diagnosis

If a disease affects 1 in 10,000 people in the general population, then the prior probability of having the disease is:
$P(\text{disease}) = \frac{1}{10{,}000} = 0.0001$
This is what we believe before any specific testing or symptoms are considered.

Likelihood: How Evidence Supports Our Hypothesis

- **Likelihood** measures how probable the observed evidence would be if our hypothesis were true.
- It's written as $P(E|H)$, the probability of evidence E given hypothesis H is true.
- Likelihood is not the same as the probability of the hypothesis being true.
- Higher likelihood means the evidence is more consistent with our hypothesis.

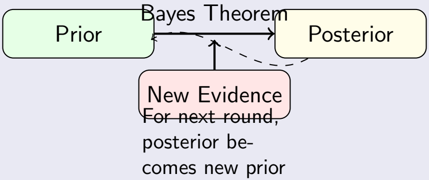
Example: Medical Test

If a test correctly identifies 95% of people who have a disease, then:
$P(\text{positive test}|\text{has disease}) = 0.95$
This is the likelihood of getting a positive test result given that you have the disease.

Posterior Probability: Our Updated Belief

- **Posterior probability** is our updated belief after considering the new evidence.
- It's written as $P(H|E)$, the probability of hypothesis H given we observed evidence E.
- The posterior becomes our new prior if additional evidence becomes available later.
- Calculating the posterior probability is the main goal of applying Bayes Theorem.

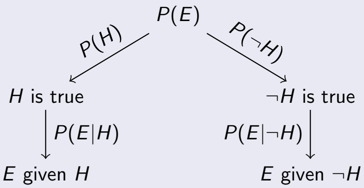
The Bayesian Learning Process



The Evidence Term P(E): Total Probability

- The denominator in Bayes Theorem, $P(E)$, represents the total probability of observing our evidence.
- We often expand this into:
  $P(E) = P(E|H) \times P(H) + P(E|\neg H) \times P(\neg H)$
- This uses the law of total probability to consider all ways the evidence might occur.
- This expansion is crucial for correctly normalizing our posterior probability.

Why We Decompose P(E)



$$P(E) = P(E|H) \times P(H) + P(E|\neg H) \times P(\neg H)$$

Bayes Theorem Formula: Putting It All Together

- Bayes Theorem combines prior probability, likelihood, and evidence to calculate posterior probability.
- The formula relates conditional probabilities in a powerful and elegant way.
- Understanding each part of the formula helps us apply it correctly to real problems.
- The denominator represents the total probability of observing our evidence.

Bayes Theorem Formula

$$P(H|E) = \frac{P(E|H) \times P(H)}{P(E)}$$

$$\text{Posterior} = \frac{\text{Likelihood} \times \text{Prior}}{\text{Evidence}}$$

Where $P(E) = P(E|H) \times P(H) + P(E|\neg H) \times P(\neg H)$

Bayes Theorem Step-by-Step: A Simple Example

- Let's walk through Bayes Theorem with a simple, intuitive example.
- Consider whether it will rain today based on observing cloudy skies.
- We'll identify each component of Bayes Theorem for this scenario.
- Following these steps will help you apply the theorem to any problem.

Weather Example

- Hypothesis (H): It will rain today
- Evidence (E): The sky is cloudy
- Prior: $P(H) = 0.3$ (30% chance of rain based on the season)
- Likelihood: $P(E|H) = 0.9$ (90% of rainy days have clouds)
- $P(E|\neg H) = 0.4$ (40% of non-rainy days have clouds)
- $P(E) = 0.9 \times 0.3 + 0.4 \times 0.7 = 0.27 + 0.28 = 0.55$
- Posterior: $P(H|E) = \frac{0.9 \times 0.3}{0.55} = \frac{0.27}{0.55} \approx 0.49$

Common Mistakes in Applying Bayes Theorem

- Confusing $P(H|E)$ with $P(E|H)$ - these are very different probabilities.
- Forgetting to calculate the total probability of evidence $P(E)$ properly.
- Using incorrect prior probabilities that don't reflect background knowledge.
- Applying Bayes Theorem when simpler methods would work better.

The Prosecutor's Fallacy

A common error in legal contexts is focusing on $P(E|H)$ instead of $P(H|E)$.
For example, saying "There's only a 1 in 10,000 chance this DNA match occurred by random chance" (which is $P(E|H)$) is not the same as saying "There's a 9,999 in 10,000 chance the defendant is guilty" (which would be $P(H|E)$).
Bayes Theorem helps us avoid this error by properly accounting for the prior probability.

Medical Testing Example: Understanding False Positives

- Medical testing provides a clear example of how Bayes Theorem works in practice.
- The problem of false positives shows why understanding conditional probability is important.
- People often incorrectly assume a positive test means they likely have the disease.
- Bayes Theorem helps us calculate the true probability of disease given a positive test.

Medical Test Parameters

Let's analyze a test for a rare disease with these characteristics:

- Disease prevalence: 1 in 10,000 people (Prior: $P(D) = 0.0001$)
- Test sensitivity: 99% (Likelihood: $P(+|D) = 0.99$)
- Test specificity: 95% (True negative rate: $P(-|\neg D) = 0.95$)
- False positive rate: 5% ($P(+|\neg D) = 0.05$)

Medical Testing Example: Calculating Real Risk

- We want to find $P(D|+)$: the probability of having the disease given a positive test.
- Using Bayes Theorem helps us calculate this accurately and avoid misinterpretation.
- The result is often surprising to people without statistical training.

Calculation

$$
\begin{aligned}
P(D|+) &= \frac{P(+|D) \times P(D)}{P(+)} \\
&= \frac{P(+|D) \times P(D)}{P(+|D) \times P(D) + P(+|\neg D) \times P(\neg D)} \\
&= \frac{0.99 \times 0.0001}{0.99 \times 0.0001 + 0.05 \times 0.9999} \\
&= \frac{0.000099}{0.000099 + 0.049995} = \frac{0.000099}{0.050094} \approx 0.002 = 0.2\%
\end{aligned}
$$

Despite the positive test, there's only about a 0.2% chance of having the disease!

Everyday Reasoning Example: Is My Friend Angry With Me?

- We use Bayesian reasoning informally in social situations all the time.
- When interpreting others' behavior, we start with prior beliefs about their feelings.
- New evidence (like a short text response) updates our probability estimates.
- Multiple possible hypotheses can be compared using Bayes Theorem.

Social Situation Analysis

You texted your friend about hanging out, and got a brief "can't today" response.

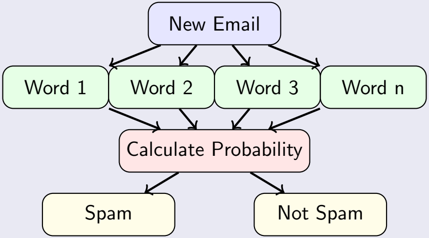
- Hypothesis A: Friend is angry with you
- Hypothesis B: Friend is just busy
- Prior: $P(A) = 0.2$ (based on recent interactions)
- Likelihood: $P(\text{brief response}|A) = 0.7$, $P(\text{brief response}|B) = 0.4$
- Posterior: $P(A|\text{brief response}) = \frac{0.7\times0.2}{0.7\times0.2+0.4\times0.8} \approx 0.30$

The probability your friend is angry increased from 20% to 30% - higher, but still not likely.

Technology Example: How Spam Filters Learn

- Email spam filters use Bayesian methods to classify incoming messages.
- The filter starts with prior probabilities for each word appearing in spam vs. legitimate emails.
- Each new email updates these probabilities based on whether you mark it as spam or not.
- The system continuously improves as it processes more data.

Naive Bayes Classifier



Philosophical Example: The Existence of God (Bayesian Perspective)

- Philosophical arguments about God's existence can be framed in Bayesian terms.
- We start with a prior probability based on existing beliefs and arguments.
- Various pieces of evidence (like the existence of suffering) update this probability.
- Different people may reach different conclusions based on their priors and how they weigh evidence.

A Simple Bayesian Approach

Consider the fine-tuning argument:

- Evidence E: The universe appears "fine-tuned" for life
- Hypothesis H: God exists
- Alternative A: Multiverse theory (many universes exist)
- Compare: $P(E|H)$ vs. $P(E|A)$
- Which hypothesis better explains the evidence?
- Different rational people can assess these likelihoods differently

Philosophical Example: The Problem of Evil as Evidence

- The problem of evil asks how a good, all-powerful God could allow suffering.
- In Bayesian terms, suffering is evidence that may affect the probability of God's existence.
- Theodicies (explanations for suffering) attempt to show why $P(E|H)$ is not actually low.
- Bayesian reasoning helps structure this debate more clearly.

Bayesian Problem of Evil

- E: Suffering exists in the world
- H: An all-good, all-powerful God exists
- $P(E|H)$ seems low (unexpected)
- Therefore, E reduces $P(H)$

Theodicy Response

- Argues that $P(E|H)$ is actually not low
- Perhaps suffering is necessary for free will
- Perhaps suffering serves a higher purpose
- If successful, E doesn't reduce $P(H)$

Bayes Factor: Comparing Hypotheses

- The **Bayes factor** is a ratio of how well two competing hypotheses predict the evidence.
- It's calculated as: $BF = \frac{P(E|H_1)}{P(E|H_2)}$
- A Bayes factor greater than 1 means the evidence supports H_1 over H_2.
- Bayes factors provide a way to quantify the strength of evidence.

Interpreting Bayes Factors

Bayes Factor	**Strength of Evidence**
1 to 3	Barely worth mentioning
3 to 10	Substantial evidence
10 to 30	Strong evidence
30 to 100	Very strong evidence
> 100	Decisive evidence

Note: The interpretation is reversed if $BF < 1$ (evidence supports H_2 over H_1)

Bayesian vs. Frequentist Approaches: Two Schools of Thought

- **Frequentist statistics** focuses on the probability of data given a hypothesis.
- **Bayesian statistics** focuses on the probability of a hypothesis given the data.
- Frequentists avoid using prior probabilities, considering only objective data.
- Bayesians incorporate prior information, allowing for subjective input.

Frequentist Approach

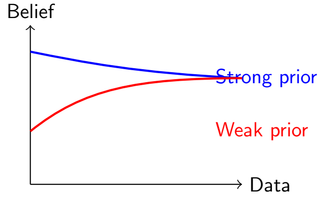
- Uses p-values and confidence intervals
- Asks: "How likely is this data if the null hypothesis is true?"
- Cannot directly calculate probability of hypothesis being true
- Rejects or fails to reject hypotheses

Bayesian Approach

- Uses posterior probabilities and credible intervals
- Asks: "How likely is this hypothesis given the observed data?"
- Directly calculates probability of hypothesis
- Updates belief in hypotheses

The Power of Prior Information: Why It Matters

- Priors allow us to incorporate existing knowledge into our analysis.
- They prevent us from treating each new problem as if we know nothing about it.
- Well-chosen priors can dramatically improve our inference, especially with limited data.
- As more evidence accumulates, the impact of the prior diminishes.



Key Insight

With large amounts of evidence, different reasonable priors will lead to similar conclusions. However, when evidence is limited, the choice of prior can significantly affect conclusions.

Overcoming Cognitive Biases with Bayesian Thinking

- Human reasoning is subject to many cognitive biases that distort our judgments.
- Bayesian thinking provides a framework to make our reasoning more objective.
- It forces us to consider both prior probabilities and new evidence explicitly.
- Regular practice with Bayes Theorem can improve decision-making in all areas of life.

Common Biases Addressed by Bayesian Thinking

- **Base rate neglect**: Ignoring prior probabilities (corrected by using proper priors)
- **Confirmation bias**: Overweighting evidence that confirms existing beliefs (corrected by consistent application of Bayes Theorem to all evidence)
- **Availability bias**: Overestimating probabilities of easily recalled events (corrected by using data rather than anecdotes)
- **Anchoring**: Being unduly influenced by initial information (corrected by updating beliefs systematically)

Limitations of Bayesian Reasoning

- Bayesian methods still require good judgment in setting priors and evaluating evidence.
- Some situations may have too much uncertainty to provide useful probability estimates.
- Computational complexity can make exact Bayesian calculations difficult for complex problems.
- The choice of prior is sometimes controversial, especially in scientific contexts.

Potential Pitfalls

- Garbage in, garbage out: Poor priors lead to poor conclusions
- Overconfidence: Thinking your probability estimates are more precise than they are
- Misapplication: Using Bayes Theorem when simpler methods would work better
- Complexity: Some real-world problems have too many variables for straightforward application

Real-World Applications: Where Bayes Theorem Is Used Today

- Bayesian methods are increasingly important in many fields and technologies.
- Understanding these applications helps appreciate the power of Bayesian reasoning.
- The principles we've learned apply across diverse areas of research and everyday life.
- These applications continue to expand as computing power increases.

Modern Applications of Bayesian Methods

- Machine learning and AI
- Medical diagnosis and testing
- Stock market prediction
- Climate science modeling
- Spam filtering
- Search engine algorithms
- DNA analysis and forensics
- Natural language processing
- Recommendation systems
- Image recognition

Connecting Inductive Logic to Your Life

- Inductive reasoning and Bayesian thinking are skills you use every day, often unconsciously.
- Becoming more explicit about how you reason can improve decision-making.
- The formal tools we've learned can be applied to both academic subjects and daily life.
- Practice updating your beliefs based on evidence, just as Bayes Theorem prescribes.

Personal Applications

- **Education**: Evaluating which study methods are most effective for you
- **Social media**: Assessing the reliability of news and information
- **Decision-making**: Choosing between options with uncertain outcomes
- **Learning**: Understanding how your beliefs should change with new information
- **Problem-solving**: Breaking down complex issues into manageable components

Conclusion: Becoming a Better Thinker with Bayes Theorem

- Inductive logic provides tools for reasoning under uncertainty, with Bayes Theorem as a cornerstone.
- The Bayesian approach formalizes how rational people should update their beliefs with new evidence.
- These concepts apply across disciplines, from science and philosophy to everyday decision-making.
- By understanding probability and Bayesian reasoning, you become a more critical and effective thinker.

Key Takeaways

- Uncertainty is inevitable, but we can reason rigorously within it
- Prior beliefs matter, but should be updated with evidence
- Probability is the language of inductive reasoning
- Bayes Theorem provides a powerful framework for learning from experience

Questions?