Risk Structured SIR Model

Background

For Andrew Berdahl's agent-based modeling class I looked at epidemic dynamics with heterogeneous risk with additional spatial/social structure.

I started with an agent-based SIR model: some number of "agents" interact and spread a disease stochastically moving between susceptible, infected and recovered states (each agent contacts every other agent daily; infected agents spread disease to susceptible agents with probability β_c; infected agents recover with probability γ)

Additionally, on a given day each individual in the population has probability p of choosing "risky" behavior (i.e. going to church or to a bar), which is fixed per individual over the life of the simulation.

Agents who choose the "risky" behavior on a given day spread the disease to one-another with probability β_r ($\beta_r > \beta_c$).

Questions that I have

Before I started this project I read a few papers about super spreading events (SSEs) and heterogeneity in epidemics:

https://journals.plos.org/plosbiology/article?id=10.1371/journal.pbio.3000897

https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0250050

My main goal was to investigate a simple and tractable mechanism for heterogeneity and consider questions/themes that came up in these papers concretely.

The two main questions:

1. How does heterogeneity affect the probability of the disease going extinct?

Basic theory of SSEs says that the more heterogeneous the population the less likely a disease is to initially take off. I figured I could come up with any of a few ways to quantify that in this model (analytically and in my simulation).

2. Basic question of intervention timing

Suppose we have access to an intervention measure like closing bars and restaurants, which in this model we could program as setting β_r to 0.

Since we may assume individuals with high riskyness (high p) are more likely to get infected earlier on, is the timing of this intervention extremely sensitive? Can I construct a scenario where an early intervention can prevent an epidemic but a slightly later intervention has little or no impact; and make some statements about the robustness of this scenario?

This is similar to questions raised in this paper: but could help investigate the actual mechanics involved in one kind of SSE.

This is similar to this paper in preprint I worked on:

https://www.medrxiv.org/content/10.1101/2020.08.21.20179473v2

DIFEQ model

I didn't actually think of how to model this deterministically for a while, but I eventually came up with the exact same integro-differential model that you and Mark Kot analyzed in your 2020 paper (in the continuous limit):

$$
\begin{aligned}
\frac{\partial S(p,t)}{\partial t} &= -\beta_c S(p,t)\int_0^1 I(u,t)du - \beta_r S(p,t)p\int_0^1 I(u,t)udu \\
\frac{\partial I(p,t)}{\partial t} &= \beta_c S(p,t)\int_0^1 I(u,t)du + \beta_r S(p,t)p\int_0^1 I(u,t)udu - \gamma I(p,t)
\end{aligned}
$$

Here S(p, t) and I(p, t) are the populations of succeptible and infected individuals at time t with riskyness parameter p. β_c and β_r are paramaters that govern community and "risky" spread. And u is the dummy variable over which I'm taking integrals.

I'm not actually sure that this leads anywhere, but I noticed that the integrals that came up looked like moments (as in probability) over the riskyness (p) dimension, i.e.:

$\int_0^1 I(u,t)udu$ is the first moment of $I(p,t)$, and $\int_0^1 I(u,t)du$ is the zeroth moment.

Let $\bar{I}$ be the zeroth moment of $I(t,p)$ over p; $\hat{I}$ the first moment, $\hat{\hat{I}}$ the second moment, etc; and the same for S.

Then:

$$
\begin{aligned}
\frac{\partial S}{\partial t} &= -\beta_c S\bar{I} - \beta_r S\hat{I} \\
\frac{\partial I}{\partial t} &= \beta_c S\bar{I} + \beta_r S\hat{I} - \gamma I
\end{aligned}
$$

Now consider

$$
\begin{aligned}
\frac{d\hat{I}}{dt} &= \frac{d}{dt}\left(\int_0^1 I(u,t)udu\right) \\
&= \int_0^1 \frac{\partial}{\partial t}I(u,t)udu \\
&= \int_0^1 (\beta_c S\bar{I} + \beta_r Su\hat{I} - \gamma I)udu \\
&= \beta_c \hat{S}\bar{I} + \beta_r \hat{\hat{S}}\hat{I} - \gamma\hat{I}
\end{aligned}
$$

and similarly

$$\begin{aligned}
\frac{d\hat{S}}{dt} &= \frac{d}{dt}\left(\int_0^1 S(u,t)udu\right) \\
&= \int_0^1 \frac{\partial}{\partial t} S(u,t)udu \\
&= \int_0^1 (-\beta_c S\bar{I} - \beta_r S u\hat{I})udu \\
&= -\beta_c \hat{S}\bar{I} - \beta_r \hat{\hat{S}}\hat{I}
\end{aligned}$$

And in general the derivative with respect to t of any moment of I or S can be expressed in terms of other moments of I and S. This allows us to consider the moment generating function $G_S(v,t) of S$:

$$G_S(v,t) = \bar{S} + u\hat{S} + \frac{u^2}{2!}\hat{\hat{S}} + ...$$

and its derivative:

$$\begin{aligned}
\frac{\partial G_S}{\partial t} &= \frac{\partial \bar{S}}{\partial t} + u\frac{\partial \hat{S}}{\partial t} + \frac{u^2}{2!}\frac{\partial \hat{\hat{S}}}{\partial t} + ... \\
&= (-\beta_c \bar{S}\bar{I} - \beta_r \hat{S}\hat{I}) + u(-\beta_c \hat{S}\bar{I} - \beta_r \hat{\hat{S}}\hat{I}) + \frac{u^2}{2!}(-\beta_c \hat{\hat{S}}\bar{I} - \beta_r \hat{\hat{\hat{S}}}\hat{I}) + ... \\
&= -\beta_c \bar{I}(\bar{S} + u\hat{S} + \frac{u^2}{2!}\hat{\hat{S}} + ...) - \beta_r \hat{I}(\hat{S} + u\hat{\hat{S}} + \frac{u^2}{2!}\hat{\hat{\hat{S}}} + ...) \\
&= -\beta_c \bar{I} G_S - \beta_r \hat{I}\frac{\partial G_S}{\partial u}
\end{aligned}$$

A similar result holds for G_I, and you can write equations for both generating functions:

$$\begin{aligned}
\frac{\partial G_S(u,t)}{\partial t} &= -\beta_c \bar{I} G_S - \beta_r \hat{I}\frac{\partial}{\partial u} G_S \\
\frac{\partial G_I(u,t)}{\partial t} &= \beta_c \bar{I} G_S + \beta_r \hat{I}\frac{\partial}{\partial u} G_S - \gamma G_I
\end{aligned}$$

So a third question I have:

3. Is there any interesting result I can get to using these generating function equations or further analyzing this model analytically?