Let $\omega = e^{2k\pi i/p}$ be a primitive p-th root of unity. Define

$$\Theta_{p,q_1,q_2}(x,y) = \prod_{j=0}^{p-1}(1 - x\omega^{q_1 j} - y\omega^{q_2 j}). \tag{1}$$

Let $\gamma : (x,y) \mapsto (x\omega^{q_1}, y\omega^{q_2})$. We claim that

$$(\Theta_{p,q_1,q_2} \circ \gamma)(x,y) = \Theta_{p,q_1,q_2}(x,y). \tag{2}$$

How do I prove the above? And how do I show that if, for example, $(p,q_1) = 1$, then considering Θ_{p,q_1,q_2} is equivalent to considering $\Theta_{p,1,q_2'}$? I know we must consider the image of Θ under the map γ, but it is unclear to me that

$$\prod_{j=0}^{p-1}(1 - x\omega^{q_1 j} - y\omega^{q_2 j}) = \prod_{j=0}^{p-1}(1 - x\omega^{q_1 + q_1 j} - y\omega^{q_2 + q_2 j}). \tag{3}$$

Is the right-hand side of (3) not the image of Θ under the given map/group action?

Let's write it out explicitly.

Let $j = 0$. Then we want to find

$$1 - x\omega^{q_1} - y\omega^{q_2} \tag{4}$$

in

$$\prod_{k=0}^{p-1}(1 - x\omega^{q_1 k} - y\omega^{q_2 k}). \tag{5}$$

Clearly $k = 1$ yields the factor we want. Now let j be arbitrary. Then we want to find

$$1 - x\omega^{q_1 + q_1 j} - y\omega^{q_2 + q_2 j} = 1 - x\omega^{q_1(j+1)} - y\omega^{q_2(j+1)} \tag{6}$$

in

$$\prod_{k=0}^{p-1}(1 - x\omega^{q_1 k} - y\omega^{q_2 k}). \tag{7}$$

Since the group of roots of unity is cyclic, we can find the desired factor. It is $k = j + 1 \mod p$.