2.1 These formulas are both consequences of Proposition 3 in Section 2.1.

$$\begin{aligned}(\chi+\chi')^2_\sigma(s) &= \tfrac{1}{2}(((\chi+\chi')(s))^2 + (\chi+\chi')(s^2))\\ &= \tfrac{1}{2}(\chi(s)^2 + 2\chi(s)\chi'(s) + \chi'(s)^2 + \chi(s^2) + \chi'(s^2))\\ &= \tfrac{1}{2}(\chi(s)^2 + \chi(s^2)) + \tfrac{1}{2}(\chi'(s)^2 + \chi'(s^2)) + \tfrac{1}{2}(2\chi(s)\chi'(s))\\ &= \chi^2_\sigma(s) + {\chi'}^2_\sigma(s) + \chi(s)\chi'(s)\end{aligned}$$

$$\begin{aligned}(\chi+\chi')^2_\alpha(s) &= \tfrac{1}{2}(((\chi+\chi')(s))^2 - (\chi+\chi')(s^2))\\ &= \tfrac{1}{2}(\chi(s)^2 + 2\chi(s)\chi'(s) + \chi'(s)^2 - \chi(s^2) - \chi'(s^2))\\ &= \tfrac{1}{2}(\chi(s)^2 - \chi(s^2)) + \tfrac{1}{2}(\chi'(s)^2 - \chi'(s^2)) + \tfrac{1}{2}(2\chi(s)\chi'(s))\\ &= \chi^2_\alpha(s) + {\chi'}^2_\alpha(s) + \chi(s)\chi'(s)\end{aligned}$$

2.2 Let $X = \{x_1, x_2, \dots, x_n\}$ and suppose the matrix of ρ_s written in the basis $\{e_{x_i} \,|\, x_i \in X\}$ is given by $(r_{ij}(s))$. Now $s \cdot x_i = x_j$ iff $(r_{ij}(s)) = 1$, so if x_i is fixed by s then $(r_{ii}(s)) = 1$, and if not then $(r_{ii}(s)) = 0$. Therefore, the number of fixed elements is $\sum_{i=1}^{n}(r_{ii}(s)) = Tr(\rho_s) = \chi_X(s)$.

2.3 Write $\langle x, f\rangle$ as $f(x)$, so that $(\rho'_s(f))(\rho_s(x)) = f(x)$ for all $x \in V$, $f \in V'$. Taking $x = \rho_{s^{-1}}(z)$, we obtain $\rho'_s(f)(z) = f(\rho_{s^{-1}}(z))$, i.e. $\rho'_s(f) = f \circ \rho_{s^{-1}}$. It is now easy to check that $\rho'_r\rho'_s(f) = \rho'_{rs}(f)$.

$$\rho'_r\rho'_s(f) = \rho'_r(f \circ \rho_{s^{-1}}) = f \circ \rho_{s^{-1}} \circ \rho_{r^{-1}} = f \circ \rho_{s^{-1}r^{-1}} = f \circ \rho_{rs^{-1}} = \rho'_{rs}(f)$$

We can also verify that ρ' is linear, hence ρ' is a representation of G.

$$\rho'_s(f+g)(x) = (f+g)(\rho_{s^{-1}}(x)) = f(\rho_{s^{-1}}(x)) + g(\rho_{s^{-1}}(x)) = \rho'_s(f)(x) + \rho'_s(g)(x)$$
$$\rho'_s(cf)(x) = (cf)(\rho_{s^{-1}}(x)) = cf(\rho_{s^{-1}}(x)) = c\rho'_s(f)(x)$$

To show the character of ρ' is χ^*, pick a basis $\beta = \{e_1, e_2, \dots, e_n\}$ for V, and let A be the matrix for ρ_s, so that $\rho_s(x) = Ax$. If we write the linear map f as a column vector $[f]$ in the dual basis, so that $f(x) = [f]^T x$, then

$$\rho'_s(f)(x) = f \circ \rho_{s^{-1}}(x) = ([f])^T A^{-1} x = ((A^{-1})^T [f])^T x$$

Thus, ρ'_s acts by mapping $[f]$ to $(A^{-1})^T[f]$, i.e. the matrix for ρ'_s is $(A^{-1})^T$. Now we can easily see that the character of ρ' is χ^*.

$$Tr(\rho'_s) = Tr((A^{-1})^T) = Tr(A^{-1}) = Tr(\rho_{s^{-1}}) = \chi(s^{-1}) = \chi(s)^*$$

2.4 To verify that ρ is a representation, we check $\rho_r\rho_s(f) = \rho_{rs}(f)$. Note that $\rho_s(f) \in \mathrm{Hom}(V_1, V_2)$, so ρ is linear by definition. In fact, in this exercise and in exercise 2.3 above, each $s \in G$ acts via composition with linear transformations. This observation alone suffices for linearity.

$$\rho_r\rho_s(f) = \rho_r(\rho_{2,s} \circ f \circ {\rho_{1,s}}^{-1}) = \rho_{2,r} \circ \rho_{2,s} \circ f \circ {\rho_{1,s}}^{-1} \circ {\rho_{1,r}}^{-1} = \rho_{2,rs} \circ f \circ {\rho_{1,rs}}^{-1} = \rho_{rs}(f)$$

Consider the natural isomorphism $\phi : {V_1}^* \otimes V_2 \to Hom(V_1, V_2)$ with $\phi(f \otimes v_2)(v_1) = f(v_1) \cdot v_2$, and extended linearly to all of ${V_1}^* \otimes V_2$. We need to show $\phi(\rho'_{1,s}(f) \otimes \rho_{2,s}(v_2)) = \rho_s(\phi(f \otimes v_2))$, i.e. that ρ and $\rho'_1 \otimes \rho_2$ are isomorphic representations, via the isomorphism ϕ. Then the character of ρ is ${\chi_1}^* \cdot \chi_2$ by the last exercise and Proposition 2 in Section 2.1.

The verification that ϕ is an intertwining map and hence an isomorphism of representations is given below. The key points are: (i) $f \circ \rho_{1,s}{}^{-1}(v_1) \in \mathbb{C}$, (ii) $\rho_{2,s}$ is linear, (iii) $\phi(f \otimes v_2)(v_1) = f(v_1) \cdot v_2$.

$$\begin{aligned}
\phi(\rho'_{1,s}(f) \otimes \rho_{2,s}(v_2))(v_1) &= f \circ \rho_{1,s}{}^{-1}(v_1) \cdot \rho_{2,s}(v_2) \\
&= \rho_{2,s}(f \circ \rho_{1,s}{}^{-1}(v_1) \cdot v_2) \\
&= \rho_{2,s} \circ \phi(f \otimes v_2) \circ \rho_{1,s}{}^{-1}(v_1) \\
&= \rho_s(\phi(f \otimes v_2))(v_1)
\end{aligned}$$

2.5 Since the character of the trivial representation has $\chi_1(s) = 1$ for all $s \in G$, this statement follows immediately from Theorem 4 and the fact that every representation decomposes as a direct sum of irreducibles.

2.6 a) The permutation representation corresponding to an action of G on X comes with the basis $\beta = \{e_x \,|\, x \in X\}$ on which ρ_s acts by $\rho_s(e_x) = e_{s \cdot x}$. This means that if $\gamma = \{x_1, x_2, \dots, x_m\}$ is an orbit, the subspace $W = \{w \,|\, w = k(e_{x_1} + e_{x_2} + \dots + e_{x_m}) \text{ with } k \in \mathbb{C}\}$ will be ρ-invariant. In fact, W is clearly one-dimensional and ρ acts by reordering the e_{x_i}, which fixes each $w \in W$.

Every orbit will yield a different such one-dimensional invariant subspace on which ρ acts as the identity. Therefore, ρ contains the trivial representation at least c times. Conversely, any trivial subrepresentation of ρ is spanned by a single vector v with $\rho_s(v) = v$. Writing v in terms of the basis $\beta = \{e_x \,|\, x \in X\}$ will give a subset $S \subseteq X$ closed under the action of ρ, i.e., an orbit. Thus, a trivial subrepresentation yields an orbit, so ρ contains the trivial representation at most c times.

b) The value of $\chi_X(s)$ is the number of elements of X fixed by s, by exercise 2.2. Thus, $\chi_{X \times X}(s)$ is the number of pairs $(x_1, x_2) \in X \times X$ fixed by s, but such a pair is fixed iff both x_1 and x_2 are fixed by s. Thus, if k elements of X are fixed by s, then each of the k^2 possible combinations of these elements in $X \times X$ are fixed by s, so $\chi_{X \times X}(s) = k^2 = \chi(s)^2$.

c) (i $\leftrightarrow$ ii) If G is doubly transitive, then $C = \{(x, y) \,|\, x \neq y \text{ and } x, y \in X\}$ constitutes an orbit. $C' = \{(x, x) \,|\, x \in X\}$ constitutes an orbit as well (this only requires transitivity). Conversely, C is an orbit exactly when G is doubly transitive, and in that case it is also transitive, so C' is an orbit.

(ii $\leftrightarrow$ iii) This is an immediate consequence of parts (a) and (b).

(iii $\leftrightarrow$ iv) Decompose $\rho = 1 + \theta$, so that $\chi(s) = 1 + \psi(s)$, where ψ is the character of θ. If we assume that $(\chi^2, 1) = 2$, then $2 = (\chi^2, 1) = ((1 + \psi)^2, 1) = (1, 1) + 2(\psi, 1) + (\psi^2, 1)$, and since $(1, 1) = 1$, we must have $(\psi, 1) = 0$ and $(\psi^2, 1) = 1$.

$$1 = (\psi^2, 1) = \frac{1}{g} \sum_{s \in G} \psi(s)\psi(s) = \frac{1}{g} \sum_{s \in G} \psi(s)\psi(s)^* = (\psi, \psi)$$

Note that $\psi(s) = \psi(s)^*$ since $\psi(s) = \chi(s) - 1$, and $\chi(s)$ is integer-valued, as it counts the number of fixed elements. Thus, $(\psi, \psi) = 1$, so θ is irreducible by Theorem 4. Note that the argument can be run backwards, starting with $(\psi, \psi) = 1$, concluding $(\psi^2, 1) = 1$, and therefore $(\chi^2, 1) = 2$ using the definition of ψ.

2.7 Let ρ be a representation of G with character χ, such that $\chi(s) = 0$ for all $s \neq 1$. Consider the trivial representation, and denote its character by χ_T, so that $\chi_T(s) = 1$ for all $s \in G$.

$$(\chi|\chi_T) = \frac{1}{g}\sum_{s\in G}\chi(s)\chi_T(s)^* = \frac{1}{g}\chi(1)$$

However, $(\chi|\chi_T)$ is the number of copies of the trivial representation in ρ, which must be an integer, and therefore $\chi(1)$ is a multiple of g, hence χ is an integral multiple of the regular character. Note that we're only concerned with $\chi(1)$ since both $\chi(s) = 0$ and $r_G(s) = 0$ for other $s \in G$.

2.8 a) First, let's show that the direct sum is the product in the category of representations of G. Given two representations ρ on U and θ on V, we have the representation $\rho \oplus \theta$ on $U \oplus V$.

Now suppose that φ is a representation on W, and $f : W \to U$ and $g : W \to V$ are intertwining maps, then $h : W \to U \oplus V$ defined by $h(w) = (f(w), g(w))$ will be an intertwining map since

$$h(\varphi_s(w)) = (f(\varphi_s(w)), g(\varphi_s(w))) = (\rho_s(f(w)), \theta_s(g(w))) = \rho_s \oplus \theta_s\,(f(w), g(w)) = \rho_s \oplus \theta_s\,(h(w))$$

In this way, any pair of intertwining maps $f : W \to U$, $g : W \to V$ determines a unique intertwining map $h : W \to U \oplus V$, and vice-versa via projection onto the direct summands.

Writing $H_i = Hom_G(W_i, V)$, choosing a decomposition $V_i = W_i \oplus ... \oplus W_i$, and using our result above, we have $H_i = Hom_G(W_i, W_i \oplus ... \oplus W_i) = Hom_G(W_i, W_i) \oplus ... \oplus Hom_G(W_i, W_i)$.

To finish, we only need to observe $Hom_G(W_i, W_i)$ is one-dimensional by Schur's lemma.

Note that the two W_i are isomorphic, but not equal, so Schur's lemma as stated in Proposition 4 does not directly apply. We need to observe that if V_1 and V_2 are isomorphic irreducibles, we can fix $\varphi : V_1 \to V_2$ any isomorphism. Composition on the right with φ is a vector space isomorphism from $Hom_G(V_1, V_1)$ to $Hom_G(V_1, V_2)$, and $Hom_G(V_1, V_1)$ is one-dimensional by Proposition 4.

b) We can verify that F is an intertwining map by the calculation below.

$$\begin{aligned} F((I \otimes \rho_s)(\sum h_\alpha \otimes w_\alpha)) &= F(\sum h_\alpha \otimes \rho_s(w_\alpha)) \\ &= \sum h_\alpha(\rho_s(w_\alpha)) \\ &= \sum \rho_s(h_\alpha(w_\alpha)) \\ &= \rho_s \sum h_\alpha(w_\alpha) \\ &= \rho_s(F(\sum h_\alpha \otimes w_\alpha)) \end{aligned}$$

Choose a decomposition $V_i = W_i \oplus ... \oplus W_i$ with n copies of W_i, and define $\beta = \{h_j \,|\, 1 \le j \le n\}$ such that $h_j : W_i \to V$ is an isomorphism onto the j^{th} summand in the decomposition. Note that the set β is easily seen to be linearly independent, since the only vector common to the images of any two of the h_j is 0. Since β contains n vectors, it is a basis of H_i by part (a).

Write $f : W_i \to V_i$ as $f = a_1\phi_1 \oplus ... \oplus a_n\phi_n$, and consider the bilinear function $\phi : H_i \times W_i \to V_i$ defined by $\phi(f, w) = a_1\phi_1(w) \oplus ... \oplus a_n\phi_n(w)$. The coefficients $a_1, ..., a_n$ determine f completely, and each ϕ_i is an isomorphism. Therefore, ϕ is an isomorphism as well (to be explicit we could choose a basis $\beta' = \{b_i\}$ for W_i and check the images of (ϕ_j, w_i) form an independent set). Thus, ϕ determines an isomorphism $F : H_i \otimes W_i \to V_i$ with $F(f \otimes w) = a_1\phi_1(w) \oplus ... \oplus a_n\phi_n(w) = f(w)$.

c) Define $h(w_1 \oplus ... \oplus w_k) = h_1(w_1) + ... + h_k(w_k)$. Then h is an intertwining map.

$$\begin{aligned}
h((\rho_s)(w_1 \oplus ... \oplus w_k)) &= h(\rho_s(w_1) \oplus ... \oplus \rho_s(w_k)) \\
&= h_1(\rho_s(w_1)) + ... + h_k(\rho_s(w_k)) \\
&= \rho_s(h_1(w_1)) + ... + \rho_s(h_k(w_k)) \\
&= \rho_s(h_1(w_1) + ... + h_k(w_k)) \\
&= \rho_s(h(w_1 \oplus ... \oplus w_k))
\end{aligned}$$

Every element of V_i can be written as $\sum h_\alpha(w_\alpha)$ by part (b), and since $\{h_1, ... h_k\}$ is a basis for H_i, we can rewrite the h_α in terms of the h_i, so everything in V_i can be written as a linear combination of the $h_i(w_\alpha)$, which can be simplified to an expression of the form $h_1(w_1)+...+h_k(w_k)$ by linearity. Thus, h is surjective, and since $dim(V_i) = dim(H_i)dim(W_i) = kdim(W_i) = dim(W_i \oplus ... \oplus W_i)$ by parts (a) and (b), h is an isomorphism.

We conclude that any choice of basis for H_i determines an isomorphism $h : W_i \oplus ... \oplus W_i \to V_i$. Conversely, given such an isomorphism h, consider the maps $p_j : W_i \to W_i \oplus ... \oplus W_i$, where each p_j is a projection onto the j^{th} summand. Let $h_j = h \circ p_j$. If we can show that the set $\{h_1, ... , h_k\}$ is a basis for H_i, then h can be recovered from $\{h_1, ... , h_k\}$ as above.

To show $\{h_1, ... , h_k\}$ is a basis, we need only check that the h_j are linearly independent, since there are k of them and $dim(H_i) = k$ by part (a). Suppose $a_1h_1 + ... + a_kh_k = 0$. Taking any nonzero $w \in W_i$, we have $0 = a_1h_1(w) + ... + a_kh_k(w) = h_1(a_1w) + ... + h_k(a_kw) = h(a_1w \oplus ... \oplus a_kw)$, and therefore the nonzero vector $a_1w \oplus ... \oplus a_kw$ is in the kernel of h, but this contradicts the fact that h is an isomorphism.

2.9 To begin, we need to show that $h(e_\alpha) \in V_{i,\alpha}$. Since $V_{i,\alpha}$ is the image of the projection $p_{\alpha\alpha}$, this means that $p_{\alpha\alpha}(h(e_\alpha)) = h(e_\alpha)$.

$$\begin{aligned}
p_{\alpha\alpha}(h(e_\alpha)) &= \frac{n}{g}\sum_{s\in G} r_{\alpha\alpha}(s^{-1})\rho_s(h(e_\alpha)) \\
&= \frac{n}{g}\sum_{s\in G} r_{\alpha\alpha}(s^{-1})h(\rho_s(e_\alpha)) \\
&= \frac{n}{g}\sum_{s\in G} h(r_{\alpha\alpha}(s^{-1})\rho_s(e_\alpha)) \\
&= h\left(\frac{n}{g}\sum_{s\in G} r_{\alpha\alpha}(s^{-1})\rho_s(e_\alpha)\right) \\
&= h\left(n\sum_{\gamma}\frac{1}{g}\sum_{s\in G} r_{\alpha\alpha}(s^{-1})r_{\gamma\alpha}(s^{-1})e_\alpha\right) \\
&= h\left(n\sum_{\gamma}\frac{1}{n}\delta_{\alpha\alpha}\delta_{\gamma\alpha}e_\alpha\right) \\
&= h(e_\alpha)
\end{aligned}$$

The second last equivalence is by Corollary 3 to Proposition 4. The map $h \mapsto h(e_\alpha)$ is injective, because if $h(e_\alpha) = h'(e_\alpha)$ then $(h - h')(e_\alpha) = 0$, so $ker(h - h') \neq 0$. However, $ker(h - h')$ is stable under ρ_s, since if $(h - h')(v) = 0$, then $(h - h')(\rho_s(v)) = \rho_s((h - h')(v)) = \rho_s(0) = 0$.

Thus, $ker(h-h')$ is a nonzero invariant subspace of W_i, so $ker(h-h') = W_i$, and $h = h'$. Therefore, the map $h \mapsto h(e_\alpha)$ is injective.

V_i is the direct sum of $V_{i,\alpha}$ for $1 \leq \alpha \leq dim(W_i)$ by Proposition 8, so $dim(V_{i,\alpha}) = dim(V_i)/dim(W_i)$. Thus, $dim(H_i) = dim(V_{i,\alpha})$ by Exercise 2.8 (a), so injectivity alone is sufficient to show $h \mapsto h(e_\alpha)$ is an isomorphism.

2.10 Using Proposition 8 (c), $W(x_1^\alpha)$ has a basis consisting of the $p_{\beta 1}(x_1^\alpha) = p_{\beta 1} \circ p_{1\alpha}(x) = p_{\beta\alpha}(x)$ and is isomorphic to W_i if $p_{1\alpha}(x) \neq 0$, trivial if $p_{1\alpha}(x) = 0$. Let $Z = W(x_1^1) + ... + W(x_1^n)$. Then Z is spanned by the $p_{\beta\alpha}(x)$, and by Proposition 8 (a), $x = \sum_\alpha p_{\alpha\alpha}(x)$, so $x \in Z$.

To show that $Z \subseteq V(x) = \{span(\rho_s(x)) \mid s \in G\}$, we need to verify that each $p_{\beta\alpha}(x)$ can be written as a linear combination of $\rho_s(x)$ for $s \in G$, but by definition $p_{\beta\alpha}(x) = \frac{n}{g}\sum_{s\in G} r_{\beta\alpha}(s^{-1})\rho_s(x)$. Thus, every element of Z is in $V(x)$, and since $V(x)$ is the smallest subrepresentation of V containing x, we must have $Z = V(x)$.

Therefore, to write $V(x)$ as a direct sum, we can take $\{p_{11}(x), ..., p_{1n}(x)\}$, reduce it to a linearly independent set, and each vector in the result will generate (as in Proposition 8 (c)) a copy of W_i in a direct sum decomposition of $V(x)$.

3.1 Let $s \in G$ and let ρ be an irreducible representation of G on V. Consider the map $\rho_s : V \to V$, and note that $\rho_s(\rho_t(v)) = \rho_t(\rho_s(v))$ because G is abelian. This means that ρ_s is an intertwining map from V to V, hence by Schur's lemma, $\rho_s = \lambda I$ for some $\lambda \in \mathbb{C}$.

Thus, each element of G acts via multiplication by a scalar, which means every subspace of V is invariant. But V has no proper invariant subspaces since it is irreducible, hence it has no proper subspaces, and we must conclude that V is one dimensional.

3.2 a) Using the same argument as in Exercise 3.1 above, each ρ_s for $s \in C$ is an intertwining map and hence $\rho_s = \lambda I$ for some $\lambda \in \mathbb{C}$ by Schur's lemma.

Since C is a finite group, we must have $(\rho_s)^c = I$, where $c = |C|$, but $(\rho_s)^c = (\lambda I)^c = \lambda^c I$. Thus, λ is a c^{th} root of unity, and therefore $|\chi(s)| = |Tr(\rho_s)| = |Tr(\lambda I)| = |n\lambda| = n|\lambda| = n$.

b) Note that since ρ is irreducible, $(\rho|\rho) = \frac{1}{g}\sum_{s\in G}\chi(s)\chi(s)^* = 1$, and recalling $\chi(s)\chi(s)^* = |\chi(s)|^2$, we obtain the formula $\sum_{s\in G}|\chi(s)|^2 = g$. Splitting the sum and using out result above,

$$\begin{aligned}
\sum_{s\in G}|\chi(s)|^2 &= g \\
\sum_{s\in C}|\chi(s)|^2 + \sum_{s\in G\setminus C}|\chi(s)|^2 &= g \\
\sum_{s\in C} n^2 + \sum_{s\in G\setminus C}|\chi(s)|^2 &= g \\
cn^2 + \sum_{s\in G\setminus C}|\chi(s)|^2 &= g
\end{aligned}$$

Therefore, since each $|\chi(s)|^2 \geq 0$, we conclude that $cn^2 \leq g$, and $n^2 \leq \dfrac{g}{c}$.

c) Using part (a), each $\rho_s = \lambda_s I$ where λ_s is a c^{th} root of unity, where $c = |C|$. If $s_1, s_2 \in C$, then $\rho_{s_1 s_2{}^{-1}} = \rho_{s_1}\rho_{s_2{}^{-1}} = \lambda_{s_1} I \lambda_{s_2}^{-1} I = \lambda_{s_1}\lambda_{s_2}^{-1} I$. We conclude that

$$\lambda_{s_1} = \lambda_{s_2} \ \rightarrow \ \rho_{s_1 s_2{}^{-1}} = I \ \rightarrow \ s_1 {s_2}^{-1} \in ker(\rho) \ \rightarrow \ s_1 {s_2}^{-1} = 1 \ \rightarrow \ s_1 = s_2$$

Therefore, the images of distinct elements of C are distinct multiples of I. There are only c roots of unity to choose from, and there are c elements of C, so some $x \in C$ must have $\rho_x = e^{2\pi i/c}$.

Since ρ_x is a primitive c^{th} root of unity, any $s \in C$ has $\rho_s = (\rho_x)^k = \rho_{x^k}$ for some k. Therefore $s = x^k$ since ρ is faithful, i.e., x generates C.

3.3 First, we must show that if χ_1 and χ_2 are irreducible characters, then so is $\chi_1\chi_2$. Recall that $|\chi_1(s)|$ and $|\chi_2(s)|$ are equal to the dimensions of their respective representations for all $s \in G$, by Exercise 3.2 (a). Since G is abelian, each irreducible representation has dimension one, so $|\chi_1(s)| = |\chi_2(s)| = 1$ for all $s \in G$.

$$\begin{aligned}
(\chi_1\chi_2 \,|\, \chi_1\chi_2) &= \frac{1}{g}\sum_{s\in G} \chi_1(s)\chi_2(s)\chi_1(s)^*\chi_2(s)^* \\
&= \frac{1}{g}\sum_{s\in G} \chi_1(s)\chi_1(s)^*\chi_2(s)\chi_2(s)^* \\
&= \frac{1}{g}\sum_{s\in G} |\chi_1(s)|^2|\chi_2(s)|^2 \\
&= \frac{1}{g}\sum_{s\in G} 1 = \frac{1}{g} g = 1
\end{aligned}$$

Thus, $\chi_1\chi_2$ is an irreducible character. In $\hat{G}$, the set of all irreducible characters, multiplication is clearly associative and commutative since these properties hold in $\mathbb{C}$. To complete the verification that $\hat{G}$ is an abelian group, we need to show the multiplicative inverse of an irreducible character χ is χ^*, and χ^* is itself irreducible.

Note that $\chi(s)\chi^*(s) = \chi(s)(\chi(s))^* = |\chi(s)|^2 = 1$, again by Exercise 3.2 (a). Therefore $\chi(s)\chi^*(s)$ is the trivial character, which is the identity in $\hat{G}$, so $\chi^{-1} = \chi^*$. Checking that χ^* is irreducible is straightforward.

$$\begin{aligned}
(\chi^* \,|\, \chi^*) &= \frac{1}{g}\sum_{s\in G} \chi(s)^*\chi(s)^{**} \\
&= \frac{1}{g}\sum_{s\in G} \chi(s)^*\chi(s) \\
&= \frac{1}{g}\sum_{s\in G} |\chi(s)|^2 \\
&= \frac{1}{g}\sum_{s\in G} 1 = \frac{1}{g} g = 1
\end{aligned}$$

Thus, $\hat{G}$ is an abelian group. The order of $\hat{G}$ is the dimension of the space of class functions on G, and since G is abelian, each element is in its own conjugacy class, so $|\hat{G}| = |G|$.

Every irreducible representation ρ has dimension one, so $\chi(s) = Tr(\rho_s) = \rho_s$. Therefore, we have that $\chi(gh) = \rho_{gh} = \rho_g \rho_h = \chi(g)\chi(h)$, i.e., each irreducible $\chi : G \to \mathbb{C}^*$ is a group homomorphism.

Consider the map $ev : G \to \hat{\hat{G}}$ defined by $ev(g)(\chi) = \chi(g)$. With the above observation in mind, it's easy to see ev is a group homomorphism.

$$ev(gh)(\chi) = \chi(gh) = \chi(g)\chi(h) = ev(g)(\chi)\, ev(h)(\chi)$$

Moreover, ev is injective since the irreducible characters form a basis for the class functions on G, and each $g \in G$ is in its own conjugacy class. Given $g, h \in G$ with $g \neq h$, there must be some irreducible χ with $\chi(g) \neq \chi(h)$, and hence $ev(g)(\chi) \neq ev(h)(\chi)$.

We showed above that $|\hat{G}| = |G|$ by noting that the dimension of the space of class functions on any abelian group is simply the number of conjugacy classes, which is equal to the size of the group.

Applying this argument once again, to the abelain group $\hat{G}$, yields $|\hat{G}| = |\hat{\hat{G}}|$, so $|G| = |\hat{\hat{G}}|$. Thus, we have an injective homomorphism ev from one group to another of the same order, and hence ev is an isomorphism.

3.4 Let $\psi_1, ..., \psi_m$ be the irreducible characters of H. Let ψ_{reg} be the character of the regular representation of H. We have $\psi_{reg} = n_1\psi_1 + ... + n_m\psi_m$. Let $\chi_{I_1}, ..., \chi_{I_m}$ be the characters of the representations of G induced by each irreducible representation of H. From Example 1 in section 3.3, we know the regular representation of G is induced by the regular representation of H. Let χ be an irreducible character of G, and apply Theorem 12 as follows.

$$\begin{aligned}
(\chi \,|\, \chi_{reg}) &= \frac{1}{g}\sum_{s\in G}\chi(s)\chi_{reg}(s)^* \\
&= \frac{1}{g}\sum_{s\in G}\chi(s)\;\frac{1}{h}\sum_{\substack{t\in G\\ t^{-1}st\in H}}\psi_{reg}(t^{-1}st)^* \\
&= \frac{1}{g}\sum_{s\in G}\chi(s)\sum_{\substack{t\in G\\ t^{-1}st\in H}}(\frac{1}{h}n_1\psi_1(t^{-1}st)^* + ... + \frac{1}{h}n_m\psi_m(t^{-1}st)^*) \\
&= \frac{1}{g}\sum_{s\in G}\chi(s)\sum_{\substack{t\in G\\ t^{-1}st\in H}}\frac{1}{h}n_1\psi_1(t^{-1}st)^* + ... + \frac{1}{g}\sum_{s\in G}\chi(s)\sum_{\substack{t\in G\\ t^{-1}st\in H}}\frac{1}{h}n_m\psi_m(t^{-1}st)^* \\
&= \frac{n_1}{g}\sum_{s\in G}\chi(s)\chi_{I_1}(s)^* + ... + \frac{n_m}{g}\sum_{s\in G}\chi(s)\chi_{I_m}(s)^* \\
&= n_1(\chi \,|\, \chi_{I_1}) + ... + n_m(\chi \,|\, \chi_{I_m})
\end{aligned}$$

Because $(\chi \,|\, \chi_{reg}) \neq 0$, at least one of the inner products above, say $(\chi \,|\, \chi_{I_k})$, is nonzero. Thus, the irreducible representation of G with character χ is contained in the representation with character χ_{I_k} which is induced by the irreducible representation of H with character ψ_k.

If (V, ρ) is an irreducible representation of G, and A is an abelian subgroup of G, then by our result above, (V, ρ) is contained in a representation (V_I, ψ) induced by some irreducible representation (W, θ) of A, which has dimension one since H is abelian. We obtain a proof of the Corollary to Theorem 9 by noting that $dim(V) \leq dim(V_I) = (G\!:\!H)dim(W) = (G\!:\!A)$.

3.5 To begin, we show $\eta : W \to W_0$ is an intertwining map. H acts via θ on W, and via ρ on W_0.

$$\begin{aligned}
\rho_t(\eta(w))(u) &= \rho_t(f_w)(u) \\
&= f_w(ut) \\
&= \theta_{ut}(w) \\
&= \theta_u\theta_t(w) \\
&= f_{\theta_t(w)}(u) \\
&= \eta(\theta_t(w))(u)
\end{aligned}$$

Now let $h : G \to W$ be a function with $h(tu) = \theta_t h(u)$, and let $v = h(e)$, where e is the identity element of H. Then we have $h = f_{h(e)}$, since $h(t) = h(te) = \theta_t(h(e)) = f_{h(e)}(t)$. Therefore, η is surjective. If $f_v = f_w$, then we have $f_v(e) = f_w(e)$, so $\theta_e(v) = \theta_e(w)$, i.e. $I(v) = I(w)$, hence $v = w$, and η is injective.

If σ is a left coset of H, and $s \in \sigma$, then $\rho_s W_0 = \{\rho_s(f) \,|\, f \in W_0\}$. The function $\rho_s(f)$ vanishes outside the right coset Hs^{-1} because $\rho_s(f)(g) = f(gs)$, so if $f(gs) \neq 0$ then $gs \in H \to g \in Hs^{-1}$.

Let $R = \{s_1, \dots, s_n\}$ be a system of representatives for the left cosets of H, and define $f_{s_i} = \rho_{s_i}(f_w)$ where $w = f(s_i{}^{-1})$. We observed above that f_{s_i} is zero outside the coset Hs_i^{-1}. Moreover, f_{s_i} is equal to f on that coset.

$$\rho_{s_i}(f_w)(hs_i{}^{-1}) = f_w(hs_i{}^{-1}s_i) = f_w(h) = \theta_h(w) = \theta_h(f(s_i{}^{-1})) = f(hs_i{}^{-1})$$

The right cosets $\{Hs_1^{-1}, \dots, Hs_n^{-1}\}$ partition G, because the inverses of a system of representatives for the left cosets of H form a system of representatives for the right cosets of H. This means $f = f_{s_1} + \dots + f_{s_n}$ and the f_{s_i} are uniquely determined by f. Therefore, $V = \rho_{s_1} W_0 \oplus \dots \oplus \rho_{s_n} W_0$.

3.5 Let $(h, k) \in G$. We identify with H the subset of elements of G of the form (h, e_K) with $h \in H$ and e_K the identity element of K, so we can, by abuse of notation, write $\chi_\theta(h, e_K) = \chi_\theta(h)$.

$$\begin{aligned}
\chi_\rho(h,k) &= \frac{1}{|H|} \sum_{\substack{(h',k')\in G \\ (h'hh'^{-1},k'kk'^{-1})\in H}} \chi_\theta(h'hh'^{-1}, k'kk'^{-1}) \\
&= \frac{1}{|H|} \sum_{\substack{(h',k')\in G \\ k'kk'^{-1}=e_K}} \chi_\theta(h'hh'^{-1}, e_K) \\
&= \frac{1}{|H|} \sum_{\substack{(h',k')\in G \\ k'kk'^{-1}=e_K}} \chi_\theta(h)
\end{aligned}$$

Observe that if $k \neq e_K$ then there are no $(h', k') \in G$ with $k'kk'^{-1} = e_K$, since the conjugacy class of e_K is a singleton. Thus, $\chi_\rho(h, k) = 0$ if $k \neq e_K$. On the other hand, if $k = e_K$, then any $k' \in K$ has $k'kk'^{-1} = e_K$.

$$\chi_\rho(h, e_K) = \frac{1}{|H|} \sum_{(h',k')\in G} \chi_\theta(h) = \frac{|G|}{|H|}\chi_\theta(h) = |K|\chi_\theta(h) = \chi_\theta(h) \cdot \chi_{reg}(e)$$

Therefore, we have $\chi_\rho(h,k) = 0 = \chi_\theta(h)\cdot 0 = \chi_\theta(h)\cdot\chi_{reg}(k)$ if $k \neq e_K$, and by the calculation above, $\chi_\rho(h, e_K) = \chi_\theta(h)\cdot\chi_{reg}(e)$. In either case, $\chi_\rho(h,k) = \chi_\theta(h)\cdot\chi_{reg}(k)$, and since the character of a representation determines it up to isomorphism, we're done.

5.1 The group D_n is generated by a rotation σ and a reflection τ with $\sigma^n = e$ and $\tau^2 = e$. The two generators are related by $\tau\sigma\tau = \sigma^{-1}$, which can also be written $\tau\sigma = \sigma^{-1}\tau$. The rotations all have the form σ^k, hence all commute, and the reflections have the form $\sigma^k\tau$.

Since the rotations commute, conjugating a rotation by a rotation has no effect. Conjugating a rotation by a reflection, we have $\sigma^\ell\tau^{-1}\sigma^k\sigma^\ell\tau = \tau\sigma^{-\ell}\sigma^k\sigma^\ell\tau = \tau\sigma^k\tau = \sigma^{-k}\tau\tau = \sigma^{-k}$. Thus, each rotation σ^k has only $\sigma^{-k} = \sigma^{n-k}$ in its conjugacy class.

If n is even, then $\{e\}$, $\{\sigma,\sigma^{n-1}\}$, $\{\sigma^2,\sigma^{n-2}\}$, ..., $\{\sigma^{\frac{n}{2}-1},\sigma^{\frac{n}{2}+1}\}$, $\{\sigma^{\frac{n}{2}}\}$ are the conjugacy classes of elements of C_n (rotations). If n is odd, then no nontrivial rotation is its own inverse, so the classes are $\{e\}$, $\{\sigma,\sigma^{n-1}\}$, $\{\sigma^2,\sigma^{n-2}\}$, ..., $\{\sigma^{\frac{n-1}{2}},\sigma^{\frac{n+1}{2}}\}$. Counting yields the desired result.

Conjugating a reflection by a rotation, $(\sigma^\ell)^{-1}\sigma^k\tau\sigma^\ell = \sigma^{-\ell+k}\sigma^{-\ell}\tau = \sigma^{k-2\ell}\tau$. Similarly, conjugating a reflection by a reflection, $(\sigma^\ell\tau)^{-1}\sigma^k\tau\sigma^\ell\tau = \tau\sigma^{-\ell}\sigma^k\tau\sigma^\ell\tau = \tau\sigma^{-\ell}\sigma^k\sigma^{-\ell}\tau\tau = \tau\sigma^{k-2\ell} = \sigma^{2\ell-k}\tau$. Thus, the conjugacy class of $\sigma^k\tau$ contains all reflections $\sigma^m\tau$ with $m \equiv 2\ell - k \pmod n$ for $\ell \in \mathbb{N}$.

If n is even, then such an ℓ can be found only when m and k have the same parity, since the condition $m \equiv 2\ell - k \pmod n$ is equivalent to $m + k = 2(\ell + p\frac{n}{2})$ for some $\ell, p \in \mathbb{N}$. So the conjugacy classes of reflections are $\{\tau, \sigma^2\tau, \sigma^4\tau, ...\}$ and $\{\sigma\tau, \sigma^3\tau, \sigma^5\tau, ...\}$. If n is odd, then $m \equiv 2\ell - k \pmod n$ for all k, m (take $\ell = \frac{k+m}{2}$ if $k+m$ is even and $\ell = \frac{k+m+n}{2}$ if not) and all reflections are conjugate.

The number of conjugacy classes of D_n for even n is thus $\frac{n}{2}+1+2 = \frac{n}{2}-1+4$, which matches the number of irreducible representations, four of degree one, and $\frac{n}{2}-1$ of degree two, as described in 5.3. If n is odd, we have $\frac{n+1}{2}+1 = \frac{n}{2}-1+2$ classes, which again matches the number of irreducible representations, two of degree one, and $\frac{n}{2}-1$ of degree two.

5.2 The characters χ_h, $\chi_{h'}$, $\chi_{h+h'}$, $\chi_{h-h'}$ are all zero on the reflections, so it suffices to check they are equal on a rotation.

$$\begin{aligned}
\chi_h\cdot\chi_{h'}(\sigma^k) &= \chi_h(\sigma^k)\chi_{h'}(\sigma^k)\\
&= (w^{hk}+w^{-hk})(w^{h'k}+w^{-h'k})\\
&= w^{(h+h')k}+w^{(h-h')k}+w^{(h'-h)k}+w^{(-h-h')k}\\
&= w^{(h+h')k}+w^{-(h+h')k}+w^{(h-h')k}+w^{-(h-h')k}\\
&= \chi_{h+h'}(\sigma^k)+\chi_{h-h'}(\sigma^k)
\end{aligned}$$

Since $\chi_h\cdot\chi_h$ is the character of $\rho^h\otimes\rho^h$, and $\chi_h\cdot\chi_h = \chi_{2h}+\psi_1+\psi_2$, it suffices to show that ψ_2 is the character of the alternating square of ρ^h, and the other result follows by Proposition 3 in section 2.1.

$$\begin{aligned}
(\chi_h)^2_\alpha(\sigma^k) &= \frac{1}{2}\left(\chi_h(\sigma^k)^2 - \chi_h(\sigma^{2k})\right)\\
&= \frac{1}{2}\left((w^{hk}+w^{-hk})^2 - (w^{2hk}+w^{-2hk}\right)\\
&= \frac{1}{2}\left(w^0+w^{2hk}+w^{-2hk}+w^0-(w^{2hk}+w^{-2hk})\right) = 1
\end{aligned}$$

$$
\begin{aligned}
(\chi_h)^2_\alpha(\sigma^k\tau) &= \frac{1}{2}\left(\chi_h(\sigma^k\tau)^2 - \chi_h((\sigma^k\tau)^2)\right)\\
&= \frac{1}{2}\left(0 - \chi_h(\sigma^k\tau\sigma^k\tau)\right)\\
&= \frac{1}{2}\left(-\chi_h(\sigma^k\sigma^{-k}\tau\tau)\right)\\
&= \frac{1}{2}\left(-\chi_h(e)\right)\\
&= \frac{1}{2}(-2) = -1
\end{aligned}
$$

5.3 In the usual realization of D_n on $\mathbb{R}^3$, σ^k and τ act on vectors written in the standard basis via multiplication on the left by the matrices given below.

$$
[\rho_{\sigma^k}] = \begin{bmatrix} \cos(\frac{2k\pi}{n}) & \sin(\frac{2k\pi}{n}) & 0 \\ \sin(\frac{2k\pi}{n}) & \cos(\frac{2k\pi}{n}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\rho_\tau] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
$$

The subspace $W_z = \{(0,0,k)\,|\,k \in \mathbb{R}\}$ is invariant under the action of D_n. All rotations act as the identity, and all reflections act as multiplication by -1, thus this subrepresentation has character ψ_2.

The complementary subspace $W_{xy} = \{(m,n,0)\,|\,m,n \in \mathbb{R}\}$ is also invariant. Using $v = (1,0,0)$ and $w = (0,1,0)$ as a basis for W_{xy}, σ^k and τ act via the matrices below.

$$
[\rho_{\sigma^k}] = \begin{bmatrix} \cos(\frac{2k\pi}{n}) & \sin(\frac{2k\pi}{n}) \\ \sin(\frac{2k\pi}{n}) & \cos(\frac{2k\pi}{n}) \end{bmatrix} \qquad [\rho_\tau] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
$$

Thus, the character of this subrepresentation, has the value $Tr([\rho_{\sigma^k}]) = 2\cos(\frac{2k\pi}{n})$ on the rotations, and as for the reflections, $Tr([\rho_{\sigma^k\tau}]) = Tr([\rho_{\sigma^k}][\rho_\tau]) = \cos(\frac{2k\pi}{n}) - \cos(\frac{2k\pi}{n}) = 0$. Therefore, this subrepresentation has character χ_1, and hence $\chi_\rho = \chi_1 + \psi_2$.

In the realization of D_n as C_{nv}, which determines a representation we'll denote θ, σ^k and τ act via multiplication on the left by the matrices given below.

$$
[\theta_{\sigma^k}] = \begin{bmatrix} \cos(\frac{2k\pi}{n}) & \sin(\frac{2k\pi}{n}) & 0 \\ \sin(\frac{2k\pi}{n}) & \cos(\frac{2k\pi}{n}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\theta_\tau] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

As above, the subspace $W_z = \{(0,0,k)\,|\,k \in \mathbb{R}\}$ is invariant under the action of D_n, however, every group element acts as the identity, so the character of this subrepresentation is ψ_1. On the subspace W_{xy}, the action of D_n is exactly as above, and so by the same argument, $\chi_\theta = \chi_1 + \psi_1$.

5.4 Consider $H = \{e, (ab)(cd), (ac)(bd), (ad)(bc)\}$ and $I = \{e, \iota\}$. Both act on $\mathbb{R}^3$ as explained in section 5.9, and under those actions, each element of the group $H \times I \simeq \{h \circ x\,|\,h \in H, x \in I\}$ either stabilizes the tetrahedron T with vertices at $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, and $(-1,-1,1)$, or sends it to T' with vertices at $(-1,-1,-1)$, $(-1,1,1)$, $(1,-1,1)$, and $(1,1,-1)$.

In fact, $H \times I$ consists of exactly the eight isomorphisms of $\mathbb{R}^3$ defined by $(x,y,z) \mapsto (\pm x, \pm y, \pm z)$. For example, $(ab)(cd)$ swaps the vertices $(1,1,1)$ and $(1,-1,-1)$ and also swaps $(-1,1,-1)$ and $(-1,-1,1)$. This isomorphism of $\mathbb{R}^3$ is given by $(x,y,z) \mapsto (x,-y,-z)$. Other elements of $H \times I$ can be checked similarly.

Thus, we have $M \simeq H \times I$. As explained in section 5.9, we will then have $G \simeq \mathfrak{S}_3 \cdot M \simeq \mathfrak{S}_3 \cdot (H \times I)$, where $\mathfrak{S}_3$ permutes coordinates. Indeed, each element of G can be written uniquely as $s \cdot m$, with $s \in \mathfrak{S}_3$ and $m \in M$ since the cube C is stable under the action of both groups, $\mathfrak{S}_3$ acts only via permuting coordinates, and M acts only via swapping signs. Concretely, the image of $(1, 1, 1)$ under $g \in G$ determines m completely, and once m is known we have $s = g \cdot m^{-1}$. There are 48 such pairs $s \cdot m$, so each element of G must be obtained in this way.

It is clear that $\mathfrak{S}_3 \cap M = \{e\}$. To finish, we need to check M is a normal subgroup of G. Since any two sign swaps commute, and $g = s \cdot m$ for all $g \in G$, this amounts to checking that $s^{-1}ms \in M$, where $s \in \mathfrak{S}_3$ and $m \in M$. It is definitely true that if one permutes coordinates, performs some sign swapping, and then undoes the permutation, the result will be a sign swap (on different coordinates determined by the permutation), so we're done.

5.5 Let $g \in G_+$ be a rotation. We will show that either $g \in S(T)$ or $g = s \circ \iota$ with $s \in S(T)$, and exactly one of these occurs. Suppose that $g \in S(T)$, then $g(T) = T$, but if $g = s \circ \iota$,

$$g(T) = s(\iota T) = \iota(sT) = T'$$

This contradicts our supposition, so g cannot be written as $s \circ \iota$ for $s \in S(T)$. Now suppose that $g \notin S(T)$. Note that since g is a symmetry of the cube, there are only two options for the image of T, thus we must have $g(T) = T'$. Taking $s = \iota \circ g$,

$$s(T) = \iota \circ g(T) = \iota(T') = T$$

Thus, $s = \iota \circ g$ is in $S(T)$, and applying ι on the left, $g = \iota \circ s$ with $s \in S(T)$.

Define $\varphi : G_+ \to S(T)$ as described above, i.e., $\varphi(g) = g$ if $g \in S(T)$ and $\varphi(g) = \iota \circ g$ if $g \notin S(T)$. It is easy to check φ is a homomorphism, though there are four cases. If $g \in S(T)$ and $h \notin S(T)$, then $\varphi(g)\varphi(h) = g \circ (\iota \circ h) = \iota \circ g \circ h = \varphi(gh)$. Note that ι commutes with any $g \in G_+$, and in this case, $gh(T) = g(T') = T'$.

All that remains is to show φ is surjective. If $s \in S(T)$ is a rotation, then $s \in G_+$ and $\varphi(s) = s$. If $s \in S(T)$ is a reflection, then $\iota \circ s(T) = T'$, and $det(\iota \circ s) = det(\iota)det(s) = 1$, so there is a $g \in G_+$ with $g = \iota \circ s$, i.e., with $\varphi(g) = s$.