Week 10 - Shortest Paths

AD 325 - Brenden West

Contents

Learning Outcomes

- Shortest path in an unweighted graph
- Edge-weighted digraphs
- Data structures for shortest paths
- Dijkstra’s Algorithm

Reading & Videos

- Carrano & Henry: Chapter 29 (Paths)
- https://www.coursera.org/learn/algorithms-part2/home/week/2 (Shortest Paths)

Reference

- https://algs4.cs.princeton.edu/44sp/
- https://www.geeksforgeeks.org/graph-data-structure-and-algorithms/#shortestPath

Shortest Path Variants

Which vertices

- **Single source**: from one vertex s to every other vertex
- **Source sink** - from one vertex s to another t
- **All pairs**: between all pairs of vertices

Edge weight restrictions?

- Unweighted
- Non-negative weights
- Euclidean (geometric distance) weights
- Arbitrary (incl. negative) weights

Cycles?

- No directed cycles
- No "negative cycles"

Shortest Paths - Unweighted Graph

In an unweighted graph, the shortest path between two vertices is the path with the fewest edges.

The path is found using a breadth-first search (BFS). Starting at the origin vertex, neighbors are placed on a queue and then the neighbors of each neighbor and so forth.

Each vertex is marked as visited and also with data about its predecessor and the path length traversed to reach it.

Once the target vertex is reached, the full path is derived by adding each predecessor vertex to a stack and returning the stack.

Shortest Paths - Weighted Graph

For a weighted graph, the shortest path between two vertices has the smallest sum of edge weights.

In general, the shortest path is found using a BFS with neighboring edges placed on a min priority queue.

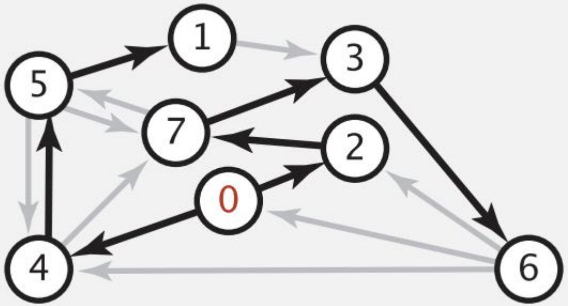
Vertices are visited according to different algorithms depending on edge weights and whether the graph has cycles:

- Dijkstra’s algorithm - no negative weights
- Bellman-Ford algorithm - no negative cycles
- Topological sorting - no directed cycles

Shortest Path Tree (SPT) data structure

Data for **shortest path tree -** path from vertex s to every other vertex - can be represented with two vertex-indexed arrays.

- distTo[v] is length of shortest path from s to v
- edgeTo[v] is last edge on shortest path from s to v



shortest-paths tree from 0

	edgeTo[]		distTo[]
0	null		0
1	5->1	0.32	1.05
2	0->2	0.26	0.26
3	7->3	0.37	0.97
4	0->4	0.38	0.38
5	4->5	0.35	0.73
6	3->6	0.52	1.49
7	2->7	0.34	0.60

parent-link representation

Single-source Shortest Path, cont.

- Distance to source vertex set to 0
- Distance from source vertex is initialized as **infinity** for all other vertices
- As each vertex is visited, it’s distance from the source vertex is **relaxed** (updated based on path used to reach it, if new distance would be lower)
- Each **relaxation** decreases distance to source vertex for some v
-

Dijkstra’s Algorithm

Computes a SPT in any edge-weighted digraph with nonnegative weights

- Consider vertices in increasing order of distance from source (s)
- Add vertex to the shortest-path tree and relax all edges pointing from that vertex
- Each edge is relaxed exactly once
- Algorithm stops when no relaxation happens (distance from source is not decreased for any vertex)

Essentially same as **Prim’s Algorithm**. Differs by choosing as next vertex the one closest to the source (via a directed path)

Dijkstra's Algorithm - performance

Performance depends on choice of priority queue implementation

Implementation	insert	Delete min	Decrease-key	Total
Unordered array	1	V	1	V^2
Binary heap	log V	log V	log V	E log V
d-way heap	$\log_d V$	$d \log_d V$	$\log_d V$	$E \log_{E/V} V$
Fibonacci heap	1	log V	1	E + V log V

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations
- Fibonacci heap best in theory, but not worth implementing

Bellman-Ford Algorithm - Negative Cycles

A **negative cycle** is a directed cycle whose sum of edge weights is negative

computes SPT in any edge-weighted digraph with no negative cycles in time proportional to $E \times V$ in worst case

Can be optimized to not relax vertices whose value didn't change in the previous iteration

Topological Sort

A Topological sort algorithm computes the SPT in any edge-weighted **directed acyclical graph (DAG)** in time proportional to E + V. Works even with negative weights

- Consider vertices in topological order
- Start from source vertex
- Relax all edges pointing from that vertex

Useful for finding longest path (e.g. **parallel job scheduling** or **critical path**):

- Negate all weights
- Find shortest paths
- Negate weights in result

Cost Summary

Algorithm	Restriction	Typical case	Worst case	Extra space
Topological sort	No directed cycles	E + V	E + V	V
Dijkstra's Algorithm (binary heap)	No negative weights	E log V	E log V	V
Bellman-Ford	No negative cycles	E V	E V	V
Bellman-Ford (queue-based)		E + V	E V	V