Music and the Fourier Transform

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Abstract

Over the years, people have discovered links between music and mathematics. One well-known connection between the two is the Fourier Transform. In this paper we will discuss the Wave Equation and the Fourier series, leading up to the Fourier Transform. From there we will cover the basics of the Fourier Transform. The paper will include a discussion of the Fourier Transform and its use in musical analysis and synthesis. Through analysis of the Fourier Transform, we will identify fundamentals and overtones of notes. Furthermore, using MatLab we will investigate and compare and contrast a variety of different instruments, such as the flute, piano, violin, tuning fork and bass.

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1. **Introduction**

This paper will review the history behind the Fourier transform. From there it will discuss the basics of sound. Then the paper will continue on to cover a brief overview of the wave equation, which deals with the mathematics behind a plucked string. Next, it will move to the Fourier series, which decomposes periodic waves into sums of sines and cosines. The Fourier series leads us into the Discrete Fourier Transform (DFT) and Fast Fourier Transform (FFT). From here, the Fourier Transform can be used to look at the spectrum for different instruments. In order to analyze these different sound .wav files, the program called Matlab will be used. Matlab uses the sound .wav files, combined with the Fourier Transform to create graphs of the waveform and the spectrum. The different instruments that will be analyzed will be a flute, violin, piano, tuning fork, and bass. Finally, the paper will demonstrate how to synthesize sounds by using Matlab.

2. **History of the Fourier Transform**

2.1. **Important People in the Discovery of the Fourier Transform.**

John Tukey began his education at Brown University where he studied chemistry. He continued to get his education in mathematics at Princeton University where he received his Ph.D. His graduate work was mainly in pure mathematics. In his sophomore year at Brown he had already began taking graduate classes in mathematics. Tukey originally attended Princeton University because of the chemistry program, but in his second year changed to math. He was interested in analysis and topology. James Cooley began his education at Manhattan College where he studied mathematics. He later received his masters in applied mathematics at Columbia University. Cooley was a member of the Digital Sound Processing Committee and was later awarded for his work on the FFT. Both Tukey and Cooley are pioneers of digital sound processing where they developed the FFT algorithm. Through mathematical application and theory is where the FFT was developed. The Cooley-Tukey FFT algorithm is the most common FFT algorithm. What their algorithm does is break down the DFT into smaller DFTs; the algorithm is now known as the FFT. Dr.Garwin presented the idea to both Tukey and Cooley and they began working on his idea. In 1965 they published their paper An algorithm for the machine calculation of complex Fourier series, where they recreated Gauss' work[6].

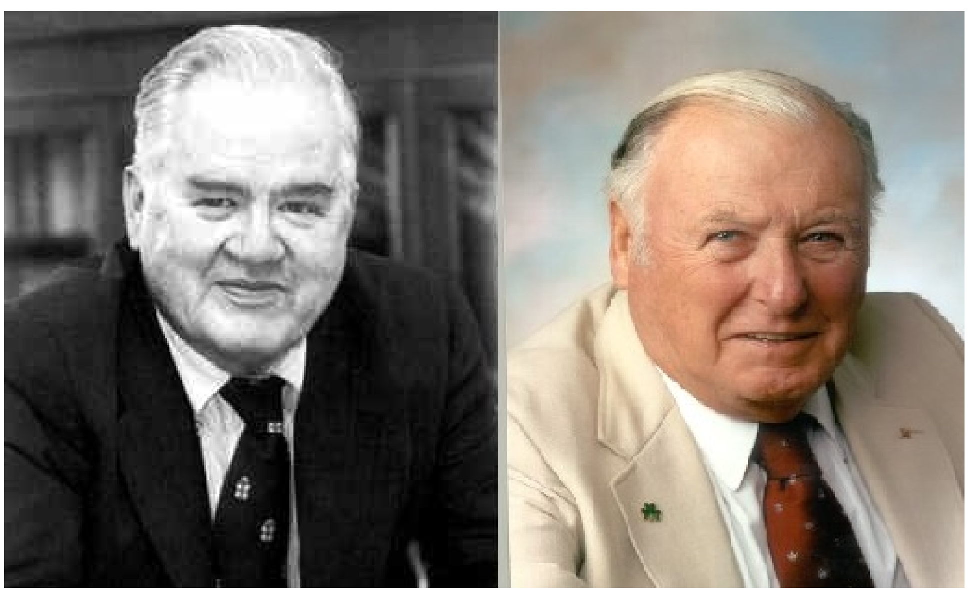

FIGURE 1. On the right is Tukey, and Cooley is on the left.

2.2. **Background on the Fourier Transform.**

In 1965, the idea of the FFT became popular because of the paper written by John Tukey and James Cooley; however as early as 1805 various algorithms were introduced. The first time the FFT was seen was by Gauss was when he observed that the Fourier series of the bandwidth $N = N_1 N_2$ can be where it can be broken up into a computation of N_2 subsampled DFTs of the length of N_1. This combines N_1 DFTs of the length of N_2; which was explained in Tukey and Cooleys paper. Outside of Gauss' collected work his algorithm was never published[6].

In 1932, statistician Yates published a less general, but still important version of the FFT used for efficient computation of the Hadamard and Walsh transform. Yates' paper 'Interaction Algorithm' discussed the fast technique for the computation of the analysis of variance for the $2^n!$ design that is brought up in almost any work on statistical design and analysis of experiments.

Danielson and Lanczos' paper Doubling Trick demonstrated how to reduce a DFT on 2N points to two DFTs using only N operations using only N points. Both Danielson and Lanczo worked in the field of x-ray crystallography[6].

3. **Basics of Sound**

There are many different types of waves. Our project will involve sound waves which are much different than water waves. Water waves are transverse, which means they are perpendicular to the direction of the energy transfer. Sound waves, however, go in the same direction as the energy transfer. These types of waves are known as longitudinal waves. There are four main attributes of sound: amplitude, pitch, timbre, and duration. Amplitude refers to loudness, pitch is the frequency of vibration, timbre involves the shape of the spectrum, and duration is the length of time of the sound. When a string is plucked, for example a guitar, a wave is created. This wave is a sum of sine waves with different amplitudes. This brings us to two important building blocks of the Fourier Transform, the Wave Equation and the Fourier Series.[2]

4. **Brief Introduction to the Wave Equation**

The Wave Equation is a part of Fourier Transforms because it deals with the mathematics behind the plucking of a string. Examples of this would be a bow on a violin or playing the guitar. The Wave Equation is a type of differential equation. More specifically, it is a second order partial differential equation, which means the function depends on several independent variables, versus just one. The Wave Equation has one dependent variable, u, that depends on the independent variables x and t. Let a string of length L be stretched tightly along the horizontal x-axis with both ends attached. Suppose it is plucked. The string is now in motion. It is vibrating along the vertical y-axis. Let $u(x,t)$ represent the point,x, at time,t.[3] Some things to know ahead of time:

$$c^2 = T/\phi$$

where c^2 is a constant, T is the tension or force of the string, ϕ is the mass per square unit length of the string material.

$$u_{tt} = c^2 u_{xx}$$

with domain 0 < x < L, where t > 0. This is known as a one-dimensional wave equation.

Some initial conditions for $u(x,t)$:

$u(0,t) = 0$ $x \in [0, L]$ and t$\geq$ 0. This tells us that the two endpoints do not move.

$u(x,0) = 0$ This means the ends of the string are attached.

Since $u(x,t)$ is a second order differential equation, it is understandable that we have two initial conditions.

Initial position $u(x,0) = f(x)$ $x \in [0, L]$, where f and g are given.

Initial velocity $u(0,t) = g(x)$

Vertical Displacement $u(x,t) : u_{tt} = c^2 u_{xx}$,

Assume $u(x,t) = X(x)T(t)$ and substitute u into $u_{tt} = c^2 u_{xx}$

Using $X'' = -\lambda X$ and $T'' = -c^2 \lambda T$, we obtain $X'' + \lambda X = 0$ and $T'' + c^2 \lambda T = 0$.

Now, $c^2 \dfrac{X''}{X} T = XT''$. Therefore, $c^2 \dfrac{X''}{X} = \dfrac{T''}{T} \rightarrow \dfrac{X''}{X} = \dfrac{1}{c^2} \dfrac{T''}{T} = -\lambda$, where -λ is a separation constant.

By this we get $X'' + \lambda X = 0$. If we let $\lambda = \dfrac{n^2 \pi^2}{L^2}$ $n = 1, 2, 3$ then $X'' + \lambda X = 0$ has non-trivial solutions.

This means X(x) is proportional to sin($\frac{nx}{L}$).

So, $T'' + c^2(\dfrac{n^2\pi^2}{L^2})T = 0$ and $T(t) = k_1 \cos(\dfrac{n\pi ct}{L}) + k_2 \sin(\dfrac{n\pi ct}{L})$ where k_1, k_2 are constants with initial condition $k_2 = 0$.

Finally, we get the homogeneous Fundamental Solution:

$$U_n(x,t) = \sin(\frac{n\pi ct}{L})\cos(\frac{n\pi ct}{L}), \text{ for n=1,2,3,...}$$

For a non-homogeneous wave equation we use $u(x,t) = \sum\limits_{\infty}^{n=1} C_n U_n(x,t)$ Where C_n are coefficients of the Fourier Sine Series, which will be discussed in the Fourier Series, of period 2L for$f$. Such that $C_n = \frac{2}{L}\int\limits_0^L f(x)\sin(\frac{n\pi x}{L})dx$, for n=1,2,3,... Hence, for n-fixed, $U_n(x,t)$ is periodic in time(t) with period $\frac{2L}{n\pi c}$. This represents a vibrating motion of string with frequency $\frac{n\pi c}{L}$. The displacement pattern is $\sin(\frac{n\pi x}{L})$. The wavelength is $\frac{2L}{n}$. The eigenvalues are $\frac{n^2\pi^2}{L^2}$ and the eigenfunction is $\sin(\frac{n\pi x}{L})$. This gives you the first three natural modes or the First Three Fundamentals [3].

5. **Fourier Series**

The Fourier Series is the breakdown of a periodic wave into a sum of sine and cosine waves. Before we jump into Fourier Series, we must first understand where Fourier Coefficients come from. Fourier Theory uses multiples of the Fundamental Frequency of a periodic wave and represents amplitude as an integral.

$f(\theta) = f(\theta + 2\pi)$. For the function f we use only a half-open interval $[0, 2\pi)$. We do this so that

$f(\theta) = f(\theta + 2\pi)$ determines the value at all other values of θ.

$f(\theta) = \frac{1}{2}a_0 + \sum\limits_{n=1}^{\infty} a_n \cos(\theta n) + b_n \sin(\theta n)$, where a_n and b_n are constants.[3]

For $m > 0$ we get

$\int_0^{2\pi} \cos(m\theta) f(\theta) d\theta = \cos(m\theta)[\frac{1}{2}a_0 + \sum\limits_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta)] =$

$\frac{1}{2}a_0 \int_0^{2\pi} \cos(m\theta) d\theta \sum\limits_{n=1}^{\infty} [a_n \int_0^{2\pi} \cos(m\theta) cos(n\theta) d\theta + b_n \int_0^{2\pi} \cos(m\theta) \sin(n\theta) d\theta = \pi a_m$

So, $a_m = \frac{1}{\pi} \int_0^{2\pi} \cos(m\theta) f(\theta) d\theta$, similarly for b_m.

Since 2π isn't time, we use $v = \frac{1}{T}(Hz)$ and $\theta = 2\pi vt$. Hence we obtain our Fourier Series,

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(2n\pi vt) + b_n \sin(2n\pi vt)]$$

with coefficients, $a_m = 2v \int_0^T \cos(2m\pi vt) F(t) dt$ and $b_m = 2v \int_0^T \sin(2m\pi vt) F(t) dt$.

A function $f(\theta)$ is even if $f(-\theta) = f(\theta)$. A function is odd if $f(-\theta) = -f(\theta)$. For example, $\cos(\theta)$ is even and $\sin(\theta)$ is odd.[3] Let f(t) represent a musical note or chord, such that $f(t) \in [0, L]$. The Fourier series representation of this looks like, $c_o + \sum\limits_{n=1}^{\infty} a_n \cos(\frac{2\pi nt}{L}) + b_n \sin(\frac{2\pi nt}{L})$ where its Fourier coefficients c_o, a_n, and b_n are

$c_o = \frac{1}{L} \int_0^L f(t)$, (Background level and constant air pressure level)

$a_n = \frac{2}{L} \int_0^L f(t) \cos(\frac{2\pi nt}{L}) dt, n = 1, 2, 3...$

$b_n = \frac{2}{L} \int_0^L f(t) \sin(\frac{2\pi nt}{L}) dt, n = 1, 2, 3...$

Each term of the series $\{a_n \cos(\frac{2\pi nt}{L}) + b_n \sin(\frac{2\pi nt}{L})\}$ has a fundamental period of $\frac{n}{L}$. and a frequency of $\frac{n}{L}$.The frequencies used in Fourier series are integral multiples of the fundamental frequency $\frac{1}{L}$. In order to simplify the Fourier series, complex analysis is used. By using Eulers formulas the Fourier series looks like,

Eulers Formulas
$e^{i\theta} = cos\theta + isin\theta$
$e^{-i\theta} = cos\theta - isin\theta$
$cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$
$sin\theta = \frac{ie^{i\theta}-e^{-i\theta}}{2}$

$c_0 + \sum_{n=1}^{\infty}\{c_n e^{\frac{i2\pi nt}{L}} + c_{-n}e^{\frac{-i2\pi nt}{L}}\}$ $c_n = \frac{1}{L}\int_0^{\infty} f(t)e^{\frac{-i2\pi nt}{L}}dt, n = 1, 2, 3..$ These are based on $c_n = \frac{a_n+ib_n}{2}$ and its conjugate $c_{-n} = \bar{c_n} = \frac{a_n-ib_n}{2}$.

5.1. **Parseval's Equality.** One major result that comes from the Fourier series is Parsevals equality.[1]

$$\frac{1}{L}\int_0^L |f(t)|^2 dt = {c_0}^2 + \sum_{n=1}^{\infty} |c_n|^2 + |c_{-n}|^2$$

If the energy of a function is represented by, $\int_0^L |g(t)|^2 dt, \forall g(t) \in [0, L]$ then the equation $L|c_n|^2$ is the energy for the complex exponential $c_n e^{\frac{i2\pi nt}{L}}$. This means that the sum of the energies of the complex exponentials is equal to the energy of the sound signal f. This includes c_0. Since $c_n = \dfrac{a_n + ib_n}{2}$ and $c_{-n} = \bar{c_n} = \dfrac{a_n - ib_n}{2}$, then $|\dfrac{a_n + ib_n}{2}|^2 = |\dfrac{a_n - ib_n}{2}|^2 \to |c_n|^2 \to |\bar{c_n}|^2 = |c_{-n}|^2$

By this, Parsevals equality can be rewritten as,

$$\frac{1}{L}\int_0^L |f(t)|^2 dt = |c_0|^2 + \sum_{n=1}^{\infty} 2|c_n|^2.$$

The spectrum is a plot of c_n, which is the complex amplitude for the nth harmonic. So here the spectrum for Parseval's equality $\{2|c_n|^2\}$ where $n \geq 1$ completely captures the energies in the frequencies that make up the audio signal. The $|c_0|^2$ is left out because it is inaudible. Now that we have a better understanding of the Fourier Series, we can proceed to the Fourier Transform.[1]

6. **The Fourier Transform**

Discrete Fourier Transform one of the most common and powerful methods people working in the digital processing fields use is the Discrete Fourier Transform (DFT). The DFT converts samples of the function that refers to a list of coefficients of a finite combination of the sine wave. That has to be done in the order of their frequencies of which has the same values. The input and output samples are both in the form of a complex number. DFT is a mathematical method used to determine frequency defined as

$$x(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

where x(t) is some continuous time domain. The DFT is a mathematical method, whereas the Fast Fourier Transform (FFT) is an algorithm for computing the DFT. There are techniques to get the data from a single note of digital music to the sound of the actual instrument. Leakage is the approximation of the true spectra of our DFT. However, there are ways we can minimize the leakage but we cannot eliminate it all together. If there was large amount of leakage it could result in minor peaks that could be incorrectly identified as sinusoids. The most efficient way is called windowing, which forces the amplitude of the beginning and the end of the input time sequence to go smoothly towards the amplitude values. There are several types of windowing including rectangle, triangle, hamming, hanning and flattop. Each windowing method has its own advantages and disadvantages, changes the shape of the leakage and also each window has a different effect on the spectrum. When windowing is not applied correctly many errors can occur in the FFT amplitude and can also lead to error in the overall spectrum shape. Rectangular windowing, which is the simplest window and is often referred to as the boxcar window focuses on the amplitude that directs change between one and zero. It does this by changing everything but the F values of the data sequence making them zero, and by doing so it makes it appears as the waveform turning on or off. The rectangular window is the first order B-spline window which will be discussed later.

With triangular windowing our values of the beginning and end of the signal seem to be the same; in this technique we must focus on the magnitude of our DFT bin to reduce side lobe levels. Triangular windowing is often referred to as Bartlett window and is also the second order B-spline window. A kth B-spline window is a piecewise polynomial with degree k-1; in the rectangular window k=1 and in the triangular window k=2.

Although the triangular window has reduced side lobe levels it is twice the width of the rectangular window.

In the Hanning window the ends of the cosine just touch zero causing the side-lobes to roll off. The Hamming window minimizes the maximum sidelobe that gives it a height of one fifth of the Hanning window. Both Hamming and Hanning windowing result in wide peaks but low sidelobes.

The flat top window is partially negative valued window. This window was created for the spectrum analyzers to measure the amplitudes of the sinusoidal frequency domain. This particular window has low amplitude measurement error by spreading the energy of a sine wave over multiple bins in the spectrum. One thing to be aware of is transforming a data set larger than the desired frequency resolution. One way to solve this issue it is to divide them into smaller sets and window them individually. Overall windows are used to improve DFT spectrum analysis accuracy[4].

The Fourier Series decomposes periodic waves into sums of sines and cosines. The Fourier Transform analyzes non-periodic functions in a similar way. Without generalized functions we use zero for large positive and negative values of time t. Since sounds are not periodic, instead of frequency analysis we want to analyze the frequency spectrum of sound. For this, we need the waveform for a time window around each point. Once we have that, we can begin analysis[4]. The function $f(t)$ is a real or complex function that contains a real variable v. To measure how much $f(t)$ there is at a specific frequency we use,

$$f(v) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ivt}dt$$

where time is t and frequency is v. The signal, $f(t)$, gets broken into periodic components, at all frequencies possible. The Inverse Fourier Transform changes the $-i$ to an $+i$, changing the signal. The FFT changes the time domain function into a frequency domain function. Conversely, the Inverse FFT turns a frequency domain function into a time domain function. For the Fourier Transform to exist we assume convergence of $f(v)$[2]. The following condition applies: $f(t)$ is L or is absolutely integrable on $(-\infty, \infty)$ if $\int_{-\infty}^{\infty} |f(t)|dt$ converges. As t$\rightarrow \infty$, $f(t)$ tends to zero.

Fourier analysis brings us a way that functions can be represented by time domains and frequency domains. When a string is plucked the function that comes from the differential equation, $f(t)$, where t represents time is the time domain representation. This shows how the string behaves over time. The Fourier Series is a basis for the Fourier Transform. The Fourier Transform provides a frequency domain representation of a time function. Since the Fourier Transform is invertible, its inverse returns the frequency domain back to a time domain function[2].

Let $f(t)$ be a Fast Fourier Transform:

$$f(v) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ivt}dt$$

Then using the Cauchy principal value of the integral, the original function $f(t)$ can be changed to the Inverse Fast Fourier Transform:

$$f(t) = \int_{-\infty}^{\infty} f(v)e^{2\pi ivt}dv$$

The values at $f(v)$ and $f(-v)$ tell us the phase and the magnitude of the frequency component v. If the function $f(t)$ contains real values, then $f(-v)$ is the complex conjugate of $f(v)$. The energy density at a specific value of v is the amplitude squared[2].

$$\text{Energy Density} = |f(v)|^2$$

In order to measure the total energy corresponding to the frequencies, we can integrate the energy density. Since v and $-v$ add to the energy, we have to double the answer when only using positive values of v. This brings us to the frequency spectrum. The frequency spectrum represents the phase(θ) and the amplitude $|f(v)|$ of $f(v)$ separately for values of v that are positive. The spectrum graphs the amplitudes of the frequencies of the sound being analyzed. Using the time domain signal, the spectrum is able to display the frequency domain.

7. **Spectrum Analysis**

There are three methods this section will describe in order to analyze the sounds of the different instruments and their frequencies. One way is by using fundamentals and overtones of individual notes. This is Fourier analysis and Spectrum analysis. Another way is by using spectrograms, which are used to analyze several notes at once. This gives a time-frequency description. The last way is by using scalograms. Scalograms focus on specific regions of spectrograms. It is possible to use all three of these to analyze music being played by different types of instruments. The first way discussed uses fundamentals and overtones. Fundamentals are base frequencies. Overtones are integral multiples of a fundamental. Fourier coefficients are used to describe fundamentals and overtones. On the other hand, Fourier series model how fundamentals and overtones play a role in producing sounds. These types of graphs are called Fourier spectra. Fourier spectra identify frequency content of individual notes. One negative aspect of the Fourier series is that is does not recognize abrupt changes in the frequency content of sound. We use spectrograms, a different method, in order to make up for what is lost using Fourier series. Spectrograms, which are used for time-frequency analysis, handle the abrupt changes over time more accurately. Spectrograms are figures that analyze sound signal using time- windows. The spectrograms allow us to tell one instrument from another instrument. However, since spectrograms do not directly deal with pitch, they can have some issues. An additional type of time-frequency analysis is called a scalogram. Scalograms are different than spectrograms; they can correlate with musical scale frequencies, since the scalograms are based on wavelets[1].

In this section we will explore how the spectrum of harmonics changes for different instruments. We will analyze plots of instruments to identify the pitch and volume of the sound. Using Matlab we will read in Microsoft .wav files and analyze the harmonics. The function we are using, analyze.m, is used to calculate the power spectrum of the .wav file. The graphs we get from analyze.m can be used to identity pitch and volume of the sounds. The waveforms that are on the left are the pressure variations with time detected by the microphone. The amplitude of the wave is a measure of its pressure oscillations and corresponds to the volume of the sound[5]. The fundamental frequency doubles with each octave; so the frequencies of octaves form a geometric series. We will look at a few examples of waveforms and spectrums from various instruments. First we have the tuning fork; the tuning fork is a very pure sound and holds a constant pitch.

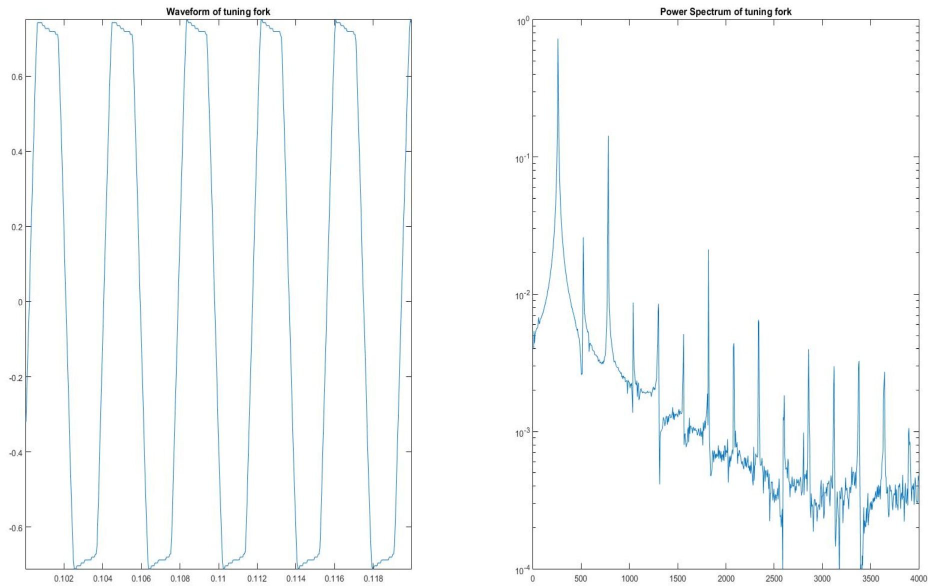

FIGURE 2. Power spectrum for a tuning fork.

The tuning forks waveform is very spread out, which translates over to the power spectrum that shows that the tuning fork is a very consistent sound.

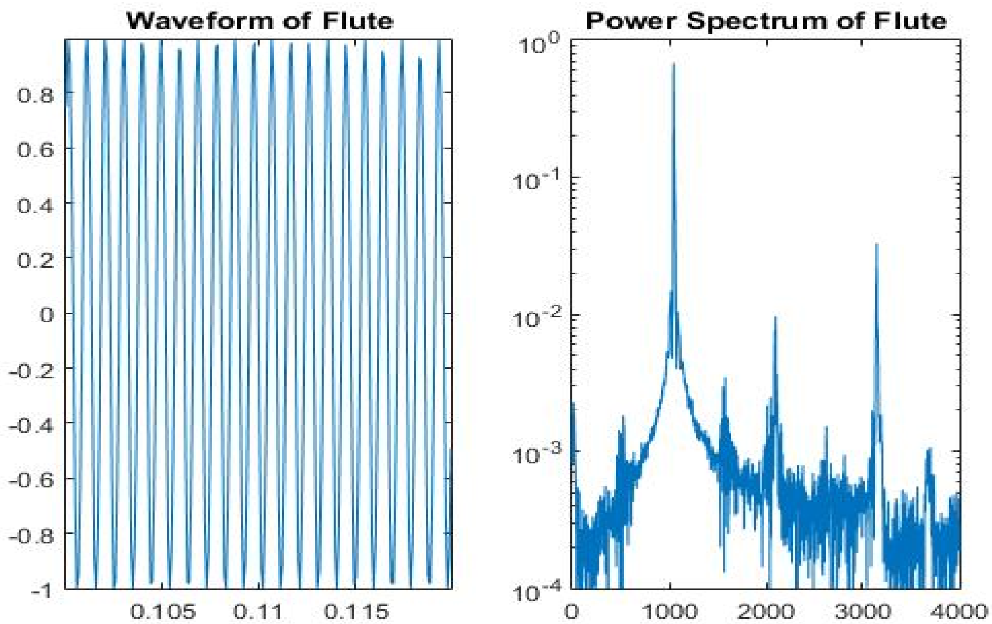

FIGURE 3. Power spectrum for the flute..

you can see that the flute has complete oscillations in the waveform. The flute has low harmonics which makes it sound like a whistle; and you can see that it has a lower power spectrum.

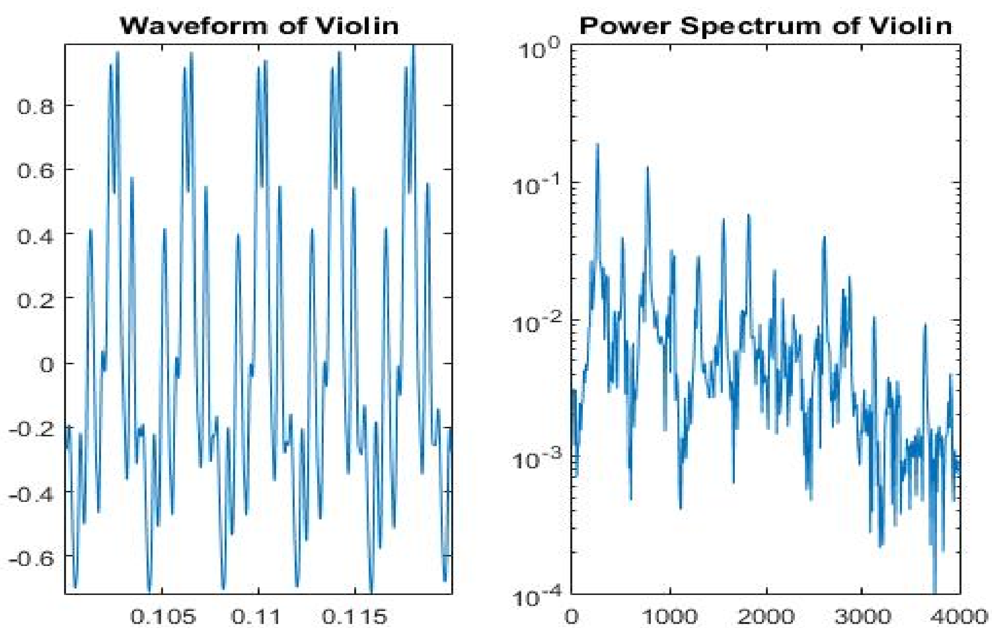

FIGURE 4. Power spectrum for the violin

The violin has very powerful harmonics giving it that rich sound. Because of the high harmonics, the violin has complex oscillations within each period.

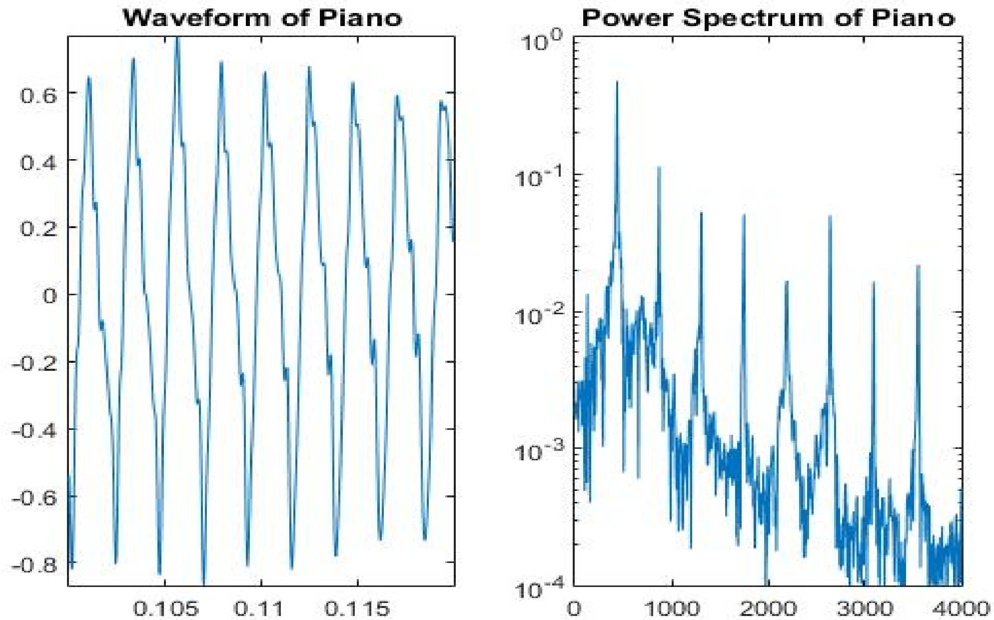

FIGURE 5. Power spectrum for the piano.

The piano is a slightly complex sound making the oscillations unique, but not as complex as a violin. The power spectrum of a piano is more consistent than any of the other instruments.

8. **Analysis of Same Instrument(the bass) but Different Chords**

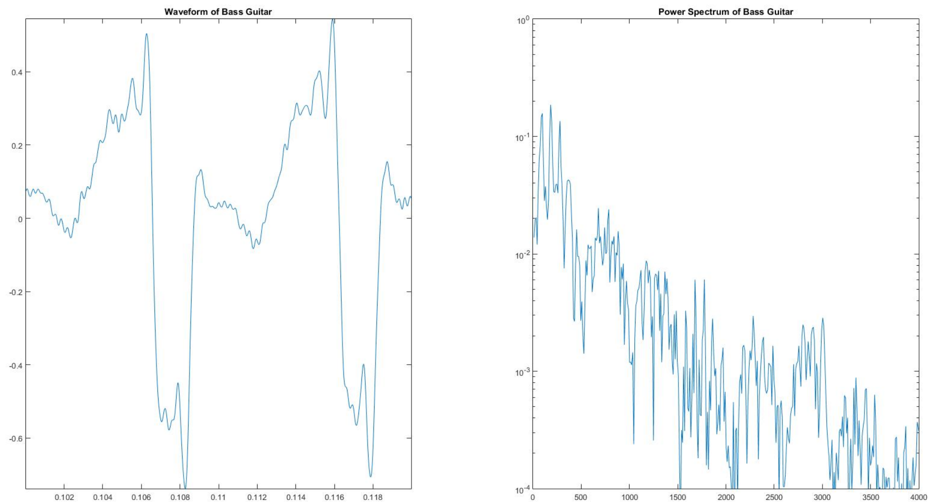

FIGURE 6. Power Spectrum for a chord for the bass.

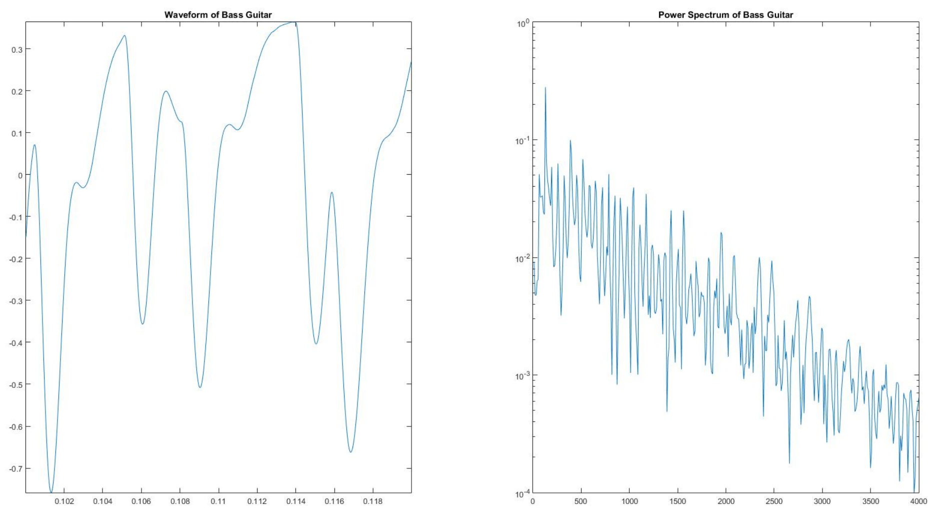

FIGURE 7. Power Spectrum for a second chord for the bass.

Now we can compare and contrast some different chords for the bass. In Figure 6 we can see that the waveform is more complex compared to Figure 7. Since the waveform is more complex, it leads to a wider variation in the power spectrum, but in Figure 7 the power spectrum stays mostly consistant and linear.

9. **Synthesis**

Synthesizing an instrument is a way to create a sound which mimics an instrument. The distribution of power in the harmonics is key to synthesizing its sound. The synthesized waveform is

$$y(t) = \sum_{n=1}^{N} p_n \cos 2\pi n f_1 t$$

Where f_1 is fundamental frequency and p_n is the power associated with the nth harmonic. When choosing which power spectrum to create a sound, the frequencies can vary. For example, a frequency between 3000 and 5000 Hz would have a tenth of the amplitude of a lower frequency for it to produce the same volume. Early music was synthesized by using a constant in time power spectrum, but that created a dull drone sound. Now the power spectrum is much more advanced and can accommodate for the attack and delay of a hit piano string or plucked banjo.[7]

Hermann Ludwig Ferdinand von Helmholtz wrote an article titled Sensation of Tones. In this article Helmholtz discusses how he discovered the idea of a resonator. A resonator is a cavity of air; Helmholtz used this to identify various musical pitches, frequencies, and other complex sounds. The way this works is by forcing air into the cavity, increasing the pressure inside. Air can flow inside and outside, but the pressure will change. The resonator can have a long neck, called the port. The diameter of the port is related to the mass of air and the volume of the cavity. If the port is too small it can halt the flow, but one that is too large will reduce the momentum of the air in the port.[7]

The way the resonator can be shown that the resonant angular frequency is given by: $\omega_H = \sqrt{\gamma \frac{A^2}{m} \frac{P_0}{V_0}}$ where:

γ is the ratio of specific heats, which is usually 1.4 for air.

A is the cross-sectional area of the neck.

m is the mass in the neck.

P_0 is the static pressure in the cavity.

V_0 is the static volume of the cavity.

For cylindrical or rectangular necks, we have $A = \frac{V_n}{L_{eq}}$ where:

L_{eq} is the equivalent length of the neck with end correction, which can be calculated as

$L_{eq} = L_n + 0.6D$, where L_n is the actual length of the neck and D is the hydraulic diameter of the neck.

V_n is the volume of the air in the neck.

Thus, $\omega_H = \sqrt{\gamma \frac{A}{m} \frac{V_N}{L_{eq}} \frac{P_0}{V_0}}$ From the definition of mass density, let's call it ρ : $\frac{V_n}{m} = \frac{1}{\rho}$ thus, $\omega_H = \sqrt{\gamma \frac{P_0}{\rho} \frac{A}{L_{eq} V_0}}$ and $f_H = \frac{\omega_H}{2\pi}$ where: f_H is the resonance frequency (Hz).

The speed of sound in a gas is given by: $v = \sqrt{\gamma \frac{P_0}{\rho}}$ Thus, the frequency of the resonance is:

$$f_H = \frac{v}{2\pi} \sqrt{\frac{A}{V_0 L_{eq}}}$$

The area of the neck is important because increasing the area will increase the inertia of the air proportionally, but will also decrease the velocity in which the air goes out. This is a very basic formula for a resonator, and can be improved analytically with similar explanations.[8]

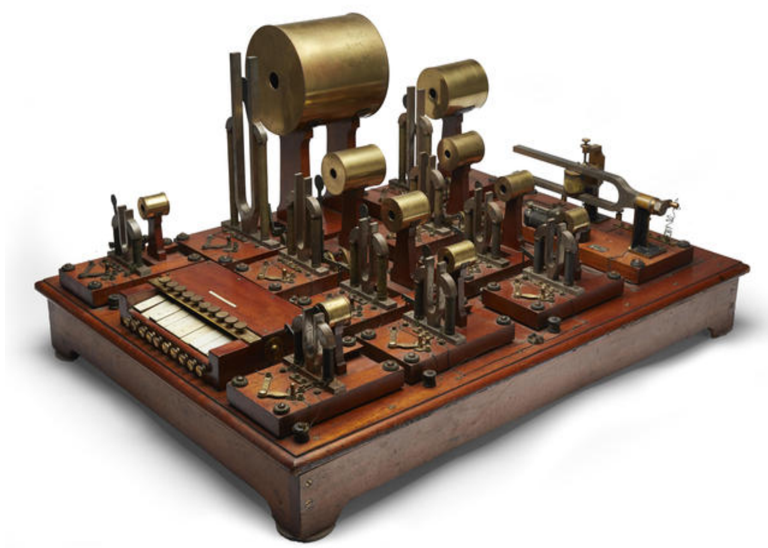

FIGURE 8. Helmholts resonator.

From here Helmholtz went into synthesizing sound. He did this by having ten electrically-driven tuning forks each facing a Helmholtz resonator tuned to the same frequency, which ran continuously, but produced little sound. Figure 8 is an example of what one of these resonators looks like. He used this apparatus to make synthesize vowel sounds. He found the harmonic content of each vowel by using Fourier Analyzer. The Fourier Analyzer is similar to what Helmholtz used to synthesize sounds but instead the holes are on the opposite side of the resonators and are connected to rubber tubes that go to flame capsules, and the rotating mirror observes the height of the flames. The strength of the each harmonic is adjusted by pushing down on the keys by various amounts. This allows the volume of the sounds of each resonator to be varied. Helmholtz was the first to synthesize sounds like this, but now we can use computer programs like we are using in this paper.[8]

Synthesized music has made great strides since the sixties, but it will never be as expressive as live music. To synthesize sounds, we will be using Matlab which has the function synthesize.m. The synthesize.m function in Matlab lets the user choose the fundamental frequency, the duration of the produced sound, and the length of the amplitudes. The fundamental frequency is the difference in frequency between any two consecutive harmonics. Using the power spectrum and Figure 10 we will synthesize vowels. The notation ff, f, mf, mp, p, and pp are used to denote how loud or soft the sound is, look at Figure 9 to see a graph of the music dynamics.[7]

For all the vowel sounds we will be using a fundamental frequency of 220 Hz and make the samples three seconds long. First, to make the U (oo) sound we used the amplitudes [2 1.2 .5]. Starting out loud with ff

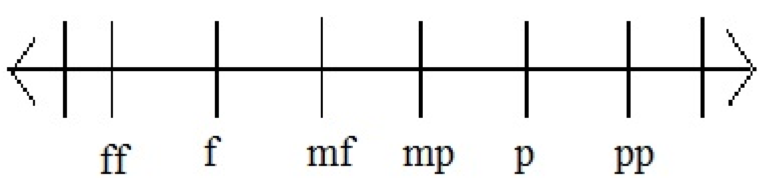

FIGURE 9. A graph of musical dynamics, from loudest to quietest.

		p_1	p_2	p_3	p_4	p_5	p_6	p_8	p_{16}
U	oo as in boot	*ff*	*mf*	*pp*					
O	oh as in no	*mf*	*f*	*mf*	*p*				
A	ah as in caught	*p*	*p*	*p*	*mf*	*mf*	*p*	*p*	
E	eh as in bed	*mf*		*mf*			*ff*		
I	ee as in see	*mf*	*p*				*p*		*mf*

FIGURE 10. Relative volume of various overtones[5].

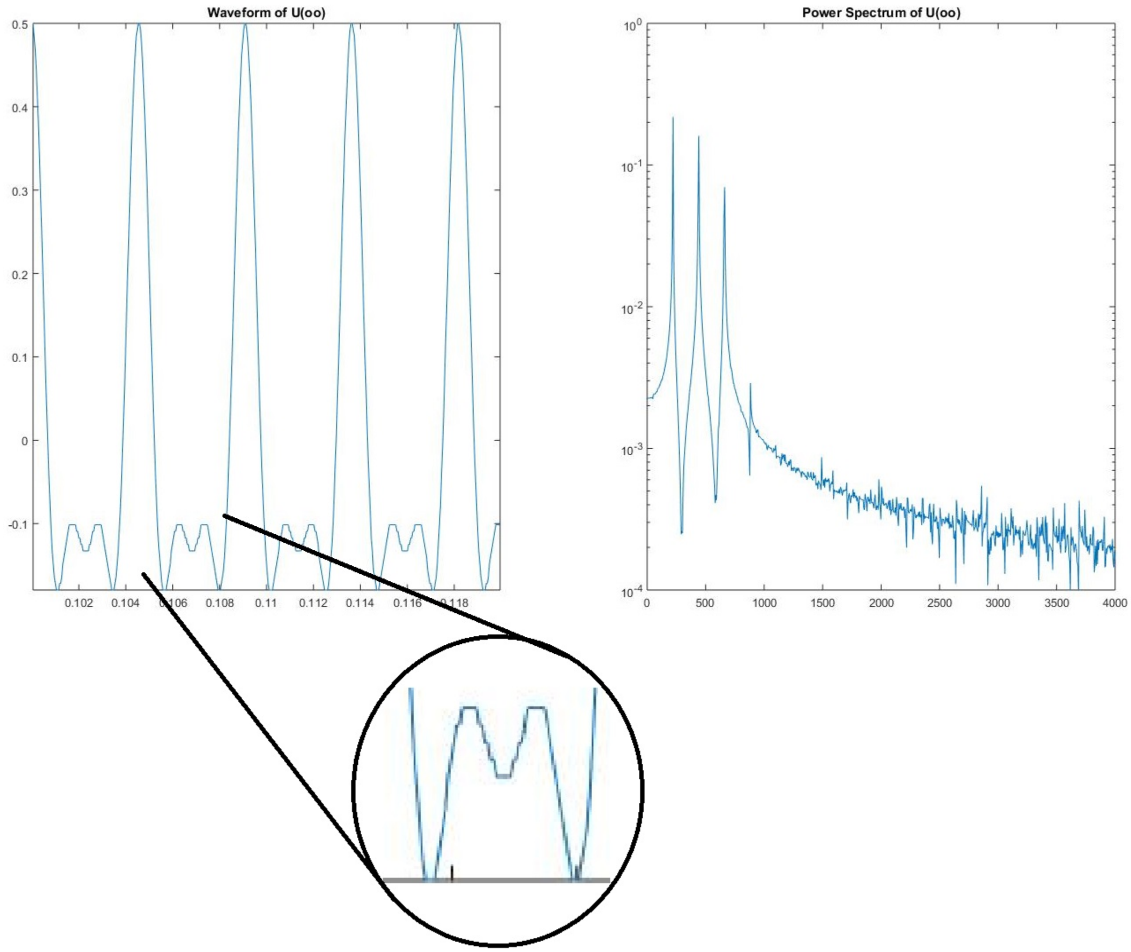

FIGURE 11. Graph for the synthesized sound for U(oo).

and then getting lower to achieve pp. Second, to make the O (oh) sound we will use amplitudes of [1.2 1.7 1.2 .7]. Third, for the A(ah) sound we use amplitudes [.1 .2 .2 1.2 1.4 .2 .2]. Fourth, the E(eh) sound uses amplitudes [1.2 0 1.2 0 0 2]. Finally, to make the I(ee) sound we must use the amplitudes [1.2 .4 0 0 0 .4 1.2]. One can use a program to synthesize a flute sound, but would have to know the exact amplitudes since it is a much more dynamic sound. With these synthesized sounds we can look at the waveform and power spectrum of these synthesized sounds. For this we will look at one of these synthesized sounds, the U(oo) sound. The waveform and power spectrum can be seen in Figure 11 on the next page.

From the graph we can see that the maximums and minimums are square, and not curved like they are in the Spectrum Analysis section for the flute and other instruments. This happens because the sound is synthesized and not natural like live music. Synthesized music may sound very similar to live music, but people are able to hear a difference if they listen closely.

10. **Conclusion**

In this paper we discussed how the wave equation leads to the fourier series and then to the fourier transform. We then examined different instruments using Matlab, looking at the power spectrum. We also synthezied vowel sounds using Matlab. We created figures to show the spectrums and what we used to synthesize vowel sounds.

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