Section 6: Sets, independence, discrete distributions

STOR 155.02, Spring '21

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What you will learn

- sets and set algebra
- probability rules of sets
- discrete distributions
- variance and expecation of discrete distributions

Resources

- Textbook: ch 3.1.2-4, 3.4 (random variables), ch 4.3.1 (Bernoulli)

Sets

Remember: General method for calculating probabilities when all choices are equally likely

$$P(X = x_i) = \frac{\text{number of ways outcome } x_i \text{ can happen}}{\text{number of choices}}$$

example: pick one random card from a briscola deck

there are **40 cards** and **four suits** with an equal number of cards: clubs, coins, cups and swords.

$$X = 1 \quad \text{if card is cups}, \quad \quad X = 0 \quad \text{if not}$$

This is such an important and useful concept that we need to make it more precise

Definition: Set

> > a set is a **collection of outcomes** for a random process

Notation

will use curly braces, e.g.

$\{1, 2, 3\}$

for a set of outcomes 1, 2, 3.

capital letters as a shorthand, e.g.

$A = \{1, 2, 3\}$

Example

draw a cups from a briscola deck

$$A = {1\text{ of cups}, 2\text{ of cups, } ... \text{ King of cups}}$$

roll an even number on six-sided die

$$A = {2, 4, 6}$$

Set algebra

unions

$$A \cup B = A \text{ or } B$$

example

$$A = \{ 2, 4, 6 \} , \quad B = \{ 1 \}$$

$$A \cup B = \{1, 2, 4, 6\}$$

intersections

$$A \cap B = A \text{ and } B$$

example

$$A = \{2, 4, 6\}, \quad B = \{2\}$$

$$A \cap B = \{2\}$$

the empty set and the set of all outcomes

$$S = \text{ all possible outcomes, a.k.a 'sample space'}$$

$$\varnothing = \text{ the empty set, the set with no outcomes}$$

$$\{2, 4, 6 \} \cap \{1\} = \varnothing$$ because these two sets have no outcomes in common

Disjoint sets

A and B are **disjoint if they have no outcomes in common**

$$A, B \text{ are disjoint if } \quad A\cap B = \varnothing$$

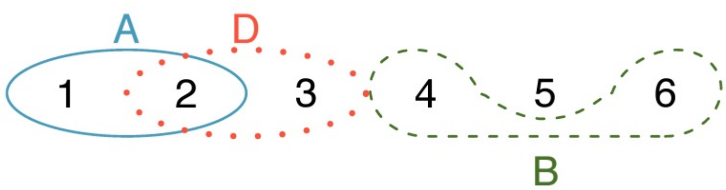


Figure 3.2: Three events, A, B, and D, consist of outcomes from rolling a die. A and B are disjoint since they do not have any outcomes in common.

image credit: textbook ch 3

Set complement

the complement of a set A is denoted A^c

and is the **collection of outcomes not in A**

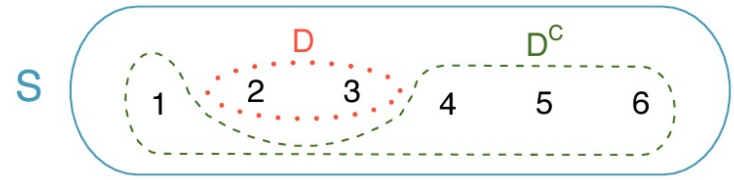


Figure 3.9: Event $D = \{2, 3\}$ and its complement, $D^c = \{1, 4, 5, 6\}$. S represents the sample space, which is the set of all possible events.

you tell me

$$A \cup A^c = ?$$

$$A \cap A^c = ?$$

Probabilities of sets of outcomes

Recap

$$P(X = x) \quad \text{is} \quad \text{the probability r.v. X takes outcome } x$$

\(X = x\) is an outcome and therefore can be written as a set

$$A = \{X = x\}$$

this is more useful when we want to collect several outcomes into a set

$$B = \{X = x_1 \text{ or } x_2 \text{ or } x_3\} = \{X = x_1\} \cup\{X = x_2\} \cup\{X = x_3\}$$

Probabilities of sets

$$P(A) = \text{ probability that some outcome in A happens}$$

Golden rule

> something will happen for sure

$$P(S) = \text{ probability that something happens} = 1$$

$$P(\varnothing) = \text{ probability that nothing happens} = 0$$

Don't double count

> for **any two sets** the following formula is true

$$P(A \cup B) = P(A) + P(B) - P(A\cap B)$$

Addition rule

'don't double count' rule simplified if sets disjoint

> > if A, B are **disjoint** sets, then the probability A or B occurs is the probability of A plus the probability of B

in mathematical terms this means

$$P(A \cup B) = P(A) + P(B) \quad \quad \text{if A, B disjoint}$$

and for multiple sets

$$P(A_1 \cup A_2 \cup \ldots A_n) = \sum_{i = 1}^n P(A_i) \quad \quad \text{if } A_i\cap A_j = \varnothing \quad i\neq j$$

you tell me

$$P(A) + P(A^c) = \text{ ? }$$

$$P(A \cap A^c) = \text{ ?}$$

Revisited: General formula when all outcomes equally likely

Say there are N possible outcomes, $S = \{1, 2 \ldots N\}$.

$$P(\{i\}) = 1/N, \quad \quad \text{if all equally likely }$$

Each outcome is disjoint, so this make sense

$$P(S) = \sum_{i=1}^N P(\{i\}) = 1$$

Now take A to be some collection of outcomes

$$P(A) = \sum_{i \, in \, A} P(\{i\}) = \frac{\text{number outcomes in A}}{N} = \frac{\text{number outcomes in A}}{\text{number of outcomes total}}$$

For example, if $A = \{1, 2, 3, 4\}$ and $N = 10$

$$P(A) = P(\{1\}) +P(\{2\}) + P(\{3\}) + P(\{4\}) = 4/10 = 2/5$$

Venn diagrams can help

You draw one card from a U.S.-standard 52-card deck $$A = \{ \text{cards with diamonds} \}, \quad B = \{ \text{cards with faces} \}$$

$$P(A) = \frac{\text{number outcomes in A}}{52}$$

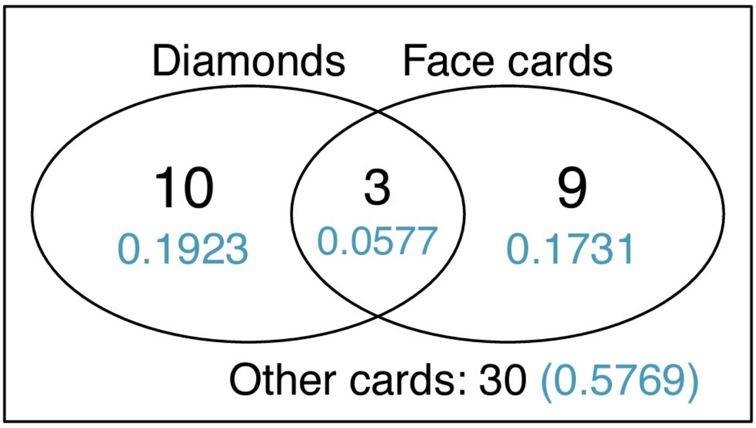


Figure 3.4: A Venn diagram for diamonds and face cards.

Independence

Conceptually

> > "Two sets are independent if knowing the outcome of one provides no useful information about the outcome of the other."

Independence for sets

sets A, B are independent if

$$P(A \cap B) = P(A) P(B)$$

Independence of random variables

random variables X, Y are independent if each collection of their outcomes is independent as sets

Example

Setup

- draw two cards from a briscola deck
- 40 cards, 4 suits (cups, swords, clubs, coins)

$$X = 1 \text{ if cups on first draw}$$

$$Y = 1 \text{ if cups on second draw}$$

Is it reasonable to think of (X, Y) as independent?

Use the conceptual definition above

If so,

$$P(\{X = 1\} \cap \{Y = 1 \}) = \text{ ? }$$

Discrete probability distributions

For a discrete random variable X

Individual outcomes are **disjoint** by definition.

$$\text{possible outcomes of X are } \quad S = \{x_1, x_2 \ldots x_N \}$$

Definition:

The **distribution of X** is the collection of probabilities

$$P(X = x_1), P(X = x_2), \ldots, P(X = x_N)$$

Remember they must add to one

$$\sum_{i=1}^N P(X = x_i) = 1$$

Example: Indicator random variable

$$1_A \text{ denotes the indicator variable for the set }A $$

$$1_A = 1 \text{ if an outcome in A happened }, \quad 1_A = 0 \text{ if not}$$

If my random variable $X = 1_A$ for some set A, how can I represent the distribution of X?

Example: Discrete uniform distribution

X is a random variable with N possible outcomes $S = \{x_1, x_2 \ldots x_N \}$.

X has the discrete uniform distribution **if all outcomes are equally likely**

$$P(X = x_i) = \text{ ? }$$

Recap: Expectation a.k.a. the mean

A weighted average of possible outcomes

$$E(X) = \sum_{i=1}^N x_iP(X = x_i)$$

Variance

Theoretical, weighted-average version of sample variance, **tells how spread out a random variable is relative to the mean.**

$$Var(X) = \sum_{i=1}^N P(X = x_i) (x_i - E(X))^2$$

Standard deviation

$$SD(X) = \sqrt{Var(X)} $$

Example: Bernoulli distribution

Definition

X is a Bernoulli distributed random variable if for some set A

$$X = 1_A$$

- a discrete distribution
- with outcomes $S = \{0, 1 \}$
- and distribution

$$P(X = 1) = P(A)$$ $$P(X = 0) = 1 - P(A)$$

Examples

Many of the examples we've seen already are Bernoulli $$A = \{\text{cups in Briscola deck}\}$$

$$X = 1_A$$ $$P(X = 1) = P(A) = 1/4$$

Can apply to any situation in which $X = 1$ marks a 'success' and $X = 0$ a failure.

Poll EV setup

Use this in the poll questions today

Sneetches are creatures that are either yellow or blue, not both, and have up to two stars painted on their bellies.

In this Sneetch village there are 56 residents, and I made the following observations:

set	number
yellow bellies	21
one star	9
blue bellies and no stars	12

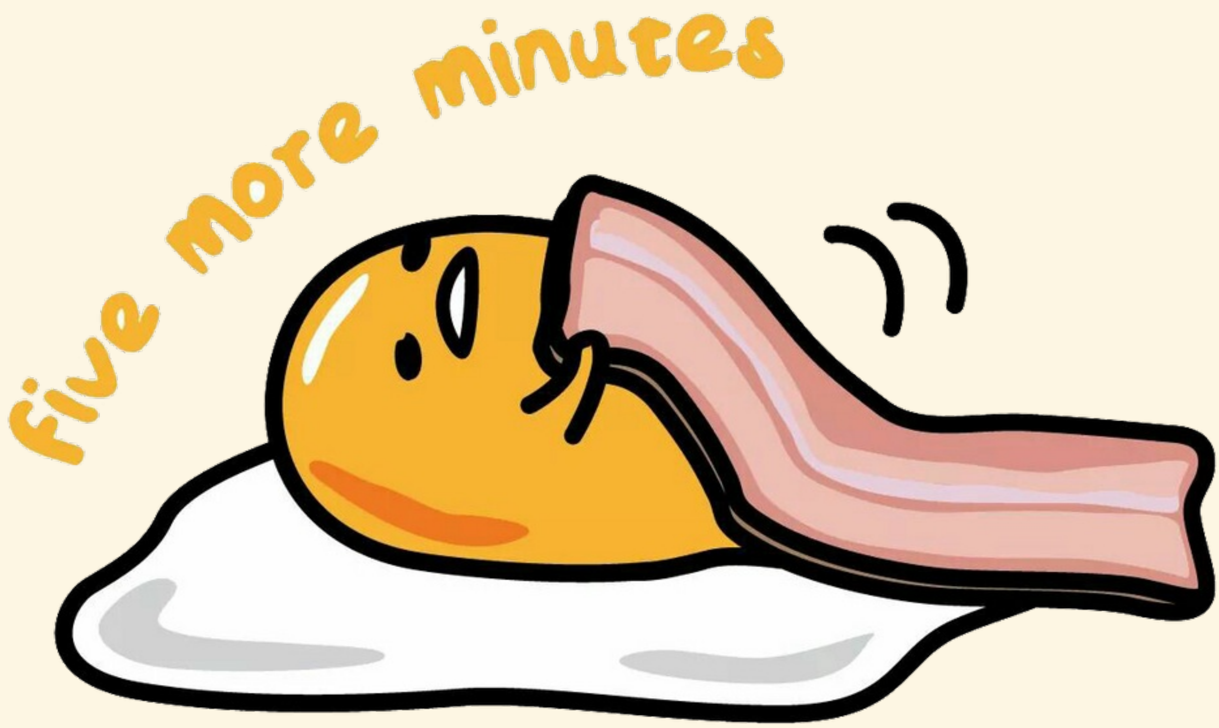
Assume belly color and number of stars are independent

by which I mean if \(A = \{\text{yellow bellies}\}\) then \(A\) is independent from any set involving number of stars.

PollEv.com/brendanbrown849

poll closes at __________

You'll need to do a little algeba using the definitions!


five more minutes