Dissipativity Theory for Evolutionary Games on Infinite Strategy Sets

by

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Abstract

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We consider evolutionary dynamics for population games in which players have a continuum of strategies at their disposal. Models in this setting amount to infinite-dimensional differential equations evolving on the manifold of probability measures. In this thesis, we generalize dissipativity theory for evolutionary games from finite to infinite strategy sets that are compact metric spaces, and derive sufficient conditions for the stability of Nash equilibria under the infinite-dimensional dynamics. The resulting analysis is applicable to a broad class of evolutionary games, and is modular in the sense that the pertinent conditions on the dynamics and the game's payoff structure can be verified independently. By specializing our theory to the class of monotone games, we recover as special cases existing stability results for the Brown-von Neumann-Nash and impartial pairwise comparison dynamics. We also extend our theory to models with dynamic payoffs, further broadening the applicability of our framework. We illustrate our theory using a variety of case studies, including a novel "continuous war of attrition" game.

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Chapter 1

Introduction

Population games are models in which a large number of agents interact strategically. Examples of such models appear ubiquitously in engineering and societal-scale problems, including traffic congestion networks, decentralized control, and economic markets (Sandholm, 2010). Within a population game, each agent employs a strategy available to them to maximize their expected payoff. When the agents are permitted to continuously revise their strategy according to some protocol, the game gives rise to an *evolutionary dynamics model* (Smith, 1982a; Sandholm, 2010). Such models have a rich history within the mathematical biology literature, as reviewed in Hofbauer and Sigmund (1998).

Traditional game-theoretic models are concerned with notions of (Nash) equilibrium states, in which no player is incentivized to choose a different strategy given knowledge of the payoffs. However, such notions of equilibria are static and incomplete, in the sense that they do not capture whether an evolutionary dynamics model dynamically converges to them when players revise their strategies according to some protocol. Indeed, static equilibria, such as Nash equilibria, need not be dynamically stable (Sato et al., 2002; Hart and Mas-Colell, 2003; Sandholm, 2010). This has led to an entire body of works concerned with assessing the dynamic stability of evolutionary games. Although many works have proven stability for specific examples of evolutionary dynamics, researchers maintain the overarching goal of proving stability for the most general classes of models (Fox and Shamma, 2013).

The aim of this thesis is to prove dynamic stability for a very broad class of evolutionary dynamics. The broadening of evolutionary stability theory has seen two notable directions: 1) generalizing the structural behavior of the dynamics and the game as much as possible while maintaining stability, and 2) generalizing prior stability results for specific dynamical structures to more abstract settings. We now discuss these two approaches in further depth.

1.1 Related Works

Potentiality, Monotonicity, and Dissipativity

Potential games constitute a class of games in which the payoff is given by the gradient of a "potential function" (Monderer and Shapley, 1996; Sandholm, 2001). It was shown in Sandholm (2001) that potential games satisfying the so-called "positive correlation property" admit the potential function as a global Lyapunov function, thereby yielding dynamic stability guarantees. Hofbauer and Sandholm (2007, 2009) introduce monotone games, also known as stable games,[1] which generalize potential games with concave potential functions to allow for payoffs that act as monotone operators (like gradients of concave functions). Zero-sum games and games with an interior globally evolutionarily stable state are also known to be special cases of monotone games (Hofbauer and Sandholm, 2007). Hofbauer and Sandholm (2007) show that common evolutionary dynamics, such as those of Brown-von Neumann-Nash and Smith, exhibit dynamic stability when coupled with monotone games. Hofbauer and Sandholm (2009) extend this result to the case of general dynamics satisfying an "integrability" condition on their revision protocols.

Generalizing even further past integrability and monotonicity, Fox and Shamma (2013) apply notions of passivity from the systems and control literature to grant stability. The authors propose to view an evolutionary game as a nonautonomous dynamical system in feedback with inputs defined by the game's payoffs. In doing so, they prove that "δ-passive" evolutionary dynamics coupled with monotone games yield dynamic stability. The core intuition is that, if the rate of change of internally stored energy of an evolutionary system is less than the rate of energy supplied to it by the game's payoffs, then the closed-loop system's total energy decreases. This approach was taken by Mabrok (2021) to analyze the stability of replicator dynamics, and was further generalized by Arcak and Martins (2021) to apply to more general "δ-dissipative" dynamics. The dissipativity theory of Arcak and Martins (2021) constitutes some of the broadest stability results, recovering many of the aforementioned prior results as special cases. Recent works have applied these broad theories to particular applications, such as distributed Nash equilibrium seeking (Martinez-Piazuelo et al., 2023) and the analysis of strategy-dependent pairwise comparison revision protocols (Kara and Martins, 2023). We emphasize that all of the works mentioned here are restricted to games defined over a finite number of strategies.

Games over Infinite Strategy Sets

Many practical games come equipped with an infinite number of strategies available to the players, e.g., pricing and generation in power systems (Park et al., 2001), games of timing such as the war of attrition (Bishop and Cannings, 1978), plant growth models in biology (Bomze

[1] We use the terminology "monotone" throughout this thesis, as it more accurately represents these games' defining property (5.6), to come later, whereas the terminology "stable" may lead to confusion when discussing dynamic stability; a stable game need not give rise to stable dynamics.

and Pötscher, 1989, Section 2.4), and, more recently, in multi-agent reinforcement learning with continuous action spaces (Mazumdar et al., 2020). Consequently, much effort has gone into abstracting dynamic stability results from the finite-strategy setting into the infinite setting. However, in doing so, the distribution of strategies being employed, termed the "population state," becomes a probability measure rather than a finite-dimensional vector in a Euclidean simplex (Myerson, 1991). This makes the analysis much more challenging, as it requires studying differential equations evolving on the manifold of probability measures within an infinite-dimensional Banach space.

Alongside this technical hurdle comes two other key challenges. First, there are multiple standard notions of convergence for probability measures on infinite sets, and evolutionary dynamics may converge in one such notion but fail to converge in another. For example, Eshel and Sansone (2003) show that the infamous replicator dynamics may exhibit dynamic instability with respect to the so-called "maximum shift topology" even when they are stable in the weak topology.[2] The second key challenge is that dynamic stability may break when moving from the finite-strategy regime to the infinite regime. For example, Oechssler and Riedel (2002) show that even strict Nash equilibria and evolutionarily stable states may be unstable under the replicator dynamics over infinite strategy sets, even though their approximations with finitely many strategies are always dynamically stable. Similarly, we show in Chapter 6 that finite-strategy approximations of the war of attrition are guaranteed to be dynamically stable via the finite-dimensional dissipativity theory of Arcak and Martins (2021), despite the fact that the underlying infinite-dimensional game is unstable. Thus, stability guarantees for evolutionary dynamics over infinite strategy sets are not automatic from their corresponding finite-dimensional counterparts. This motivates our work in directly considering the infinite strategy set setting.

A handful of related works have directly analyzed the stability of infinite-dimensional evolutionary dynamics. Some of the first such work was Bomze (1990, 1991), which considered replicator dynamics with respect to the strong topology. A line of follow-up works on the replicator dynamics has emerged, many of which come to the consensus that convergence in the sense of the weak topology is most appropriate for evolutionary dynamics, as it better respects notions of distance between strategies (Oechssler and Riedel, 2001, 2002; Cressman, 2005; Cressman and Hofbauer, 2005; Hingu et al., 2018, 2020). These works also propose alternative notions of equilibria (beyond Nash) to ensure dynamic stability. Beyond the replicator dynamics, stability (typically of Nash equilibria) with respect to the weak topology has been assessed for the Brown-von Neumann-Nash, pairwise comparison, logit, general imitative, and perturbed best response dynamics (Hofbauer et al., 2009; Cheung, 2014; Lahkar and Riedel, 2015; Cheung, 2016; Lahkar et al., 2022). However, despite the applicability of these results to quite general strategy sets, all of these works are restricted to specific evolutionary dynamics and are proven in a case-by-case fashion. In comparison, the approach in this thesis is to keep in the spirit of broadening stability guarantees, and to derive results for infinite-strategy games applicable to general classes of dissipative dynamics.

[2]Formal definitions of the weak and strong topologies are given in Chapter 2.

1.2 Contributions

In this thesis, we unify the two above generalization approaches to achieve the following primary contributions:

1. We introduce novel notions of dissipativity for evolutionary dynamics over infinite strategy sets. This extension from the finite to infinite strategy sets requires new technicalities, since our models are defined on Banach spaces, and the weak topology in which we seek dynamic convergence is not equivalent to the topology induced by the total variation norm on our population states, unlike the finite-dimensional Hilbert space setting in which they are equivalent.

2. In our main result (Theorem 1), we prove a new stability theorem showing that δ-dissipative evolutionary dynamics on infinite strategy sets weakly converge to Nash equilibria under decreasing energy supply rates induced by the game's payoffs and some technical regularity conditions. Our complete identification of such technical conditions is nontrivial, as, again, our infinite-dimensional setting breaks down the topological equivalence between notions of convergence and notions of norm.

3. We specialize our framework to prove a new stability theorem for the class of monotone games (our Theorem 2), and prove that this specialization recovers the main stability results of Hofbauer et al. (2009, Theorem 3) and Cheung (2014, Theorem 4) as special cases (our Corollary 1).

4. We further extend the generality of our stability theory to the case in which the game's payoffs exhibit dynamic behavior (Theorem 3).

5. We show that the classical war of attrition game on infinite strategy sets simultaneously fails to converge to Nash equilibria while its finite-strategy approximations succeed in convergence to Nash, and we use our theoretical framework to identify the technical stability conditions being violated. We subsequently propose and verify the stability of a new "continuous" variant of the war of attrition game, and we illustrate the generality of our framework on an example of a monotone game with smoothed payoff dynamics.

1.3 Outline

This thesis is organized as follows. In Chapter 2, we introduce our notations and review relevant mathematical definitions and results. Population games and evolutionary dynamics are formally introduced in Chapter 3 and Chapter 4, along with associated definitions and results for both static and dynamic stability. Our primary contributions are given in Chapter 5 and Chapter 6. Namely, in Chapter 5, we present our dissipativity theory for infinite strategy sets and our stability theorems. In Chapter 6, we provide case studies illustrating our framework and results. We conclude in Chapter 7. The proofs of our primary results (the theorems) are given in the main text. To streamline presentation, all other proofs (the propositions and corollaries) are deferred to Appendix A. Supplementary definitions, results, and discussions are given in Appendix B.

Chapter 2

Mathematical Preliminaries

2.1 Notations and Basic Definitions

The set of nonnegative real numbers is denoted by $\mathbb{R}_+$. We define $\operatorname{sign}\colon \mathbb{R} \to \mathbb{R}$ by $\operatorname{sign}(x) = 1$ for $x > 0$, $\operatorname{sign}(x) = 0$ for $x = 0$, and $\operatorname{sign}(x) = -1$ for $x < 0$. The dual space of a normed vector space X (i.e., the space of bounded linear functionals on X) is denoted by X^*. Let S be a compact metric space. The Banach space of bounded continuous real-valued functions on S endowed with the supremum norm is denoted by $(C_b(S), \|\cdot\|_\infty)$. Since S is compact, $C_b(S)$ equals the set of all continuous real-valued functions on S, denoted by $C(S)$. The Borel σ-algebra on S is denoted by $\mathcal{B}(S)$, and the Banach space of finite signed Borel measures on S endowed with the total variation norm is denoted by $(\mathcal{M}(S), \|\cdot\|_{\mathrm{TV}})$. Recall that $\|\mu\|_{\mathrm{TV}} := |\mu|(S) = \sup_{f \text{ measurable}: \|f\|_\infty \leq 1} \int_S f d\mu$, where $|\mu|$ is the total variation measure of μ. The support of a measure $\mu \in \mathcal{M}(S)$ is denoted by $\operatorname{supp}(\mu)$.

We denote the set of probability measures on $(S, \mathcal{B}(S))$ by $\mathcal{P}(S) = \{\mu \in \mathcal{M}_+(S) : \mu(X) = 1\}$, where $\mathcal{M}_+(S) \subseteq \mathcal{M}(S)$ is the set of positive Borel measures on S. The tangent space of $\mathcal{P}(S)$ is given by $T\mathcal{P}(S) = \{\nu \in \mathcal{M}(S) : \nu(S) = 0\}$, which is a linear subspace of $\mathcal{M}(S)$. The Dirac measure at $s \in S$ is denoted by $\delta_s \in \mathcal{P}(S)$. We define the bilinear form $\langle \cdot, \cdot \rangle : C(S) \times \mathcal{M}(S) \to \mathbb{R}$ by $\langle f, \mu \rangle = \int_S f d\mu$, which is well-defined and satisfies $\left| \int_S f d\mu \right| \leq \|f\|_\infty \|\mu\|_{\mathrm{TV}}$ for all $f \in C(S)$ and all $\mu \in \mathcal{M}(S)$. Recall that $\mathcal{M}(S)$ is isometrically isomorphic to the dual space of $C(S)$ (Folland, 1999, Theorem 7.17), and therefore every element of $\mathcal{M}(S)$ can be uniquely identified with a bounded linear functional on $C(S)$. Thus, for all bounded linear functionals $I \in C(S)^*$, there exists a unique $\mu \in \mathcal{M}(S)$ such that $I(f) = \langle f, \mu \rangle$ for all $f \in C(S)$.

2.2 Topologies and Convergence of Measures

Two types of convergence in $\mathcal{M}(S)$ will be of use. Recall that a sequence $\{\mu_n \in \mathcal{M}(S) : n \in \mathbb{N}\}$ *converges weakly to* $\mu \in \mathcal{M}(S)$ if $\lim_{n\to\infty} \int_S f d\mu_n = \int_S f d\mu$ for all $f \in C(S)$, and *converges strongly to* $\mu \in \mathcal{M}(S)$ if $\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathrm{TV}} = 0$. Recall that strong convergence implies weak convergence. Strong and weak convergence induce topologies on $\mathcal{M}(S)$, termed the *strong topology* and *weak topology*, respectively.[1] We will also need the following product topology.

Definition 1. Consider $\mathcal{M}(S)$ endowed with the weak topology and $C(S)$ endowed with its usual topology induced by $\|\cdot\|_\infty$. We call the corresponding product topology on $\mathcal{M}(S) \times C(S)$ the *weak-∞ topology.*

We use the following fact throughout our analyses.

Lemma 1 (Parthasarathy, 1967, Theorem 6.4)**.** *It holds that $\mathcal{P}(S)$ is weakly compact.*

2.3 Notions of Differentiability

We also need various notions of differentiability. Consider Banach spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $(Z, \|\cdot\|_Z)$, and open sets $U \subseteq X$ and $V \subseteq Y$. The Fréchet derivative of a map $f\colon U \to Y$ at $x \in U$, if it exists, is denoted by $Df(x)$. Recall that $Df(x)\colon X \to Y$ is a bounded linear operator. If $f\colon U \times V \to Z$ is a map defined on $U \times V$, its first partial Fréchet derivative at $(x, y) \in U \times V$, if it exists, is given by $\partial_1 f(x, y) \coloneqq D(f(\cdot, y))(x)$. The second partial Fréchet derivative of such a map f is similarly given by $\partial_2 f(x, y) \coloneqq D(f(x, \cdot))(y)$. Recall that a map $x\colon [0, \infty) \to X$ is said to be differentiable at $t \in (0, t)$ if there exists $\dot{x}(t) \in X$ such that

$$\lim_{\epsilon \to 0} \left\| \frac{x(t+\epsilon) - x(t)}{\epsilon} - \dot{x}(t) \right\| = 0,$$

and in this case $\dot{x}(t)$ is called the derivative of x at t. We call a map $\mu\colon [0, \infty) \to \mathcal{M}(S)$ strongly differentiable at t if it is differentiable at t, to emphasize the underlying topology on $\mathcal{M}(S)$ induced by $\|\cdot\|_{\mathrm{TV}}$. In Appendix B, these notions of differentiability are defined more formally and are discussed further.

[1] The weak topology is sometimes called the "narrow topology." Since S is compact, the weak topology coincides with the weak-$*$ topology on $\mathcal{M}(S) = C(S)^*$ (i.e., the weakest topology on $C(S)^*$ making every element $f \in C(S) \subseteq C(S)^{**}$ a continuous linear functional on $C(S)^*$). In functional analysis the term "weak topology" on $\mathcal{M}(S)$ would refer to the weakest topology on $\mathcal{M}(S)$ making every element of the dual space $\mathcal{M}(S)^* = C(S)^{**}$ continuous. We stick with our definitions to remain consistent with related works.

Chapter 3

Population Games

We now describe the game-theoretic aspects of our problem. The compact set S represents the (infinite) set of pure strategies of the game, and is hence called the *strategy set*.[1] A *population state* is a distribution $\mu \in \mathcal{P}(S)$, which encodes how strategies in S are being employed across the game's population. Thus, $\mathcal{P}(S)$ is termed the *population state space*. To every population state $\mu \in \mathcal{P}(S)$ associates a *mean payoff function* $F_\mu \in C(S)$ such that $F_\mu(s)$ quantifies the average payoff to strategy s when the population is at state μ. We refer to the mapping $F\colon \mathcal{P}(S) \to C(S)$ defined by $F(\mu) = F_\mu$ as the *population game*, or simply the *game*. One of the primary quantities of interest when analyzing population games over infinite strategy sets is the *average mean payoff* $E_F(\nu, \mu) \in \mathbb{R}$ to a population state $\nu \in \mathcal{P}(S)$ relative to a population state $\mu \in \mathcal{P}(S)$, which is given by

$$E_F(\nu, \mu) \coloneqq \langle F(\mu), \nu \rangle = \int_S F_\mu d\nu.$$

The average mean payoff gives rise to a simple definition for Nash equilibria of population games.

Definition 2. A population state $\mu \in \mathcal{P}(S)$ is a *Nash equilibrium of the game* $F\colon \mathcal{P}(S) \to C(S)$ if

$$E_F(\nu, \mu) \leq E_F(\mu, \mu) \tag{3.1}$$

for all $\nu \in \mathcal{P}(S)$. If, additionally, the inequality (3.1) holds strictly for all $\nu \in \mathcal{P}(S) \setminus \{\mu\}$, then μ is a *strict Nash equilibrium of the game* F. The set of all Nash equilibria of the game F is denoted by $\mathrm{NE}(F)$.

[1]The compactness of the strategy set S is standard in the literature on infinite-dimensional evolutionary games. Although this compactness is a technical condition needed for our use of Lyapunov theory, it is also an important qualitative requirement in our context of games, as it ensures that evolutionary dynamics move the population state towards distributions of strategies that are actually available to the players. For example, compactness avoids cases where there exists a "hidden Nash equilibrium" at a probability measure with support at strategies on the boundary of S or "at infinity" that are inaccessible by the players.

Intuitively, a population state $\mu \in \mathcal{P}(S)$ is a Nash equilibrium if the average mean payoff to the population cannot be increased by moving to any other state $\nu \in \mathcal{P}(S)$ given the current payoffs defined by F_μ. The notion of a Nash equilibrium is static in the sense that it does not depend on any dynamical behavior endowed to the game. Other types of relevant static equilibria are discussed in Appendix B. The following result gives equivalent characterizations of Nash equilibria, which are used throughout our proofs. Such characterizations are sometimes taken as alternative definitions in the literature, e.g., in Hofbauer et al. (2009); Cheung (2014), albeit without proof of equivalence.

Proposition 1. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$, and let $\mu \in \mathcal{P}(S)$. The following are equivalent:*

1. *μ is a Nash equilibrium of the game F.*
2. *$E_F(\delta_s, \mu) \leq E_F(\mu, \mu)$ for all $s \in S$.*
3. *$F_\mu(s) \leq F_\mu(s')$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$.*

Proposition 1 shows that at a Nash equilibrium state $\mu \in \mathcal{P}(S)$, every strategy $s' \in S$ that is in use (meaning that $s' \in \operatorname{supp}(\mu)$) must have maximal average payoff $F_\mu(s')$ compared to all other possible strategies $s \in S$. From the contrapositive viewpoint, this shows that a strategy $s' \in S$ whose average payoff $F_\mu(s')$ is strictly less than that of some other strategy will not be employed at a Nash equilibrium state μ.

In general, there may be more than one Nash equilibrium of the game F. Even in this case, the following result unveils advantageous topological characteristics of $\mathrm{NE}(F)$.

Proposition 2. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$. If $\theta_\nu\colon \mathcal{P}(S) \to \mathbb{R}$ defined by $\theta_\nu(\mu) = E_F(\nu, \mu) - E_F(\mu, \mu)$ is weakly continuous for all $\nu \in \mathcal{P}(S)$, then $\mathrm{NE}(F)$ is weakly compact.*

Together with Proposition 2, the following result shows that $\mathrm{NE}(F)$ is weakly compact whenever the game F is weakly continuous. This result is of particular technical importance in our stability proofs of Chapter 5.

Proposition 3. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$. If F is weakly continuous, then $\theta_\nu\colon \mathcal{P}(S) \to \mathbb{R}$ defined by $\theta_\nu(\mu) = E_F(\nu, \mu) - E_F(\mu, \mu)$ is weakly continuous for all $\nu \in \mathcal{P}(S)$.*

See Proposition 9, Corollary 4, and Proposition 11 in Appendix B for conditions under which the (Nash) equilibria of a game are unique or constitute a convex set.

Chapter 4

Evolutionary Dynamics

In this section, the population game F is endowed with dynamical behavior. Such dynamics are used to model the evolutionary aspects of a population playing out a game, wherein players revise their strategies over time according to the game's current payoff profile. Our infinite-strategy analogue of the evolutionary dynamics models considered in Fox and Shamma (2013) and Arcak and Martins (2021) is formalized as follows.

Definition 3. Consider a game $F\colon \mathcal{P}(S) \to C(S)$. Let $\mu_0 \in \mathcal{P}(S)$ and let $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$. The differential equation

$$\begin{aligned} \dot{\mu}(t) &= v(\mu(t), \rho(t)), \\ \rho(t) &= F(\mu(t)), \\ \mu(0) &= \mu_0, \end{aligned} \tag{4.1}$$

is called an *evolutionary dynamics model (EDM)*. The measure μ_0 is called the *initial state* and the mapping v is called the *dynamics map*. A strongly differentiable mapping $\mu\colon [0, \infty) \to \mathcal{P}(S)$ satisfying (4.1) is called a *solution to the EDM*.

We emphasize that, although the overall EDM (4.1) defines an autonomous system, the nonautonomous dynamics term $\dot{\mu}(t) = v(\mu(t), \rho(t))$ may be studied in isolation from the feedback term $\rho(t) = F(\mu(t))$. In particular, this viewpoint lends itself towards control theoretic analyses, where the dynamics map v defines the system to be controlled, and the game F defines the feedback controller. This approach was proposed in Fox and Shamma (2013) and further studied in Arcak and Martins (2021) as a means to derive finite-strategy stability results based on the idea that interconnections of energy-dissipating systems results in a closed-loop system that is dynamically stable. This allows one to prove stability of the overall evolutionary dynamics model by studying the dissipativity properties of the (nonautonomous) system and input in a modular fashion. To the best of our knowledge, our work is the first to generalize this modular dissipativity approach to evolutionary games with infinite strategy sets—prior works on infinite strategy sets primarily prove stability in a closed-loop black-box fashion on a case-by-case basis, e.g., Oechssler and Riedel (2001); Hingu et al.

(2020) for replicator dynamics, Hofbauer et al. (2009) for Brown-von Neumann-Nash dynamics, and Cheung (2014) for pairwise comparison dynamics, as well as the references therein and subsequent works.

Before moving on to our main results in Chapter 5, we give examples of some of the most commonly studied evolutionary dynamics models, and also formalize the notions of dynamic stability to be considered.

Example 1. Let $\lambda \in \mathcal{P}(S)$ be a fixed reference probability measure with full support. This reference measure is commonly taken as that of a uniform distribution, but more general probability measures may be used to model the case where strategies are chosen at nonuniform revision rates. The *Brown-von Neumann-Nash (BNN) dynamics* are given by the EDM (4.1) with closed-loop dynamics defined by

$$v(\mu, F(\mu))(B) = \int_B \sigma_+(s,\mu) d\lambda(s) - \mu(B) \int_S \sigma_+(s,\mu) d\lambda(s)$$

for all $B \in \mathcal{B}(S)$, where

$$\sigma_+(s,\mu) = \max\{0, E_F(\delta_s,\mu) - E_F(\mu,\mu)\}$$

is the excess average mean payoff to population state $\delta_s \in \mathcal{P}(S)$ relative to the population state $\mu \in \mathcal{P}(S)$. Here, the mean payoff function associated to $\mu \in \mathcal{P}(S)$ takes the form

$$F_\mu(s) = \int_S f(s,s') d\mu(s'),$$

with $f\colon S \times S \to \mathbb{R}$ being a bounded measurable function that gives the payoff $f(s,s')$ to a player choosing strategy s when an opponent chooses strategy s'. In this case, it is easy to see that the BNN dynamics may be written in the interconnected form (4.1) with the dynamics map given by

$$v(\mu,\rho)(B) = \int_B \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s) - \mu(B) \int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s).$$

See Hofbauer et al. (2009) for a thorough study on the BNN dynamics over infinite strategy sets.

Example 2. Let λ be a fixed reference measure as in Example 1. Furthermore, let $\gamma\colon S \times S \times C(S) \to \mathbb{R}_+$ be a continuous and bounded map, termed the *conditional switch rate*, such that $\gamma(s,s',\rho)$ encodes the rate at which players switch from strategy $s \in S$ to strategy $s' \in S$ whenever the strategies' payoffs are described by the function $\rho \in C(S)$. Assume that the conditional switch rate satisfies *sign-preservation*, given by

$$\operatorname{sign}(\gamma(s,s',\rho)) = \operatorname{sign}(\max\{0, \rho(s') - \rho(s)\})$$

for all $s, s' \in S$ and all $\rho \in C(S)$. Sign-preservation ensures that the conditional switch rate from strategy s to strategy s' is positive if and only if s' has higher payoff than s according to the function ρ. The *pairwise comparison dynamics* are given by the EDM (4.1) with

$$v(\mu,\rho)(B) = \int_S \int_B \gamma(s,s',\rho) d\lambda(s') d\mu(s) - \int_S \int_B \gamma(s',s,\rho) d\mu(s') d\lambda(s)$$

for all $B \in \mathcal{B}(S)$. Notice that the nonautonomous portion of the dynamics, defined by this dynamics map v, is entirely determined by the conditional switch rate γ.

When γ takes the form $\gamma(s,s',\rho) = \max\{0, \rho(s') - \rho(s)\}$, the pairwise comparison dynamics reduce to the famous *Smith dynamics*, introduced in Smith (1984). If, for all $s' \in S$, there exists some continuous function $\phi_{s'} \colon \mathbb{R} \to \mathbb{R}_+$ such that the conditional switch rate satisfies

$$\gamma(s,s',\rho) = \phi_{s'}(\rho(s') - \rho(s))$$

for all $s \in S$ and all $\rho \in C(S)$, then the pairwise comparison dynamics are said to be *impartial*. The Smith dynamics are seen to be impartial by taking $\phi_{s'}(\cdot) = \max\{0, \cdot\}$ for all s'. See Cheung (2014) for a thorough study on the pairwise comparison dynamics over infinite strategy sets.

We make use of the following existence and uniqueness assumption throughout the remainder of the thesis.

Assumption 1. Consider a dynamics map $v \colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$. For every initial state $\mu_0 \in \mathcal{P}(S)$, there exists a unique solution to the EDM (4.1) with initial state μ_0 and dynamics map v.

Assumption 1 holds for the BNN dynamics of Example 1 (Hofbauer et al., 2009, Theorem 1), and also holds for the pairwise comparison dynamics of Example 2 (Cheung, 2014, Theorem 1) under some mild regularity conditions on F. It is also easy to see that, for these dynamics, $v(\mu,\rho)(S) = 0$ for all $\mu \in \mathcal{P}(S)$ and all $\rho \in C(S)$, and hence the codomain of these dynamics maps can be taken to be $T\mathcal{P}(S)$. For more discussion on the characteristics and existence of solutions to EDMs, see Appendix B.

4.1 Dynamic Notions of Stability

We now formally introduce the notions of dynamic equilibria and stability with which we are concerned.

Definition 4. A population state $\mu \in \mathcal{P}(S)$ is a *rest point of the EDM* (4.1) if $v(\mu, F(\mu)) = 0$.

The following condition, which is solely a property of the nonautonomous dynamics defined by the dynamics map v, ensures that the rest points and Nash equilibria coincide for the EDM under the feedback interconnection (4.1).

Definition 5. A map $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$ is *Nash stationary* if, for all $\mu \in \mathcal{P}(S)$ and all $\rho \in C(S)$, it holds that

$$v(\mu, \rho) = 0$$

if and only if

$$\langle \rho, \nu \rangle \leq \langle \rho, \mu \rangle \text{ for all } \nu \in \mathcal{P}(S).$$

Proposition 4. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$ and let $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$. If v is Nash stationary, then the set of rest points of the EDM* (4.1) *equals* $\mathrm{NE}(F)$.

Proposition 4 shows that if an evolutionary game's population state converges to a rest point under the EDM with a Nash stationary dynamics map, then the population state converges to a Nash equilibrium. In other words, Nash stationarity ensures that all dynamically stable rest points have game-theoretic importance. We now recall that the popular BNN dynamics and pairwise comparison dynamics both satisfy Nash stationarity.[1]

Proposition 5 (Hofbauer et al., 2009; Cheung, 2014)**.** *If $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$ is the dynamics map for either the BNN dynamics of Example 1 or the pairwise comparison dynamics of Example 2, then v is Nash stationary.*

Although a population state being at a rest point ensures that the population's distribution of strategies remains constant for all time, the definition of rest point does not itself come equipped with an adequate notion of stability. For this, we turn to the classical definitions of Lyapunov stability and attraction. Since ultimately we are interested in convergence of a game's dynamics to *some* Nash equilibrium, we deal with such definitions in the sense of sets. We present the definitions for general Banach spaces with topologies on them that may not be induced by the space's norm. This level of abstraction will be needed for our extension of dissipativity theory to the case of dynamic payoff models in Section 5.2.

Definition 6. Consider a Banach space X and a topology τ on X. Let $Y \subseteq X$ and let $v\colon Y \to X$. A τ-compact set $P \subseteq Y$ is *τ-Lyapunov stable under v* if, for all relatively τ-open sets $Q \subseteq Y$ containing P, there exists a relatively τ-open set $R \subseteq Y$ containing P such that every solution $x\colon [0, \infty) \to Y$ to the differential equation $\dot{x}(t) = v(x(t))$ with $x(0) = x_0 \in Y$ satisfies $x(t) \in Q$ for all $t \in [0, \infty)$ whenever $x(0) \in R$.

Definition 7. Consider a Banach space X and a topology τ on X. Let $Y \subseteq X$ and let $v\colon Y \to X$. A τ-compact set $P \subseteq Y$ is *τ-attracting under v from $x_0 \in Y$* if, for all relatively τ-open sets $Q \subseteq Y$ containing P and for all solutions $x\colon [0, \infty) \to Y$ to the differential equation $\dot{x}(t) = v(x(t))$ with $x(0) = x_0$, there exists $T \in [0, \infty)$ such that

$$x(t) \in Q \text{ for all } t \in [T, \infty).$$

[1] Technically, our definition of Nash stationarity, which is a property of the dynamics map v viewed as a nonautonomous system, is slightly different than the definitions used in Hofbauer et al. (2009) and Cheung (2014), which are properties of the closed-loop interconnection (4.1). For self-containedness, we prove Proposition 5 in Appendix A using our definition.

The τ-compact set P is *globally τ-attracting under v* if it is τ-attracting under v from x_0 for all $x_0 \in Y$.

In our main results of Theorem 1 and Theorem 2, we are concerned with stability with respect to the weak topology on $\mathcal{P}(S)$. Therein, τ coincides with the weak topology, which in our setting is strictly weaker than the topology induced by $\|\cdot\|_{\mathrm{TV}}$. For this case, we specialize the above stability definitions to the setting of evolutionary dynamics of the form (4.1).

Definition 8. Consider a game $F\colon \mathcal{P}(S) \to C(S)$ and let $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$. A weakly compact set $P \subseteq \mathcal{P}(S)$ is *weakly Lyapunov stable under the EDM* (4.1) if P is τ-Lyapunov stable under $\mu \mapsto v(\mu, F(\mu))$ according to definition Definition 6 with $X = \mathcal{M}(S)$, $Y = \mathcal{P}(S)$, and τ being the weak topology.

Definition 9. Consider a game $F\colon \mathcal{P}(S) \to C(S)$ and let $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$. A weakly compact set $P \subseteq \mathcal{P}(S)$ is *weakly attracting under the EDM* (4.1) *from initial state* $\mu_0 \in \mathcal{P}(S)$ if P is τ-attracting under $\mu \mapsto v(\mu, F(\mu))$ from μ_0 according to Definition 7 with $X = \mathcal{M}(S)$, $Y = \mathcal{P}(S)$, and τ being the weak topology. The weakly compact set P is *globally weakly attracting under the EDM* (4.1) if it is weakly attracting under the EDM (4.1) with initial state μ_0 for all $\mu_0 \in \mathcal{P}(S)$.

Definition 8 and Definition 9 are equivalent to the dynamic stability notions used in Cheung (2014), which are defined in terms of the Prokhorov metric on $\mathcal{P}(S)$ that metrizes the weak topology.

Chapter 5

Dissipativity Theory

In this section, we present our main theoretical contributions. We begin by defining notions of dissipativity in our setting of infinite strategy sets, which we then use to prove our main results characterizing the dynamic stability of Nash equilibria. To the best of our knowledge, the following definition is new to the literature on evolutionary games over infinite strategy sets.

Definition 10. A map $v\colon \mathcal{P}(S)\times C(S) \to \mathcal{M}(S)$ is *δ-dissipative with supply rate $w\colon \mathcal{M}(S)\times C(S) \to \mathbb{R}$* if there exist $\sigma\colon \mathcal{P}(S)\times C(S) \to \mathbb{R}_+$ and $\Sigma\colon \mathcal{P}(S)\times C(S) \to \mathbb{R}_+$ that extends to a map $\overline{\Sigma}\colon U\times C(S) \to \mathbb{R}$ with strongly open $U \subseteq \mathcal{M}(S)$ containing $\mathcal{P}(S)$, such that the following conditions hold:

1. $\overline{\Sigma}$ is weak-∞-continuous.
2. $\overline{\Sigma}$ is Fréchet differentiable.
3. For all $\mu\in\mathcal{P}(S)$, all $\rho\in C(S)$, and all $\eta\in C(S)$, it holds that
$$\partial_1\overline{\Sigma}(\mu,\rho)v(\mu,\rho) + \partial_2\overline{\Sigma}(\mu,\rho)\eta \le -\sigma(\mu,\rho) + w(v(\mu,\rho),\eta). \tag{5.1}$$
4. For all $\mu\in\mathcal{P}(S)$ and all $\rho\in C(S)$, it holds that
$$\Sigma(\mu,\rho) = 0 \text{ if and only if } v(\mu,\rho) = 0. \tag{5.2}$$

If, additionally, $(\mu,\rho)\mapsto \partial_1\overline{\Sigma}(\mu,\rho)$ and $(\mu,\rho)\mapsto \partial_2\overline{\Sigma}(\mu,\rho)$ are weak-∞-continuous, every partial Fréchet derivative $\partial_1\overline{\Sigma}(\mu,\rho)$ is weakly continuous, and
$$\sigma(\mu,\rho) = 0 \text{ if and only if } v(\mu,\rho) = 0 \tag{5.3}$$
for all $\mu\in\mathcal{P}(S)$ and all $\rho\in C(S)$, then v is *strictly δ-dissipative with supply rate w*.

The "δ" in the definition of δ-dissipative is short for "differentially;" it does not refer to a quantity δ. Also note that δ-dissipativity is solely a property of the nonautonomous dynamics defined by the dynamics map v. As mentioned in Example 2, the dynamics map v for the pairwise comparison dynamics is entirely determined by some conditional switch rate function $\gamma\colon S \times S \times C(S) \to \mathbb{R}_+$, and therefore δ-dissipativity may be viewed as a property of the conditional switch rate function in such a setting.

We also remark that (5.1), (5.2), and (5.3) are the key criteria for δ-dissipativity; the continuity and differentiability conditions in Definition 10 are regularity conditions that are indispensable technical criteria for our stability proofs that follow. In addition, we need the following regularity conditions on the game F and the dynamics map v, which, as we will see in the case study of Section 6.1, are of utmost importance in ensuring that dynamic stability actually holds.

Assumption 2. Consider a game $F\colon \mathcal{P}(S) \to C(S)$. The following hold:

1. F is weakly continuous.

2. F extends to a weakly continuous Fréchet differentiable map $\overline{F}\colon U' \to C(S)$ defined on a strongly open set $U' \subseteq \mathcal{M}(S)$ containing $\mathcal{P}(S)$.

Assumption 3. Consider a game $F\colon \mathcal{P}(S) \to C(S)$ that satisfies Assumption 2. It holds that $D\overline{F}$ and every Fréchet derivative $D\overline{F}(\mu)$ are weakly continuous.

Assumption 4. The dynamics map $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$ is $\|\cdot\|_{\mathrm{TV}}$-bounded on weak-∞ compact subsets of $\mathcal{P}(S) \times C(S)$, and is continuous with respect to the weak-∞ topology on its domain and the weak topology on its codomain.

We now present our main result, which shows that, when the nonautonomous portion of the dynamics is Nash stationary and δ-dissipative, and when the feedback portion of the dynamics induces decreasing energy supply rates, the interconnected closed-loop evolutionary dynamics model has dynamically stable Nash equilibria.

Theorem 1 (Main Result)**.** *Consider a game $F\colon \mathcal{P}(S) \to C(S)$, let $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$, and assume that Assumption 1 and Assumption 2 both hold. If v is Nash stationary and δ-dissipative with supply rate $w\colon \mathcal{M}(S) \times C(S) \to \mathbb{R}$ and it holds that*

$$w\left(\nu, D\overline{F}(\mu)\nu\right) \leq 0 \text{ for all } \mu \in \mathcal{P}(S) \text{ and all } \nu \in T\mathcal{P}(S), \tag{5.4}$$

then $\mathrm{NE}(F)$ *is weakly Lyapunov stable under the EDM* (4.1)*. If, additionally, Assumption 3 and Assumption 4 both hold and v is strictly δ-dissipative, then* $\mathrm{NE}(F)$ *is globally weakly attracting under the EDM* (4.1)*.*

Proof. Since v is δ-dissipative with supply rate $w\colon \mathcal{M}(S) \times C(S) \to \mathbb{R}$, there exist $\sigma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}_+$ and $\Sigma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}_+$ with Σ having an appropriate extension $\overline{\Sigma}\colon U \times$

$C(S) \to \mathbb{R}$ as in Definition 10. Define $V : \mathcal{P}(S) \to \mathbb{R}_+$ by $V(\mu) = \Sigma(\mu, F(\mu))$. By Proposition 2 and Proposition 3, NE(F) is weakly compact. Thus, by Lemma 7, it suffices to show that V is a global Lyapunov function for NE(F) under $\mu \mapsto v(\mu, F(\mu))$ (according to Definition 25). Let $\overline{V} : U \cap U' \to \mathbb{R}$ be defined by $\overline{V}(\mu) = \overline{\Sigma}(\mu, \overline{F}(\mu))$. Note that $U \cap U'$ is strongly open and contains $\mathcal{P}(S)$, and that $\overline{V}$ is weakly continuous and Fréchet differentiable since $\overline{\Sigma}$ is weak-∞-continuous and $\overline{F}$ is weakly continuous, and both $\overline{\Sigma}$ and $\overline{F}$ are Fréchet differentiable. Also note that $\overline{V}(\mu) = V(\mu)$ for all $\mu \in \mathcal{P}(S)$. Also, if $\mu \in$ NE(F), then $v(\mu, F(\mu)) = 0$ by Proposition 4, and therefore $\overline{V}(\mu) = V(\mu) = \Sigma(\mu, F(\mu)) = 0$ by (5.2). Furthermore, if $\mu \in \mathcal{P}(S) \setminus \text{NE}(F)$, then again by Proposition 4 we have that $v(\mu, F(\mu)) \neq 0$, so $\overline{V}(\mu) = V(\mu) = \Sigma(\mu, F(\mu)) > 0$ by (5.2). Therefore, the first two conditions from Definition 25 on V to be a global Lyapunov function for NE(F) under $\mu \mapsto v(\mu, F(\mu))$ are satisfied.

Next, since $\overline{\Sigma}$ and $\overline{F}$ are Fréchet differentiable,

$$D\overline{V}(\mu) = \partial_1 \overline{\Sigma}(\mu, \overline{F}(\mu)) + \partial_2 \overline{\Sigma}(\mu, \overline{F}(\mu)) \circ D\overline{F}(\mu)$$

for all $\mu \in U \cap U'$, and therefore, since $\overline{F}(\mu) = F(\mu)$ for all $\mu \in \mathcal{P}(S)$, it holds for all $\mu \in \mathcal{P}(S)$ that

$$\begin{aligned} D\overline{V}(\mu)v(\mu, F(\mu)) &= \partial_1 \overline{\Sigma}(\mu, F(\mu))v(\mu, F(\mu)) + \partial_2 \overline{\Sigma}(\mu, F(\mu)) D\overline{F}(\mu)v(\mu, F(\mu)) \\ &\leq -\sigma(\mu, F(\mu)) + w(v(\mu, F(\mu)), D\overline{F}(\mu)v(\mu, F(\mu))) \\ &\leq -\sigma(\mu, F(\mu)) \\ &\leq 0, \end{aligned} \tag{5.5}$$

where the first inequality follows from (5.1), and the second inequality follows from (5.4) together with the fact that $v(\mu, F(\mu)) \in T\mathcal{P}(S)$. Hence, V is indeed a global Lyapunov function for NE(F) under $\mu \mapsto v(\mu, F(\mu))$, so NE$(F)$ is weakly Lyapunov stable under the EDM (4.1).

Now suppose that the δ-dissipativity of v is strict and that the additional hypotheses of Assumption 3 and Assumption 4 both hold. By Lemma 8, it suffices to show that V is a strict global Lyapunov function for NE(F) under $\mu \mapsto v(\mu, F(\mu))$ (according to Definition 25). This amounts to proving that $\mu \mapsto D\overline{V}(\mu)v(\mu, F(\mu))$ is weakly continuous and that $D\overline{V}(\mu)v(\mu, F(\mu)) < 0$ for all $\mu \in \mathcal{P}(S) \setminus \text{NE}(F)$. Indeed, the continuity condition holds by Lemma 4, which we state and prove in Appendix A.

Next, if $\mu \in \mathcal{P}(S) \setminus \text{NE}(F)$, then Proposition 4 gives that $v(\mu, F(\mu)) \neq 0$ so $\sigma(\mu, F(\mu)) > 0$ by (5.3), implying that $D\overline{V}(\mu)v(\mu, F(\mu)) < 0$ for all such μ by (5.5). Hence, V is indeed a strict global Lyapunov function for NE(F) under $\mu \mapsto v(\mu, F(\mu))$, so NE$(F)$ is globally weakly attracting under the EDM (4.1). □

It is easy to see that Theorem 1 generalizes the first main result in Arcak and Martins (2021), i.e., our Theorem 1 recovers Theorem 1 in Arcak and Martins (2021) when S is finite. We will see in Section 5.1 that Theorem 1 also recovers other recent stability results for special types of games.

5.1 Specialization to Monotone Games

In this section, we consider the special class of "monotone games," which are sometimes also referred to as "stable games," "contractive games," and "negative semidefinite games" in the literature. For a thorough analysis of monotone games over finite strategy sets, see Hofbauer and Sandholm (2009); Fox and Shamma (2013); Park et al. (2019), and for works considering monotone games with an infinite number of strategies, see Hofbauer et al. (2009); Cheung (2014); Lahkar and Riedel (2015); Lahkar et al. (2022). The latter works are all restricted to special types of dynamics, e.g., BNN, pairwise comparison, logit, and perturbed best response dynamics. In contrast, our stability result (Theorem 2) for monotone games derived in this section holds more broadly for the class of δ-passive dynamics (see Definition 13 to come), which constitutes a property that may be verified modularly for various instances of particular dynamics.

Definition 11. A game $F\colon \mathcal{P}(S) \to C(S)$ is *monotone* if

$$\langle F(\mu) - F(\nu), \mu - \nu \rangle \leq 0 \tag{5.6}$$

for all $\mu, \nu \in \mathcal{P}(S)$. If, additionally, the inequality (5.6) holds strictly for all $\mu, \nu \in \mathcal{P}(S)$ such that $\mu \neq \nu$, then F is *strictly monotone*.

Many games in practice are monotone, e.g., random matching in two-player symmetric zero-sum games (Cheung, 2014, Example 4), contests (Hofbauer et al., 2009, Example 5), and the war of attrition (Hofbauer and Sandholm, 2009, Example 2.4). Characterizations of the equilibria of monotone games are given in Appendix B, e.g., the convexity of $\mathrm{NE}(F)$. The following notion is closely related to that of monotonicity, as we will see in Lemma 2, and will serve as the link between monotonicity and the inequality (5.4).

Definition 12. A game $F\colon \mathcal{P}(S) \to C(S)$ that extends to a continuously Fréchet differentiable map $\overline{F}\colon U' \to C(S)$ defined on a strongly open set $U' \subseteq \mathcal{M}(S)$ containing $\mathcal{P}(S)$ is said to *satisfy self-defeating externalities* if

$$\left\langle D\overline{F}(\mu)\nu, \nu \right\rangle \leq 0 \text{ for all } \mu \in \mathcal{P}(S) \text{ and all } \nu \in T\mathcal{P}(S).$$

Lemma 2 (Cheung, 2014, Lemma 3)**.** *Consider a game $F\colon \mathcal{P}(S) \to C(S)$ that extends to a continuously Fréchet differentiable map $\overline{F}\colon U' \to C(S)$ defined on a strongly open set $U' \subseteq \mathcal{M}(S)$ containing $\mathcal{P}(S)$. It holds that F is monotone if and only if F satisfies self-defeating externalities.*

We now show that our general dissipativity theory can be applied to monotone games to recover recent stability results in the literature. We start with the following specialization of δ-dissipativity, which generalizes the notion of δ-passivity introduced in Fox and Shamma (2013) for the finite-strategy setting.

Definition 13. A map $v\colon \mathcal{P}(S)\times C(S) \to \mathcal{M}(S)$ is *δ-passive* if it is δ-dissipative with supply rate $w\colon (\mu,\eta) \mapsto \langle \eta, \mu\rangle = \int_S \eta d\mu$. The map v is *strictly δ-passive* if it is strictly δ-dissipative with such supply rate w.

As is the case with δ-dissipativity, we see that δ-passivity is solely a property of the nonautonomous portion of the evolutionary dynamics defined by the dynamics map v. We remark that δ-passivity is common in practice, as the following result shows.

Proposition 6. *If $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$ is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2, then v is strictly δ-passive.*

The proof of Proposition 6 relies on generalizing and combining the proof techniques of Fox and Shamma (2013, Theorem 4.5), Hofbauer et al. (2009, Theorem 3), and Cheung (2014, Theorem 4). We write the proof in full detail in Appendix A.

Finally, we give our reduction of Theorem 1 to the case of monotone games.

Theorem 2. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$, let $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$, and assume that Assumption 1 holds. Furthermore, assume that Assumption 2 holds and that the extension $\overline{F}$ is continuously Fréchet differentiable. If v is Nash stationary, v is δ-passive, and F is monotone, then* $\mathrm{NE}(F)$ *is weakly Lyapunov stable under the EDM* (4.1). *If, additionally, Assumption 3 and Assumption 4 both hold and v is strictly δ-passive, then* $\mathrm{NE}(F)$ *is globally weakly attracting under the EDM* (4.1).

Proof. Suppose that v is Nash stationary, v is δ-passive, and F is monotone. Let $w\colon \mathcal{M}(S)\times C(S) \to \mathbb{R}$ be defined by $w(\mu,\eta) = \langle \eta, \mu\rangle$. Then it holds that v is δ-dissipative with supply rate w. Furthermore, by Lemma 2, F satisfies self-defeating externalities, and therefore

$$w(\nu, D\overline{F}(\mu)\nu) = \left\langle D\overline{F}(\mu)\nu, \nu\right\rangle \leq 0 \text{ for all } \mu \in \mathcal{P}(S) \text{ and all } \nu \in T\mathcal{P}(S).$$

Hence, by Theorem 1, it holds that $\mathrm{NE}(F)$ is weakly Lyapunov stable under the EDM (4.1). The fact that $\mathrm{NE}(F)$ is globally weakly attracting under the EDM (4.1) given the additional hypotheses of Assumption 3 and Assumption 4 is immediate from Theorem 1. □

Notice the modularity of Theorem 2: to prove stability of the interconnected EDM (4.1), we may analyze the Nash stationarity and δ-passivity of the nonautonomous portion of the dynamics defined by the dynamics map v independently from the monotonicity of the system's feedback defined by the game F. This allows for the direct proof of stability for the entire class of monotone games F given some dynamics map v that is known to be Nash stationary and δ-passive. For example, Theorem 2 together with Proposition 6 recovers the key stability results for BNN dynamics and impartial pairwise comparison dynamics over infinite strategy sets, proven in Hofbauer et al. (2009, Theorem 3) and Cheung (2014, Theorem 4), respectively. This recovery is formally stated below.

Corollary 1. *Consider a game* $F\colon \mathcal{P}(S) \to C(S)$*, let* $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$*, and assume that Assumption 1 holds. Furthermore, assume that Assumption 2 holds and that the extension* $\overline{F}$ *is continuously Fréchet differentiable. If* F *is monotone and* v *is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2, then* $\mathrm{NE}(F)$ *is weakly Lyapunov stable under the EDM* (4.1)*. If, additionally, Assumption 3 holds, then* $\mathrm{NE}(F)$ *is globally weakly attracting under the EDM* (4.1)*.*

5.2 Extension to Dynamic Payoff Models

In this section, we consider the case where, instead of static payoff feedback given by $\rho(t) = F(\mu(t))$, as in the EDM (4.1), the payoff itself has dynamics. In doing so, we will consider the derivatives $\dot{\rho}(t)$ of a payoff $\rho\colon [0,\infty) \to C(S)$ (see Section 2.3). Since $C(S)$ is a Banach space, it holds that $\dot{\rho}(t) \in C(S)$ whenever it exists.

Definition 14. Let $\mu_0 \in \mathcal{P}(S)$, $\rho_0 \in C(S)$, $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$, and $u\colon \mathcal{P}(S) \times C(S) \to C(S)$. The differential equation

$$\begin{aligned} \dot{\mu}(t) &= v(\mu(t), \rho(t)), \\ \dot{\rho}(t) &= u(\mu(t), \rho(t)), \\ \mu(0) &= \mu_0, \\ \rho(0) &= \rho_0, \end{aligned} \tag{5.7}$$

is called a *dynamic payoff evolutionary dynamics model (DPEDM)*. The measure μ_0 is called the *initial state*, the function ρ_0 is called the *initial payoff*, the mapping v is called the *dynamics map*, and the mapping u is called the *payoff map*. A pair (μ, ρ) with strongly differentiable $\mu\colon [0,\infty) \to \mathcal{P}(S)$ and differentiable $\rho\colon [0,\infty) \to C(S)$ satisfying (5.7) is called a *solution to the DPEDM*.

Similar to the case for general EDMs, we will assume that unique solutions to the DPEDM (5.7) exist.

Assumption 5. Consider a dynamics map $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$ and a payoff map $u\colon \mathcal{P}(S) \times C(S) \to C(S)$. For every initial state $\mu_0 \in \mathcal{P}(S)$ and initial payoff $\rho_0 \in C(S)$, there exists a unique solution (μ, ρ) to the DPEDM (5.7).

For games $F\colon \mathcal{P}(S) \to C(S)$ that extend to a Fréchet differentiable map $\overline{F}\colon U' \to C(S)$ defined on a strongly open set $U' \subseteq \mathcal{M}(S)$ containing $\mathcal{P}(S)$, the EDM (4.1) is a special case of the DPEDM (5.7) with $u\colon (\mu, \rho) \mapsto D\overline{F}(\mu)v(\mu, \rho)$. Since the payoff map in a DPEDM is no longer defined by a static game, the inequality (5.4) and notions of monotonicity are no longer applicable when characterizing the "energy supplied" to the population by the payoff. Instead, we turn to notions of "antidissipativity." The following definition extends such notions from those introduced in Fox and Shamma (2013) for finite strategy sets to the setting of infinite S.

Definition 15. A map $u\colon \mathcal{P}(S)\times C(S) \to C(S)$ is *δ-antidissipative with supply rate $\tilde{w}\colon \mathcal{M}(S)\times C(S) \to \mathbb{R}$* if there exist $\gamma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}_+$ and $\Gamma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}_+$ that extends to a map $\overline{\Gamma}\colon \tilde{U} \times C(S) \to \mathbb{R}$ with strongly open $\tilde{U} \subseteq \mathcal{M}(S)$ containing $\mathcal{P}(S)$, such that the following conditions hold:

1. $\overline{\Gamma}$ is weak-∞-continuous.
2. $\overline{\Gamma}$ is Fréchet differentiable.
3. For all strongly differentiable $\mu\colon [0,\infty) \to \mathcal{P}(S)$, all $\rho_0 \in C(S)$, all solutions $\rho\colon [0,\infty) \to C(S)$ to the differential equation $\dot{\rho}(t) = u(\mu(t), \rho(t))$ with $\rho(0) = \rho_0$, and all $t \in [0,\infty)$, it holds that
$$\partial_1\overline{\Gamma}(\mu(t),\rho(t))\dot{\mu}(t)+\partial_2\overline{\Gamma}(\mu(t),\rho(t))u(\mu(t),\rho(t)) \leq -\gamma(\mu(t),\rho(t))-\tilde{w}(\dot{\mu}(t),u(\mu(t),\rho(t))). \tag{5.8}$$
4. For all $\mu \in \mathcal{P}(S)$ and all $\rho \in C(S)$, it holds that
$$\Gamma(\mu,\rho) = 0 \text{ if and only if } u(\mu,\rho) = 0. \tag{5.9}$$

If, additionally, $(\mu,\rho) \mapsto \partial_1\overline{\Gamma}(\mu,\rho)$ and $(\mu,\rho) \mapsto \partial_2\overline{\Gamma}(\mu,\rho)$ are weak-∞-continuous, every partial Fréchet derivative $\partial_1\overline{\Gamma}(\mu,\rho)$ is weakly continuous, and
$$\gamma(\mu,\rho) = 0 \text{ if and only if } u(\mu,\rho) = 0 \tag{5.10}$$
for all $\mu \in \mathcal{P}(S)$ and all $\rho \in C(S)$, then u is *strictly δ-antidissipative with supply rate $\tilde{w}$.*

Notice that δ-antidissipativity is a property solely of a payoff map u, and not of any particular dynamics map v. One may intuitively think of δ-antidissipativity with supply rate $\tilde{w}$ as δ-dissipativity with supply rate $-\tilde{w}$, albeit the notions are defined for maps with different codomains. We may also define a similar notion that is analogous to δ-passivity.

Definition 16. A map $u\colon \mathcal{P}(S) \times C(S) \to C(S)$ is *δ-antipassive* if it is δ-antidissipative with supply rate $\tilde{w}\colon (\mu,\eta) \mapsto \langle \eta,\mu\rangle = \int_S \eta d\mu$. The map u is *strictly δ-antipassive* if it is strictly δ-antidissipative with such supply rate $\tilde{w}$.

Fox and Shamma (2013, Theorem 4.3) show that every monotone game over a finite strategy set induces δ-antipassive payoff dynamics,[1] so δ-antipassivity may be viewed as a generalization of monotonicity to the dynamic payoff setting. Before moving on to our generalization of Theorem 1 to the setting of DPEDMs, we remark that Definition 14 does not immediately come equipped with any notion of a game, and hence has no inherent game-theoretic notion of equilibria. The following definition serves to link dynamic payoffs to games, namely, by ensuring that payoffs represent a valid static game at steady state.

[1] Technically, they show δ-antipassivity in the sense of input-output mappings, which slightly differs from the notion of δ-antipassivity of payoff maps used in our thesis.

Definition 17. Consider a game $F\colon \mathcal{P}(S) \to C(S)$. A map $u\colon \mathcal{P}(S)\times C(S) \to C(S)$ is *F-payoff stationary* if, for all $\mu \in \mathcal{P}(S)$ and all $\rho \in C(S)$,

$$u(\mu,\rho) = 0$$

implies that

$$\rho = F(\mu).$$

As was the case in the static payoff setting, we need some technical regularity conditions in order to apply Lyapunov theory. The appropriate conditions are distinct from those for static payoffs (i.e., we no longer need to assume Assumption 2, Assumption 3, or Assumption 4 in what follows). We list the new conditions below, and then state our stability theorem for DPEDMs.

Assumption 6. Consider a dynamics map $v\colon \mathcal{P}(S)\times C(S) \to \mathcal{M}(S)$ and a payoff map $u\colon \mathcal{P}(S)\times C(S)\to C(S)$. There exists a compact set $K \subseteq C(S)$ such that, for all initial states $\mu_0 \in \mathcal{P}(S)$ and all initial payoffs $\rho_0 \in K$, the solution (μ,ρ) to the DPEDM (5.7) satisfies $\rho(t) \in K$ for all $t \in [0,\infty)$.

Assumption 6 can be viewed as a “positive invariance” condition on the payoff dynamics. Such an assumption on the bounded evolutions of the payoffs is standard in related works (c.f., Kara and Martins 2023) and is necessary to employ Lyapunov theory.

Assumption 7. The dynamics map $v\colon \mathcal{P}(S)\times C(S)\to\mathcal{M}(S)$ is continuous with respect to the weak-∞ topology on its domain and the weak topology on its codomain. Furthermore, the payoff map $u\colon \mathcal{P}(S)\times C(S)\to C(S)$ is weak-∞-continuous.

Theorem 3. *Consider a weakly continuous game $F\colon \mathcal{P}(S)\to C(S)$, let $v\colon \mathcal{P}(S)\times C(S) \to T\mathcal{P}(S)$, and let $u\colon \mathcal{P}(S)\times C(S)\to C(S)$. Assume that Assumption 5 holds. Furthermore, assume that Assumption 6 holds with some compact $K\subseteq C(S)$ containing $F(\mathrm{NE}(F))$, and that Assumption 7 holds. If v is Nash stationary and δ-dissipative with supply rate $w\colon \mathcal{M}(S)\times C(S)\to\mathbb{R}$ and u is F-payoff stationary and δ-antidissipative with supply rate $\tilde{w}\geq w$, then*

$$P \coloneqq \{(\mu,\rho)\in\mathcal{P}(S)\times C(S) : v(\mu,\rho)=0,\ u(\mu,\rho)=0\}$$

is a subset of $\mathrm{NE}(F)\times F(\mathrm{NE}(F))$ and is weak-∞-Lyapunov stable under the DPEDM (5.7)*. If, additionally, the δ-dissipativity of v and the δ-antidissipativity of u are both strict and v is $\|\cdot\|_{\mathrm{TV}}$-bounded on $\mathcal{P}(S)\times K$, then P is weak-∞-attracting under the DPEDM* (5.7) *from every $(\mu_0,\rho_0)\in\mathcal{P}(S)\times K$.*

Proof. Since v is δ-dissipative with supply rate $w\colon \mathcal{M}(S)\times C(S)\to\mathbb{R}$, there exist $\sigma\colon \mathcal{P}(S)\times C(S)\to\mathbb{R}_+$ and $\Sigma\colon \mathcal{P}(S)\times C(S)\to\mathbb{R}_+$ with Σ having an appropriate extension $\overline{\Sigma}\colon U\times C(S)\to\mathbb{R}$ as in Definition 10. Furthermore, since u is δ-antidissipative with supply rate $\tilde{w}\colon \mathcal{M}(S)\times C(S)\to\mathbb{R}$, there exist $\gamma\colon \mathcal{P}(S)\times C(S)\to\mathbb{R}_+$ and $\Gamma\colon \mathcal{P}(S)\times C(S)\to\mathbb{R}_+$ with Γ having an appropriate extension $\overline{\Gamma}\colon \tilde{U}\times C(S)\to\mathbb{R}$ as in Definition 15. Define

$V \colon \mathcal{P}(S) \times K \to \mathbb{R}_+$ by $V(\mu,\rho) = \Sigma(\mu,\rho) + \Gamma(\mu,\rho)$. Consider $P = \{(\mu,\rho) \in \mathcal{P}(S) \times C(S) : v(\mu,\rho) = 0,\ u(\mu,\rho) = 0\}$. If $(\mu,\rho) \in P$, then $u(\mu,\rho) = 0$, implying that $\rho = F(\mu)$ by F-payoff stationarity, and hence $v(\mu, F(\mu)) = 0$, so $\mu \in \mathrm{NE}(F)$ by Nash stationarity. Thus, $P \subseteq \mathrm{NE}(F) \times F(\mathrm{NE}(F)) \subseteq \mathcal{P}(S) \times K$. By Proposition 2 and Proposition 3, $\mathrm{NE}(F)$ is weakly compact, and hence $F(\mathrm{NE}(F))$ is compact as F is weakly continuous. Since v is continuous with respect to the weak-∞ topology on its domain and the weak topology on its codomain, and u is weak-∞-continuous, it holds that $P = v^{-1}(\{0\}) \cap u^{-1}(\{0\})$ is weak-∞-closed, and hence must be weak-∞-compact as well as $\mathrm{NE}(F) \times F(\mathrm{NE}(F))$ is. Thus, by Lemma 7, it suffices to show that V is a global Lyapunov function for P under $(\mu,\rho) \mapsto (v(\mu,\rho), u(\mu,\rho))$ (according to Definition 25). Let $\overline{V} \colon U \cap \tilde{U} \times C(S) \to \mathbb{R}$ be defined by $\overline{V}(\mu,\rho) = \overline{\Sigma}(\mu,\rho) + \overline{\Gamma}(\mu,\rho)$. Note that $U \cap \tilde{U}$ is strongly open and contains $\mathcal{P}(S)$, and that $\overline{V}$ is weak-∞-continuous and Fréchet differentiable since $\overline{\Sigma}$ and $\overline{\Gamma}$ are. Also note that $\overline{V}(\mu,\rho) = V(\mu,\rho)$ for all $(\mu,\rho) \in \mathcal{P}(S) \times K$. Also, if $(\mu,\rho) \in P$, then $v(\mu,\rho) = 0$ and $u(\mu,\rho) = 0$, so $\overline{V}(\mu,\rho) = 0$ by (5.2) and (5.9). Furthermore, if $(\mu,\rho) \in (\mathcal{P}(S) \times K) \setminus P$, then again by (5.2) and (5.9) we have that $\overline{V}(\mu,\rho) > 0$. Therefore, the first two conditions from Definition 25 on V to be a global Lyapunov function for P under (v,u) are satisfied.

Next, it holds for all $(\mu,\rho) \in \mathcal{P}(S) \times K$ that

$$
\begin{aligned}
D\overline{V}(\mu,\rho)(v(\mu,\rho), u(\mu,\rho)) &= \partial_1 \overline{\Sigma}(\mu,\rho) v(\mu,\rho) + \partial_2 \overline{\Sigma}(\mu,\rho) u(\mu,\rho) \\
&\quad + \partial_1 \overline{\Gamma}(\mu,\rho) v(\mu,\rho) + \partial_2 \overline{\Gamma}(\mu,\rho) u(\mu,\rho) \\
&\le -\sigma(\mu,\rho) + w(v(\mu,\rho), u(\mu,\rho)) - \gamma(\mu,\rho) - \tilde{w}(v(\mu,\rho), u(\mu,\rho)) \\
&\le -\sigma(\mu, F(\mu)) - \gamma(\mu,\rho) \\
&\le 0,
\end{aligned}
\tag{5.11}
$$

where the first inequality follows from (5.1) and (5.8), and the second inequality follows from the fact that $\tilde{w} \ge w$. Hence, V is indeed a global Lyapunov function for P under (v,u), so P is weak-∞-Lyapunov stable under the DPEDM (5.7).

Now suppose that the δ-dissipativity of v and the δ-antidissipativity of u are both strict, and that v is $\|\cdot\|_{\mathrm{TV}}$-bounded on $\mathcal{P}(S) \times K$. By Lemma 8, it suffices to show that V is a strict global Lyapunov function for P under $(\mu,\rho) \mapsto (v(\mu,\rho), u(\mu,\rho))$ (according to Definition 25). This amounts to proving that $(\mu,\rho) \mapsto D\overline{V}(\mu,\rho)(v(\mu,\rho), u(\mu,\rho))$ is weak-∞-continuous and that $D\overline{V}(\mu,\rho)(v(\mu,\rho), u(\mu,\rho)) < 0$ for all $(\mu,\rho) \in (\mathcal{P}(S) \times K) \setminus P$. Indeed, the continuity condition holds by Lemma 5, which we state and prove in Appendix A.

Next, if $(\mu,\rho) \in (\mathcal{P}(S) \times K) \setminus P$, then $v(\mu,\rho) \neq 0$ or $u(\mu,\rho) \neq 0$, so $\sigma(\mu,\rho) > 0$ or $\gamma(\mu,\rho) > 0$ by (5.3) and (5.10), implying that $D\overline{V}(\mu,\rho)(v(\mu,\rho), u(\mu,\rho)) < 0$ for all such (μ,ρ) by (5.11). Hence, V is indeed a strict global Lyapunov function for P under (v,u), so P is globally weak-∞-attracting under the DPEDM (5.7) from K. □

The set P in Theorem 3 corresponds to the set of rest points of the DPEDM (5.7). The result shows that, under the appropriate regularity conditions, the DPEDM has dynamically stable rest points whenever the dynamics map is δ-dissipative and the payoff map is δ-antidissipative, and the incoming energy supply rate to the dynamics is less than that of the payoffs. Since, under the hypotheses of the theorem, it holds that $P \subseteq \{(\mu,\rho) \in \mathcal{P}(S) \times C(S) : \rho = F(\mu),\ \mu \in \mathrm{NE}(F)\} \subseteq \mathrm{NE}(F) \times F(\mathrm{NE}(F))$, the result shows convergence of the μ-component of the trajectory (μ,ρ) to $\mathrm{NE}(F)$, and convergence of the ρ-component to the corresponding static payoff given by the game F.

Similar to the static payoff setting, it is easy to see that our dissipativity-based result Theorem 3 may be specialized to the case of δ-passive dynamics maps coupled with δ-antipassive payoff maps, resulting in analogues to Theorem 2 and Corollary 1 for the dynamic payoff setting. In particular, the latter specialization yields the following result, which is stronger than Hofbauer et al. (2009, Theorem 3) and Cheung (2014, Theorem 4), as it allows for δ-antipassive dynamic payoffs.

Corollary 2. *Consider a weakly continuous game $F\colon \mathcal{P}(S) \to C(S)$, let $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$, and let $u\colon \mathcal{P}(S) \times C(S) \to C(S)$. Assume that Assumption 5 holds. Furthermore, assume that Assumption 6 holds with some compact $K \subseteq C(S)$ containing $F(\mathrm{NE}(F))$, and that Assumption 7 holds. If v is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2 and u is F-payoff stationary and strictly δ-antipassive, then*

$$P \coloneqq \{(\mu,\rho) \in \mathcal{P}(S) \times C(S) : v(\mu,\rho) = 0,\ u(\mu,\rho) = 0\}$$

is a subset of $\mathrm{NE}(F) \times F(\mathrm{NE}(F))$ and is weak-∞-Lyapunov stable under the DPEDM (5.7) and weak-∞-attracting under the DPEDM (5.7) from every $(\mu_0,\rho_0) \in \mathcal{P}(S) \times K$.

Chapter 6

Case Studies

6.1 War of Attrition—Failure of Finite Approximations

In this section, we tie together various results from the literature to generate an example in which approximations of an infinite-dimensional evolutionary game are proven to be dynamically stable via finite-dimensional dissipativity analysis, yet the true underlying dynamics do not weakly converge to the set of Nash equilibria. This finding shows that one cannot in general use finite-dimensional dissipativity theory to assess the stability of evolutionary games over infinite strategy sets, further motivating our studies directly concerned with the infinite-dimensional regime.

Consider the famous "war of attrition" game, which is motivated by animal conflict and studied at length in Smith (1982b, Chapter 3). We adopt the formalism of the game from Bishop and Cannings (1978) and Hofbauer et al. (2009, Example 6). Consider a contest being carried out on a time interval $S := [0, T] \subseteq \mathbb{R}$, with a common value of $V \in \mathbb{R}$ awarded to the winner. The winner is the one who decides to compete in the contest for the longest amount of time. The game is given by $F_\mu(s) = \int_S f(s, s') d\mu(s')$, where

$$f(s, s') = \begin{cases} V - s' & \text{if } s' < s, \\ \frac{V}{2} - s & \text{if } s' = s, \\ -s & \text{if } s' > s, \end{cases}$$

defines the payoff to a player employing strategy s when their opponent employs strategy s'. It is assumed that $T > V/2$, so that there may be incentive to resigning from the contest before time T. The game F is monotone and has a unique Nash equilibrium $\mu^\star \in \mathcal{P}(S)$ given by

$$\mu^\star([0, s]) = \begin{cases} 1 - e^{-s/V} & \text{if } s \in [0, s^\star), \\ 1 - e^{-s^\star/V} & \text{if } s \in [s^\star, T), \\ 1 & \text{if } s = T, \end{cases} \tag{6.1}$$

where $s^\star = T - V/2$ (c.f., Bishop and Cannings 1978; Hofbauer et al. 2009).

Consider endowing the game F with the BNN dynamics of Example 1, where the reference measure λ is Lebesgue. We will now show that the prior dissipativity results of Arcak and Martins (2021) guarantee that finite-strategy approximations of this evolutionary game asymptotically converge to their unique Nash equilibrium. Despite this, we will find that the infinite-dimensional dynamics do not weakly converge to the unique Nash equilibrium $\mu^\star$, justifying the need for direct consideration of dissipativity theory over infinite strategy sets, as we have done in this thesis.

Let $n \in \mathbb{N}$ and consider a finite approximation of the strategy set given by $S_n = \{s_1, \dots, s_n\} \subseteq S$, with $s_1 < s_2 < \cdots < s_n$. Restricting the game F to the set of measures

$$\mathcal{D}(S_n) \coloneqq \left\{ \sum_{i=1}^n x_i \delta_{s_i} \in \mathcal{P}(S) : x \in \Delta^{n-1} \right\}$$

with $\Delta^{n-1} \coloneqq \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i,\ \sum_{i=1}^n x_i = 1\}$ yields the finite-dimensional approximation $\hat{F}_n \colon \Delta^{n-1} \to \mathbb{R}^n$ given by

$$(\hat{F}_n(x))_i \coloneqq F\left(\sum_{j=1}^n x_j \delta_{s_j} \right)(s_i) = \sum_{j=1}^n x_j f(s_i, s_j).$$

Thus, the finite-dimensional game may be written as

$$\hat{F}_n(x) = A_n x,$$

where

$$A_n \coloneqq \begin{bmatrix} f(s_1,s_1) & f(s_1,s_2) & \cdots & f(s_1,s_n) \\ f(s_2,s_1) & f(s_2,s_2) & \cdots & f(s_2,s_n) \\ \vdots & \vdots & \ddots & \vdots \\ f(s_n,s_1) & f(s_n,s_2) & \cdots & f(s_n,s_n) \end{bmatrix} = \begin{bmatrix} \frac{V}{2} - s_1 & -s_1 & \cdots & -s_1 \\ V - s_1 & \frac{V}{2} - s_2 & \cdots & -s_2 \\ \vdots & \vdots & \ddots & \vdots \\ V - s_1 & V - s_2 & \cdots & \frac{V}{2} - s_n \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

This finite-dimensional game $\hat{F}_n$ is monotone (Hofbauer and Sandholm, 2009). The corresponding finite-dimensional BNN dynamics are given by

$$\dot{x}_i(t) = \max\{0, (\hat{F}_n(x(t)))_i - x(t)^\top \hat{F}_n(x(t))\} - x_i(t) \sum_{i=1}^n \max\{0, (\hat{F}_n(x(t)))_i - x(t)^\top \hat{F}_n(x(t))\}$$

for all $i \in \{1, \dots, n\}$ (c.f., Sandholm 2010, Example 4.3.4). These finite-dimensional BNN dynamics are Nash stationary and δ-passive (Arcak and Martins, 2021), and therefore since $\hat{F}_n$ is monotone and admits a continuously differentiable extension defined on $\mathbb{R}^n$ (given by the linear map defined by A_n), Arcak and Martins (2021, Theorem 1) asserts that $\mathrm{NE}(\hat{F}_n)$ is globally asymptotically stable under these dynamics. Based on the above analysis, one may hope that $\mathrm{NE}(F) = \{\mu^\star\}$ is also globally weakly attracting under the infinite-dimensional

EDM (4.1). However, despite F being monotone and v being Nash stationary and strictly δ-passive, this is not the case, as Hofbauer et al. (2009, Example 6) shows that this infinite-dimensional dynamic does not weakly converge to $\mu^\star$ for initial states with density. The intuition for this lack of convergence given by Hofbauer et al. (2009) is that the BNN dynamics under Lebesgue measure cannot generate mass at $s = T$, but that a point mass at $s = T$ is present in the equilibrium distribution $\mu^\star$. Our theoretical results pinpoint two key underlying technical conditions being violated in this example. In particular, F is not weakly continuous since f is not continuous, and furthermore, there exists $\mu \in \mathcal{P}(S)$ such that $F(\mu) \notin C(S)$, implying that F does not even have codomain $C(S)$. Such continuity conditions are key assumptions in our stability results. This breakdown of dissipativity-based stability guarantees when moving "from finite to infinite" demonstrates the importance in carefully identifying the technical conditions under which infinite-dimensional stability may be guaranteed, as we have done in our main results of Chapter 5.

6.2 Continuous War of Attrition

The function f defining the war of attrition game in Section 6.1 can be equivalently written as

$$f(s, s') = V\Theta(s - s') - s\Theta(s' - s) - s'\Theta(s - s'),$$

where $\Theta\colon \mathbb{R} \to \mathbb{R}$ is the step function given by

$$\Theta(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{2} & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Iyer and Killingback (2016) propose a smoothed variant of the war of attrition by replacing the discontinuous step function Θ by the logistic function $\Theta_\alpha\colon \mathbb{R} \to \mathbb{R}$ given by

$$\Theta_\alpha(x) = \frac{1}{1 + e^{-\alpha x}},$$

where $\alpha > 0$ is the smoothing parameter. However, in doing so, it is unclear whether the resulting game is monotone, where the difficulty arises when analyzing the values of $\int_S \int_S (s\Theta_\alpha(s' - s) + s'\Theta_\alpha(s - s'))d\mu(s')d\nu(s)$ for various $\mu, \nu \in \mathcal{P}(S)$.

In this example, we propose a relaxed variant of the game in Iyer and Killingback (2016) in which we only modify the war of attrition to be continuous, rather than smooth. This is accomplished by noting that

$$s\Theta(s' - s) + s'\Theta(s - s') = \min\{s, s'\}$$

is already a continuous function of $(s, s') \in S \times S$, and therefore the only term that should be replaced in $f(s, s')$ is $V\Theta(s - s')$, as it is where the discontinuity appears. To do so, let

$\tilde{\Theta}\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function such that $0 \leq \tilde{\Theta}(X) \leq 1$ and $\tilde{\Theta}(x)+\tilde{\Theta}(-x) = 1$ for all $x \in \mathbb{R}$. For example, one may use the logistic function $\tilde{\Theta} = \Theta_\alpha$, or even a piecewise linear approximation of the step function Θ given by

$$\tilde{\Theta}(x) = \begin{cases} 0 & \text{if } x < x_0, \\ \frac{x}{2x_0} + \frac{1}{2} & \text{if } x \in [-x_0, x_0], \\ 1 & \text{if } x > x_0. \end{cases}$$

Then, we consider the game given by

$$\begin{aligned} F_\mu(s) &\coloneqq \int_S \tilde{f}(s,s')d\mu(s'), \\ \tilde{f}(s,s') &\coloneqq V\tilde{\Theta}(s-s') - \min\{s,s'\}. \end{aligned} \tag{6.2}$$

We refer to our variant F as the "continuous war of attrition." The closer $\tilde{\Theta}$ approximates the step function Θ, the closer the continuous war of attrition approximates the original form of the war of attrition. We give two new results: 1) the continuous war of attrition is a monotone game, and 2) the continuous war of attrition is weakly Lyapunov stable and globally weakly attracting under the BNN and impartial pairwise comparison dynamics. It is easily verified that indeed $F(\mu) \in C(S)$ for all $\mu \in \mathcal{P}(S)$, that F satisfies both Assumption 2 and Assumption 3 with the extension $\overline{F}$ being defined by $\tilde{f}$ as well, and that $\overline{F}$ is continuously Fréchet differentiable. We now present our results.

Theorem 4. *The continuous war of attrition game $F\colon \mathcal{P}(S) \to C(S)$ defined by* (6.2) *is monotone.*

Proof. In this proof, we denote the indicator function on a set $A \subseteq \mathbb{R}$ by $\chi_A\colon \mathbb{R} \to \mathbb{R}$, where

$$\chi_A(t) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{if } t \notin A. \end{cases}$$

Let $\mu, \nu \in \mathcal{P}(S)$. It holds that

$$\begin{aligned} 2\int_S\int_S &\tilde{\Theta}(s-s')d(\mu-\nu)(s')d(\mu-\nu)(s) \\ &= \int_S\int_S \tilde{\Theta}(s-s')d(\mu-\nu)(s')d(\mu-\nu)(s) + \int_S\int_S \tilde{\Theta}(s'-s)d(\mu-\nu)(s')d(\mu-\nu)(s) \\ &= \int_S\int_S \left(\tilde{\Theta}(s-s') + \tilde{\Theta}(s'-s)\right) d(\mu-\nu)(s')d(\mu-\nu)(s) \\ &= \int_S\int_S d(\mu-\nu)(s')d(\mu-\nu)(s) \\ &= ((\mu-\nu)(S))^2 \\ &= 0, \end{aligned}$$

since $(\mu-\nu)(S)=\mu(S)-\nu(S)=0$. Therefore,

$$\int_S\int_S \tilde{\Theta}(s-s')d(\mu-\nu)(s')d(\mu-\nu)(s)=0.$$

Next, we note that

$$\begin{aligned}\int_S\int_S \min\{s,s'\}d\mu(s')d\nu(s) &= \int_S\int_S\int_{[0,\min\{s,s'\}]} dtd\mu(s')d\nu(s)\\ &= \int_S\int_S\int_{[0,\infty)} \chi_{\{t'\in\mathbb{R}:t'\leq\min\{s,s'\}\}}(t)dtd\mu(s')d\nu(s)\\ &= \int_S\int_S\int_{[0,\infty)} \chi_{\{t'\in\mathbb{R}:t'\leq s\}}(t)\chi_{\{t'\in\mathbb{R}:t\leq s'\}}(t)dtd\mu(s')d\nu(s)\\ &= \int_S\int_S\int_{[0,\infty)} \chi_{\{\tilde{s}\in S:\tilde{s}\geq t\}}(s)\chi_{\{\tilde{s}\in S:\tilde{s}\geq t\}}(s')dtd\mu(s')d\nu(s)\\ &= \int_{[0,\infty)}\int_S \chi_{\{\tilde{s}\in S:\tilde{s}\geq t\}}(s')d\mu(s')\int_S \chi_{\{\tilde{s}\in S:\tilde{s}\geq t\}}(s)d\nu(s)dt\\ &= \int_{[0,\infty)} \mu(S\cap[t,\infty))\nu(S\cap[t,\infty))dt.\end{aligned}$$

Therefore, we find that

$$\begin{aligned}&\int_S\int_S \min\{s,s'\}d(\mu-\nu)(s')d(\mu-\nu)(s)\\ &\quad= \int_{[0,\infty)} \left(\mu(S\cap[t,\infty))^2-2\mu(S\cap[t,\infty))\nu(S\cap[t,\infty))+\nu(S\cap[t,\infty))^2\right)dt\\ &\quad= \int_{[0,\infty)} \left(\mu(S\cap[t,\infty))-\nu(S\cap[t,\infty))\right)^2 dt.\end{aligned}$$

Thus, overall, it holds that

$$\begin{aligned}\langle F(\mu)-F(\nu),\mu-\nu\rangle &= \int_S (F_\mu(s)-F_\nu(s))d(\mu-\nu)(s)\\ &= \int_S\int_S \tilde{f}(s,s')d(\mu-\nu)(s')d(\mu-\nu)(s)\\ &= V\int_S\int_S \tilde{\Theta}(s-s')d(\mu-\nu)(s')d(\mu-\nu)(s)\\ &\qquad -\int_S\int_S \min\{s,s'\}d(\mu-\nu)(s')d(\mu-\nu)(s)\\ &= -\int_{[0,\infty)} \left(\mu(S\cap[t,\infty))-\nu(S\cap[t,\infty))\right)^2 dt\\ &\leq 0.\end{aligned}$$

Hence, F is monotone. $\square$

Theorem 4 allows us to immediately apply our dissipativity theory to conclude that indeed the continuous war of attrition exhibits global stability on the infinite strategy set S, unlike the original version of the game:

Corollary 3. *Consider the continuous war of attrition game* $F\colon \mathcal{P}(S) \to C(S)$ *defined by* (6.2). *If* $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$ *is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2 and if Assumption 1 holds, then* $\mathrm{NE}(F)$ *is weakly Lyapunov stable and globally weakly attracting under the EDM* (4.1).

In Figure 6.1, we display a computer simulation illustrating the stability of the continuous war of attrition game (6.2) with $T = 2$, $V = 1$, $\tilde{\Theta} = \Theta_\alpha$, and $\alpha = 100$, under the BNN dynamics. The simulation is carried out using the discretization technique described in Section 6.1, which we know respects the true stability of the infinite-dimensional dynamics due to Corollary 3. The initial population state in Figure 6.1 is the uniform distribution on $S = [0, 2]$. We see that the distribution function values $\mu(t)([0, s])$ converge in time towards those of a distribution closely resembling $\mu^\star$, the unique Nash equilibrium of the (discontinuous) war of attrition given in (6.1). Upon increasing α, this limiting distribution function even more closely approximates that of $\mu^\star$. The simulation is repeated in Figure 6.2 using a Guassian initial population state with mean 1 and variance 0.1. The same convergent behavior is observed.

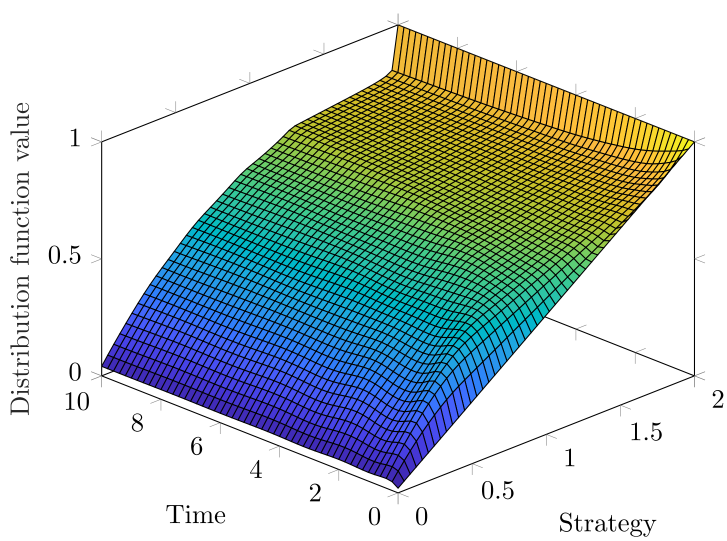

Figure 6.1: Evolution of the distribution function $s \mapsto \mu(t)([0, s])$ for continuous war of attrition on $S = [0, 2]$ under BNN dynamics with uniform initial distribution μ_0.

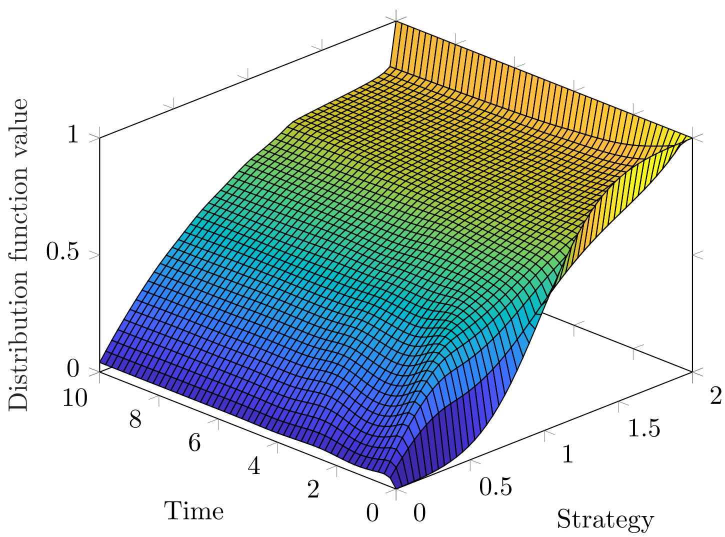

Figure 6.2: Evolution of the distribution function $s \mapsto \mu(t)([0, s])$ for continuous war of attrition on $S = [0, 2]$ under BNN dynamics with Gaussian initial distribution μ_0 (mean 1, variance 0.1).

6.3 Smoothing Dynamics

In this section, we consider the DPEDM (5.7) with dynamic payoffs. Specifically, we consider smoothing dynamics, which occur when short-term variations in an evolutionary game's payoffs are suppressed, e.g., by the time delay between when a player receives payoff information and when they revise their strategy (Fox and Shamma, 2013; Arcak and Martins, 2021). Formally, the smoothing dynamics DPEDM corresponding to a game $F \colon \mathcal{P}(S) \to C(S)$ is given by

$$\begin{aligned}\dot{\mu}(t) &= v(\mu(t), \rho(t)),\\ \dot{\rho}(t) &= \lambda \left(F(\mu(t)) - \rho(t) \right),\\ \mu(0) &= \mu_0,\\ \rho(0) &= \rho_0,\end{aligned}$$

where $\lambda > 0$ is the smoothing parameter. Notice that $u(\mu, \rho) = \lambda \left(F(\mu) - \rho \right) = 0$ if and only if $\rho = F(\mu)$, so u is F-payoff stationary.

Even in the case of finite strategy sets, the incorporation of smoothing dynamics may turn a dynamically stable evolutionary process into an unstable one (Park et al., 2019); smoothing the payoff dynamics does not necessarily help with closed-loop stability. This may also be the case in our setting of infinite strategy sets. Indeed, for the continuous war of attrition game of Section 6.2 with $T = 2$, $V = 1$, $\tilde{\Theta} = \Theta_\alpha$, and $\alpha = 100$, together with the BNN dynamics map v and $\lambda = 1$, we see in Figure 6.3 that the smoothing has caused the population state to become unstable (the persistent oscillations are verified numerically at times $t > 10^4$).

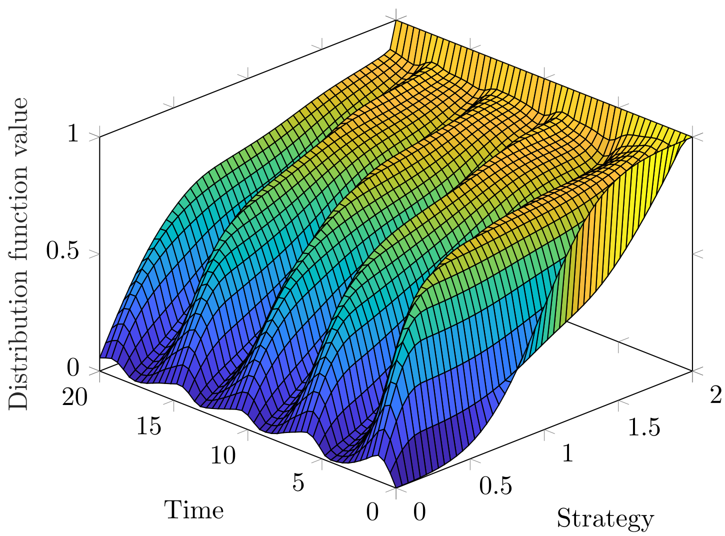

Figure 6.3: Evolution of the distribution function $s \mapsto \mu(t)([0, s])$ for continuous war of attrition game under BNN dynamics with smoothing, together with Gaussian initial distribution μ_0 (mean 1, variance 0.1) and initial payoff $\rho_0 = F(\mu_0)$.

Next, we consider the smoothing dynamics corresponding to a different game, namely, that given by

$$F_\mu(s) := \int_S f(s, s') d\mu(s'),$$
$$f(s, s') = \cos(2\pi s) - \cos(2\pi s').$$

We will refer to this as the "cosine game." It is easy to see that $\langle F_\nu, \nu \rangle = 0$ for all $\nu \in \mathcal{M}(S)$, and in particular this shows that F is monotone. For finite S, Fox and Shamma (2013) show that the smoothing dynamics corresponding to games satisfying $\langle F_\nu, \nu \rangle \leq 0$ for all $\nu \in \mathcal{M}(S)$ are δ-antipassive under an invertibility condition. Therefore, one may suspect based on our Theorem 3 and Corollary 2 that this DPEDM with a δ-passive dynamics map (such as that of BNN or impartial pairwise comparison) results in closed-loop dynamic stability. We numerically find that this is indeed the case for simulated dynamics with smoothing parameter $\lambda = 0.5$ and Gaussian initial population state μ_0 with mean 1 and variance 0.1. Figure 6.4 shows the evolution of the population state without smoothing (i.e., for the EDM (4.1) with static feedback), Figure 6.5 shows the evolution for smoothing with initial payoff $\rho_0 = F(\mu_0)$, and Figure 6.6 shows the evolution for smoothing with initial payoff given by $\rho_0(s) = -s^2$.

All evolutions appear to exhibit asymptotic stability towards a Nash equilibrium; it is easy to verify that δ_0, δ_1, and δ_2 are all Nash equilibria of F, and hence the convex combination

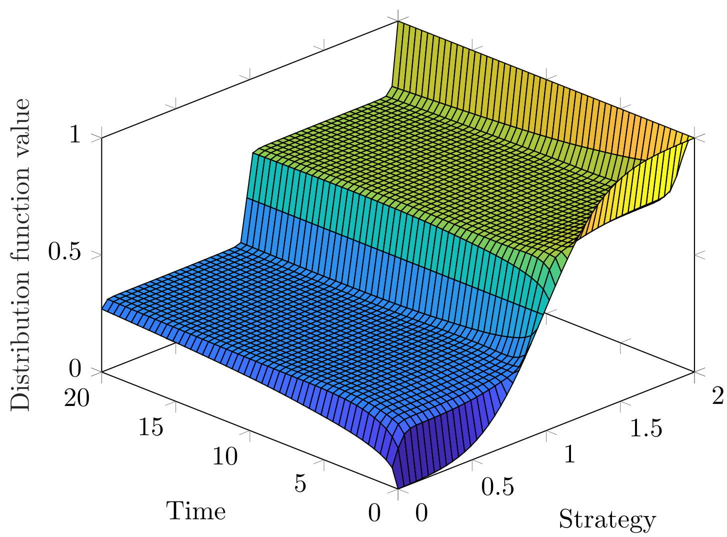

Figure 6.4: Evolution of the distribution function $s \mapsto \mu(t)([0, s])$ for the cosine game under BNN dynamics with static feedback, together with Gaussian initial distribution μ_0 (mean 1, variance 0.1).

$\frac{1}{3}\delta_0 + \frac{1}{3}\delta_1 + \frac{1}{3}\delta_2$ is as well. Interestingly, in the case of Figure 6.6 where the initial payoff is uninformative of the game's structure, the population state initially approaches a different Nash equilibrium, namely δ_0, before the system overcomes the time delay of smoothing and begins approaching $\frac{1}{3}\delta_0 + \frac{1}{3}\delta_1 + \frac{1}{3}\delta_2$. However, running the simulations for a longer time horizon shows that all of these evolutions actually end up adjusting their mass distributions to coincide with an even different Nash equilibrium, that being $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$ (c.f., Figure 6.7 for the static feedback case).

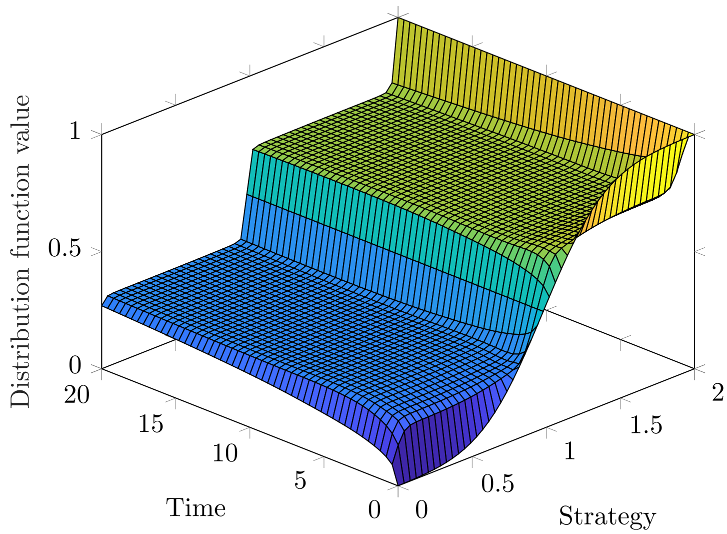

Figure 6.5: Evolution of the distribution function $s \mapsto \mu(t)([0, s])$ for the cosine game under BNN dynamics with smoothing, together with Gaussian initial distribution μ_0 (mean 1, variance 0.1) and initial payoff $\rho_0 = F(\mu_0)$.

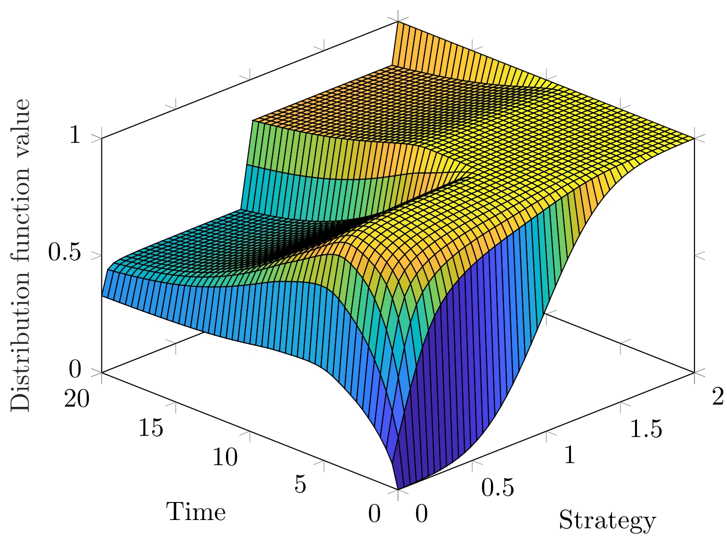

Figure 6.6: Evolution of the distribution function $s \mapsto \mu(t)([0, s])$ for the cosine game under BNN dynamics with smoothing, together with Gaussian initial distribution μ_0 (mean 1, variance 0.1) and initial payoff $\rho_0(s) = -s^2$.

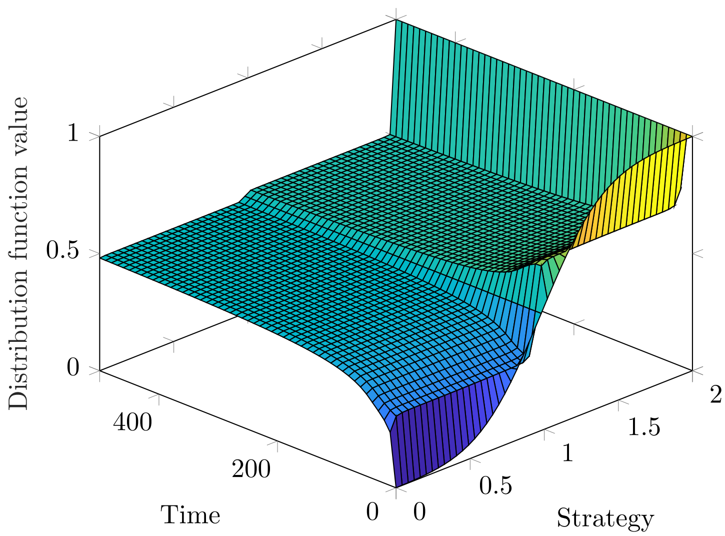

Figure 6.7: Long-time evolution of the distribution function $s \mapsto \mu(t)([0, s])$ for the cosine game under BNN dynamics with static feedback, together with Gaussian initial distribution μ_0 (mean 1, variance 0.1).

Chapter 7

Conclusions

In this thesis, we extend notions from dissipativity theory to evolutionary games with an infinite number of strategies. Our novel dynamic stability results for games evolving under δ-dissipative evolutionary dynamics provide a complete characterization of the technical conditions under which such stability is guaranteed. We both specialize our theory to monotone games, and extend our theory to δ-dissipative evolutionary dynamics coupled with δ-antidissipative dynamic feedback payoffs. Our new framework and results are applicable to much broader classes of games and dynamics than past works, recovering a handful of prior stability guarantees as special cases. This breadth is illustrated through case studies including a newly proposed variant of the classical war of attrition game. Interesting directions for future research include the development of sufficient conditions for δ-dissipativity and δ-antidissipativity from properties of a system's finite-strategy approximations, the identification and analysis of game-theoretic models and applications falling within the scope of our framework, and extensions to games with multiple populations. Another open problem of interest is the generalization of the invertibility requirement used in Fox and Shamma (2013, Theorem 4.6) to our setting of maps between Banach spaces in order to prove δ-antipassivity of payoff maps generated by smoothing of monotone games.

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Appendix A

Proofs

Proposition 1. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$, and let $\mu \in \mathcal{P}(S)$. The following are equivalent:*

1. *μ is a Nash equilibrium of the game F.*
2. *$E_F(\delta_s, \mu) \leq E_F(\mu, \mu)$ for all $s \in S$.*
3. *$F_\mu(s) \leq F_\mu(s')$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$.*

Proof of Proposition 1. Suppose that the third condition holds, so that $F_\mu(s) \leq F_\mu(s')$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$. Then, for all $s \in S$, it holds that $E_F(\delta_s, \mu) = F_\mu(s) \leq F_\mu(s')$ for all $s' \in \operatorname{supp}(\mu)$ and consequently that $E_F(\delta_s, \mu) = \int_S F_\mu(s) d\mu(s') \leq \int_S F_\mu(s') d\mu(s') = E_F(\mu, \mu)$. Thus, the second condition holds. Furthermore, if $\nu \in \mathcal{P}(S)$, then $E_F(\nu, \mu) = \int_S F_\mu(s) d\nu(s) = \int_S E_F(\delta_s, \mu) d\nu(s) \leq \int_S E_F(\mu, \mu) d\nu(s) = E_F(\mu, \mu)$, so the first condition holds as well.

To complete the proof, we show that the first condition implies the third. Suppose that the first condition holds, so that $E_F(\nu, \mu) \leq E_F(\mu, \mu)$ for all $\nu \in \mathcal{P}(S)$. Notice that $\sup_{s \in \operatorname{supp}(\mu)} E_F(\delta_s, \mu) \leq E_F(\mu, \mu)$, and also that

$$
\begin{aligned}
\sup_{s \in \operatorname{supp}(\mu)} E_F(\delta_s, \mu) &= \int_S \left(\sup_{s \in \operatorname{supp}(\mu)} E_F(\delta_s, \mu) \right) d\mu(s') \\
&= \int_{\operatorname{supp}(\mu)} \left(\sup_{s \in \operatorname{supp}(\mu)} E_F(\delta_s, \mu) \right) d\mu(s') \\
&\geq \int_{\operatorname{supp}(\mu)} E_F(\delta_{s'}, \mu) d\mu(s') \\
&= \int_S F_\mu(s') d\mu(s') \\
&= E_F(\mu, \mu).
\end{aligned}
$$

Hence, $\sup_{s\in\operatorname{supp}(\mu)} E_F(\delta_s,\mu) = E_F(\mu,\mu)$. Suppose for the sake of contradiction that there exists $s' \in \operatorname{supp}(\mu)$ such that $E_F(\delta_{s'},\mu) < \sup_{s\in\operatorname{supp}(\mu)} E_F(\delta_s,\mu) = E_F(\mu,\mu)$. Since F_μ is a continuous real-valued function on S, the preimage $U \coloneqq F_\mu^{-1}((-\infty, E_F(\mu,\mu))) = \{s \in S : F_\mu(s) < E_F(\mu,\mu)\}$ is open and contains s', and hence it must be the case that $\mu(U) > 0$ by definition of $\operatorname{supp}(\mu)$. Thus, since the Lebesgue integral of a positive function over a set of positive measure is positive, we find that

$$\begin{aligned}
0 &= E_F(\mu,\mu) - E_F(\mu,\mu) \\
&= \int_S (E_F(\mu,\mu) - F_\mu(s))d\mu(s) \\
&= \int_U (E_F(\mu,\mu) - F_\mu(s))d\mu(s) + \int_{S\setminus U} (E_F(\mu,\mu) - E_F(\delta_s,\mu))d\mu(s) \\
&\geq \int_U (E_F(\mu,\mu) - F_\mu(s))d\mu(s) \\
&> 0,
\end{aligned}$$

which is a contradiction. Hence, it must be the case that $E_F(\delta_{s'},\mu) = \sup_{s\in\operatorname{supp}(\mu)} E_F(\delta_s,\mu) = E_F(\mu,\mu)$ for all $s' \in \operatorname{supp}(\mu)$. Therefore, $F_\mu(s') = E_F(\delta_{s'},\mu) = E_F(\mu,\mu) \geq E_F(\nu,\mu)$ for all $\nu \in \mathcal{P}(S)$ and all $s' \in \operatorname{supp}(\mu)$, and in particular, we find that $F_\mu(s') \geq E_F(\delta_s,\mu) = F_\mu(s)$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$, so the third condition holds. □

Proposition 2. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$. If $\theta_\nu\colon \mathcal{P}(S) \to \mathbb{R}$ defined by $\theta_\nu(\mu) = E_F(\nu,\mu) - E_F(\mu,\mu)$ is weakly continuous for all $\nu \in \mathcal{P}(S)$, then* $\mathrm{NE}(F)$ *is weakly compact.*

Proof of Proposition 2. It holds that

$$\begin{aligned}
\mathrm{NE}(F) &= \{\mu \in \mathcal{P}(S) : E_F(\nu,\mu) - E_F(\mu,\mu) \leq 0 \text{ for all } \nu \in \mathcal{P}(S)\} \\
&= \bigcap_{\nu\in\mathcal{P}(S)} \{\mu \in \mathcal{P}(S) : E_F(\nu,\mu) - E_F(\mu,\mu) \leq 0\}.
\end{aligned}$$

For all $\nu \in \mathcal{P}(S)$, the set $\{\mu \in \mathcal{P}(S) : E_F(\nu,\mu) - E_F(\mu,\mu) \leq 0\}$ is the preimage of the closed set $(-\infty, 0]$ under the map θ_ν. Hence, if this map is weakly continuous, then $\mathrm{NE}(F)$ is weakly closed. Since $\mathcal{P}(S)$ is weakly compact by Lemma 1, the weakly closed subset $\mathrm{NE}(F) \subseteq \mathcal{P}(S)$ must also be weakly compact. □

Proposition 3. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$. If F is weakly continuous, then $\theta_\nu\colon \mathcal{P}(S) \to \mathbb{R}$ defined by $\theta_\nu(\mu) = E_F(\nu,\mu) - E_F(\mu,\mu)$ is weakly continuous for all $\nu \in \mathcal{P}(S)$.*

Proof of Proposition 3. Suppose that F is weakly continuous and let $\nu \in \mathcal{P}(S)$. Since S is a metric space, the weak topology on $\mathcal{P}(S)$ is metrizable (Dudley, 2002, Theorem 11.3.3). Therefore, the weak topology on $\mathcal{P}(S)$ is first-countable and hence functions with domain $\mathcal{P}(S)$ are weakly continuous if they are weakly sequentially continuous. Thus, to prove the claim, it suffices to show that θ_ν is weakly sequentially continuous. To this end, let $\{\mu_n \in \mathcal{P}(S) : n \in \mathbb{N}\}$ be a sequence that converges weakly to $\mu \in \mathcal{P}(S)$. Then we have that

$$\begin{aligned} |E_F(\nu,\mu_n) - E_F(\nu,\mu)| &= |\,\langle F(\mu_n),\nu\rangle - \langle F(\mu),\nu\rangle\,| \\ &= |\,\langle F(\mu_n) - F(\mu),\nu\rangle\,| \\ &\le \|F(\mu_n) - F(\mu)\|_\infty \|\nu\|_{\mathrm{TV}} \\ &\to 0 \end{aligned}$$

since $\|\nu\|_{\mathrm{TV}} = 1$ and $F(\mu_n) \to F(\mu)$ in $C(S)$ with the topology induced by $\|\cdot\|_\infty$ due to weak continuity of F. Furthermore, we have that

$$\begin{aligned} |E_F(\mu_n,\mu_n) - E_F(\mu,\mu)| &= |\,\langle F(\mu_n),\mu_n\rangle - \langle F(\mu),\mu\rangle\,| \\ &\le |\,\langle F(\mu_n),\mu_n\rangle - \langle F(\mu),\mu_n\rangle\,| + |\,\langle F(\mu),\mu_n\rangle - \langle F(\mu),\mu\rangle\,| \\ &= |\,\langle F(\mu_n) - F(\mu),\mu_n\rangle\,| + |\,\langle F(\mu),\mu_n - \mu\rangle\,| \\ &\le \|F(\mu_n) - F(\mu)\|_\infty \|\mu_n\|_{\mathrm{TV}} + |\,\langle F(\mu),\mu_n - \mu\rangle\,| \\ &= \|F(\mu_n) - F(\mu)\|_\infty + |\,\langle F(\mu),\mu_n - \mu\rangle\,| \\ &\to 0 \end{aligned}$$

since again $F(\mu_n) \to F(\mu)$ by weak continuity of F, and since $\langle F(\mu),\mu_n - \mu\rangle \to 0$ by definition of weak convergence of μ_n to μ. Therefore, we conclude that

$$\theta_\nu(\mu_n) = E_F(\nu,\mu_n) - E_F(\mu_n,\mu_n) \to E_F(\nu,\mu) - E_F(\mu,\mu) = \theta_\nu(\mu),$$

which proves the claim. □

Proposition 4. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$ and let $v\colon \mathcal{P}(S)\times C(S) \to \mathcal{M}(S)$. If v is Nash stationary, then the set of rest points of the EDM* (4.1) *equals* $\mathrm{NE}(F)$.

Proof of Proposition 4. Suppose that v is Nash stationary. Let $\mu \in \mathcal{P}(S)$ be a rest point of the EDM (4.1) with dynamics map v. Then $v(\mu, F(\mu)) = 0$. Since v is Nash stationary, this implies that $E_F(\nu,\mu) - E_F(\mu,\mu) = \langle F(\mu),\nu\rangle - \langle F(\mu),\mu\rangle \le 0$ for all $\nu \in \mathcal{P}(S)$. Thus, $\mu \in \mathrm{NE}(F)$. On the other hand, if $\mu \in \mathrm{NE}(F)$, then $\langle F(\mu),\nu\rangle - \langle F(\mu),\mu\rangle = E_F(\nu,\mu) - E_F(\mu,\mu) \le 0$, so $v(\mu,F(\mu)) = 0$ since v is Nash stationary. Thus, μ is a rest point of the EDM (4.1) with dynamics map v. □

Proposition 5 (Hofbauer et al., 2009; Cheung, 2014)**.** *If $v\colon \mathcal{P}(S)\times C(S) \to \mathcal{M}(S)$ is the dynamics map for either the BNN dynamics of Example 1 or the pairwise comparison dynamics of Example 2, then v is Nash stationary.*

Proof of Proposition 5. Let $\mu \in \mathcal{P}(S)$ and let $\rho \in C(S)$. First consider the BNN dynamics of Example 1. We have that

$$v(\mu,\rho)(B) = \int_B \max\{0, \langle \rho,\delta_s\rangle - \langle\rho,\mu\rangle\} d\lambda(s) - \mu(B)\int_S \max\{0, \langle \rho,\delta_s\rangle - \langle\rho,\mu\rangle\} d\lambda(s)$$

for all $B \in \mathcal{B}(S)$. If $\langle \rho, \nu \rangle \leq \langle \rho, \mu \rangle$ for all $\nu \in \mathcal{P}(S)$, then it follows immediately that $v(\mu, \rho)(B) = 0$ for all $B \in \mathcal{B}(S)$, and hence $v(\mu, \rho) = 0$.

On the other hand, suppose that $v(\mu, \rho) = 0$. Suppose that

$$\int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s) = 0.$$

Then we find that

$$\int_B \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s) = 0$$

for all $B \in \mathcal{B}(S)$. Hence, $\max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} = 0$ for λ-almost every $s \in S$. Since λ has full support by assumption and $s \mapsto \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\}$ is continuous, this shows that $\max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} = 0$ for all $s \in S$. Hence, $\langle \rho, \delta_s \rangle \leq \langle \rho, \mu \rangle$ for all $s \in S$. Since S is compact and ρ is continuous, the optimization $\sup_{s \in S} \rho(s)$ is attained by some $s' \in S$. Therefore, for all $\nu \in \mathcal{P}(S)$ it holds that $\langle \rho, \nu \rangle = \int_S \rho(s) d\nu(s) \leq \int_S \rho(s') d\nu(s) = \rho(s') = \langle \rho, \delta_{s'} \rangle \leq \langle \rho, \mu \rangle$. Now suppose that the other case holds, namely, that $\int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s) > 0$. Then it holds that

$$\mu(B) = \int_B \frac{\max\{0, \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle\}}{\int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s)} d\lambda(s')$$

for all $B \in \mathcal{B}(S)$. Suppose for the sake of contradiction that there exists $\tilde{s} \in S$ such that $\langle \rho, \delta_{\tilde{s}} \rangle - \langle \rho, \mu \rangle > 0$. Then, by continuity of $s' \mapsto \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle$, the preimage $\{s' \in S : \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle > 0\}$ is open and contains $\tilde{s}$, and hence it must be the case that $\lambda(\{s' \in S : \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle > 0\}) > 0$ by definition of $\operatorname{supp}(\lambda)$ and the fact that λ has full support. Therefore, since the Lebesgue integral of a positive function over a set of positive measure is positive, we find that

$$\begin{aligned}
\langle \rho, \mu \rangle &= \int_S \rho d\mu \\
&= \int_S \rho(s') \frac{\max\{0, \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle\}}{\int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s)} d\lambda(s') \\
&= \int_{\{s' \in S : \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle > 0\}} \langle \rho, \delta_{s'} \rangle \frac{\max\{0, \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle\}}{\int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s)} d\lambda(s') \\
&> \int_{\{s' \in S : \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle > 0\}} \langle \rho, \mu \rangle \frac{\max\{0, \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle\}}{\int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s)} d\lambda(s') \\
&= \langle \rho, \mu \rangle,
\end{aligned}$$

which is a contradiction. Hence, it must be that $\langle \rho, \delta_{\tilde{s}} \rangle \leq \langle \rho, \mu \rangle$ for all $\tilde{s} \in S$. Arguing as in the prior case, this yields that $\langle \rho, \nu \rangle \leq \langle \rho, \mu \rangle$ for all $\nu \in \mathcal{P}(S)$. Since this exhausts all cases to be considered, we conclude that indeed v is Nash stationary.

Now consider the pairwise comparison dynamics of Example 2. We have that

$$v(\mu,\rho)(B)=\int_S\int_B \gamma(s,s',\rho)d\lambda(s')d\mu(s)-\int_S\int_B \gamma(s',s,\rho)d\mu(s')d\lambda(s)$$

for all $B\in\mathcal{B}(S)$. By Lemma 3, which we prove after completing the current proof, it holds that $\langle\rho,\nu\rangle\le\langle\rho,\mu\rangle$ for all $\nu\in\mathcal{P}(S)$ if and only if $\rho(s)\le\rho(s')$ for all $s\in S$ and all $s'\in\operatorname{supp}(\mu)$. Thus, if $\langle\rho,\nu\rangle\le\langle\rho,\mu\rangle$ for all $\nu\in\mathcal{P}(S)$, then $\max\{0,\rho(s)-\rho(s')\}=0$ for all $s\in S$ and all $s'\in\operatorname{supp}(\mu)$, implying that $\operatorname{sign}(\max\{0,\rho(s)-\rho(s')\})=0$ for all $s\in S$ and all $s'\in\operatorname{supp}(\mu)$. Hence, since the conditional switch rate γ satisfies sign-preservation by assumption, we find that

$$\operatorname{sign}(\gamma(s',s,\rho))=0$$

for all $s\in S$ and all $s'\in\operatorname{supp}(\mu)$. This implies that $v(\mu,\rho)(B)=0$ for all $B\in\mathcal{B}(S)$, and therefore that $v(\mu,\rho)=0$.

On the other hand, suppose that $v(\mu,\rho)=0$. Define the measures $v_1(\mu,\rho),v_2(\mu,\rho)\in\mathcal{M}(S)$ by

$$\begin{aligned}
v_1(\mu,\rho)(B)&\coloneqq\int_S\int_B\gamma(s,s',\rho)d\lambda(s')d\mu(s)=\int_B\int_S\gamma(s,s',\rho)d\mu(s)d\lambda(s'),\\
v_2(\mu,\rho)(B)&\coloneqq\int_S\int_B\gamma(s',s,\rho)d\mu(s')d\lambda(s)=\int_B\int_S\gamma(s',s,\rho)d\lambda(s)d\mu(s'),
\end{aligned}$$

so that $v(\mu,\rho)=v_1(\mu,\rho)-v_2(\mu,\rho)$. Since $v(\mu,\rho)=0$, it holds that $v_1(\mu,\rho)=v_2(\mu,\rho)$, and hence $\langle\rho,v_1(\mu,\rho)\rangle=\langle\rho,v_2(\mu,\rho)\rangle$. Therefore,

$$\int_S\rho(s')\int_S\gamma(s,s',\rho)d\mu(s)d\lambda(s')=\int_S\rho(s')\int_S\gamma(s',s,\rho)d\lambda(s)d\mu(s').$$

Hence,

$$\int_S\int_S\rho(s')\gamma(s,s',\rho)d\mu(s)d\lambda(s')=\int_S\int_S\rho(s')\gamma(s',s,\rho)d\lambda(s)d\mu(s'),$$

implying that

$$\int_S\int_S(\rho(s)-\rho(s'))\gamma(s',s,\rho)d\mu(s')d\lambda(s)=0.$$

By sign-preservation of the conditional switch rate γ, it holds that

$$\operatorname{sign}(\gamma(s',s,\rho))=\operatorname{sign}(\max\{0,\rho(s)-\rho(s')\})$$

for all $s,s'\in S$, and therefore, if $\rho(s)\ge\rho(s')$, we find that $\gamma(s',s,\rho)\ge 0$ so that $(\rho(s)-\rho(s'))\gamma(s',s,\rho)\ge 0$, and similarly if $\rho(s)\le\rho(s')$, we find that $\gamma(s',s,\rho)=0$ so that $(\rho(s)-\rho(s'))\gamma(s',s,\rho)=0$. Hence, $(\rho(s)-\rho(s'))\gamma(s',s,\rho)\ge 0$ for all $s,s'\in S$, and also $\int_S(\rho(s)-\rho(s'))\gamma(s',s,\rho)d\mu(s')\ge 0$ for all $s\in S$. Since $s\mapsto\int_S(\rho(s)-\rho(s'))\gamma(s',s,\rho)d\mu(s')$ is continuous (which follows from compactness of S and continuity of $s'\mapsto(\rho(s)-\rho(s'))\gamma(s',s,\rho)$, together with the dominated convergence theorem), the preimage $\{s \in S : \int_S(\rho(s) - \rho(s'))\gamma(s', s, \rho)d\mu(s') > 0\}$ is open and therefore must be empty, for otherwise $\int_S\int_S(\rho(s) - \rho(s'))\gamma(s', s, \rho)d\mu(s')d\lambda(s) > 0$ as λ has full support. Hence,

$$\int_S (\rho(s) - \rho(s'))\gamma(s', s, \rho)d\mu(s') = 0 \text{ for all } s \in S.$$

Similarly, since $s' \mapsto (\rho(s) - \rho(s'))\gamma(s', s, \rho)$ is continuous for all $s \in S$, the preimage $\{s' \in S : (\rho(s) - \rho(s'))\gamma(s', s, \rho) > 0\}$ is open for all $s \in S$, and hence for all $s' \in \operatorname{supp}(\mu)$ it must be the case that

$$(\rho(s) - \rho(s'))\gamma(s', s, \rho) = 0$$

for all $s \in S$. Thus, for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$, either $\rho(s) = \rho(s')$, or $\gamma(s', s, \rho) = 0$. In the latter case, we see by sign-preservation of the conditional switch rate that

$$\operatorname{sign}(\max\{0, \rho(s) - \rho(s')\}) = \operatorname{sign}(\gamma(s', s, \rho)) = 0,$$

and hence $\rho(s) \leq \rho(s')$. Therefore, we conclude that

$$\rho(s) \leq \rho(s') \text{ for all } s \in S \text{ and all } s' \in \operatorname{supp}(\mu).$$

By Lemma 3, this proves that $\langle \rho, \nu \rangle \leq \langle \rho, \mu \rangle$ for all $\nu \in \mathcal{P}(S)$, and consequently that v is Nash stationary. □

Lemma 3. *It holds that $\langle \rho, \nu \rangle \leq \langle \rho, \mu \rangle$ for all $\nu \in \mathcal{P}(S)$ if and only if $\rho(s) \leq \rho(s')$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$.*

Proof of Lemma 3. Suppose first that $\rho(s) \leq \rho(s')$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$. Then, it holds that

$$\rho(s) = \int_S \rho(s)d\mu(s') \leq \int_S \rho(s')d\mu(s') = \langle \rho, \mu \rangle$$

for all $s \in S$. Therefore, for all $\nu \in \mathcal{P}(S)$, we conclude that

$$\langle \rho, \nu \rangle = \int_S \rho(s)d\nu(s) \leq \int_S \langle \rho, \mu \rangle \, d\nu(s) = \langle \rho, \mu \rangle \,,$$

which proves one direction of the lemma.

On the other hand, suppose that $\langle \rho, \nu \rangle \leq \langle \rho, \mu \rangle$ for all $\nu \in \mathcal{P}(S)$. Then we have that $\rho(s) = \langle \rho, \delta_s \rangle \leq \langle \rho, \mu \rangle$ for all $s \in S$. Furthermore,

$$
\begin{aligned}
\sup_{s \in \operatorname{supp}(\mu)} \rho(s) &= \int_S \left(\sup_{s \in \operatorname{supp}(\mu)} \rho(s) \right) d\mu(s') \\
&= \int_{\operatorname{supp}(\mu)} \left(\sup_{s \in \operatorname{supp}(\mu)} \rho(s) \right) d\mu(s') \\
&\geq \int_{\operatorname{supp}(\mu)} \rho(s') d\mu(s') \\
&= \int_S \rho(s') d\mu(s') \\
&= \langle \rho, \mu \rangle .
\end{aligned}
$$

Hence, $\sup_{s \in \operatorname{supp}(\mu)} \rho(s) = \langle \rho, \mu \rangle$. Suppose for the sake of contradiction that there exists $s' \in \operatorname{supp}(\mu)$ such that $\rho(s') < \sup_{s \in \operatorname{supp}(\mu)} \rho(s) = \langle \rho, \mu \rangle$. Since ρ is a continuous real-valued function on S, the preimage $U \coloneqq \rho^{-1}((-\infty, \langle \rho, \mu \rangle)) = \{s \in S : \rho(s) < \langle \rho, \mu \rangle\}$ is open and contains s', and hence it must be the case that $\mu(U) > 0$ by definition of $\operatorname{supp}(\mu)$. Thus, since the Lebesgue integral of a positive function over a set of positive measure is positive, we find that

$$
\begin{aligned}
0 &= \langle \rho, \mu \rangle - \langle \rho, \mu \rangle \\
&= \int_S (\langle \rho, \mu \rangle - \rho(s)) d\mu(s) \\
&= \int_U (\langle \rho, \mu \rangle - \rho(s)) d\mu(s) + \int_{S \setminus U} (\langle \rho, \mu \rangle - \rho(s)) d\mu(s) \\
&\geq \int_U (\langle \rho, \mu \rangle - \rho(s)) d\mu(s) \\
&> 0,
\end{aligned}
$$

which is a contradiction. Hence, it must be the case that $\rho(s') = \sup_{s \in \operatorname{supp}(\mu)} \rho(s) = \langle \rho, \mu \rangle$ for all $s' \in \operatorname{supp}(\mu)$. Therefore, $\rho(s') = \langle \rho, \mu \rangle \geq \langle \rho, \nu \rangle$ for all $\nu \in \mathcal{P}(S)$ and all $s' \in \operatorname{supp}(\mu)$, and in particular, we find that $\rho(s') \geq \langle \rho, \delta_s \rangle = \rho(s)$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$. This concludes the proof. □

Lemma 4. *The map $\mu \mapsto D\overline{V}(\mu) v(\mu, F(\mu))$ is weakly continuous.*

Proof of Lemma 4. Since S is a metric space, the weak topology on $\mathcal{P}(S)$ is metrizable (Dudley, 2002, Theorem 11.3.3). Therefore, the weak topology on $\mathcal{P}(S)$ is first-countable and hence functions with domain $\mathcal{P}(S)$ are weakly continuous if they are weakly sequentially continuous. Thus, to prove the claim, it suffices to show that

$$
D\overline{V}(\mu_n) v(\mu_n, F(\mu_n)) \to D\overline{V}(\mu) v(\mu, F(\mu))
$$

whenever $\mu_n \to \mu$ weakly. To this end, let $\{\mu_n \in \mathcal{P}(S) : n \in \mathbb{N}\}$ be a sequence that converges weakly to $\mu \in \mathcal{P}(S)$. Then we have that

$$\begin{aligned}|D\overline{V}(\mu_n)v(\mu_n, F(\mu_n)) - D\overline{V}(\mu)v(\mu, F(\mu))| &= |D\overline{V}(\mu_n)v(\mu_n, \overline{F}(\mu_n)) - D\overline{V}(\mu)v(\mu, \overline{F}(\mu))| \\ &\leq |D\overline{V}(\mu)(v(\mu_n, \overline{F}(\mu_n)) - v(\mu, \overline{F}(\mu)))| \\ &\quad + |(D\overline{V}(\mu_n) - D\overline{V}(\mu))v(\mu, \overline{F}(\mu))| \\ &\quad + |(D\overline{V}(\mu_n) - D\overline{V}(\mu))(v(\mu_n, \overline{F}(\mu_n)) \\ &\qquad - v(\mu, \overline{F}(\mu)))|.\end{aligned} \tag{A.1}$$

Assume for the time being that every Fréchet derivative $D\overline{V}(\nu)$ is weakly continuous, and that $D\overline{V}$ is weakly continuous on $\mathcal{P}(S)$. Then, under this assumption, it holds that

$$|D\overline{V}(\mu)(v(\mu_n, \overline{F}(\mu_n)) - v(\mu, \overline{F}(\mu)))| \to 0,$$

since $v(\mu_n, \overline{F}(\mu_n)) \to v(\mu, \overline{F}(\mu))$ weakly, as $\overline{F}$ is weakly continuous and v is continuous with respect to the weak-∞ topology on its domain and the weak topology on its codomain. Furthermore,

$$|(D\overline{V}(\mu_n) - D\overline{V}(\mu))v(\mu, \overline{F}(\mu))| \leq \|D\overline{V}(\mu_n) - D\overline{V}(\mu)\|_{\mathcal{M}(S)^*}\|v(\mu, \overline{F}(\mu))\|_{\mathrm{TV}} \to 0,$$

since $D\overline{V}(\mu_n) \to D\overline{V}(\mu)$ in the dual space $\mathcal{M}(S)^*$ with associated operator norm $\|\cdot\|_{\mathcal{M}(S)^*}$ induced by the total variation norm on $\mathcal{M}(S)$, as $D\overline{V} \colon U \cap U' \to \mathcal{M}(S)^*$ is weakly continuous on $\mathcal{P}(S)$ by our above assumption. Finally,

$$\begin{aligned}&|(D\overline{V}(\mu_n) - D\overline{V}(\mu))(v(\mu_n, \overline{F}(\mu_n)) - v(\mu, \overline{F}(\mu)))| \\ &\quad\leq \|D\overline{V}(\mu_n) - D\overline{V}(\mu)\|_{\mathcal{M}(S)^*}\|v(\mu_n, \overline{F}(\mu_n)) - v(\mu, \overline{F}(\mu))\|_{\mathrm{TV}} \\ &\quad\leq 2\|D\overline{V}(\mu_n) - D\overline{V}(\mu)\|_{\mathcal{M}(S)^*} \sup_{\nu \in \mathcal{P}(S)} \|v(\nu, \overline{F}(\nu))\|_{\mathrm{TV}} \\ &\quad\to 0,\end{aligned}$$

since again $D\overline{V}(\mu_n) \to D\overline{V}(\mu)$ in $\mathcal{M}(S)^*$ by the weak continuity assumption on $D\overline{V}$, and $\sup_{\nu \in \mathcal{P}(S)} \|v(\nu, \overline{F}(\nu))\|_{\mathrm{TV}} \leq \sup_{(\nu, g) \in \mathcal{P}(S) \times \overline{F}(\mathcal{P}(S))} \|v(\nu, g)\|_{\mathrm{TV}} \leq M$ for some finite $M \in [0, \infty)$ by the $\|\cdot\|_{\mathrm{TV}}$-boundedness of v on weak-∞ compact subsets of $\mathcal{P}(S) \times C(S)$. Therefore, under the above assumptions, it must be that

$$D\overline{V}(\mu_n)v(\mu_n, F(\mu_n)) \to D\overline{V}(\mu)v(\mu, F(\mu)),$$

which is what was to be proven. Thus, it remains to prove the above assumptions, namely, that every Fréchet derivative $D\overline{V}(\nu)$ is weakly continuous, and that $D\overline{V}$ is weakly continuous on $\mathcal{P}(S)$.

Let us first prove that $D\overline{V}(\mu) \colon \mathcal{M}(S) \to \mathbb{R}$ is weakly continuous for all $\mu \in U \cap U'$. Let $\mu \in U \cap U'$. Since $\partial_2\overline{\Sigma}(\mu, \rho) \colon C(S) \to \mathbb{R}$ is continuous (with respect to the topology on

$C(S)$ induced by $\|\cdot\|_\infty$) for all $\rho \in C(S)$ by definition of the Fréchet derivative, and since $D\overline{F}(\mu)\colon \mathcal{M}(S) \to C(S)$ is weakly continuous under the hypotheses of the theorem, it holds that the composition $\partial_2\overline{\Sigma}(\mu, \overline{F}(\mu)) \circ D\overline{F}(\mu)\colon \mathcal{M}(S) \to \mathbb{R}$ is weakly continuous. Since we also have that $\partial_1\overline{\Sigma}(\mu, \overline{F}(\mu))\colon \mathcal{M}(S) \to \mathbb{R}$ is also weakly continuous under the hypotheses of the theorem, we conclude that

$$D\overline{V}(\mu) = \partial_1\overline{\Sigma}(\mu, \overline{F}(\mu)) + \partial_2\overline{\Sigma}(\mu, \overline{F}(\mu)) \circ D\overline{F}(\mu)$$

is weakly continuous, which proves the first assumption to be proven.

Finally, let us prove the remaining assumption, namely, that $D\overline{V}\colon U \cap U' \to \mathcal{M}(S)^*$ is weakly continuous on $\mathcal{P}(S)$ (that is, continuous with respect to the weak topology on its domain $U \cap U' \subseteq \mathcal{M}(S)$ and the topology on its codomain $\mathcal{M}(S)^*$ induced by the operator norm $\|\cdot\|_{\mathcal{M}(S)^*}$). Once again, since we are considering weak continuity of a function on $\mathcal{P}(S)$, where the weak topology is first-countable, it suffices to prove weak sequential continuity. Let $\{\mu_n \in \mathcal{P}(S) : n \in \mathbb{N}\}$ be a sequence that converges weakly to $\mu \in \mathcal{P}(S)$. Then

$$\begin{aligned}\|D\overline{V}(\mu_n) - D\overline{V}(\mu)\|_{\mathcal{M}(S)^*} \leq{}& \left\|\partial_1\overline{\Sigma}(\mu_n, \overline{F}(\mu_n)) - \partial_1\overline{\Sigma}(\mu, \overline{F}(\mu))\right\|_{\mathcal{M}(S)^*} \\ &+ \left\|\partial_2\overline{\Sigma}(\mu_n, \overline{F}(\mu_n)) \circ D\overline{F}(\mu_n) - \partial_2\overline{\Sigma}(\mu, \overline{F}(\mu)) \circ D\overline{F}(\mu)\right\|_{\mathcal{M}(S)^*}.\end{aligned}$$

It is clear that the first term in the above upper bound converges to 0 due to the weak-∞ continuity of $(\nu, \rho) \mapsto \partial_1\overline{\Sigma}(\nu, \rho)$ together with the weak continuity of $\overline{F}$. Further upper-bounding the second term in a similar manner to the bound (A.1) and appealing to the finiteness of $\|\varphi\|_{\mathrm{TV}}$ and $\|\psi\|_{\mathcal{M}(S)^*}$ for $\varphi \in C(S)^* = \mathcal{M}(S)$ and $\psi \in \mathcal{M}(S)^*$ together with the weak continuity of $\overline{F}$ and $D\overline{F}$ as well as the weak-∞ continuity of $(\nu, \rho) \mapsto \partial_2\overline{\Sigma}(\nu, \rho)$ yields that the second term converges to 0 as well. Thus, $D\overline{V}(\mu_n) \to D\overline{V}(\mu)$ in $\mathcal{M}(S)^*$, so $D\overline{V}$ is indeed weakly continuous on $\mathcal{P}(S)$. □

Proposition 6. *If $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$ is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2, then v is strictly δ-passive.*

Proof of Proposition 6. We prove the result for the two dynamics separately.

BNN dynamics. Consider the BNN dynamics of Example 1. We have that

$$v(\mu, \rho)(B) = \int_B \max\{0, \langle \rho, \delta_s\rangle - \langle \rho, \mu\rangle\} d\lambda(s) - \mu(B) \int_S \max\{0, \langle \rho, \delta_s\rangle - \langle \rho, \mu\rangle\} d\lambda(s)$$

for all $\mu \in \mathcal{P}(S)$, all $\rho \in C(S)$, and all $B \in \mathcal{B}(S)$. Define $\overline{\Sigma}\colon \mathcal{M}(S) \times C(S) \to \mathbb{R}$ and $\sigma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}$ by

$$\begin{aligned}\overline{\Sigma}(\mu, \rho) &= \frac{1}{2}\int_S \max\{0, \langle \rho, \delta_s\rangle - \langle \rho, \mu\rangle\}^2 d\lambda(s), \\ \sigma(\mu, \rho) &= \langle \rho, v(\mu, \rho)\rangle \int_S \max\{0, \langle \rho, \delta_s\rangle - \langle \rho, \mu\rangle\} d\lambda(s).\end{aligned}$$

Notice that $\overline{\Sigma}(\mu,\rho)$ and $\sigma(\mu,\rho)$ are finite for all $\mu \in \mathcal{M}(S)$ and all $\rho \in C(S)$, since $s \mapsto \max\{0, \langle \rho, \delta_s\rangle - \langle \rho, \mu\rangle\}$ and $s \mapsto \max\{0, \langle \rho, \delta_s\rangle - \langle \rho, \mu\rangle\}^2$ are continuous and S is compact. Also notice that $\overline{\Sigma}(\mu,\rho) \geq 0$ for all $\mu \in \mathcal{M}(S)$ and all $\rho \in C(S)$. Thus, we may define $\Sigma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}_+$ by the restriction of $\overline{\Sigma}$ to the domain $\mathcal{P}(S) \times C(S) \subseteq \mathcal{M}(S) \times C(S)$. We claim that σ and Σ are appropriate maps to prove the strict δ-passivity of v.

To this end, first note that $\mathcal{M}(S)$ is strongly open, $\overline{\Sigma}$ is weak-∞-continuous, $\overline{\Sigma}$ is Fréchet differentiable, $(\mu,\rho) \mapsto \partial_1\overline{\Sigma}(\mu,\rho)$ and $(\mu,\rho) \mapsto \partial_2\overline{\Sigma}(\mu,\rho)$ are weak-∞-continuous, and every partial Fréchet derivative $\partial_1\overline{\Sigma}(\mu,\rho)$ is weakly continuous. All that remains to prove are (5.1) with $w\colon (\mu,\eta) \mapsto \langle \eta, \mu\rangle$, (5.2), (5.3), and that $\sigma \geq 0$.

Let $\mu \in \mathcal{P}(S)$ and $\rho \in C(S)$. It holds that $\Sigma(\mu,\rho) = 0$ if and only if

$$\int_S \max\{0, \langle \rho, \delta_s\rangle - \langle \rho, \mu\rangle\}^2 d\lambda(s) = 0. \tag{A.2}$$

Since $s \mapsto \max\{0, \langle \rho, \delta_s\rangle - \langle \rho, \mu\rangle\}^2$ is a continuous real-valued function on S, the preimage $U \coloneqq \{s \in S : \max\{0, \langle \rho, \delta_s\rangle - \langle \rho, \mu\rangle\}^2 > 0\}$ is open. Therefore, if U is nonempty, it contains some $s' \in S$, and hence since λ has full support, s' must be an element of $\mathrm{supp}(\lambda)$, implying that $\lambda(U) > 0$. This in turn would imply that $\int_U \max\{0, \langle \rho, \delta_s\rangle - \langle \rho, \mu\rangle\}^2 d\lambda(s) > 0$ as the Lebesgue integral of a positive function over a set of positive measure is positive. However, this would contradict (A.2). Thus, $\Sigma(\mu,\rho) = 0$ if and only if

$$\max\{0, \langle \rho, \delta_s\rangle - \langle \rho, \mu\rangle\}^2 = 0 \text{ for all } s \in S,$$

which holds if and only if

$$\rho(s) \leq \langle \rho, \mu\rangle \text{ for all } s \in S. \tag{A.3}$$

It is clear that, if $\langle \rho, \nu\rangle \leq \langle \rho, \mu\rangle$ for all $\nu \in \mathcal{P}(S)$, then (A.3) holds. Conversely, if (A.3) holds, then $\langle \rho, \nu\rangle = \int_S \rho(s) d\nu(s) \leq \int_S \langle \rho, \mu\rangle\, d\nu(s) = \langle \rho, \mu\rangle$ for all $\nu \in \mathcal{P}(S)$, and thus by Nash stationarity of v (Proposition 5) we conclude that $\Sigma(\mu,\rho) = 0$ if and only if

$$v(\mu,\rho) = 0,$$

which proves (5.2).

Again let $\mu \in \mathcal{P}(S)$ and $\rho \in C(S)$. If $v(\mu,\rho) = 0$, then certainly $\sigma(\mu,\rho) = 0$ due to linearity of $\langle \rho, \cdot\rangle$. Notice that

$$\begin{aligned}
\langle \rho, v(\mu,\rho)\rangle &= \int_S \rho(s') d(v(\mu,\rho))(s') \\
&= \int_S \rho(s') \max\{0, \langle \rho, \delta_{s'}\rangle - \langle \rho, \mu\rangle\} d\lambda(s') \\
&\qquad - \int_S \max\{0, \langle \rho, \delta_s\rangle - \langle \rho, \mu\rangle\} d\lambda(s) \int_S \rho(s') d\mu(s') \\
&= \int_S \left(\rho(s') - \int_S \rho(\tilde{s}) d\mu(\tilde{s})\right) \max\{0, \langle \rho, \delta_{s'}\rangle - \langle \rho, \mu\rangle\} d\lambda(s') \\
&= \int_S (\langle \rho, \delta_{s'}\rangle - \langle \rho, \mu\rangle) \max\{0, \langle \rho, \delta_{s'}\rangle - \langle \rho, \mu\rangle\} d\lambda(s').
\end{aligned}$$

Notice that $(\langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle) \max\{0, \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle\} \geq 0$ for all $s' \in S$ and hence $\langle \rho, v(\mu, \rho) \rangle \geq 0$. Furthermore, by the usual arguments based on continuity and nonnegativity of the integrand together with full support of λ, we see that

$$\langle \rho, v(\mu, \rho) \rangle = 0$$

if and only if

$$\rho(s') = \langle \rho, \delta_{s'} \rangle \leq \langle \rho, \mu \rangle \ \text{ for all } s' \in S,$$

which, as shown above, holds true if and only if $v(\mu, \rho) = 0$. Furthermore, notice that by the same arguments,

$$\int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s) \geq 0,$$

with equality holding if and only if $v(\mu, \rho) = 0$. Thus,

$$\sigma(\mu, \rho) = \langle \rho, v(\mu, \rho) \rangle \int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s) \geq 0,$$

with equality holding if and only if $v(\mu, \rho) = 0$. This proves (5.3).

All that remains to be proven is (5.1) with $w\colon (\mu, \eta) \mapsto \langle \eta, \mu \rangle$. Let $\mu \in \mathcal{P}(S)$, $\rho \in C(S)$, and $\eta \in C(S)$. Define $\tau\colon \mathbb{R} \to \mathbb{R}_+$ by $\tau(r) = \max\{0, r\}^2$, so that $\tau'(r) = 2\max\{0, r\}$ and $\overline{\Sigma}(\mu, \rho) = \frac{1}{2} \int_S \tau(\langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle) d\lambda(s)$. Computing the first partial Fréchet derivative of $\overline{\Sigma}$ using the chain rule yields that

$$\begin{aligned}
\partial_1 \overline{\Sigma}(\mu, \rho) v(\mu, \rho) &= \frac{1}{2} \int_S \tau'(\langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle)(- \langle \rho, v(\mu, \rho) \rangle) d\lambda(s) \\
&= - \langle \rho, v(\mu, \rho) \rangle \int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s) \\
&= -\sigma(\mu, \rho).
\end{aligned}$$

Computing the second partial Fréchet derivative of $\overline{\Sigma}$ using the chain rule yields that

$$\begin{aligned}
\partial_2\overline{\Sigma}(\mu,\rho)\eta &= \frac{1}{2}\int_S \tau'(\langle\rho,\delta_s\rangle - \langle\rho,\mu\rangle)(\langle\eta,\delta_s\rangle - \langle\eta,\mu\rangle)d\lambda(s) \\
&= \int_S (\langle\eta,\delta_s\rangle - \langle\eta,\mu\rangle)\max\{0,\langle\rho,\delta_s\rangle - \langle\rho,\mu\rangle\}d\lambda(s) \\
&= \int_S \left(\eta(s) - \int_S \eta(\tilde{s})d\mu(\tilde{s})\right)\max\{0,\langle\rho,\delta_s\rangle - \langle\rho,\mu\rangle\}d\lambda(s) \\
&= \int_S \eta(s)\max\{0,\langle\rho,\delta_s\rangle - \langle\rho,\mu\rangle\}d\lambda(s) \\
&\quad - \int_S \eta(\tilde{s})d\mu(\tilde{s})\int_S \max\{0,\langle\rho,\delta_s\rangle - \langle\rho,\mu\rangle\}d\lambda(s) \\
&= \int_S \eta(s)d(v(\mu,\rho))(s) \\
&= \langle\eta, v(\mu,\rho)\rangle \\
&= w(v(\mu,\rho),\eta).
\end{aligned}$$

Thus, altogether we find that

$$\partial_1\overline{\Sigma}(\mu,\rho)v(\mu,\rho) + \partial_2\overline{\Sigma}(\mu,\rho)\eta = -\sigma(\mu,\rho) + w(v(\mu,\rho),\eta),$$

which shows that (5.1) holds and hence concludes the proof for the BNN dynamics.

Impartial pairwise comparison dynamics. Consider the impartial pairwise comparison dynamics of Example 2. We have that

$$v(\mu,\rho)(B) = \int_B\int_S \gamma(s,s',\rho)d\mu(s)d\lambda(s') - \int_B\int_S \gamma(s',s,\rho)d\lambda(s)d\mu(s')$$

for all $\mu \in \mathcal{P}(S)$, all $\rho \in C(S)$, and all $B \in \mathcal{B}(S)$. Since the pairwise comparison dynamics under consideration are impartial, it holds that for all $s' \in S$, there exists some continuous function $\phi_{s'} \colon \mathbb{R} \to \mathbb{R}_+$ such that

$$\gamma(s,s',\rho) = \phi_{s'}(\rho(s') - \rho(s))$$

for all $s \in S$ and all $\rho \in C(S)$. For all $s' \in S$, define $\tau_{s'} \colon \mathbb{R} \to \mathbb{R}_+$ by

$$\tau_{s'}(r) = \int_{[0,r]} \phi_{s'}(u)du,$$

where we see that $\tau_{s'}(r) = 0$ whenever $r < 0$, since we take $[0,r] = \emptyset$ in such cases. Notice that $\tau_{s'}$ is strictly increasing on $[0,\infty)$ since $\phi_{s'}(u) > 0$ for all $u > 0$: let $u > 0$, let $s, s' \in S$ be such that $s \neq s'$, and let $\rho \in C(S)$ be such that $\rho(s') - \rho(s) = u > 0$ (which exists by Urysohn's lemma and the fact that S is a metric space and hence normal), so that, by sign-preservation, we have that $\operatorname{sign}(\phi_{s'}(u)) = \operatorname{sign}(\phi_{s'}(\rho(s') - \rho(s))) = \operatorname{sign}(\gamma(s, s', \rho)) = \operatorname{sign}(\max\{0, \rho(s') - \rho(s)\}) = 1$. Define $\overline{\Sigma}\colon \mathcal{M}(S) \times C(S) \to \mathbb{R}$ and $\sigma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}$ by

$$\begin{aligned}\overline{\Sigma}(\mu, \rho) &= \int_S \int_S \tau_s(\rho(s) - \rho(s'))d\lambda(s)d\mu(s'),\\ \sigma(\mu, \rho) &= -\overline{\Sigma}(v(\mu, \rho), \rho).\end{aligned}$$

Notice that $\overline{\Sigma}(\mu, \rho)$ is finite for all $\mu \in \mathcal{M}(S)$ and all $\rho \in C(S)$ since $(s, s') \mapsto \tau_s(\rho(s) - \rho(s'))$ is continuous and S is compact. Also notice that $\overline{\Sigma}(\mu, \rho) \geq 0$ for all $\mu \in \mathcal{P}(S)$ and all $\rho \in C(S)$. Thus, we may define $\Sigma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}_+$ by the restriction of $\overline{\Sigma}$ to the domain $\mathcal{P}(S) \times C(S) \subseteq \mathcal{M}(S) \times C(S)$. We claim that σ and Σ are appropriate maps to prove the strict δ-passivity of v.

To this end, first note that $\mathcal{M}(S)$ is strongly open, $\overline{\Sigma}$ is weak-∞-continuous, $\overline{\Sigma}$ is Fréchet differentiable, $(\mu, \rho) \mapsto \partial_1 \overline{\Sigma}(\mu, \rho)$ and $(\mu, \rho) \mapsto \partial_2 \overline{\Sigma}(\mu, \rho)$ are weak-∞-continuous, and every partial Fréchet derivative $\partial_1 \overline{\Sigma}(\mu, \rho)$ is weakly continuous. All that remains to prove are (5.1) with $w\colon (\mu, \eta) \mapsto \langle \eta, \mu \rangle$, (5.2), (5.3), and that $\sigma \geq 0$.

Let $\mu \in \mathcal{P}(S)$ and $\rho \in C(S)$. It holds that $\Sigma(\mu, \rho) = 0$ if and only if

$$\int_S \int_S \tau_s(\rho(s) - \rho(s'))d\lambda(s)d\mu(s') = 0,$$

which holds if and only if

$$\tau_s(\rho(s) - \rho(s')) = 0 \text{ for all } s \in S \text{ and all } s' \in \operatorname{supp}(\mu),$$

since λ has full support, $s \mapsto \tau_s(\rho(s) - \rho(s'))$ is nonnegative and continuous for all $s' \in S$, and $s' \mapsto \int_S \tau_s(\rho(s) - \rho(s'))d\lambda(s)$ is nonnegative and continuous (which follows from compactness of S together with the dominated convergence theorem). Since, for all $s \in S$, it holds that τ_s is strictly increasing on $[0, \infty)$ and $\tau_s(0) = 0$, it must be that $\Sigma(\mu, \rho) = 0$ if and only if

$$\rho(s) \leq \rho(s') \text{ for all } s \in S \text{ and all } s' \in \operatorname{supp}(\mu).$$

Therefore, by Lemma 3, it holds that $\langle \rho, \nu \rangle \leq \langle \rho, \mu \rangle$ for all $\nu \in \mathcal{P}(S)$. Hence, by Nash stationarity of v (Proposition 5), it holds that $\Sigma(\mu, \rho) = 0$ if and only if

$$v(\mu, \rho) = 0,$$

which proves (5.2).

Again let $\mu \in \mathcal{P}(S)$ and $\rho \in C(S)$. If $v(\mu,\rho)=0$, then certainly $\sigma(\mu,\rho) = -\overline{\Sigma}(v(\mu,\rho),\rho) = 0$ due to linearity of $\overline{\Sigma}(\cdot,\rho)$. Writing out $\sigma(\mu,\rho)$, we find that

$$\begin{aligned}
\sigma(\mu,\rho) &= -\int_S\int_S \tau_s(\rho(s)-\rho(s'))d\lambda(s)d(v(\mu,\rho))(s') \\
&= -\int_S \left(\int_S \tau_s(\rho(s)-\rho(s'))d\lambda(s)\right)\left(\int_S \gamma(s,s',\rho)d\mu(s)\right)d\lambda(s') \\
&\quad + \int_S \left(\int_S \tau_s(\rho(s)-\rho(s'))d\lambda(s)\right)\left(\int_S \gamma(s',s,\rho)d\lambda(s)\right)d\mu(s') \\
&= \int_S\int_S \gamma(s',s,\rho)\int_S \left(\tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s')) - \tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s))\right)d\lambda(\tilde{s})d\lambda(s)d\mu(s').
\end{aligned}$$

For all $s,s' \in S$ such that $\rho(s) \le \rho(s')$, it holds by sign-preservation that $\operatorname{sign}(\gamma(s',s,\rho)) = \operatorname{sign}(\max\{0,\rho(s)-\rho(s')\}) = 0$, and therefore $\gamma(s',s,\rho)=0$ for all such s,s'. On the other hand, if $s,s' \in S$ are such that $\rho(s) > \rho(s')$, then $\operatorname{sign}(\gamma(s',s,\rho)) = \operatorname{sign}(\max\{0,\rho(s)-\rho(s')\}) = 1$, implying that $\gamma(s',s,\rho)>0$. Furthermore, in this case with $\rho(s)>\rho(s')$, we see that $\rho(\tilde{s})-\rho(s) < \rho(\tilde{s})-\rho(s')$ for all $\tilde{s}\in S$, and therefore $\tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s')) \ge \tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s))$ for all $\tilde{s}\in S$ by the fact that every $\tau_{\tilde{s}}$ is nondecreasing. Thus, we immediately see that

$$\sigma(\mu,\rho) \ge 0.$$

We furthermore see that if $\sigma(\mu,\rho)=0$, then

$$\gamma(s',s,\rho)\int_S \left(\tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s')) - \tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s))\right)d\lambda(\tilde{s}) = 0 \text{ for all } s\in S \text{ and all } s'\in\operatorname{supp}(\mu)$$

by the usual arguments based on continuity and nonnegativity of the integrand together with full support of λ. Thus, let $s\in S$ and $s'\in\operatorname{supp}(\mu)$. Either $\gamma(s',s,\rho)=0$, or $\int_S \left(\tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s')) - \tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s))\right)d\lambda(\tilde{s}) = 0$. In the former case, it must be that $\rho(s) \le \rho(s')$, for otherwise $\phi_s(\rho(s)-\rho(s'))>0$, which would contradict the fact that $\phi_s(\rho(s)-\rho(s')) = \gamma(s',s,\rho) = 0$. Suppose that the latter case holds. Then either $\rho(s)\le\rho(s')$ or $\rho(s)>\rho(s')$. If $\rho(s)>\rho(s')$, then, as argued above, we find that $\tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s')) - \tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s)) \ge 0$ for all $\tilde{s}\in S$, and hence by the usual arguments based on continuity and nonnegativity of the integrand together with the full support of λ, we conclude that $\tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s')) = \tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s))$ for all $\tilde{s}\in S$. In this case, by the fact that every $\tau_{\tilde{s}}$ is strictly increasing on $[0,\infty)$ and $\rho(s')\ne\rho(s)$, it must be the case that, for all $\tilde{s}\in S$, we have that $\rho(\tilde{s})-\rho(s')\le 0$ and $\rho(\tilde{s})-\rho(s)\le 0$. But these two inequalities cannot hold simultaneously, as they would imply that $\rho(s)\le\rho(s')$ and $\rho(s')\le\rho(s)$, which contradicts the fact that $\rho(s)>\rho(s')$ in the case under consideration. Hence, we conclude that, when $\sigma(\mu,\rho)=0$, it must hold that

$$\rho(s)\le\rho(s') \text{ for all } s\in S \text{ and all } s'\in\operatorname{supp}(\mu).$$

Thus, by Lemma 3, we find that $\langle \rho, \nu \rangle \leq \langle \rho, \mu \rangle$ for all $\nu \in \mathcal{P}(S)$, and therefore by Nash stationarity of v (Proposition 5), it holds that $v(\mu, \rho) = 0$ whenever $\sigma(\mu, \rho) = 0$. This proves (5.3).

All that remains to be proven is (5.1) with $w \colon (\mu, \eta) \mapsto \langle \eta, \mu \rangle$. Let $\mu \in \mathcal{P}(S)$, $\rho \in C(S)$, and $\eta \in C(S)$. Since $\overline{\Sigma}(\cdot, \rho)$ is linear, it is immediate that $D(\overline{\Sigma}(\cdot, \rho))(\mu) = \overline{\Sigma}(\cdot, \rho)$, which implies that

$$\partial_1 \overline{\Sigma}(\mu, \rho) v(\mu, \rho) = \overline{\Sigma}(v(\mu, \rho), \rho) = -\sigma(\mu, \rho).$$

Furthermore, computing the second partial Fréchet derivative of $\overline{\Sigma}$ using the chain rule yields that

$$\partial_2 \overline{\Sigma}(\mu, \rho)\eta = \int_S \int_S \tau_s'(\rho(s) - \rho(s'))(\eta(s) - \eta(s')) d\lambda(s) d\mu(s'),$$

where the derivatives of the functions $\tau_s \colon \mathbb{R} \to \mathbb{R}_+$ are computed via the fundamental theorem of calculus:

$$\tau_s'(r) = \frac{d}{dr} \int_{[0,r]} \phi_s(u) du = \phi_s(r).$$

By impartiality of the pairwise comparison dynamics under consideration, we find that

$$\begin{aligned}
\partial_2 \overline{\Sigma}(\mu, \rho)\eta &= \int_S \int_S \gamma(s', s, \rho)(\eta(s) - \eta(s')) d\lambda(s) d\mu(s') \\
&= \int_S \eta(s) \int_S \gamma(s', s, \rho) d\mu(s') d\lambda(s) - \int_S \eta(s') \int_S \gamma(s', s, \rho) d\lambda(s) d\mu(s') \\
&= \int_S \eta(s') \int_S \gamma(s, s', \rho) d\mu(s) d\lambda(s') - \int_S \eta(s') \int_S \gamma(s', s, \rho) d\lambda(s) d\mu(s') \\
&= \int_S \eta(s') d(v(\mu, \rho))(s') \\
&= \langle \eta, v(\mu, \rho) \rangle \\
&= w(v(\mu, \rho), \eta).
\end{aligned}$$

Thus, altogether we find that

$$\partial_1 \overline{\Sigma}(\mu, \rho) v(\mu, \rho) + \partial_2 \overline{\Sigma}(\mu, \rho)\eta = -\sigma(\mu, \rho) + w(v(\mu, \rho), \eta),$$

which shows that (5.1) holds and hence concludes the proof. □

Corollary 1. *Consider a game $F \colon \mathcal{P}(S) \to C(S)$, let $v \colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$, and assume that Assumption 1 holds. Furthermore, assume that Assumption 2 holds and that the extension $\overline{F}$ is continuously Fréchet differentiable. If F is monotone and v is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2, then* $\mathrm{NE}(F)$ *is weakly Lyapunov stable under the EDM* (4.1)*. If, additionally, Assumption 3 holds, then* $\mathrm{NE}(F)$ *is globally weakly attracting under the EDM* (4.1).

Proof of Corollary 1. Notice that weak Lyapunov stability of $\mathrm{NE}(F)$ follows immediately from Theorem 2 together with Proposition 5 and Proposition 6. Furthermore, global weak attraction of $\mathrm{NE}(F)$ under Assumption 3 follows by additionally noting that, for both the BNN dynamics and the impartial pairwise comparison dynamics, v satisfies the appropriate continuity conditions of Assumption 4 and v is $\|\cdot\|_{\mathrm{TV}}$-bounded on weak-∞ compact subsets of $\mathcal{P}(S)\times C(S)$ (the latter condition of which follows from the fact that $v(\mu,\rho)(B)\le 4\|\rho\|_\infty$ for all $\mu\in\mathcal{P}(S)$ and all $\rho\in C(S)$ for the BNN dynamics and that the conditional switch rate γ is assumed bounded for the pairwise comparison dynamics). $\square$

Lemma 5. *The map $\mu\mapsto D\overline{V}(\mu)v(\mu,F(\mu))$ is weakly continuous.*

Proof of Lemma 5. The result follows from a nearly identical analysis as in the proof of Lemma 4 with minor changes. In particular, it follows from the $\|\cdot\|_{\mathrm{TV}}$-boundedness of v on $\mathcal{P}(S)\times K$, the weak-∞-to-weak continuity of v, the weak-∞ continuity of u, the weak continuity of every $\partial_1\overline{\Sigma}(\mu,\rho)$ and every $\partial_1\overline{\Gamma}(\mu,\rho)$, and the weak-$\infty$ continuity of the maps $(\mu,\rho)\mapsto\partial_1\overline{\Sigma}(\mu,\rho)$, $(\mu,\rho)\mapsto\partial_2\overline{\Sigma}(\mu,\rho)$, $(\mu,\rho)\mapsto\partial_1\overline{\Gamma}(\mu,\rho)$, and $(\mu,\rho)\mapsto\partial_2\overline{\Gamma}(\mu,\rho)$. $\square$

Corollary 2. *Consider a weakly continuous game $F\colon\mathcal{P}(S)\to C(S)$, let $v\colon\mathcal{P}(S)\times C(S)\to T\mathcal{P}(S)$, and let $u\colon\mathcal{P}(S)\times C(S)\to C(S)$. Assume that Assumption 5 holds. Furthermore, assume that Assumption 6 holds with some compact $K\subseteq C(S)$ containing $F(\mathrm{NE}(F))$, and that Assumption 7 holds. If v is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2 and u is F-payoff stationary and strictly δ-antipassive, then*

$$P\coloneqq\{(\mu,\rho)\in\mathcal{P}(S)\times C(S): v(\mu,\rho)=0,\ u(\mu,\rho)=0\}$$

is a subset of $\mathrm{NE}(F)\times F(\mathrm{NE}(F))$ and is weak-∞-Lyapunov stable under the DPEDM (5.7) and weak-∞-attracting under the DPEDM (5.7) from every $(\mu_0,\rho_0)\in\mathcal{P}(S)\times K$.

Proof of Corollary 2. The proof follows analogously to that of Corollary 1. $\square$

Corollary 3. *Consider the continuous war of attrition game $F\colon\mathcal{P}(S)\to C(S)$ defined by (6.2). If $v\colon\mathcal{P}(S)\times C(S)\to T\mathcal{P}(S)$ is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2 and if Assumption 1 holds, then $\mathrm{NE}(F)$ is weakly Lyapunov stable and globally weakly attracting under the EDM (4.1).*

Proof of Corollary 3. This is immediate from Corollary 1 together with Theorem 4 and the fact that F satisfies all of the appropriate regularity conditions. $\square$

Appendix B

Supplementary Definitions and Results

B.1 Differentiation in Banach Spaces

Here, we review formal definitions for the notions of differentiability used throughout this thesis.

Definition 18. Let $(X, \|\cdot\|)$ be a Banach space. A mapping $x\colon [0,\infty) \to X$ is *differentiable at $t = 0$* if there exists $\dot{x}(0) \in X$ such that

$$\lim_{\epsilon \downarrow 0} \left\| \frac{x(\epsilon) - x(0)}{\epsilon} - \dot{x}(0) \right\| = 0,$$

and is *differentiable at $t \in (0,\infty)$* if there exists $\dot{x}(t) \in X$ such that

$$\lim_{\epsilon \to 0} \left\| \frac{x(t+\epsilon) - x(t)}{\epsilon} - \dot{x}(t) \right\| = 0,$$

and in either of these cases, $\dot{x}(t)$ is called the *derivative of x at t*. A mapping $x\colon [0,\infty) \to X$ that is differentiable at $t = 0$ and at every $t \in (0,\infty)$ is called *differentiable*.

Definition 19. A mapping $\mu\colon [0,\infty) \to \mathcal{M}(S)$ is *strongly differentiable at $t \in [0,\infty)$* if μ is differentiable at t with respect to the norm $\|\cdot\|_{\mathrm{TV}}$ on the Banach space $\mathcal{M}(S)$.

A strong derivative $\dot{\mu}(t)$ of μ at t, if it exists, is necessarily unique. The qualifier "strong" is used to emphasize that $\dot{\mu}(t)$ is defined in terms of convergence with respect to the strong topology. Note that if μ is strongly differentiable, then it is continuous with respect to the strong topology. In this case, since every weakly open set is strongly open, it must also be that μ is weakly continuous.

Definition 20. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $U \subseteq X$ be open. A mapping $f\colon U \to Y$ is called *Fréchet differentiable at* $x \in U$ if there exists a bounded linear operator $Df(x)\colon X \to Y$ such that

$$\lim_{\epsilon \to 0} \frac{\|f(x+\epsilon) - f(x) - Df(x)\epsilon\|_Y}{\|\epsilon\|_X} = 0,$$

and in this case $Df(x)$ is called the *Fréchet derivative of* f *at* x. A mapping $f\colon U \to Y$ that is Fréchet differentiable at every $x \in U$ is called *Fréchet differentiable*.

Throughout this work, we consider maps $f\colon U \to Y$ with $U \subseteq X$ where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ may be $(\mathbb{R}, |\cdot|)$, $(\mathcal{M}(S), \|\cdot\|_{\mathrm{TV}})$, or $(C(S), \|\cdot\|_\infty)$. Fréchet differentiability is always with respect to one of the norms $|\cdot|$, $\|\cdot\|_{\mathrm{TV}}$, or $\|\cdot\|_\infty$ in this work. The particular norm is clear from context. We remark that $\mu\colon [0,\infty) \to \mathcal{M}(S)$ is strongly differentiable on $(0,\infty)$ if and only if it is Fréchet differentiable on $(0,\infty)$. In this case, the strong derivative coincides with the Fréchet derivative under the identification of $\dot{\mu}(t)$ with the linear map $D\mu(t)\colon \mathbb{R} \to \mathcal{M}(S)$ defined by usual multiplication; $D\mu(t)\colon \epsilon \mapsto \dot{\mu}(t)\epsilon$. We similarly identify $D\mu(0)$ with $\dot{\mu}(0)$ when μ is strongly differentiable at 0. As is the case with strong derivatives, Fréchet derivatives are unique when they exist.

Partial Fréchet differentiation is defined as follows:

Definition 21. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $(Z, \|\cdot\|_Z)$ be Banach spaces and let $U \subseteq X$ and $V \subseteq Y$ be open. Let $(x,y) \in U \times Y$ and assume that $f(\cdot, y)\colon U \to Z$ and $f(x,\cdot)\colon V \to Z$ are Fréchet differentiable. The *first partial Fréchet derivative of* f *at* (x,y) is the bounded linear operator $\partial_1 f(x,y)\colon X \to Z$ defined by

$$\partial_1 f(x,y) = D(f(\cdot,y))(x).$$

Similarly, the *second partial Fréchet derivative of* f *at* (x,y) is the bounded linear operator $\partial_2 f(x,y)\colon Y \to Z$ defined by

$$\partial_2 f(x,y) = D(f(x,\cdot))(y).$$

B.2 Alternative Notions of Equilibrium in Population Games

Aside from the notion of a Nash equilibrium, another commonly used notion of static stability within evolutionary game theory is the following, due to Smith (1974).

Definition 22. A population state $\mu \in \mathcal{P}(S)$ is an *evolutionarily stable state (ESS) of the game* $F\colon \mathcal{P}(S) \to C(S)$ if, for all $\nu \in \mathcal{P}(S) \setminus \{\mu\}$, there exists $\epsilon(\nu) \in (0,1]$ such that for all $\eta \in (0, \epsilon(\nu)]$ it holds that

$$h^F_{\nu:\mu}(\eta) \coloneqq E_F(\nu, (1-\eta)\mu + \eta\nu) - E_F(\mu, (1-\eta)\mu + \eta\nu) < 0. \tag{B.1}$$

The function $h^F_{\nu:\mu}$ is called the *score function of* ν *against* μ, and the value $\epsilon(\nu)$ is called an *invasion barrier for* μ *against* ν.

Intuitively, a population state $\mu \in \mathcal{P}(S)$ is evolutionarily stable whenever the average mean payoff to a mutated population ν is lower given payoffs defined by a small mutation $(1-\eta)\mu+\eta\nu$ towards it, i.e., the population is not incentivized to continue evolving towards any mutant population given a small fluctuation towards it. Perhaps less commonly used is the following relaxation of evolutionary stability—yet, it becomes important in the study of monotone games to be defined later.

Definition 23. A population state $\mu \in \mathcal{P}(S)$ is a *neutrally stable state (NSS) of the game* $F\colon \mathcal{P}(S) \to C(S)$ if, for all $\nu \in \mathcal{P}(S)$, there exists $\epsilon(\nu) \in (0,1]$ such that for all $\eta \in (0,\epsilon(\nu)]$ it holds that

$$h^F_{\nu:\mu}(\eta) \leq 0.$$

Such a value $\epsilon(\nu)$ is called a *neutrality barrier for* μ *against* ν.

The following proposition shows that, under a mild condition, neutral stability (and hence evolutionary stability) is stronger than stability in the sense of Nash.

Proposition 7. *Let* $\mu \in \mathcal{P}(S)$ *be a NSS of the game* $F\colon \mathcal{P}(S) \to C(S)$*. If* $h^F_{\nu:\mu}$ *is right-continuous at* 0 *for all* $\nu \in \mathcal{P}(S)$*, then* μ *is a Nash equilibrium of the game* F*.*

Proof. Let $\mu \in \mathcal{P}(S)$ be a NSS of the game $F\colon \mathcal{P}(S) \to C(S)$. Suppose that $h^F_{\nu:\mu}$ is right-continuous at 0 for all $\nu \in \mathcal{P}(S)$. Let $\nu \in \mathcal{P}(S)$. Then, there exists $\epsilon(\nu) \in (0,1]$ such that

$$h^F_{\nu:\mu}(\eta) = E_F(\nu,(1-\eta)\mu+\eta\nu) - E_F(\mu,(1-\eta)\mu+\eta\nu) \leq 0$$

for all $\eta \in (0,\epsilon(\nu)]$. Thus, by the right-continuity of $h^F_{\nu:\mu}$, it holds that

$$E_F(\nu,\mu) - E_F(\mu,\mu) = h^F_{\nu:\mu}(0) = \lim_{\eta\downarrow 0} h^F_{\nu:\mu}(\eta) \leq 0.$$

Since ν is arbitrary, this proves the claim. □

Notice that $h^F_{\nu:\mu}$ is right-continuous at 0 for all $\mu,\nu \in \mathcal{P}(S)$ whenever F is weakly continuous. The converse of Proposition 7 is not true in general. However, it can be shown that a Nash equilibrium is an ESS (and hence a NSS) under additional conditions; see, e.g., Proposition 10 and Bomze and Pötscher (1989, Theorem 21).

Notice that the notions of ESS and NSS are local ones. They can be extended into global notions as follows.

Definition 24. A population state $\mu \in \mathcal{P}(S)$ is a *globally neutrally stable state (GNSS) of the game* $F\colon \mathcal{P}(S) \to C(S)$ if

$$E_F(\nu,\nu) \leq E_F(\mu,\nu) \tag{B.2}$$

for all $\nu \in \mathcal{P}(S)$. If, additionally, the inequality (B.2) holds strictly for all $\nu \in \mathcal{P}(S)\setminus\{\mu\}$, then μ is a *globally evolutionarily stable state (GESS) of the game* F.

As one should expect, every GNSS is a NSS, and every GESS is an ESS, as the following result shows.

Proposition 8. *Let $\mu \in \mathcal{P}(S)$. If μ is a GNSS of the game $F\colon \mathcal{P}(S) \to C(S)$, then it is a NSS of the game F. If μ is a GESS of the game F, then it is an ESS of the game F.*

Proof. Suppose that $\mu \in \mathcal{P}(S)$ is a GNSS of the game $F\colon \mathcal{P}(S) \to C(S)$. Let $\nu \in \mathcal{P}(S)$. Then, since μ is a GNSS of the game F, it holds that

$$E_F((1-\eta)\mu + \eta\nu, (1-\eta)\mu + \eta\nu) - E_F(\mu, (1-\eta)\mu + \eta\nu) \leq 0$$

for all $\eta \in (0, 1]$. By linearity of E_F in its first argument, we find that

$$\eta E_F(\nu, (1-\eta)\mu + \eta\nu) - \eta E_F(\mu, (1-\eta)\mu + \eta\nu) \leq 0$$

for all $\eta \in (0, 1]$. Dividing by η proves that μ is a NSS of the game F. The proof that μ being a GESS implies that μ is an ESS is identical as above with strict inqualities when considering $\nu \in \mathcal{P}(S) \setminus \{\mu\}$. □

If a GESS exists, it must necessarily be the unique Nash equilibrium under a mild regularity condition, as the following proposition shows. Hence, globally evolutionarily stable states are stable in a very strong sense.

Proposition 9. *Let $\mu \in \mathcal{P}(S)$ be a GESS of the game $F\colon \mathcal{P}(S) \to C(S)$, and suppose that $h^F_{\nu:\mu}$ is right-continuous at 0 for all $\nu \in \mathcal{P}(S)$. Then, it holds that* $\mathrm{NE}(F) = \{\mu\}$.

Proof. Since μ is a GESS of the game F, it holds that μ is a NSS of the game F, and therefore $\mu \in \mathrm{NE}(F)$ by Proposition 7, as $h^F_{\nu:\mu}$ is right-continuous at 0. For all $\nu \in \mathcal{P}(S) \setminus \{\mu\}$, it holds that $E_F(\nu, \nu) < E(\mu, \nu)$ since μ is a GESS of the game F, and therefore such ν are not Nash equilibria of the game F. This proves that indeed $\mathrm{NE}(F) = \{\mu\}$. □

Equilibria of Monotone Games

The following results show that the added structure of monotone games yields more information about the game's equilibria.

Proposition 10. *Suppose that the game $F\colon \mathcal{P}(S) \to C(S)$ is monotone. Then the following all hold:*

1. *Every Nash equilibrium of the game F is a GNSS of the game F.*
2. *Every strict Nash equilibrium of the game F is a GESS of the game F.*
3. *If F is strictly monotone, then every Nash equilibrium of the game F is a GESS of the game F.*

Proof. Let $\mu \in \mathcal{P}(S)$ be a Nash equilibrium of the game F. Then $\int_S F_\mu d\nu \leq \int_S F_\mu d\mu$ for all $\nu \in \mathcal{P}(S)$, so by monotonicity it holds that

$$\begin{aligned} E_F(\nu,\nu) - E_F(\mu,\nu) &= \int_S F_\nu d\nu - \int_S F_\nu d\mu \\ &= \int_S F_\mu(\nu - \mu) + \int_S (F_\nu - F_\mu) d(\nu - \mu) \\ &\leq 0 \end{aligned}$$

for all $\nu \in \mathcal{P}(S)$. Hence, μ is a GNSS of the game F. It is clear that if μ is a strict Nash equilibrium or if F is strictly monotone, then the above inequality becomes strict for $\nu \in \mathcal{P}(S) \setminus \{\mu\}$ and hence μ is a GESS of the game F in these cases. □

Proposition 10 shows that we can ensure a sort of "global evolutionary stability" for Nash equilibria in the case of monotone games, whereas in more general games Nash equilibria may only be "locally" neutrally or evolutionarily stable, or they may not be neutrally or evolutionarily stable at all.

Corollary 4. *Suppose that the game $F\colon \mathcal{P}(S) \to C(S)$ is monotone, let $\mu \in \mathrm{NE}(F)$, and assume that $h^F_{\nu:\mu}$ is right-continuous at 0 for all $\nu \in \mathcal{P}(S)$. If either μ is a strict Nash equilibrium of F or F is strictly monotone, then μ is the unique Nash equilibrium of the game F.*

Proof. This follows directly from Proposition 10 together with Proposition 9. □

Lemma 6. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$ and let $N \subseteq \mathcal{P}(S)$ be an arbitrary set of population states. Let $\mathfrak{S}^F_N \subseteq \mathcal{P}(S)$ denote the set of all population states $\mu \in \mathcal{P}(S)$ such that, for all $\nu \in N$, it holds that*

$$E_F(\nu,\nu) \leq E_F(\mu,\nu).$$

Then, it holds that $\mathfrak{S}^F_N$ is a convex set.

Proof. It holds that

$$\begin{aligned} \mathfrak{S}^F_N &= \{\mu \in \mathcal{P}(S) : E_F(\nu,\nu) \leq E_F(\mu,\nu) \text{ for all } \nu \in N\} \\ &= \bigcap_{\nu \in N} \{\mu \in \mathcal{P}(S) : E_F(\nu,\nu) \leq E_F(\mu,\nu)\}. \end{aligned}$$

Since E_F is linear in its first argument, the set $\{\mu \in \mathcal{P}(S) : E_F(\nu,\nu) \leq E_F(\mu,\nu)\}$ is convex for all $\nu \in N$, and therefore the set $\mathfrak{S}^F_N$, being the intersection of convex sets, is also a convex set. □

We now show in Proposition 11 that the set of Nash equilibria of a monotone game is a convex set under a mild regularity condition. The convexity of $\mathrm{NE}(F)$ rules out the case of isolated Nash equilibria. This result is similar to Hofbauer et al. (2009, Lemma 2), but allows for general nonlinear maps F (whereas their result is derived in the special case that $F(\mu)(s) = \int_S f(s,s')d\mu(s')$ for some function $f\colon S \times S \to \mathbb{R}$).

Proposition 11. *Suppose that the game $F\colon \mathcal{P}(S) \to C(S)$ is monotone. If $h^F_{\nu:\mu}$ is right-continuous at 0 for every GNSS $\mu \in \mathcal{P}(S)$ of the game F and for all $\nu \in \mathcal{P}(S)$, then $\mathrm{NE}(F)$ is a convex set.*

Proof. By Proposition 10, every Nash equilibrium of the game F is a GNSS of the game F, and by Proposition 7 every GNSS of the game F is a Nash equilibrium of the game F. Hence, the set of Nash equilibria of the game F equals the set of globally neutrally stable states of the game F, so $\mathrm{NE}(F) = \{\mu \in \mathcal{P}(S) : E_F(\nu,\nu) \leq E_F(\mu,\nu) \text{ for all } \nu \in \mathcal{P}(S)\}$. Applying Lemma 6 with $N = \mathcal{P}(S)$ proves the claim. □

B.3 Characteristics and Existence of Solutions to Evolutionary Dynamics

Since the population states of our evolutionary game are probability measures, we are primarily concerned with the case where the image of the mapping $\mu\colon [0,\infty) \to \mathcal{M}(S)$ is a subset of $\mathcal{P}(S)$ (so that the curve $t \mapsto \mu(t)$ evolves on the manifold of probability measures). In fact, for such maps, we can characterize their strong derivatives using the tangent space $T\mathcal{P}(S)$.

Proposition 12. *Let $\mu\colon [0,\infty) \to \mathcal{M}(S)$ be strongly differentiable. If $\mu([0,\infty)) \subseteq \mathcal{P}(S)$, then $\dot{\mu}(t) \in T\mathcal{P}(S)$ for all $t \in [0,\infty)$.*

Proof. Suppose that $\mu([0,\infty)) \subseteq \mathcal{P}(S)$. Let $t \in (0,\infty)$. Since $\frac{\mu(t+\epsilon)-\mu(t)}{\epsilon}$ converges strongly to $\dot{\mu}(t)$ as $\epsilon \to 0$, it also converges weakly to $\dot{\mu}(t)$ as $\epsilon \to 0$, so

$$\lim_{\epsilon\to 0} \int_S f d\left(\frac{\mu(t+\epsilon)-\mu(t)}{\epsilon}\right) = \int_S f d\dot{\mu}(t)$$

for all $f \in C(S)$. In particular, taking f to be the function that is identically 1 on S yields that

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(\mu(t+\epsilon)(S) - \mu(t)(S)\right) = \dot{\mu}(t)(S).$$

Since $\mu(t)$ and $\mu(t+\epsilon)$ are probability measures for all $\epsilon \in [-t,\infty)$, it holds that $\mu(t+\epsilon)(S) = \mu(t)(S) = 1$ for all such ϵ, and hence $\frac{1}{\epsilon}\left(\mu(t+\epsilon)(S) - \mu(t)(S)\right) = 0$ for all $\epsilon \in [-t,\infty) \setminus \{0\}$. Therefore, it must be that

$$\dot{\mu}(t)(S) = 0,$$

so indeed $\dot{\mu}(t) \in T\mathcal{P}(S)$. The case for $t = 0$ follows similarly. □

Remark 1. The proof of Proposition 12 shows that, upon fixing an arbitrary time t, the condition $\dot{\mu}(t) \in T\mathcal{P}(S)$ still holds under a weaker hypothesis. In particular, if μ is strongly differentiable and $t \in (0,\infty)$ is such that there exists $\epsilon \in (0,t)$ such that $\mu((t-\epsilon, t+\epsilon)) \subseteq \mathcal{P}(S)$, then $\dot{\mu}(t) \in T\mathcal{P}(S)$.

We now briefly discuss characteristics and existence of solutions to the EDM (4.1). Proposition 12 shows that if a solution $\mu\colon [0,\infty) \to \mathcal{P}(S)$ to the EDM (4.1) exists, then its strong derivative must satisfy $\dot{\mu}(t) = v(\mu(t), F(\mu(t))) \in T\mathcal{P}(S)$ for all $t \in [0,\infty)$, since the mapping's image satisfies $\mu([0,\infty)) \subseteq \mathcal{P}(S)$. The intuition in this case is that the curve μ, which remains in $\mathcal{P}(S)$ for all time, must necessarily have instantaneous velocity vectors that are "tangent" to $\mathcal{P}(S)$. This is analogous to the case where $S = \{1,2\}$ so that $\mathcal{P}(S)$ corresponds to the probability simplex $\{\mu \in \mathbb{R}^2_+ : \mu_1 + \mu_2 = 1\}$ in $\mathbb{R}^2$—in this setting it is geometrically obvious that a curve $\mu\colon [0,\infty) \to \mathcal{P}(S)$ must always have velocity vectors in $\{\nu \in \mathbb{R}^2 : \nu_1 + \nu_2 = 0\}$ that keep $\mu(t)$ on the probability simplex.

Natural questions to ask are when a solution to the EDM (4.1) exists, and when such a solution is unique. These questions have simple answers in the case that the EDM is defined on the entire Banach space $\mathcal{M}(S)$ (Zeidler, 1986, Corollary 3.9), but our restriction of solutions to the subset $\mathcal{P}(S)$ makes things more difficult. In the case that S is finite, Sandholm (2010, Theorem 4.4.1) shows that a unique solution exists when $V_F\colon \mathcal{P}(S) \to \mathcal{M}(S)$ defined by $V_F(\mu) = v(\mu, F(\mu))$ is Lipschitz continuous and satisfies that $V_F(\mu)$ is in the tangent cone of $\mathcal{P}(S)$ at μ for all $\mu \in \mathcal{P}(S)$. However, this proof cannot be directly generalized to the case where S is infinite, as it relies on the existence and uniqueness of closest point projections onto $\mathcal{P}(S)$ (which fails to hold due to non-uniqueness of solutions to $\inf_{\mu\in\mathcal{P}(S)} \|\mu - \nu\|_{\mathrm{TV}}$ for general $\nu \in \mathcal{M}(S)$, e.g., $\arg\min_{\mu\in\mathcal{P}(S)} \|\mu\|_{\mathrm{TV}} = \mathcal{P}(S)$). Despite these difficulties, some related existence and uniqueness conditions have been proven for differential equations defined on closed subsets of Banach spaces, albeit, they are reliant on technically cumbersome conditions (Martin, 1973). Since our work is focused on the development of dynamic stability conditions for general EDMs that do in fact possess solutions, we make use of the existence and uniqueness conditions granted by Assumption 1 throughout this thesis.

B.4 Dynamical Systems in Banach Spaces

Definition 25. Consider a Banach space X and a topology τ on X. Let $Y \subseteq X$, let $v\colon Y \to X$, and let $P \subseteq Y$ be τ-compact. A map $V\colon Y \to \mathbb{R}_+$ is a *global Lyapunov function for P under v* if it extends to a τ-continuous Fréchet differentiable map $\overline{V}\colon U \to \mathbb{R}$ defined on a norm-open set $U \subseteq X$ containing Y that satisfies the following conditions:

1. $\overline{V}(x) = 0$ for all $x \in P$.

2. $\overline{V}(x) > 0$ for all $x \in Y \setminus P$.

3. $D\overline{V}(x)v(x) \leq 0$ for all $x \in Y$.

If, additionally, the map $x \mapsto D\overline{V}(x)v(x)$ is τ-continuous and $D\overline{V}(x)v(x) < 0$ for all $x \in Y \setminus P$, then V is a *strict global Lyapunov function for P under v*.

Notice that the topology τ in Definition 25 need not coincide with the topology induced by the norm on X. Indeed, our dissipativity results for static feedback $\rho(t) = F(\mu(t))$ rely on taking $X = \mathcal{M}(S)$ with τ being the weak topology and $Y = \mathcal{P}(S)$.

Lemma 7. *Consider a Banach space X and a topology τ on X. Let $Y \subseteq X$, let $v\colon Y \to X$, and let $P \subseteq Y$ be τ-compact. If τ is weaker than the norm topology, Y is τ-compact, and there exists a global Lyapunov function for P under v, then P is τ-Lyapunov stable under v.*

Proof. Suppose that there exists a global Lyapunov function $V\colon Y \to \mathbb{R}_+$ for P under v, and let $\overline{V}\colon U \to \mathbb{R}$ be an appropriate extension as in Definition 25. Let $Q \subseteq Y$ be relatively τ-open and contain P. Then $Q = Y \cap O$ for some τ-open set $O \subseteq X$. Define $\partial_Y Q \coloneqq Y \cap \partial O$, where ∂O is the boundary of O in X with respect to τ. It holds that $\partial_Y Q$ is τ-compact since Y is τ-compact and ∂O is τ-closed. Therefore,

$$m \coloneqq \min_{x \in \partial_Y Q} \overline{V}(x)$$

exists, since $\overline{V}$ is τ-continuous. Notice that, since $\partial O \cap O = \emptyset$, it must be that $\partial_Y Q \cap Q = \emptyset$, and therefore $\partial_Y Q \cap P = \emptyset$. Hence, since $\overline{V}(x) > 0$ for all $x \in Y \setminus P$, it must be that $\overline{V}(x) > 0$ for all $x \in \partial_Y Q$ and thus $m > 0$.

Now, let

$$R = \{x \in Q : \overline{V}(x) \in (-\infty, m)\}.$$

Since $\overline{V}$ is τ-continuous and $(-\infty, m)$ is open, the preimage $\overline{V}^{-1}((-\infty, m))$ is τ-open, and hence $R = Q \cap \overline{V}^{-1}((-\infty, m)) = Y \cap O \cap \overline{V}^{-1}((0, m)) \subseteq Y$ is relatively τ-open. Furthermore, since $P \subseteq Q$ and $P \subseteq V^{-1}((-\infty, m))$ as $\overline{V}(x) = 0$ for all $x \in P$, it holds that $P \subseteq R \subseteq Y$. Let $x\colon [0,\infty) \to Y$ be a solution to the differential equation $\dot{x}(t) = v(x(t))$ with $x(0) = x_0 \in Y$. Suppose that $x_0 \in R$. Then, since the Fréchet derivative of real-valued functions on $\mathbb{R}$ recovers the usual derivative, we have that

$$\frac{d\overline{V} \circ x}{dt}(t)\epsilon = D(\overline{V} \circ x)(t)\epsilon = (D\overline{V}(x(t)) \circ Dx(t))\epsilon = D\overline{V}(x(t))(\epsilon \dot{x}(t)) = \epsilon D\overline{V}(x(t))v(x(t))$$

for all $t \in [0, \infty)$ and all $\epsilon \in \mathbb{R}$, where we have used the chain rule for Fréchet differentiation, linearity of Fréchet derivatives. Hence,

$$\frac{d\overline{V} \circ x}{dt}(t) = D\overline{V}(x(t))v(x(t)) \leq 0$$

for all $t \in [0, \infty)$. Since $\overline{V}$ is τ-continuous and x is τ-continuous since it is necessarily norm-continuous and τ is weaker than the norm topology, we may apply the mean value theorem to find that $\overline{V}(x(t)) \leq \overline{V}(x(0)) < m$ for all $t \in [0, \infty)$. Since $R \subseteq Q$, we conclude that $x(t) \in Q$ for all $t \in [0, \infty)$, so indeed P is τ-Lyapunov stable under v. □

Lemma 8. *Consider a Banach space X and a topology τ on X. Let $Y \subseteq X$, let $v\colon Y \to X$, and let $P \subseteq Y$ be τ-compact. Suppose that, for every $x_0 \in Y$, there exists a unique solution $x\colon [0,\infty) \to Y$ to the differential equation $\dot{x}(t) = v(x(t))$ with $x(0) = x_0$. If τ is weaker than the norm topology, Y is τ-compact, and there exists a strict global Lyapunov function for P under v, then P is globally τ-attracting under v.*

Proof. In this proof, we denote the complement of a subset $M \subseteq X$ by M^c.

Suppose that there exists a strict global Lyapunov function $V\colon Y \to \mathbb{R}_+$ for P under v, and let $\overline{V}\colon U \to \mathbb{R}$ be an appropriate extension as in Definition 25. Let $x_0 \in Y$ be arbitrary. Let $Q \subseteq Y$ be relatively τ-open and contain P, and let $x\colon [0,\infty) \to Y$ be the unique solution to the differential equation $\dot{x}(t) = v(x(t))$ with $x(0) = x_0$. It suffices to show that there exists $T \in [0,\infty)$ such that

$$x(t) \in Q \text{ for all } t \in [T,\infty). \tag{B.3}$$

Since V is a global Lyapunov function, Lemma 7 gives that P is τ-Lyapunov stable under v, which implies that there exists a relatively τ-open set $R \subseteq Y$ containing P such that $x(t) \in Q$ for all $t \in [0,\infty)$ whenever $x(0) \in R$. By time-invariance of the ordinary differential equation $\dot{x}(t) = v(x(t))$ with $x(0) = x_0$ and uniqueness of its solutions, if there exists $T \in [0,\infty)$ such that $x(T) \in R$, this implies that $x(t) \in Q$ for all $t \in [T,\infty)$. Thus, to prove (B.3), it suffices to prove that there exists $T \in [0,\infty)$ such that $x(T) \in R$.

For the sake of contradiction, suppose that $x(t) \notin R$ for all $t \in [0,\infty)$. Since R is relatively τ-open, $R = Y \cap O$ for some τ-open set $O \subseteq X$, and therefore $Y \setminus R = Y \cap (Y \cap O)^c = Y \cap (Y^c \cup O^c) = Y \cap O^c$ is τ-compact since O^c is τ-closed and Y is τ-compact. Hence,

$$m \coloneqq \max_{y \in Y \setminus R} D\overline{V}(y)v(y)$$

exists, since $y \mapsto D\overline{V}(y)v(y)$ is τ-continuous. Since $Y \setminus R \subseteq Y \setminus P$, it must hold that $m < 0$ as V is a strict global Lyapunov function. Furthermore, since $x(t) \in Y \setminus R$ for all $t \in [0,\infty)$, it holds that

$$\frac{d\overline{V} \circ x}{dt}(t) = D\overline{V}(x(t))v(x(t)) \leq m$$

for all $t \in [0,\infty)$. Since $\overline{V}$ is τ-continuous and x is τ-continuous since it is necessarily norm-continuous and τ is weaker than the norm topology, we may apply the mean value theorem to conclude that, for all $\tau \in (0,\infty)$, there exists $t \in (0,\tau)$ such that

$$\frac{\overline{V}(x(\tau)) - \overline{V}(x(0))}{\tau} = \frac{d\overline{V} \circ x}{dt}(t) \leq m,$$

and hence

$$\overline{V}(x(\tau)) \leq m\tau + \overline{V}(x(0))$$

for all $\tau \in (0,\infty)$. Since $m < 0$, $m\tau + \overline{V}(x(0)) \to -\infty$ as $\tau \to \infty$, which implies that there exists $\tau \in (0,\infty)$ such that $\overline{V}(x(\tau)) < 0$. Since, for such τ, it holds that $x(\tau) \in Y \setminus R \subseteq Y \setminus P$, this contradicts the property of the global Lyapunov function V that $\overline{V}(y) > 0$ for all

$y \in Y \setminus P$. Therefore, the supposition that $x(t) \notin R$ for all $t \in [0, \infty)$ is false, and we conclude that indeed there exists $T \in [0, \infty)$ such that $x(T) \in R$, which completes the proof. $\square$