Dissipativity Theory for Evolutionary Games on Infinite Strategy Sets

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Abstract

In this work, we consider evolutionary dynamics for population games in which players may have a continuum of strategies at their disposal. Models in this setting amount to infinite-dimensional differential equations evolving on the manifold of probability measures. Due to the technical difficulties in analyzing such models, prior works in this setting are restricted to studying specific evolutionary dynamics models on a case-by-case basis. Taking a broader approach, recent work by Arcak and Martins (2021) uses system-theoretic notions of dissipativity to derive sufficient conditions for general evolutionary games to converge to Nash equilibria, but such results are restricted to finite strategy sets. In this paper, we generalize dissipativity theory for evolutionary games to infinite strategy sets that are compact metric spaces. Our main result is a general sufficient condition for dynamic stability of Nash equilibria, and is modular in the sense that the pertinent conditions on the dynamics and the game's payoff structure can be verified independently. By specializing our theory to the class of monotone games, we recover as special cases the key stability results from Hofbauer et al. (2009, Theorem 3) and Cheung (2014, Theorem 4) for the Brown-von Neumann-Nash and impartial pairwise comparison dynamics. We also extend our theory to models with dynamic payoffs, further broadening the applicability of our framework over past works. We illustrate the scope of our theory using examples, including a continuous war of attrition game and a monotone game subject to smoothing dynamics.

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1 Introduction and Description of Main Results

In this section, we describe the problem and our results in a non-rigorous fashion. Consider an evolutionary game with strategies in some set S, and with payoffs defined by some function F, which encodes the payoff to each strategy when the game's population is at a certain state μ. We are interested in an evolutionary dynamics model of the form

$$\dot{\mu}(t) = v(\mu(t), F(\mu(t))), \tag{1}$$

which is a differential equation governing how the population state μ changes over time. The map v may be viewed as the vector field describing the population's dynamics, which is dependent on the current state of the population as well as an "input" $F(\mu(t))$. In other words, the payoff function F acts like a feedback controller, and steers the population dynamics according to how each strategy is currently paying off. *The general problem at hand is to determine when this dynamics model converges to a Nash equilibrium of the game.*

Inspired by Arcak and Martins (2021), we study the dynamic stability of this closed-loop system by independently studying properties of the system itself (i.e., the function v), and the control input (i.e., the payoff function F). The primary notion of interest is that of δ-dissipative systems.

Definition 1 (Informal)**.** The system v is *δ-dissipative with supply rate w* if the rate of change of its internally stored energy is less than or equal to the energy supply rate w. If the rate of change in stored energy is strictly less than the energy supply rate w, then v is *strictly δ-dissipative with supply rate w*.

Definition 1 is made formal in Definition 21. Intuitively, if the energy supply rate is negative, then the energy supplied to the system is decreasing. Thus, if the supply rate remains negative, we expect the internal energy of a (strictly) δ-dissipative system to decrease and for the state to converge to some "lowest energy" point, which is hopefully a Nash equilibrium. Our main result is that, under some technical stationarity, continuity, and differentiability conditions, this is indeed the case when the strategy set is compact and the payoff function F induces negative supply rates along the system trajectories.

Theorem 1 (Main Result, Informal)**.** *If the strategy set S is compact, the system v is δ-dissipative with supply rate w, and the payoff function F induces negative supply rates, then the set of Nash equilibria is dynamically stable under the evolutionary dynamics model* (1) *under appropriate stationarity, continuity, and differentiability conditions. If, additionally, v is strictly δ-dissipative with supply rate w, then the set of Nash equilibria is asymptotically stable (also under under appropriate related technical conditions).*

The payoff function F inducing negative supply rates can be intuitively thought of as the energy imparted by the feedback controller never increasing along system trajectories.

This write-up is organized as follows. In Section 2, we recall some mathematical definitions and results that are of use to us. Population games and evolutionary dynamics are formally introduced in Section 3 and Section 4, along with associated definitions and results for both static and dynamic stability. Our generalization of dissipativity theory to the infinite strategy set setting as well as our main result are presented in Section 5. Due to space constraints, we defer all proofs to our online technical report Anderson et al. (2023).

2 Mathematical Preliminaries

The set of nonnegative real numbers is denoted by $\mathbb{R}_+$. We define $\operatorname{sign}\colon \mathbb{R} \to \mathbb{R}$ by $\operatorname{sign}(x) = 1$ for $x > 0$, $\operatorname{sign}(x) = 0$ for $x = 0$, and $\operatorname{sign}(x) = -1$ for $x < 0$. The dual space of a normed vector space X (i.e., the space of bounded linear functionals on X) is denoted by X^*. Let S be a compact metric space. The

Banach space of bounded continuous real-valued functions on S endowed with the supremum norm is denoted by $(C_b(S), \|\cdot\|_\infty)$. Since S is compact, $C_b(S)$ equals the set of all continuous real-valued functions on S, denoted by $C(S)$. The Borel σ-algebra on S is denoted by $\mathcal{B}(S)$, and the Banach space of finite signed Borel measures on S endowed with the total variation norm is denoted by $(\mathcal{M}(S), \|\cdot\|_{\mathrm{TV}})$. Recall that $\|\mu\|_{\mathrm{TV}} \coloneqq |\mu|(S) = \sup_{f \text{ measurable}:\|f\|_\infty \leq 1} \int_S f d\mu$, where $|\mu|$ is the total variation measure of μ. The support of a measure $\mu \in \mathcal{M}(S)$ is denoted by $\mathrm{supp}(\mu)$.

We denote the set of probability measures on $(S, \mathcal{B}(S))$ by $\mathcal{P}(S) = \{\mu \in \mathcal{M}_+(S) : \mu(X) = 1\}$, where $\mathcal{M}_+(S) \subseteq \mathcal{M}(S)$ is the set of positive Borel measures on S. The Dirac measure at $s \in S$ is denoted by $\delta_s \in \mathcal{P}(S)$. We define the bilinear form $\langle \cdot, \cdot \rangle : C(S) \times \mathcal{M}(S) \to \mathbb{R}$ by $\langle f, \mu \rangle = \int_S f d\mu$, which is well-defined and satisfies $\left|\int_S f d\mu\right| \leq \|f\|_\infty \|\mu\|_{\mathrm{TV}}$ for all $f \in C(S)$ and all $\mu \in \mathcal{M}(S)$. Recall that $\mathcal{M}(S)$ is isometrically isomorphic to the dual space of $C(S)$ (Folland, 1999, Theorem 7.17), and therefore every element of $\mathcal{M}(S)$ can be uniquely identified with a bounded linear functional on $C(S)$. Thus, for all bounded linear functionals $I \in C(S)^*$, there exists a unique $\mu \in \mathcal{M}(S)$ such that $I(f) = \langle f, \mu \rangle$ for all $f \in C(S)$.

2.1 Topologies and Convergence of Measures

Two types of convergence in $\mathcal{M}(S)$ will be of use:

Definition 2. A sequence $\{\mu_n \in \mathcal{M}(S) : n \in \mathbb{N}\}$ *converges strongly to* $\mu \in \mathcal{M}(S)$ if $\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathrm{TV}} = 0$.

Definition 3. A sequence $\{\mu_n \in \mathcal{M}(S) : n \in \mathbb{N}\}$ *converges weakly to* $\mu \in \mathcal{M}(S)$ if $\lim_{n\to\infty} \int_S f d\mu_n = \int_S f d\mu$ for all $f \in C(S)$.

It is easy to see that strong convergence implies weak convergence. Strong and weak convergence induce topologies on $\mathcal{M}(S)$, termed the *strong topology* and *weak topology*, respectively. The weak topology is sometimes called the "narrow topology." Since S is compact, the weak topology coincides with the weak-$*$ topology on $\mathcal{M}(S) = C(S)^*$ (i.e., the weakest topology on $C(S)^*$ making every element $f \in C(S) \subseteq C(S)^{**}$ a continuous linear functional on $C(S)^*$). In functional analysis the term "weak topology" on $\mathcal{M}(S)$ would refer to the weakest topology on $\mathcal{M}(S)$ making every element of the dual space $\mathcal{M}(S)^* = C(S)^{**}$ continuous. We stick with our definitions to remain consistent with related works.

Recall the following well-known fact.

Lemma 1. *It holds that $\mathcal{P}(S)$ is weakly compact.*

2.2 Notions of Differentiability

We also need various notions of differentiability.

Definition 4. Let $(X, \|\cdot\|)$ be a Banach space. A mapping $x\colon [0,\infty) \to X$ is *differentiable at* $t = 0$ if there exists $\dot{x}(0) \in X$ such that

$$\lim_{\epsilon \downarrow 0} \left\| \frac{x(\epsilon) - x(0)}{\epsilon} - \dot{x}(0) \right\| = 0,$$

and is *differentiable at* $t \in (0,\infty)$ if there exists $\dot{x}(t) \in X$ such that

$$\lim_{\epsilon \to 0} \left\| \frac{x(t+\epsilon) - x(t)}{\epsilon} - \dot{x}(t) \right\| = 0,$$

and in either of these cases, $\dot{x}(t)$ is called the *derivative of* x *at* t. A mapping $x\colon [0,\infty) \to X$ that is differentiable at $t = 0$ and at every $t \in (0,\infty)$ is called *differentiable*.

Definition 5. A mapping $\mu\colon [0,\infty) \to \mathcal{M}(S)$ is *strongly differentiable at* $t \in [0,\infty)$ if μ is differentiable at t with respect to the norm $\|\cdot\|_{\mathrm{TV}}$ on the Banach space $\mathcal{M}(S)$.

It is easy to see that a strong derivative $\dot{\mu}(t)$ of μ at t, if it exists, is necessarily unique. The qualifier "strong" is used to emphasize that $\dot{\mu}(t)$ is defined in terms of convergence with respect to the strong topology. Note that if μ is strongly differentiable, then it is continuous with respect to the strong topology. In this case, since every weakly open set is strongly open, it must also be that μ is weakly continuous.

Definition 6. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $U \subseteq X$ be open. A mapping $f : U \to Y$ is called *Fréchet differentiable at* $x \in U$ if there exists a bounded linear operator $Df(x) : X \to Y$ such that

$$\lim_{\epsilon \to 0} \frac{\|f(x+\epsilon) - f(x) - Df(x)\epsilon\|_Y}{\|\epsilon\|_X} = 0,$$

and in this case $Df(x)$ is called the *Fréchet derivative of* f *at* x. A mapping $f : U \to Y$ that is Fréchet differentiable at every $x \in U$ is called *Fréchet differentiable.*

Throughout, we will consider maps $f : U \to Y$ with $U \subseteq X$ where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ may be $(\mathbb{R}, |\cdot|)$, $(\mathcal{M}(S), \|\cdot\|_{\mathrm{TV}})$, or $(C(S), \|\cdot\|_\infty)$. Fréchet differentiability will always be with respect to one of the norms $|\cdot|$, $\|\cdot\|_{\mathrm{TV}}$, or $\|\cdot\|_\infty$. The particular norm will be clear from context. We remark that $\mu : [0, \infty) \to \mathcal{M}(S)$ is strongly differentiable on $(0, \infty)$ if and only if it is Fréchet differentiable on $(0, \infty)$, and in this case the strong derivative coincides with the Fréchet derivative by identifying $\dot{\mu}(t)$ with the linear map $D\mu(t) : \mathbb{R} \to \mathcal{M}(S)$ defined by usual multiplication; $D\mu(t) : \epsilon \mapsto \dot{\mu}(t)\epsilon$. We similarly identify $D\mu(0)$ with $\dot{\mu}(0)$ when μ is strongly differentiable at 0. As is the case with strong derivatives, Fréchet derivatives are unique when they exist.

Definition 7. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $(Z, \|\cdot\|_Z)$ be Banach spaces and let $U \subseteq X$ and $V \subseteq Y$ be open. Let $(x, y) \in U \times Y$ and assume that $f(\cdot, y) : U \to Z$ and $f(x, \cdot) : V \to Z$ are Fréchet differentiable. The *first partial Fréchet derivative of* f *at* (x, y) is the bounded linear operator $\partial_1 f(x, y) : X \to Z$ defined by

$$\partial_1 f(x, y) = D(f(\cdot, y))(x).$$

The *second partial Fréchet derivative of* f *at* (x, y) is the bounded linear operator $\partial_2 f(x, y) : Y \to Z$ defined by

$$\partial_2 f(x, y) = D(f(x, \cdot))(y).$$

3 Population Games

We now move on to describing the game-theoretic aspects of our problem. The set S represents the (infinite) set of pure strategies of the game, and is hence called the *strategy set.* A *population state* is a distribution $\mu \in \mathcal{P}(S)$, which encodes how strategies in S are being employed across the game's population. Thus, $\mathcal{P}(S)$ is termed the *population state space.* To every population state $\mu \in \mathcal{P}(S)$ associates a *mean payoff function* $F_\mu \in C(S)$ such that $F_\mu(s)$ quantifies the average payoff to strategy s when the population is at state μ. We refer to the mapping $F : \mathcal{P}(S) \to C(S)$ defined by $F(\mu) = F_\mu$ as the *population game*, or simply the *game.* One of the primary quantities of interest when analyzing population games over infinite strategy sets is the *average mean payoff* $E_F(\nu, \mu) \in \mathbb{R}$ to a population state $\nu \in \mathcal{P}(S)$ relative to a population state $\mu \in \mathcal{P}(S)$, which is given by

$$E_F(\nu, \mu) := \langle F(\mu), \nu \rangle = \int_S F_\mu d\nu.$$

3.1 Static Notions of Stability

We now move on to introducing and characterizing equilibrium states and stable states of the game F. The stability concepts studied in this section are static in the sense that they do not depend on any dynamical behavior endowed to the game. However, these static notions of stability will still play an important role in our study of evolutionary dynamics to come.

Definition 8. A population state $\mu \in \mathcal{P}(S)$ is a *Nash equilibrium of the game* $F : \mathcal{P}(S) \to C(S)$ if

$$E_F(\nu, \mu) \leq E_F(\mu, \mu) \tag{2}$$

for all $\nu \in \mathcal{P}(S)$. If, additionally, the inequality (2) holds strictly for all $\nu \in \mathcal{P}(S) \setminus \{\mu\}$, then μ is a *strict Nash equilibrium of the game* F. The set of all Nash equilibria of the game F is denoted by $\mathrm{NE}(F)$.

Intuitively, a population state $\mu \in \mathcal{P}(S)$ is a Nash equilibrium if the average mean payoff to the population cannot be increased by moving to any other state $\nu \in \mathcal{P}(S)$ given the current payoffs defined by F_μ. The following result gives equivalent characterizations of Nash equilibria, which are sometimes used as alternative definitions in the literature (see, e.g., Hofbauer et al. (2009),Cheung (2014)).

Proposition 1. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$, and let $\mu \in \mathcal{P}(S)$. The following are equivalent:*

1. *μ is a Nash equilibrium of the game F.*
2. *$E_F(\delta_s, \mu) \leq E_F(\mu, \mu)$ for all $s \in S$.*
3. *$F_\mu(s) \leq F_\mu(s')$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$.*

Proposition 1 shows that at a Nash equilibrium state $\mu \in \mathcal{P}(S)$, every strategy $s' \in S$ that is in use (meaning that $s' \in \operatorname{supp}(\mu)$) must have maximal average payoff $F_\mu(s')$ compared to all other possible strategies $s \in S$. From the contrapositive viewpoint, this shows that a strategy $s' \in S$ whose average payoff $F_\mu(s')$ is strictly less than that of some other strategy will not be employed at a Nash equilibrium state μ.

Another commonly used notion of static stability within evolutionary game theory is the following, due to Smith (1974).

Definition 9. A population state $\mu \in \mathcal{P}(S)$ is an *evolutionarily stable state (ESS) of the game $F\colon \mathcal{P}(S) \to C(S)$* if, for all $\nu \in \mathcal{P}(S) \setminus \{\mu\}$, there exists $\epsilon(\nu) \in (0, 1]$ such that for all $\eta \in (0, \epsilon(\nu)]$ it holds that

$$h^F_{\nu:\mu}(\eta) \coloneqq E_F(\nu, (1-\eta)\mu + \eta\nu) - E_F(\mu, (1-\eta)\mu + \eta\nu) < 0. \tag{3}$$

The function $h^F_{\nu:\mu}$ is called the *score function of ν against μ*, and the value $\epsilon(\nu)$ is called an *invasion barrier for μ against ν.*

Intuitively, a population state $\mu \in \mathcal{P}(S)$ is evolutionarily stable whenever the average mean payoff to a mutated populated ν is lower given payoffs defined by a small mutation $(1-\eta)\mu + \eta\nu$ towards it, i.e., the population is not incentivized to continue evolving towards any mutant population given a small fluctation towards it. Perhaps less commonly used is the following relaxation of evolutionary stability—yet, it becomes important in the study of monotone games to be defined later.

Definition 10. A population state $\mu \in \mathcal{P}(S)$ is a *neutrally stable state (NSS) of the game $F\colon \mathcal{P}(S) \to C(S)$* if, for all $\nu \in \mathcal{P}(S)$, there exists $\epsilon(\nu) \in (0, 1]$ such that for all $\eta \in (0, \epsilon(\nu)]$ it holds that

$$h^F_{\nu:\mu}(\eta) \leq 0.$$

Such a value $\epsilon(\nu)$ is called a *neutrality barrier for μ against ν.*

The following proposition shows that, under a mild condition, neutral stability (and hence evolutionary stability) is stronger than stability in the sense of Nash.

Proposition 2. *Let $\mu \in \mathcal{P}(S)$ be a NSS of the game $F\colon \mathcal{P}(S) \to C(S)$. If $h^F_{\nu:\mu}$ is right-continuous at* 0 *for all $\nu \in \mathcal{P}(S)$, then μ is a Nash equilibrium of the game F.*

Notice that $h^F_{\nu:\mu}$ is right-continuous at 0 for all $\mu, \nu \in \mathcal{P}(S)$ whenever F is weakly continuous. The converse of Proposition 2 is not true in general. However, it can be shown that a Nash equilibrium is an ESS (and hence a NSS) under an additional technical condition; see Bomze and Pötscher (1989, Theorem 21).

Notice that the notions of ESS and NSS are local ones. They can be extended into global notions as follows.

Definition 11. A population state $\mu \in \mathcal{P}(S)$ is a *globally neutrally stable state (GNSS) of the game $F\colon \mathcal{P}(S) \to C(S)$* if

$$E_F(\nu, \nu) \leq E_F(\mu, \nu) \tag{4}$$

for all $\nu \in \mathcal{P}(S)$. If, additionally, the inequality (4) holds strictly for all $\nu \in \mathcal{P}(S) \setminus \{\mu\}$, then μ is a *globally evolutionarily stable state (GESS) of the game F.*

As one should expect, every GNSS is a NSS, and every GESS is an ESS, as the following result shows.

Proposition 3. *Let $\mu \in \mathcal{P}(S)$. If μ is a GNSS of the game $F\colon \mathcal{P}(S) \to C(S)$, then it is a NSS of the game F. If μ is a GESS of the game F, then it is an ESS of the game F.*

If a GESS exists, it must necessarily be the unique Nash equilibrium under a mild regularity condition, as the following proposition shows. Hence, globally evolutionarily stable states are stable in a very strong sense.

Proposition 4. *Let $\mu \in \mathcal{P}(S)$ be a GESS of the game $F\colon \mathcal{P}(S) \to C(S)$, and suppose that $h^F_{\nu:\mu}$ is right-continuous at 0 for all $\nu \in \mathcal{P}(S)$. Then, it holds that $\mathrm{NE}(F) = \{\mu\}$.*

In the more general case where there may be more than one Nash equilibrium, the following result still unveils advantageous topological characteristics of $\mathrm{NE}(F)$.

Proposition 5. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$. If $\theta_\nu\colon \mathcal{P}(S) \to \mathbb{R}$ defined by $\theta_\nu(\mu) = E_F(\nu,\mu) - E_F(\mu,\mu)$ is weakly continuous for all $\nu \in \mathcal{P}(S)$, then $\mathrm{NE}(F)$ is weakly compact.*

Together with Proposition 5, the following result shows that $\mathrm{NE}(F)$ is weakly compact whenever the game F is weakly continuous.

Proposition 6. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$. If F is weakly continuous, then $\theta_\nu\colon \mathcal{P}(S) \to \mathbb{R}$ defined by $\theta_\nu(\mu) = E_F(\nu,\mu) - E_F(\mu,\mu)$ is weakly continuous for all $\nu \in \mathcal{P}(S)$.*

4 Evolutionary Dynamics

In this section, we endow the population game F with dynamical behavior. Such dynamics are used to model the evolutionary aspects of a population playing out a game, wherein players revise their strategies over time according to the game's current payoff profile.

Definition 12. Consider a game $F\colon \mathcal{P}(S) \to C(S)$. Let $\mu_0 \in \mathcal{P}(S)$ and let $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$. The differential equation

$$\begin{aligned} \dot\mu(t) &= v(\mu(t), \rho(t)), \\ \rho(t) &= F(\mu(t)), \\ \mu(0) &= \mu_0, \end{aligned} \tag{5}$$

is called an *evolutionary dynamics model (EDM)*. The measure μ_0 is called the *initial state* and the mapping v is called the *dynamics map*. A strongly differentiable mapping $\mu\colon [0,\infty) \to \mathcal{P}(S)$ satisfying (5) is called a *solution to the EDM*.

We emphasize that, although the overall EDM (5) defines an autonomous system, the nonautonomous dynamics term $\dot\mu(t) = v(\mu(t), \rho(t))$ may be studied in isolation from the feedback term $\rho(t) = F(\mu(t))$. In particular, this viewpoint lends itself towards control theoretic analyses, where the dynamics map v defines the system to be controlled, and the game F defines the feedback controller. This approach was proposed in Fox and Shamma (2013) and further studied in Arcak and Martins (2021) as a means to derive general stability results based on the idea that interconnections of energy-dissipating systems results in a closed-loop system that is dynamically stable. This allows one to prove stability of the overall evolutionary dynamics model by studying the dissipativity properties of the (nonautonomous) system and input in a modular fashion. To the best of our knowledge, our work is the first to generalize this modular dissipativity approach to evolutionary games with infinite strategy sets—prior works on infinite strategy sets primarily prove stability in a closed-loop black-box fashion on a case-by-case basis, e.g., Oechssler and Riedel (2001); Hingu et al. (2020) for replicator dynamics, Hofbauer et al. (2009) for Brown-von Neumann-Nash dynamics, and Cheung (2014) for pairwise comparison dynamics, as well as the references therein and subsequent works.

Before moving on to our main results in Section 5, we give examples of some of the most commonly studied evolutionary dynamics models, and also formalize the notions of dynamic stability to be considered.

Example 1. Let $\lambda \in \mathcal{P}(S)$ be a fixed reference probability measure with full support. The *Brown-von Neumann-Nash (BNN) dynamics* are given by the EDM (5) with closed-loop dynamics defined by

$$v(\mu, F(\mu))(B) = \int_B \sigma_+(s,\mu) d\lambda(s) - \mu(B) \int_S \sigma_+(s,\mu) d\lambda(s)$$

for all $B \in \mathcal{B}(S)$, where

$$\sigma_+(s,\mu) = \max\{0, E_F(\delta_s, \mu) - E_F(\mu,\mu)\}$$

is the excess average mean payoff to population state $\delta_s \in \mathcal{P}(S)$ relative to the population state $\mu \in \mathcal{P}(S)$, and where, for all $\mu \in \mathcal{P}(S)$, the associated mean payoff function takes the form

$$F_\mu(s) = \int_S f(s,s') d\mu(s')$$

for some bounded measurable function $f\colon S \times S \to \mathbb{R}$. In this case, it is easy to see that the BNN dynamics may be written in the interconnected form (5) with the dynamics map given by

$$v(\mu,\rho)(B) = \int_B \max\{0, \langle \rho, \delta_s\rangle - \langle \rho, \mu\rangle\} d\lambda(s) - \mu(B) \int_S \max\{0, \langle \rho, \delta_s\rangle - \langle \rho, \mu\rangle\} d\lambda(s).$$

See Hofbauer et al. (2009) for a thorough study on the BNN dynamics over infinite strategy sets.

Example 2. Let λ be a fixed reference measure as in Example 1. Furthermore, let $\gamma\colon S \times S \times C(S) \to \mathbb{R}_+$ be a continuous and bounded map, termed the *conditional switch rate*, such that $\gamma(s,s',\rho)$ encodes the rate at which players switch from strategy $s \in S$ to strategy $s' \in S$ whenever the strategies' payoffs are described by the function $\rho \in C(S)$. Assume that the conditional switch rate satisfies *sign-preservation*, given by

$$\operatorname{sign}(\gamma(s,s',\rho)) = \operatorname{sign}(\max\{0, \rho(s') - \rho(s)\})$$

for all $s, s' \in S$ and all $\rho \in C(S)$. Sign-preservation ensures that the conditional switch rate from strategy s to strategy s' is positive if and only if s' has higher payoff than s according to the function ρ. The *pairwise comparison dynamics* are given by the EDM (5) with

$$v(\mu,\rho)(B) = \int_S \int_B \gamma(s,s',\rho) d\lambda(s') d\mu(s) - \int_S \int_B \gamma(s',s,\rho) d\mu(s') d\lambda(s)$$

for all $B \in \mathcal{B}(S)$. Notice that the nonautonomous portion of the dynamics, defined by this dynamics map v, is entirely determined by the conditional switch rate γ.

When γ takes the form $\gamma(s,s',\rho) = \max\{0, \rho(s') - \rho(s)\}$, the pairwise comparison dynamics reduce to the famous *Smith dynamics*, introduced in Smith (1984). If, for all $s' \in S$, there exists some continuous function $\phi_{s'}\colon \mathbb{R} \to \mathbb{R}_+$ such that the conditional switch rate satisfies

$$\gamma(s,s',\rho) = \phi_{s'}(\rho(s') - \rho(s))$$

for all $s \in S$ and all $\rho \in C(S)$, then the pairwise comparison dynamics are said to be *impartial*. Note that the Smith dynamics are impartial. See Cheung (2014) for a thorough study on the pairwise comparison dynamics over infinite strategy sets.

Since the population states of our evolutionary game are probability measures, we will be primarily concerned with the case where the image of the mapping μ is a subset of $\mathcal{P}(S)$ (so that the curve $t \mapsto \mu(t)$ evolves on the manifold of probability measures). In fact, for such maps, we can characterize their strong derivatives using the notion of a tangent space.

Definition 13. The *tangent space of* $\mathcal{P}(S)$ is defined to be the set

$$T\mathcal{P}(S) \coloneqq \{\nu \in \mathcal{M}(S) : \nu(S) = 0\}.$$

Obviously, $T\mathcal{P}(S)$ is a linear subspace of $\mathcal{M}(S)$.

Proposition 7. *Let $\mu\colon [0,\infty) \to \mathcal{M}(S)$ be strongly differentiable. If $\mu([0,\infty)) \subseteq \mathcal{P}(S)$, then $\dot{\mu}(t) \in T\mathcal{P}(S)$ for all $t \in [0,\infty)$.*

Remark 1. The proof of Proposition 7, found in our online technical report Anderson et al. (2023), shows that, upon fixing an arbitrary time t, the condition $\dot{\mu}(t) \in T\mathcal{P}(S)$ still holds under a weaker hypothesis. In particular, if μ is strongly differentiable and $t \in (0,\infty)$ is such that there exists $\epsilon \in (0,t)$ such that $\mu((t-\epsilon, t+\epsilon)) \subseteq \mathcal{P}(S)$, then $\dot{\mu}(t) \in T\mathcal{P}(S)$.

We now briefly discuss characteristics and existence of solutions to the EDM (5). Proposition 7 shows that if a solution $\mu\colon [0,\infty) \to \mathcal{P}(S)$ to the EDM (5) exists, then its strong derivative must satisfy $\dot{\mu}(t) = v(\mu(t), F(\mu(t))) \in T\mathcal{P}(S)$ for all $t \in [0,\infty)$, since the mapping's image satisfies $\mu([0,\infty)) \subseteq \mathcal{P}(S)$. The intuition in this case is that the curve μ, which remains in $\mathcal{P}(S)$ for all time, must necessarily have instantaneous velocity vectors that are "tangent" to $\mathcal{P}(S)$. This is analogous to the case where $S = \{1,2\}$ so that $\mathcal{P}(S)$ corresponds to the probability simplex $\{\mu \in \mathbb{R}^2_+ : \mu_1 + \mu_2 = 1\}$ in $\mathbb{R}^2$—in this setting it is geometrically obvious that a curve $\mu\colon [0,\infty) \to \mathcal{P}(S)$ must always have velocity vectors in $\{\nu \in \mathbb{R}^2 : \nu_1 + \nu_2 = 0\}$ that keep $\mu(t)$ on the probability simplex.

Natural questions to ask are when a solution to the EDM (5) exists, and when such a solution is unique. These questions have simple answers in the case that the EDM is defined on the entire Banach space $\mathcal{M}(S)$ (Zeidler, 1986, Corollary 3.9), but our restriction of solutions to the subset $\mathcal{P}(S)$ makes things more difficult. In the case that S is finite, Sandholm (2010, Theorem 4.4.1) shows that a unique solution exists when $V_F\colon \mathcal{P}(S) \to \mathcal{M}(S)$ defined by $V_F(\mu) = v(\mu, F(\mu))$ is Lipschitz continuous and satisfies that $V_F(\mu)$ is in the tangent cone of $\mathcal{P}(S)$ at μ for all $\mu \in \mathcal{P}(S)$. However, this proof cannot be directly generalized to the case where S is infinite, as it relies on the existence and uniqueness of closest point projections onto $\mathcal{P}(S)$ (which fails to hold due to non-uniqueness of solutions to $\inf_{\mu\in\mathcal{P}(S)} \|\mu - \nu\|_{\mathrm{TV}}$ for general $\nu \in \mathcal{M}(S)$, e.g., $\arg\min_{\mu\in\mathcal{P}(S)} \|\mu\|_{\mathrm{TV}} = \mathcal{P}(S)$). Despite these difficulties, some related existence and uniqueness conditions have been proven for differential equations defined on closed subsets of Banach spaces, albeit, they are reliant on technically cumbersome conditions (Martin, 1973). Since our work is focused on the development of dynamic stability conditions for general EDMs that do in fact possess solutions, we will make use of the following existence and uniqueness assumption throughout the remainder of the paper.

Assumption 1. Consider a dynamics map $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$. For every initial state $\mu_0 \in \mathcal{P}(S)$, there exists a unique solution to the EDM (5) with initial state μ_0 and dynamics map v.

We recall that Assumption 1 holds for the BNN dynamics of Example 1 (Hofbauer et al., 2009, Theorem 1), and also holds for the pairwise comparison dynamics of Example 2 (Cheung, 2014, Theorem 1) under some mild regularity conditions on F. It is also easy to see that, for these dynamics, $v(\mu,\rho)(S) = 0$ for all $\mu \in \mathcal{P}(S)$ and all $\rho \in C(S)$, and hence the codomain of these dynamics maps can be taken to be $T\mathcal{P}(S)$.

4.1 Dynamic Notions of Stability

We now formally introduce the notions of dynamic stability with which we are concerned.

Definition 14. A population state $\mu \in \mathcal{P}(S)$ is a *rest point of the EDM* (5) if $v(\mu, F(\mu)) = 0$.

The following condition, which is solely a property of the nonautonomous dynamics defined by the dynamics map v, ensures that the rest points and Nash equilibria coincide for the EDM under the feedback interconnection (5).

Definition 15. A map $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$ is *Nash stationary* if, for all $\mu \in \mathcal{P}(S)$ and all $\rho \in C(S)$, it holds that

$$v(\mu,\rho) = 0$$

if and only if

$$\langle \rho, \nu \rangle \le \langle \rho, \mu \rangle \text{ for all } \nu \in \mathcal{P}(S).$$

Proposition 8. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$ and let $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$. If v is Nash stationary, then the set of rest points of the EDM (5) equals $\mathrm{NE}(F)$.*

Proposition 8 shows that if an evolutionary game's population state converges to a rest point under the EDM with a Nash stationary dynamics map, then the population state converges to a Nash equilibrium. In other words, Nash stationarity ensures that all dynamically stable rest points have game-theoretic importance. We now recall that the popular BNN dynamics and pairwise comparison dynamics both satisfy Nash stationarity.[1]

Proposition 9 (Hofbauer et al., 2009; Cheung, 2014)**.** *If $v\colon \mathcal{P}(S)\times C(S)\to\mathcal{M}(S)$ is the dynamics map for either the BNN dynamics of Example 1 or the pairwise comparison dynamics of Example 2, then v is Nash stationary.*

Although a population state being at a rest point ensures that the population's distribution of strategies remains constant for all time, the definition of rest point does not itself come equipped with an adequate notion of stability. For this, we turn to the classical definitions of Lyapunov stability and attraction. Since ultimately we are interested in convergence of a game's dynamics to *some* Nash equilibrium, we deal with such definitions in the sense of sets. We begin with the definitions for general Banach spaces.

Definition 16. Consider a Banach space X and a topology τ on X. Let $Y\subseteq X$ and let $v\colon Y\to X$. A τ-compact set $P\subseteq Y$ is *τ-Lyapunov stable under v* if, for all relatively τ-open sets $Q\subseteq Y$ containing P, there exists a relatively τ-open set $R\subseteq Y$ containing P such that every solution $x\colon[0,\infty)\to Y$ to the differential equation $\dot{x}(t)=v(x(t))$ with $x(0)=x_0\in Y$ satisfies $x(t)\in Q$ for all $t\in[0,\infty)$ whenever $x(0)\in R$.

Definition 17. Consider a Banach space X and a topology τ on X. Let $Y\subseteq X$ and let $v\colon Y\to X$. A τ-compact set $P\subseteq Y$ is *τ-attracting under v from $x_0\in Y$* if, for all relatively τ-open sets $Q\subseteq Y$ containing P and for all solutions $x\colon[0,\infty)\to Y$ to the differential equation $\dot{x}(t)=v(x(t))$ with $x(0)=x_0$, there exists $T\in[0,\infty)$ such that

$$x(t)\in Q \text{ for all } t\in[T,\infty).$$

The τ-compact set P is *globally τ-attracting under v* if it is τ-attracting under v from x_0 for all $x_0\in Y$.

We now specialize the above stability definitions to the setting of evolutionary dynamics.

Definition 18. Consider a game $F\colon\mathcal{P}(S)\to C(S)$ and let $v\colon\mathcal{P}(S)\times C(S)\to\mathcal{M}(S)$. A weakly compact set $P\subseteq\mathcal{P}(S)$ is *weakly Lyapunov stable under the EDM* (5) if P is τ-Lyapunov stable under $\mu\mapsto v(\mu,F(\mu))$ according to definition Definition 16 with $X=\mathcal{M}(S)$, $Y=\mathcal{P}(S)$, and τ being the weak topology.

Definition 19. Consider a game $F\colon\mathcal{P}(S)\to C(S)$ and let $v\colon\mathcal{P}(S)\times C(S)\to\mathcal{M}(S)$. A weakly compact set $P\subseteq\mathcal{P}(S)$ is *weakly attracting under the EDM* (5) *from initial state $\mu_0\in\mathcal{P}(S)$* if P is τ-attracting under $\mu\mapsto v(\mu,F(\mu))$ from μ_0 according to Definition 17 with $X=\mathcal{M}(S)$, $Y=\mathcal{P}(S)$, and τ being the weak topology. The weakly compact set P is *globally weakly attracting under the EDM* (5) if it is weakly attracting under the EDM (5) with initial state μ_0 for all $\mu_0\in\mathcal{P}(S)$.

Definition 18 and Definition 19 are equivalent to the dynamic stability notions introduced in Cheung (2014), which are defined in terms of the Prokhorov metric on $\mathcal{P}(S)$ that metrizes the weak topology.

5 Dissipativity Theory

We now move on to introducing notions of dissipativity to characterize the dynamic stability of Nash equilibria.

Definition 20. Consider $\mathcal{M}(S)$ endowed with the weak topology and $C(S)$ endowed with its usual topology induced by $\|\cdot\|_\infty$. We call the corresponding product topology on $\mathcal{M}(S)\times C(S)$ the *weak-∞ topology.*

Definition 21. A map $v\colon\mathcal{P}(S)\times C(S)\to\mathcal{M}(S)$ is *δ-dissipative with supply rate $w\colon\mathcal{M}(S)\times C(S)\to\mathbb{R}$* if there exist $\sigma\colon\mathcal{P}(S)\times C(S)\to\mathbb{R}_+$ and $\Sigma\colon\mathcal{P}(S)\times C(S)\to\mathbb{R}_+$ that extends to a map $\overline{\Sigma}\colon U\times C(S)\to\mathbb{R}$ with strongly open $U\subseteq\mathcal{M}(S)$ containing $\mathcal{P}(S)$, such that the following conditions hold:

[1]Technically, our definition of Nash stationarity, which is a property of the dynamics map v viewed as a nonautonomous system, is slightly different than the definitions used in Hofbauer et al. (2009) and Cheung (2014), which are properties of the closed-loop interconnection (5). For self-containedness, we prove Proposition 9 in the online technical report Anderson et al. (2023) using our definition.

1. $\overline{\Sigma}$ is weak-∞-continuous.

2. $\overline{\Sigma}$ is Fréchet differentiable.

3. For all $\mu \in \mathcal{P}(S)$, all $\rho \in C(S)$, and all $\eta \in C(S)$, it holds that

$$\partial_1 \overline{\Sigma}(\mu, \rho) v(\mu, \rho) + \partial_2 \overline{\Sigma}(\mu, \rho) \eta \leq -\sigma(\mu, \rho) + w(v(\mu, \rho), \eta). \tag{6}$$

4. For all $\mu \in \mathcal{P}(S)$ and all $\rho \in C(S)$, it holds that

$$\Sigma(\mu, \rho) = 0 \text{ if and only if } v(\mu, \rho) = 0. \tag{7}$$

If, additionally, it holds that $(\mu, \rho) \mapsto \partial_1 \overline{\Sigma}(\mu, \rho)$ and $(\mu, \rho) \mapsto \partial_2 \overline{\Sigma}(\mu, \rho)$ are weak-∞-continuous, every partial Fréchet derivative $\partial_1 \overline{\Sigma}(\mu, \rho)$ is weakly continuous, and

$$\sigma(\mu, \rho) = 0 \text{ if and only if } v(\mu, \rho) = 0 \tag{8}$$

for all $\mu \in \mathcal{P}(S)$ and all $\rho \in C(S)$, then v is *strictly δ-dissipative with supply rate w*.

Note that the "δ" in the definition of δ-dissipative is short for "differentially;" it does not refer to a quantity δ. Also note that δ-dissipativity is solely a property of the nonautonomous dynamics defined by the dynamics map v. As mentioned in Example 2, the dynamics map v for the pairwise comparison dynamics is entirely determined by some conditional switch rate function $\gamma\colon S \times S \times C(S) \to \mathbb{R}_+$, and therefore δ-dissipativity may be viewed as a property of the conditional switch rate function in such a setting.

We also remark that (6), (7), and (8) are the "dynamically meaningful" conditions for δ-dissipativity; the continuity and differentiability conditions in Definition 21 are technical regularity conditions. These regularity conditions are needed to rigorously prove our main result, in addition to some regularity conditions on the game F and the dynamics map v under consideration. We formalize the latter conditions as follows.

Assumption 2. Consider a game $F\colon \mathcal{P}(S) \to C(S)$. The following all hold:

1. F is weakly continuous.

2. F extends to a weakly continuous Fréchet differentiable map $\overline{F}\colon U' \to C(S)$ defined on a strongly open set $U' \subseteq \mathcal{M}(S)$ containing $\mathcal{P}(S)$.

Assumption 3. Consider a game $F\colon \mathcal{P}(S) \to C(S)$ that satisfies Assumption 2. It holds that $D\overline{F}$ and every Fréchet derivative $D\overline{F}(\mu)$ are weakly continuous.

Assumption 4. Consider a dynamics map $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$. It holds that v is $\|\cdot\|_{\mathrm{TV}}$-bounded on weak-∞ compact subsets of $\mathcal{P}(S) \times C(S)$, and is continuous with respect to the weak-∞ topology on its domain and the weak topology on its codomain.

We now present our main result, which shows that, when the nonautonomous portion of the dynamics is Nash stationary and δ-dissipative, and when the feedback portion of the dynamics satisfies an appropriate supply rate inequality, the interconnected closed-loop evolutionary dynamics model is stable.

Theorem 2 (Main Result)**.** *Consider a game $F\colon \mathcal{P}(S) \to C(S)$, let $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$, and assume that Assumption 1 and Assumption 2 both hold. If v is Nash stationary and δ-dissipative with supply rate $w\colon \mathcal{M}(S) \times C(S) \to \mathbb{R}$ and it holds that*

$$w\left(\nu, D\overline{F}(\mu)\nu\right) \leq 0 \text{ for all } \mu \in \mathcal{P}(S) \text{ and all } \nu \in T\mathcal{P}(S), \tag{9}$$

then $\mathrm{NE}(F)$ *is weakly Lyapunov stable under the EDM* (5)*. If, additionally, Assumption 3 and Assumption 4 both hold and v is strictly δ-dissipative, then* $\mathrm{NE}(F)$ *is globally weakly attracting under the EDM* (5)*.*

It is easy to see that Theorem 2 generalizes the first main result in Arcak and Martins (2021), i.e., Theorem 2 recovers Theorem 1 in Arcak and Martins (2021) when S is finite. We will see in Section 5.1 that Theorem 2 also recovers other recent stability results for special types of games.

Let us now discuss the hypotheses of Theorem 2. Although all of the hypotheses of Theorem 2 are used in its proof, it is important to clarify which hypotheses are contextually meaningful ones and which are merely technical regularity conditions. The most pertinent hypotheses are those of Nash stationarity, δ-dissipativity, and (9), which serve to quantitatively characterize the informal descriptions in Definition 1 and Theorem 1. In particular, we may view the main result Theorem 2 as asserting that, under regularity conditions, a Nash stationary and δ-dissipative evolutionary dynamics model with payoff function satisfying the supply rate inequality (9) has dynamically stable Nash equilibria over compact strategy sets. On the other hand, the continuity and differentiability requirements can be viewed as regularity conditions ensuring the applicability of Lyapunov theory. The $\|\cdot\|_{\mathrm{TV}}$-boundedness of v on weak-∞ compacts sets can be replaced by v being continuous with respect to the weak-∞ topology on its domain and the strong topology on its codomain, although this may be a strong assumption in general. The compactness of the strategy set S, although a technical condition needed for the use of Lyapunov theory, is also an important qualitative requirement in our context of games, as it avoids cases where there exists a "hidden Nash equilibrium" at a probability measure with support at strategies on the boundary of S or "at infinity" that are inaccessible by the players. In other words, the compactness of S ensures that the dynamics always move the population state towards distributions of strategies that are actually available to the players.

5.1 Specialization to Monotone Games

In this section, we consider the special class of "monotone games," which are sometimes also referred to as "stable games," "contractive games," and "negative semidefinite games" in the literature. For a thorough analysis of monotone games over finite strategy sets, see Hofbauer and Sandholm (2009), and for works considering monotone games with an infinite number of strategies, see Hofbauer et al. (2009); Cheung (2014); Lahkar and Riedel (2015); Lahkar et al. (2022).

Definition 22. A game $F\colon \mathcal{P}(S) \to C(S)$ is *monotone* if

$$\langle F(\mu) - F(\nu), \mu - \nu \rangle \leq 0 \tag{10}$$

for all $\mu, \nu \in \mathcal{P}(S)$. If, additionally, the inequality (10) holds strictly for all $\mu, \nu \in \mathcal{P}(S)$ such that $\mu \neq \nu$, then F is *strictly monotone*.

Many games in practice are monotone, e.g., random matching in two-player symmetric zero-sum games (Cheung, 2014, Example 4), contests (Hofbauer et al., 2009, Example 5), and the War of Attrition (Hofbauer and Sandholm, 2009, Example 2.4). The following results show that the added structure of monotone games yields more information about the game's equilibria.

Proposition 10. *Suppose that the game $F\colon \mathcal{P}(S) \to C(S)$ is monotone. Then the following all hold:*

1. *Every Nash equilibrium of the game F is a GNSS of the game F.*
2. *Every strict Nash equilibrium of the game F is a GESS of the game F.*
3. *If F is strictly monotone, then every Nash equilibrium of the game F is a GESS of the game F.*

Proposition 10 shows that we can ensure a sort of "global evolutionary stability" for Nash equilibria in the case of monotone games, whereas in more general games Nash equilibria may only be "locally" neutrally or evolutionarily stable.

Corollary 1. *Every strict Nash equilibrium $\mu \in \mathcal{P}(S)$ of a monotone game $F\colon \mathcal{P}(S) \to C(S)$ with $h^F_{\nu:\mu}$ right-continuous at 0 for all $\nu \in \mathcal{P}(S)$ is necessarily the unique Nash equilibrium of the game F. Every Nash equilibrium $\mu \in \mathcal{P}(S)$ of a strictly monotone game $F\colon \mathcal{P}(S) \to C(S)$ with $h^F_{\nu:\mu}$ right-continuous at 0 for all $\nu \in \mathcal{P}(S)$ is necessarily unique.*

We now show in Proposition 11 that the set of Nash equilibria of a monotone game is a convex set under a mild regularity condition. The convexity of $\mathrm{NE}(F)$ rules out the case of isolated Nash equilibria. This result is similar to Hofbauer et al. (2009, Lemma 2), but allows for general nonlinear maps F (whereas their result is derived in the special case that $F(\mu)(s) = \int_S f(s,s')d\mu(s')$ for some function $f\colon S \times S \to \mathbb{R}$).

Proposition 11. *Suppose that the game $F\colon \mathcal{P}(S) \to C(S)$ is monotone. Then, if $h^F_{\nu:\mu}$ is right-continuous at 0 for every GNSS $\mu \in \mathcal{P}(S)$ of the game F and for all $\nu \in \mathcal{P}(S)$, then $\mathrm{NE}(F)$ is a convex set.*

The following notion is closely related to that of monotonicity, as we will see in Lemma 2, and will serve as the link between monotonicity and the supply rate inequality (9).

Definition 23. A game $F\colon \mathcal{P}(S) \to C(S)$ that extends to a continuously Fréchet differentiable map $\overline{F}\colon U' \to C(S)$ defined on a strongly open set $U' \subseteq \mathcal{M}(S)$ containing $\mathcal{P}(S)$ is said to *satisfy self-defeating externalities* if

$$\langle D\overline{F}(\mu)\nu, \nu \rangle \leq 0 \text{ for all } \mu \in \mathcal{P}(S) \text{ and all } \nu \in T\mathcal{P}(S).$$

Lemma 2 (Cheung, 2014, Lemma 3)**.** *Consider a game $F\colon \mathcal{P}(S) \to C(S)$ that extends to a continuously Fréchet differentiable map $\overline{F}\colon U' \to C(S)$ defined on a strongly open set $U' \subseteq \mathcal{M}(S)$ containing $\mathcal{P}(S)$. It holds that F is monotone if and only if F satisfies self-defeating externalities.*

We now show that our general dissipativity theory can be applied to monotone games to recover recent stability results in the literature. We start with the following specialization of δ-dissipativity, which generalizes the notion of δ-passivity introduced in Fox and Shamma (2013) for the finite strategy set setting.

Definition 24. A map $v\colon \mathcal{P}(S)\times C(S) \to \mathcal{M}(S)$ is *δ-passive* if it is δ-dissipative with supply rate $w\colon (\mu,\eta) \mapsto \langle \eta, \mu \rangle = \int_S \eta d\mu$. The map v is *strictly δ-passive* if it is strictly δ-dissipative with such supply rate w.

As is the case with δ-dissipativity, we see that δ-passivity is solely a property of the nonautonomous portion of the evolutionary dynamics defined by the dynamics map v. We remark that δ-passivity is common in practice, as the following result shows.

Proposition 12. *If $v\colon \mathcal{P}(S)\times C(S) \to \mathcal{M}(S)$ is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2, then v is strictly δ-passive.*

The proof of Proposition 12 relies on generalizing and combining the proof techniques of Fox and Shamma (2013, Theorem 4.5), Hofbauer et al. (2009, Theorem 3), and Cheung (2014, Theorem 4). We write the proof in full detail in our online technical report Anderson et al. (2023).

Finally, we give our reduction of Theorem 2 to the case of monotone games.

Theorem 3. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$, let $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$, and assume that Assumption 1 holds. Furthermore, assume that Assumption 2 holds and that the extension $\overline{F}$ is continuously Fréchet differentiable. If v is Nash stationary, v is δ-passive, and F is monotone, then $\mathrm{NE}(F)$ is weakly Lyapunov stable under the EDM (5). If, additionally, Assumption 3 and Assumption 4 both hold and v is strictly δ-passive, then $\mathrm{NE}(F)$ is globally weakly attracting under the EDM (5).*

Notice the modularity of Theorem 3: to prove stability of the interconnected EDM (5), we may analyze the Nash stationarity and δ-passivity of the nonautonomous portion of the dynamics defined by the dynamics map v independently from the monotonicity of the system's feedback defined by the game F. This allows for the direct proof of stability for the entire class of monotone games F given some dynamics map v that is known to be Nash stationary and δ-passive. For example, our unifying result in Theorem 3 together with Proposition 12 recovers the key stability results for BNN dynamics and impartial pairwise comparison dynamics over infinite strategy sets, proven in Hofbauer et al. (2009, Theorem 3) and Cheung (2014, Theorem 4), respectively. This recovery is formally stated below.

Corollary 2. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$, let $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$, and assume Assumption 1 holds. Furthermore, assume that Assumption 2 holds and that the extension $\overline{F}$ is continuously Fréchet differentiable. If F is monotone and v is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2, then $\mathrm{NE}(F)$ is weakly Lyapunov stable under the EDM (5). If, additionally, Assumption 3 holds, then $\mathrm{NE}(F)$ is globally weakly attracting under the EDM (5).*

5.2 Extension to Dynamic Payoff Models

In this section, we consider the case where, instead of static payoff feedback given by $\rho(t) = F(\mu(t))$, as in the EDM (5), the payoff itself has dynamics. In doing so, we will consider the derivatives $\dot{\rho}(t)$ of a payoff $\rho\colon [0,\infty) \to C(S)$ (see Definition 4). Since $C(S)$ is a Banach space, it holds that $\dot{\rho}(t) \in C(S)$ whenever it exists.

Definition 25. Let $\mu_0 \in \mathcal{P}(S)$, $\rho_0 \in C(S)$, $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$, and $u\colon \mathcal{P}(S) \times C(S) \to C(S)$. The differential equation

$$\begin{aligned} \dot{\mu}(t) &= v(\mu(t), \rho(t)), \\ \dot{\rho}(t) &= u(\mu(t), \rho(t)), \\ \mu(0) &= \mu_0, \\ \rho(0) &= \rho_0, \end{aligned} \tag{11}$$

is called a *dynamic payoff evolutionary dynamics model (DPEDM).* The measure μ_0 is called the *initial state*, the function ρ_0 is called the *initial payoff*, the mapping v is called the *dynamics map*, and the mapping u is called the *payoff map*.

Similar to the case for general EDMs, we will assume that unique solutions to the DPEDM (11) exist.

Assumption 5. Consider a dynamics map $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$ and a payoff map $u\colon \mathcal{P}(S) \times C(S) \to C(S)$. For every initial state $\mu_0 \in \mathcal{P}(S)$ and initial payoff $\rho_0 \in C(S)$, there exists a unique solution (μ, ρ) to the DPEDM (11).

For games $F\colon \mathcal{P}(S) \to C(S)$ that extend to a Fréchet differentiable map $\overline{F}\colon U' \to C(S)$ defined on a strongly open set $U' \subseteq \mathcal{M}(S)$ containing $\mathcal{P}(S)$, the EDM (5) is a special case of the DPEDM (11) with $u\colon (\mu, \rho) \mapsto D\overline{F}(\mu)v(\mu,\rho)$. Since the payoff map in a DPEDM is no longer defined by a static game, our supply rate inequality (9) and notions of monotonicity are no longer applicable when characterizing the "energy supplied" to the population by the payoff. Instead, we turn to notions of "antidissipativity." The following definition extends such notions from those introduced in Fox and Shamma (2013) for finite strategy sets to the setting of infinite S.

Definition 26. A map $u\colon \mathcal{P}(S) \times C(S) \to C(S)$ is *δ-antidissipative with supply rate $\tilde{w}\colon \mathcal{M}(S) \times C(S) \to \mathbb{R}$* if there exist $\gamma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}_+$ and $\Gamma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}_+$ that extends to a map $\overline{\Gamma}\colon \tilde{U} \times C(S) \to \mathbb{R}$ with strongly open $\tilde{U} \subseteq \mathcal{M}(S)$ containing $\mathcal{P}(S)$, such that the following conditions hold:

1. $\overline{\Gamma}$ is weak-∞-continuous.
2. $\overline{\Gamma}$ is Fréchet differentiable.
3. For all strongly differentiable $\mu\colon [0,\infty) \to \mathcal{P}(S)$, all $\rho_0 \in C(S)$, all solutions $\rho\colon [0,\infty) \to C(S)$ to the differential equation $\dot{\rho}(t) = u(\mu(t), \rho(t))$ with $\rho(0) = \rho_0$, and all $t \in [0,\infty)$, it holds that
$$\partial_1\overline{\Gamma}(\mu(t),\rho(t))\dot{\mu}(t) + \partial_2\overline{\Gamma}(\mu(t),\rho(t))u(\mu(t),\rho(t)) \le -\gamma(\mu(t),\rho(t)) - \tilde{w}(\dot{\mu}(t), u(\mu(t),\rho(t))). \tag{12}$$
4. For all $\mu \in \mathcal{P}(S)$ and all $\rho \in C(S)$, it holds that
$$\Gamma(\mu,\rho) = 0 \text{ if and only if } u(\mu,\rho) = 0. \tag{13}$$

If, additionally, it holds that $(\mu,\rho) \mapsto \partial_1\overline{\Gamma}(\mu,\rho)$ and $(\mu,\rho) \mapsto \partial_2\overline{\Gamma}(\mu,\rho)$ are weak-∞-continuous, every partial Fréchet derivative $\partial_1\overline{\Gamma}(\mu,\rho)$ is weakly continuous, and

$$\gamma(\mu,\rho) = 0 \text{ if and only if } u(\mu,\rho) = 0 \tag{14}$$

for all $\mu \in \mathcal{P}(S)$ and all $\rho \in C(S)$, then u is *strictly δ-antidissipative with supply rate $\tilde{w}$.*

Notice that δ-antidissipativity is a property solely of a payoff map u, and not of any particular dynamics map v. One may intuitively think of δ-antidissipativity with supply rate $\tilde{w}$ as δ-dissipativity with supply rate $-\tilde{w}$, albeit the notions are defined for maps with different codomains. We may also define a similar notion that is analogous to δ-passivity.

Definition 27. A map $u\colon \mathcal{P}(S) \times C(S) \to C(S)$ is *δ-antipassive* if it is δ-antidissipative with supply rate $\tilde{w}\colon (\mu, \eta) \mapsto \langle \eta, \mu \rangle = \int_S \eta d\mu$. The map u is *strictly δ-antipassive* if it is strictly δ-antidissipative with such supply rate $\tilde{w}$.

Fox and Shamma (2013, Theorem 4.3) show that every monotone game over a finite strategy set induces δ-antipassive payoff dynamics,[2] so δ-antipassivity may be viewed as a generalization of monotonicity to the dynamic payoff setting. Before moving on to our generalization of Theorem 2 to the setting of DPEDMs, we remark that Definition 25 does not immediately come equipped with any notion of a game, and hence has no inherent game-theoretic notion of equilibria. The following definition serves to link dynamic payoffs to games, namely, by ensuring that payoffs represent a valid static game once they stop evolving.

Definition 28. Consider a game $F\colon \mathcal{P}(S) \to C(S)$. A map $u\colon \mathcal{P}(S) \times C(S) \to C(S)$ is *F-payoff stationary* if, for all $\mu \in \mathcal{P}(S)$ and all $\rho \in C(S)$,

$$u(\mu, \rho) = 0$$

implies that

$$\rho = F(\mu).$$

As was the case in the static payoff setting, we need some technical regularity conditions in order to apply Lyapunov theory. We list these below, and then state our main result for DPEDMs.

Assumption 6. Consider a dynamics map $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$ and a payoff map $u\colon \mathcal{P}(S) \times C(S) \to C(S)$. There exists a compact set $K \subseteq C(S)$ such that, for all initial states $\mu_0 \in \mathcal{P}(S)$ and all initial payoffs $\rho_0 \in K$, the solution (μ, ρ) to the DPEDM (11) satisfies $\rho(t) \in K$ for all $t \in [0, \infty)$.

Assumption 6 can be viewed as a "positive invariance" condition on the payoff dynamics. Such an assumption on the bounded evolutions of the payoffs is standard in related works (c.f., Kara and Martins 2023) and is necessary to employ Lyapunov theory.

Assumption 7. The game $F\colon \mathcal{P}(S) \to C(S)$ is weakly continuous.

Assumption 8. The dynamics map $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$ is continuous with respect to the weak-∞ topology on its domain and the weak topology on its codomain. Furthermore, the payoff map $u\colon \mathcal{P}(S) \times C(S) \to C(S)$ is weak-∞-continuous.

Theorem 4. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$, let $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$, and let $u\colon \mathcal{P}(S) \times C(S) \to C(S)$. Assume that Assumption 5 holds and that Assumption 6 holds with some compact $K \subseteq C(S)$ containing $F(\mathrm{NE}(F))$. If v is Nash stationary and δ-dissipative with supply rate $w\colon \mathcal{M}(S) \times C(S) \to \mathbb{R}$ and u is F-payoff stationary and δ-antidissipative with supply rate $\tilde{w} \geq w$, then $P \coloneqq \{(\mu, \rho) \in \mathcal{P}(S) \times C(S) : v(\mu, \rho) = 0,\ u(\mu, \rho) = 0\}$ is a subset of $\mathrm{NE}(F) \times F(\mathrm{NE}(F))$ and is weak-∞-Lyapunov stable under the DPEDM (11). If, additionally, the δ-dissipativity of v and the δ-antidissipativity of u are both strict and v is $\|\cdot\|_{\mathrm{TV}}$-bounded on $\mathcal{P}(S) \times K$, then P is weak-∞-attracting under the DPEDM (11) from every $(\mu_0, \rho_0) \in \mathcal{P}(S) \times K$.*

The set P in Theorem 4 corresponds to the set of rest points of the DPEDM (11). The result shows that, under the appropriate regularity conditions, the DPEDM has dynamically stable rest points whenever the dynamics map is δ-dissipative and the payoff map is δ-antidissipative, and the incoming energy supply rate to the dynamics is less than that of the payoffs. Since, under the hypotheses of the theorem, it holds that $P \subseteq \{(\mu, \rho) \in \mathcal{P}(S) \times C(S) : \rho = F(\mu),\ \mu \in \mathrm{NE}(F)\} \subseteq \mathrm{NE}(F) \times F(\mathrm{NE}(F))$, the result shows convergence of the μ-component of the trajectory (μ, ρ) to $\mathrm{NE}(F)$, and convergence of the ρ-component to the corresponding static payoff given by the game F.

Similar to the static payoff setting, it is easy to see that our dissipativity-based result Theorem 4 may be specialized to the case of δ-passive dynamics maps coupled with δ-antipassive payoff maps, resulting in analogues to Theorem 3 and Corollary 2 for the dynamic payoff setting. In particular, the latter specialization yields the following result, which is stronger than Hofbauer et al. (2009, Theorem 3) and Cheung (2014, Theorem 4), as it allows for δ-antipassive dynamic payoffs.

[2] Technically, they show δ-antipassivity in the sense of input-output mappings, which slightly differs from the notion of δ-antipassivity of payoff maps used in our paper.

Corollary 3. *Consider a game* $F\colon \mathcal{P}(S) \to C(S)$*, let* $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$*, and let* $u\colon \mathcal{P}(S) \times C(S) \to C(S)$*. Assume that Assumption 5 holds and that Assumption 6 holds with some compact* $K \subseteq C(S)$ *containing* $F(\mathrm{NE}(F))$*. If* v *is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2 and* u *is* F*-payoff stationary and strictly* δ*-antipassive, then* $P \coloneqq \{(\mu,\rho) \in \mathcal{P}(S) \times C(S) : v(\mu,\rho) = 0,\ u(\mu,\rho) = 0\}$ *is a subset of* $\mathrm{NE}(F) \times F(\mathrm{NE}(F))$ *and is weak-*∞*-Lyapunov stable under the DPEDM* (11) *and weak-*∞*-attracting under the DPEDM* (11) *from every* $(\mu_0,\rho_0) \in \mathcal{P}(S) \times K$*.*

6 Examples

6.1 War of Attrition—Failure of Finite Approximations

In this section, we consider the famous "war of attrition" game, the formalism of which we adopt from Bishop and Cannings (1978) and Hofbauer et al. (2009, Example 6). Consider a contest being carried out on a time interval $S \coloneqq [0,T] \subseteq \mathbb{R}$, with a common value of $V \in \mathbb{R}$ awarded to the winner. The winner is the one who decides to compete in the contest for the longest amount of time. The game is given by $F_\mu(s) = \int_S f(s,s')d\mu(s')$, where

$$f(s,s') = \begin{cases} V - s' & \text{if } s' < s, \\ \frac{V}{2} - s & \text{if } s' = s, \\ -s & \text{if } s' > s, \end{cases}$$

defines the payoff to a player employing strategy s when their opponent employs strategy s'. It is assumed that $T > V/2$, so that there may be incentive to resigning from the contest before time T. The game F is monotone and has a unique Nash equilibrium $\mu^\star \in \mathcal{P}(S)$ given by

$$\mu^\star([0,s]) = \begin{cases} 1 - e^{-s/V} & \text{if } s \in [0,s^\star), \\ 1 - e^{-s^\star/V} & \text{if } s \in [s^\star, T), \\ 1 & \text{if } s = T, \end{cases} \tag{15}$$

where $s^\star = T - V/2$ (Bishop and Cannings, 1978; Hofbauer et al., 2009).

Consider endowing the game F with the BNN dynamics of Example 1, where the reference measure λ is Lebesgue. We will now show that the prior dissipativity results of Arcak and Martins (2021) guarantee that finite-strategy approximations of this evolutionary game asymptotically converge to their unique Nash equilibrium. Despite this, we will find that the infinite-dimensional dynamics do not converge to the unique Nash equilibrium $\mu^\star$, further motivating the need for direct consideration of dissipativity theory over infinite strategy sets, as we have done in this paper.

Let $n \in \mathbb{N}$ and consider a finite approximation of the strategy set given by $S_n = \{s_1,\dots,s_n\} \subseteq S$, with $s_1 < s_2 < \cdots < s_n$. Restricting the game F to the set of measures

$$\mathcal{D}(S_n) \coloneqq \left\{ \sum_{i=1}^n x_i \delta_{s_i} \in \mathcal{P}(S) : x \in \Delta^{n-1} \right\}$$

with $\Delta^{n-1} \coloneqq \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i,\ \sum_{i=1}^n x_i = 1\}$ yields the finite-dimensional approximation $\hat{F}_n\colon \Delta^{n-1} \to \mathbb{R}^n$ given by

$$(\hat{F}_n(x))_i \coloneqq F\left(\sum_{j=1}^n x_j \delta_{s_j}\right)(s_i) = \sum_{j=1}^n x_j f(s_i,s_j).$$

Thus, the finite-dimensional game may be written as

$$\hat{F}_n(x) = A_n x,$$

where

$$A_n := \begin{bmatrix} f(s_1,s_1) & f(s_1,s_2) & \cdots & f(s_1,s_n) \\ f(s_2,s_1) & f(s_2,s_2) & \cdots & f(s_2,s_n) \\ \vdots & \vdots & \ddots & \vdots \\ f(s_n,s_1) & f(s_n,s_2) & \cdots & f(s_n,s_n) \end{bmatrix} = \begin{bmatrix} \frac{V}{2}-s_1 & -s_1 & \cdots & -s_1 \\ V-s_1 & \frac{V}{2}-s_2 & \cdots & -s_2 \\ \vdots & \vdots & \ddots & \vdots \\ V-s_1 & V-s_2 & \cdots & \frac{V}{2}-s_n \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

The finite-dimensional game $\hat{F}_n$ is monotone (Hofbauer and Sandholm, 2009). The corresponding finite-dimensional BNN dynamics are given by

$$\dot{x}_i(t) = \max\{0, (\hat{F}_n(x(t)))_i - x(t)^\top \hat{F}_n(x(t))\} - x_i(t)\sum_{i=1}^n \max\{0, (\hat{F}_n(x(t)))_i - x(t)^\top \hat{F}_n(x(t))\}$$

for all $i \in \{1,\dots,n\}$ (c.f., Sandholm 2010, Example 4.3.4). These finite-dimensional BNN dynamics are Nash stationary and δ-passive (Arcak and Martins, 2021), and therefore since $\hat{F}_n$ is monotone and admits a continuously differentiable extension defined on $\mathbb{R}^n$ (given by the linear map defined by A_n), Arcak and Martins (2021, Theorem 1) asserts that $\mathrm{NE}(\hat{F}_n)$ is globally asymptotically stable under these dynamics. Based on the above analysis, one may hope that $\mathrm{NE}(F) = \{\mu^\star\}$ is also globally weakly attracting under the infinite-dimensional EDM (5). However, despite F being monotone and v being Nash stationary and δ-passive, this is not the case, as the authors in Hofbauer et al. (2009, Example 6) show that this infinite-dimensional dynamic does not weakly converge to $\mu^\star$ for some initial states. We remark that there exists $\mu \in \mathcal{P}(S)$ such that $F(\mu) \notin C(S)$, and furthermore that F is not weakly continuous since f is not continuous. Such continuity conditions are key assumptions in our stability results. This breakdown of dissipativity-based stability guarantees when moving "from finite to infinite" demonstrates the importance in carefully identifying the technical conditions under which infinite-dimensional stability may be guaranteed, as we have done in our main results.

6.2 Continuous War of Attrition

The function f defining the war of attrition game in Section 6.1 can be equivalently written as

$$f(s,s') = V\Theta(s-s') - s\Theta(s'-s) - s'\Theta(s-s'),$$

where $\Theta\colon \mathbb{R}\to\mathbb{R}$ is the step function given by

$$\Theta(x) = \begin{cases} 0 & \text{if } x<0, \\ \frac{1}{2} & \text{if } x=0, \\ 1 & \text{if } x>0. \end{cases}$$

The authors of Iyer and Killingback (2016) propose a smoothed variant of the war of attrition by replacing the discontinuous step function Θ by the logistic function $\Theta_\alpha\colon \mathbb{R}\to\mathbb{R}$ given by

$$\Theta_\alpha(x) = \frac{1}{1+e^{-\alpha x}},$$

where $\alpha > 0$ is the smoothing parameter. However, in doing so, it is unclear whether the resulting game is monotone, where the difficulty arises when analyzing the values of $\int_S\int_S (s\Theta_\alpha(s'-s)+s'\Theta_\alpha(s-s'))d\mu(s')d\nu(s)$ for various $\mu,\nu \in \mathcal{P}(S)$. In this example, we study a relaxed variant of the game in Iyer and Killingback (2016) in which we only modify the war of attrition to be continuous, rather than smooth. This is accomplished by noting that $s\Theta(s'-s)+s'\Theta(s-s') = \min\{s,s'\}$ is already a continuous function of $(s,s')\in S\times S$, and therefore the only term that should be replaced in $f(s,s')$ is $V\Theta(s-s')$, as it is where the discontinuity appears. To do so, let $\tilde{\Theta}\colon \mathbb{R}\to\mathbb{R}$ be a Lipschitz continuous function such that $0 \le \tilde{\Theta}(X) \le 1$ and $\tilde{\Theta}(x)+\tilde{\Theta}(-x) = 1$ for all $x\in\mathbb{R}$. For example, one may use the logistic function $\tilde{\Theta} = \Theta_\alpha$, or even a piecewise linear approximation of the step function Θ given by

$$\tilde{\Theta}(x) = \begin{cases} 0 & \text{if } x < x_0, \\ \frac{x}{2x_0}+\frac{1}{2} & \text{if } x\in[-x_0,x_0], \\ 1 & \text{if } x > x_0. \end{cases}$$

Then, we consider the game given by

$$\begin{aligned} F_\mu(s) &\coloneqq \int_S \tilde{f}(s,s')d\mu(s'), \\ \tilde{f}(s,s') &\coloneqq V\tilde{\Theta}(s-s') - \min\{s,s'\}. \end{aligned} \tag{16}$$

We refer our variant F as the "continuous war of attrition." The closer $\tilde{\Theta}$ approximates the step function Θ, the closer the continuous war of attrition approximates the original form of the war of attrition. We give two new results: 1) the continuous war of attrition is a monotone game, and 2) the continuous war of attrition is weakly Lyapunov stable and globally weakly attracting under the BNN and impartial pairwise comparison dynamics. It is easily verified that indeed $F(\mu) \in C(S)$ for all $\mu \in \mathcal{P}(S)$, that F satisfies both Assumption 2 and Assumption 3 with the extension $\overline{F}$ being defined by $\tilde{f}$ as well, and that $\overline{F}$ is continuously Fréchet differentiable. We now present our results.

Proposition 13. *It holds that the continuous war of attrition game $F\colon \mathcal{P}(S) \to C(S)$ defined by* (16) *is monotone.*

Proposition 13 allows us to immediately apply our dissipativity theory to conclude that indeed the continuous war of attrition exhibits global stability on the infinite strategy set S, unlike the original version of the game:

Corollary 4. *Consider the continuous war of attrition game $F\colon \mathcal{P}(S) \to C(S)$ defined by* (16)*. If $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$ is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2 and if Assumption 1 holds, then* $\mathrm{NE}(F)$ *is weakly Lyapunov stable and globally weakly attracting under the EDM* (5)*.*

In Figure 1, we display a computer simulation illustrating the stability of the continuous war of attrition game (16) with $T = 2$, $V = 1$, $\tilde{\Theta} = \Theta_\alpha$, and $\alpha = 100$, under the BNN dynamics. The initial population state in Figure 1 is the uniform distribution on $S = [0, 2]$. We see that the distribution function values $\mu(t)([0, s])$ converge in time towards those of some distribution closely resembling $\mu^\star$, the unique Nash equilibrium of the (discontinuous) war of attrition given in (15). Upon increasing α, this limiting distribution function even more closely approximates that of $\mu^\star$. The simulation is repeated in Figure 2 using a Guassian initial population state with mean 1 and variance 0.1. The same convergent behavior is observed.

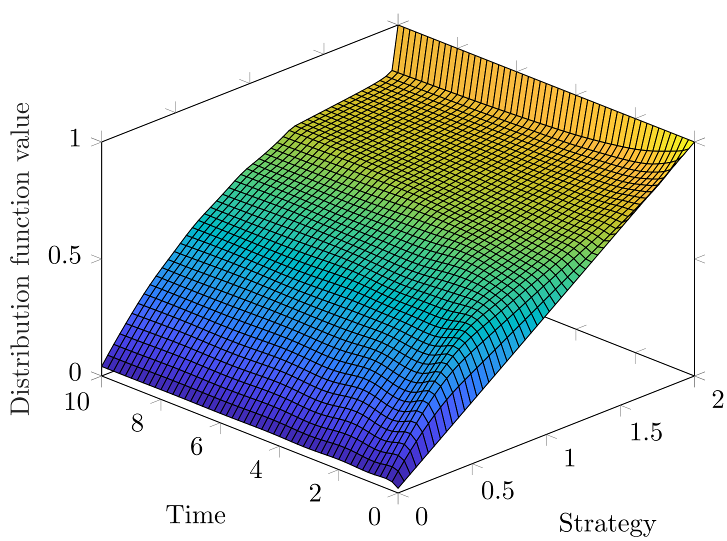

Figure 1: Evolution of the distribution function $s \mapsto \mu(t)([0, s])$ for continuous war of attrition on $S = [0, 2]$ under BNN dynamics with uniform initial distribution $\mu(0)$.

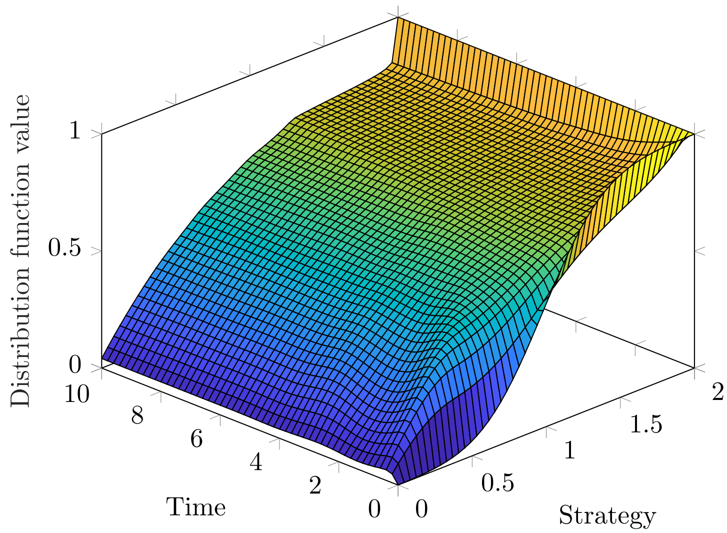

Figure 2: Evolution of the distribution function $s \mapsto \mu(t)([0,s])$ for continuous war of attrition on $S = [0,2]$ under BNN dynamics with Gaussian initial distribution $\mu(0)$ (mean 1, variance 0.1).

6.3 Smoothing Dynamics

In this section, we consider the DPEDM (11) with dynamic payoffs. Specifically, we consider smoothing dynamics, which occur when short-term variations in an evolutionary game's payoffs are suppressed, e.g., by the time delay between when a player receives payoff information and when they revise their strategy (Fox and Shamma, 2013; Arcak and Martins, 2021). Formally, the smoothing dynamics DPEDM corresponding to a game $F\colon \mathcal{P}(S) \to C(S)$ is given by

$$\begin{aligned}\dot{\mu}(t) &= v(\mu(t), \rho(t)), \\ \dot{\rho}(t) &= \lambda\left(F(\mu(t)) - \rho(t)\right), \\ \mu(0) &= \mu_0, \\ \rho(0) &= \rho_0,\end{aligned}$$

where $\lambda > 0$ is the smoothing parameter. Notice that $u(\mu, \rho) = \lambda\left(F(\mu) - \rho\right) = 0$ if and only if $\rho = F(\mu)$, so u is F-payoff stationary.

Even in the case of finite strategy sets, the incorporation of smoothing dynamics may turn a dynamically stable evolutionary process into an unstable one (Park et al., 2019); smoothing the payoff dynamics does not necessarily help with closed-loop stability. This may also be the case in our setting of infinite strategy sets. Indeed, for the continuous war of attrition game of Section 6.2 with $T = 2$, $V = 1$, $\tilde{\Theta} = \Theta_\alpha$, and $\alpha = 100$, together with the BNN dynamics map v and $\lambda = 1$, we see in Figure 3 that the smoothing has caused the population state to become unstable. The initial population state μ_0 in Figure 3 is the Gaussian distribution with mean 1 and variance 0.1, and the initial payoff is $\rho_0 = F(\mu_0)$.

Next, we consider the smoothing dynamics corresponding to a different game, namely, that given by

$$\begin{aligned}F_\mu(s) &\coloneqq \int_S f(s,s')d\mu(s'), \\ f(s,s') &= \cos(2\pi s) - \cos(2\pi s').\end{aligned}$$

We will refer to this as the "cosine game." It is easy to see that $\langle F_\nu, \nu\rangle = 0$ for all $\nu \in \mathcal{M}(S)$, and in particular this shows that F is monotone. For finite S, Fox and Shamma (2013) show that the smoothing dynamics corresponding to games satisfying $\langle F_\nu, \nu\rangle \leq 0$ for all $\nu \in \mathcal{M}(S)$ are δ-antipassive under an invertibility condition. Therefore, one may suspect based on our Theorem 4 and Corollary 3 that this DPEDM with a δ-passive dynamics map (such as that of BNN or impartial pairwise comparison) results in closed-loop

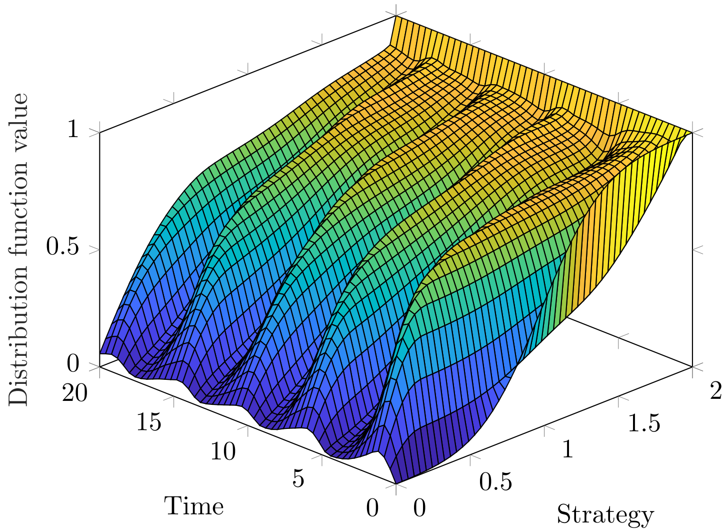

Figure 3: Evolution of the distribution function $s \mapsto \mu(t)([0,s])$ for continuous war of attrition game under BNN dynamics with smoothing.

dynamic stability. We numerically find that this is indeed the case for simulated dynamics with smoothing parameter $\lambda = 0.5$ and Gaussian initial population state μ_0 with mean 1 and variance 0.1. Figure 4 shows the evolution of the population state without smoothing (i.e., for the EDM (5) with static feedback), Figure 5 shows the evolution for smoothing with initial payoff $\rho_0 = F(\mu_0)$, and Figure 6 shows the evolution for smoothing with initial payoff given by $\rho_0(s) = -s^2$. All evolutions appear to exhibit asymptotic stability towards a Nash equilibrium; it is easy to verify that δ_0, δ_1, and δ_2 are all Nash equilibria of F, and hence the convex combination $\frac{1}{3}\delta_0 + \frac{1}{3}\delta_1 + \frac{1}{3}\delta_2$ is as well. Interestingly, in the case of Figure 6 where the initial payoff is uninformative of the game's structure, the population state initially approaches a different Nash equilibrium, namely δ_0, before the system overcomes the time delay of smoothing and begins approaching $\frac{1}{3}\delta_0 + \frac{1}{3}\delta_1 + \frac{1}{3}\delta_2$. However, running the simulations for a longer time horizon shows that all of these evolutions actually end up adjusting their mass distributions to coincide with an even different Nash equilibrium, that being $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$ (c.f., Figure 7 for the static feedback case).

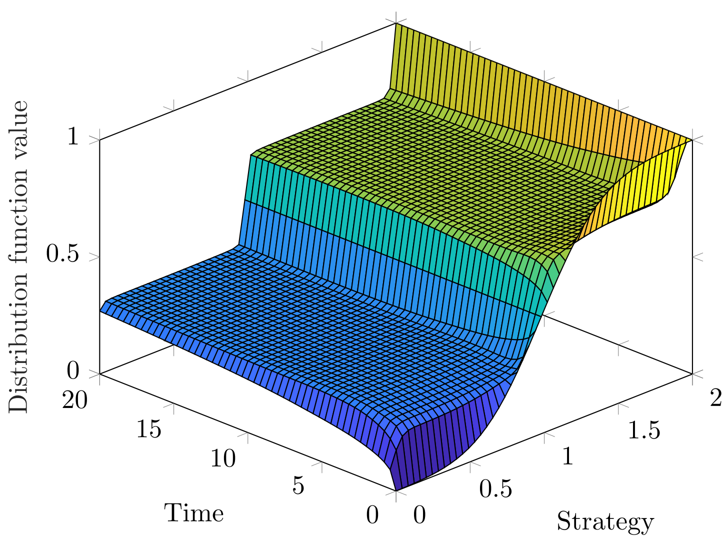

Figure 4: Evolution of the distribution function $s \mapsto \mu(t)([0,s])$ for the cosine game under BNN dynamics with static feedback.

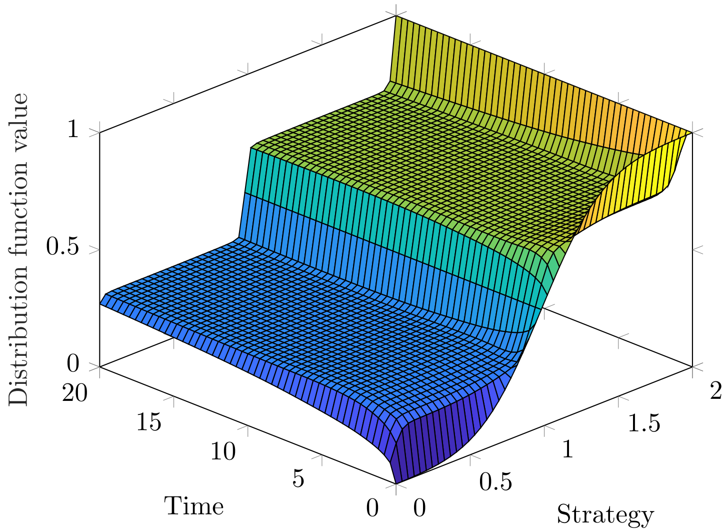

Figure 5: Evolution of the distribution function $s \mapsto \mu(t)([0, s])$ for the cosine game under BNN dynamics with smoothing and $\rho_0 = F(\mu_0)$.

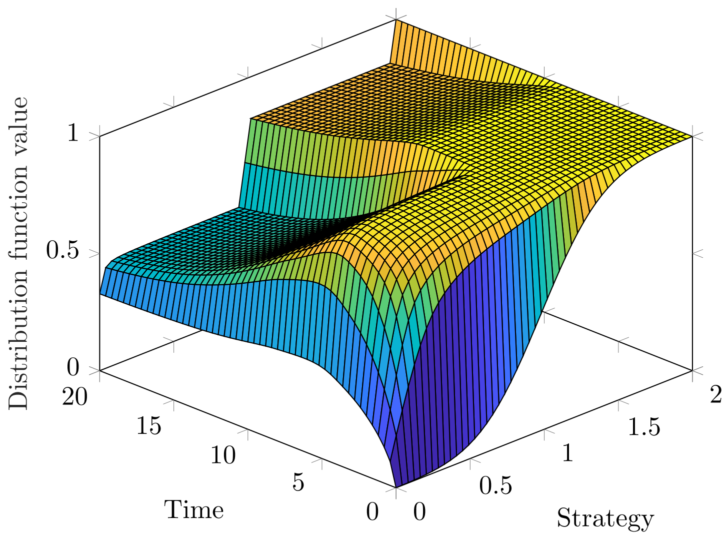

Figure 6: Evolution of the distribution function $s \mapsto \mu(t)([0, s])$ for the cosine game under BNN dynamics with smoothing and $\rho_0(s) = -s^2$.

We leave as future work the rigorous application of our unifying δ-antipassivity framework to further examples of DPEDMs over infinite strategy sets. In particular, an interesting direction for future research would be to generalize the invertibility requirement used in Fox and Shamma (2013, Theorem 4.6) to our setting of maps between Banach spaces in order to prove δ-antipassivity of payoff maps generated by smoothing of monotone games.

7 Conclusions

In this paper, we extend notions from dissipativity theory to evolutionary games with an infinite number of strategies. Our general dynamic stability results for games evolving under δ-dissipative evolutionary dynamics provide a complete characterization of the technical conditions under which such stability

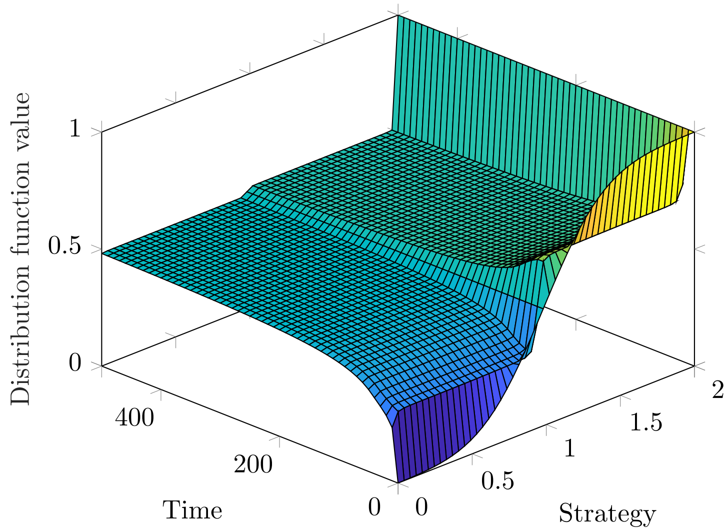

Figure 7: Long-time evolution of the distribution function $s \mapsto \mu(t)([0, s])$ for the cosine game under BNN dynamics with static feedback.

is guaranteed. We both specialize our theory to monotone games and extend our theory to δ-dissipative evolutionary dynamics coupled with δ-passive dynamic feedback payoffs, culminating into a unifying theory that recovers a handful of prior works as special cases. Our framework and results are broadly applicable, as illustrated through examples including a newly proposed variant of the classical the war of attrition. Interesting directions for future research include the development of sufficient conditions for δ-dissipativity and δ-antidissipativity from properties of a system's finite strategy approximations, and the identification and analysis of novel game-theoretic models and applications falling within the scope of our framework.

A Proofs

Lemma 1. *It holds that $\mathcal{P}(S)$ is weakly compact.*

Proof of Lemma 1. Since S is a metric space, the weak topology on $\mathcal{P}(S)$ is metrizable (Dudley, 2002, Theorem 11.3.3). Therefore, the weak topology on $\mathcal{P}(S)$ is first-countable, and hence this is a sequential space. Therefore, $\mathcal{P}(S)$ is weakly closed if and only if it is weakly sequentially closed. It is clear that if $\{\mu_n \in \mathcal{P}(S) : n \in \mathbb{N}\}$ is a sequence in $\mathcal{P}(S)$ that weakly converges to $\mu \in \mathcal{M}(S)$, then $\mu \in \mathcal{P}(S)$ since it must be nonnegative and since the constant function 1 is an element of $C(S)$. Hence, $\mathcal{P}(S)$ is weakly closed. By the Banach-Alaoglu theorem, the unit ball $\{\mu \in \mathcal{M}(S) : \|\mu\|_{\mathrm{TV}} \leq 1\}$ is weak-$*$ compact and hence weakly compact as these two topologies coincide since S is compact. Thus, the weakly closed subset $\mathcal{P}(S)$ of the weakly compact unit ball must also be weakly compact. $\square$

Proposition 1. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$, and let $\mu \in \mathcal{P}(S)$. The following are equivalent:*

1. *μ is a Nash equilibrium of the game F.*
2. *$E_F(\delta_s, \mu) \leq E_F(\mu, \mu)$ for all $s \in S$.*
3. *$F_\mu(s) \leq F_\mu(s')$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$.*

Proof of Proposition 1. Suppose that the third condition holds, so that $F_\mu(s) \leq F_\mu(s')$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$. Then, for all $s \in S$, it holds that $E_F(\delta_s, \mu) = F_\mu(s) \leq F_\mu(s')$ for all $s' \in \operatorname{supp}(\mu)$ and consequently that $E_F(\delta_s, \mu) = \int_S F_\mu(s) d\mu(s') \leq \int_S F_\mu(s') d\mu(s') = E_F(\mu, \mu)$. Thus, the second condition holds. Furthermore, if $\nu \in \mathcal{P}(S)$, then $E_F(\nu, \mu) = \int_S F_\mu(s) d\nu(s) = \int_S E_F(\delta_s, \mu) d\nu(s) \leq \int_S E_F(\mu, \mu) d\nu(s) = E_F(\mu, \mu)$, so the first condition holds as well.

To complete the proof, we show that the first condition implies the third. Suppose that the first condition holds, so that $E_F(\nu,\mu) \leq E_F(\mu,\mu)$ for all $\nu \in \mathcal{P}(S)$. Notice that $\sup_{s\in \mathrm{supp}(\mu)} E_F(\delta_s,\mu) \leq E_F(\mu,\mu)$, and also that

$$\begin{aligned}
\sup_{s\in\mathrm{supp}(\mu)} E_F(\delta_s,\mu) &= \int_S \left(\sup_{s\in\mathrm{supp}(\mu)} E_F(\delta_s,\mu) \right) d\mu(s') \\
&= \int_{\mathrm{supp}(\mu)} \left(\sup_{s\in\mathrm{supp}(\mu)} E_F(\delta_s,\mu) \right) d\mu(s') \\
&\geq \int_{\mathrm{supp}(\mu)} E_F(\delta_{s'},\mu) d\mu(s') \\
&= \int_S F_\mu(s') d\mu(s') \\
&= E_F(\mu,\mu).
\end{aligned}$$

Hence, $\sup_{s\in\mathrm{supp}(\mu)} E_F(\delta_s,\mu) = E_F(\mu,\mu)$. Suppose for the sake of contradiction that there exists $s' \in \mathrm{supp}(\mu)$ such that $E_F(\delta_{s'},\mu) < \sup_{s\in\mathrm{supp}(\mu)} E_F(\delta_s,\mu) = E_F(\mu,\mu)$. Since F_μ is a continuous real-valued function on S, the preimage $U \coloneqq F_\mu^{-1}((-\infty, E_F(\mu,\mu))) = \{s \in S : F_\mu(s) < E_F(\mu,\mu)\}$ is open and contains s', and hence it must be the case that $\mu(U) > 0$ by definition of $\mathrm{supp}(\mu)$. Thus, since the Lebesgue integral of a positive function over a set of positive measure is positive, we find that

$$\begin{aligned}
0 &= E_F(\mu,\mu) - E_F(\mu,\mu) \\
&= \int_S (E_F(\mu,\mu) - F_\mu(s)) d\mu(s) \\
&= \int_U (E_F(\mu,\mu) - F_\mu(s)) d\mu(s) + \int_{S\setminus U} (E_F(\mu,\mu) - E_F(\delta_s,\mu)) d\mu(s) \\
&\geq \int_U (E_F(\mu,\mu) - F_\mu(s)) d\mu(s) \\
&> 0,
\end{aligned}$$

which is a contradiction. Hence, it must be the case that $E_F(\delta_{s'},\mu) = \sup_{s\in\mathrm{supp}(\mu)} E_F(\delta_s,\mu) = E_F(\mu,\mu)$ for all $s' \in \mathrm{supp}(\mu)$. Therefore, $F_\mu(s') = E_F(\delta_{s'},\mu) = E_F(\mu,\mu) \geq E_F(\nu,\mu)$ for all $\nu \in \mathcal{P}(S)$ and all $s' \in \mathrm{supp}(\mu)$, and in particular, we find that $F_\mu(s') \geq E_F(\delta_s,\mu) = F_\mu(s)$ for all $s \in S$ and all $s' \in \mathrm{supp}(\mu)$, so the third condition holds. □

Proposition 2. *Let $\mu \in \mathcal{P}(S)$ be a NSS of the game $F\colon \mathcal{P}(S) \to C(S)$. If $h^F_{\nu:\mu}$ is right-continuous at 0 for all $\nu \in \mathcal{P}(S)$, then μ is a Nash equilibrium of the game F.*

Proof of Proposition 2. Let $\mu \in \mathcal{P}(S)$ be a NSS of the game $F\colon \mathcal{P}(S) \to C(S)$. Suppose that $h^F_{\nu:\mu}$ is right-continuous at 0 for all $\nu \in \mathcal{P}(S)$. Let $\nu \in \mathcal{P}(S)$. Then, there exists $\epsilon(\nu) \in (0,1]$ such that

$$h^F_{\nu:\mu}(\eta) = E_F(\nu, (1-\eta)\mu + \eta\nu) - E_F(\mu, (1-\eta)\mu + \eta\nu) \leq 0$$

for all $\eta \in (0, \epsilon(\nu)]$. Thus, by the right-continuity of $h^F_{\nu:\mu}$, it holds that

$$E_F(\nu,\mu) - E_F(\mu,\mu) = h^F_{\nu:\mu}(0) = \lim_{\eta\downarrow 0} h^F_{\nu:\mu}(\eta) \leq 0.$$

Since ν is arbitrary, this proves the claim. □

Proposition 3. *Let $\mu \in \mathcal{P}(S)$. If μ is a GNSS of the game $F\colon \mathcal{P}(S) \to C(S)$, then it is a NSS of the game F. If μ is a GESS of the game F, then it is an ESS of the game F.*

Proof of Proposition 3. Suppose that $\mu \in \mathcal{P}(S)$ is a GNSS of the game $F\colon \mathcal{P}(S) \to C(S)$. Let $\nu \in \mathcal{P}(S)$. Then, since μ is a GNSS of the game F, it holds that

$$E_F((1-\eta)\mu + \eta\nu, (1-\eta)\mu + \eta\nu) - E_F(\mu, (1-\eta)\mu + \eta\nu) \leq 0$$

for all $\eta \in (0,1]$. By linearity of E_F in its first argument, we find that

$$\eta E_F(\nu, (1-\eta)\mu + \eta\nu) - \eta E_F(\mu, (1-\eta)\mu + \eta\nu) \leq 0$$

for all $\eta \in (0,1]$. Dividing by η proves that μ is a NSS of the game F. The proof that μ being a GESS implies that μ is an ESS is identical as above with strict inqualities when considering $\nu \in \mathcal{P}(S) \setminus \{\mu\}$. ☐

Proposition 4. *Let $\mu \in \mathcal{P}(S)$ be a GESS of the game $F\colon \mathcal{P}(S) \to C(S)$, and suppose that $h^F_{\nu:\mu}$ is right-continuous at 0 for all $\nu \in \mathcal{P}(S)$. Then, it holds that $\mathrm{NE}(F) = \{\mu\}$.*

Proof of Proposition 4. Since μ is a GESS of the game F, it holds that μ is a NSS of the game F, and therefore $\mu \in \mathrm{NE}(F)$ by Proposition 2, as $h^F_{\nu:\mu}$ is right-continuous at 0. For all $\nu \in \mathcal{P}(S) \setminus \{\mu\}$, it holds that $E_F(\nu,\nu) < E(\mu,\nu)$ since μ is a GESS of the game F, and therefore such ν are not Nash equilibria of the game F. This proves that indeed $\mathrm{NE}(F) = \{\mu\}$. ☐

Proposition 5. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$. If $\theta_\nu\colon \mathcal{P}(S) \to \mathbb{R}$ defined by $\theta_\nu(\mu) = E_F(\nu,\mu) - E_F(\mu,\mu)$ is weakly continuous for all $\nu \in \mathcal{P}(S)$, then $\mathrm{NE}(F)$ is weakly compact.*

Proof of Proposition 5. It holds that $\mathrm{NE}(F) = \{\mu \in \mathcal{P}(S) : E_F(\nu,\mu) - E_F(\mu,\mu) \leq 0 \text{ for all } \nu \in \mathcal{P}(S)\} = \bigcap_{\nu\in\mathcal{P}(S)}\{\mu \in \mathcal{P}(S) : E_F(\nu,\mu) - E_F(\mu,\mu) \leq 0\}$. For all $\nu \in \mathcal{P}(S)$, the set $\{\mu \in \mathcal{P}(S) : E_F(\nu,\mu) - E_F(\mu,\mu) \leq 0\}$ is the preimage of the closed set $(-\infty, 0]$ under the map θ_ν. Hence, if this map is weakly continuous, then $\mathrm{NE}(F)$ is weakly closed. Since $\mathcal{P}(S)$ is weakly compact by Lemma 1, the weakly closed subset $\mathrm{NE}(F) \subseteq \mathcal{P}(S)$ must also be weakly compact. ☐

Proposition 6. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$. If F is weakly continuous, then $\theta_\nu\colon \mathcal{P}(S) \to \mathbb{R}$ defined by $\theta_\nu(\mu) = E_F(\nu,\mu) - E_F(\mu,\mu)$ is weakly continuous for all $\nu \in \mathcal{P}(S)$.*

Proof of Proposition 6. Suppose that F is weakly continuous and let $\nu \in \mathcal{P}(S)$. Since S is a metric space, the weak topology on $\mathcal{P}(S)$ is metrizable (Dudley, 2002, Theorem 11.3.3). Therefore, the weak topology on $\mathcal{P}(S)$ is first-countable and hence functions with domain $\mathcal{P}(S)$ are weakly continuous if they are weakly sequentially continuous. Thus, to prove the claim, it suffices to show that θ_ν is weakly sequentially continuous. To this end, let $\{\mu_n \in \mathcal{P}(S) : n \in \mathbb{N}\}$ be a sequence that converges weakly to $\mu \in \mathcal{P}(S)$. Then we have that

$$|E_F(\nu,\mu_n) - E_F(\nu,\mu)| = |\langle F(\mu_n),\nu\rangle - \langle F(\mu),\nu\rangle| = |\langle F(\mu_n) - F(\mu),\nu\rangle| \leq \|F(\mu_n) - F(\mu)\|_\infty \|\nu\|_{\mathrm{TV}} \to 0$$

since $\|\nu\|_{\mathrm{TV}} = 1$ and $F(\mu_n) \to F(\mu)$ in $C(S)$ with the topology induced by $\|\cdot\|_\infty$ due to weak continuity of F. Furthermore, we have that

$$\begin{aligned}
|E_F(\mu_n,\mu_n) - E_F(\mu,\mu)| &= |\langle F(\mu_n),\mu_n\rangle - \langle F(\mu),\mu\rangle| \\
&\leq |\langle F(\mu_n),\mu_n\rangle - \langle F(\mu),\mu_n\rangle| + |\langle F(\mu),\mu_n\rangle - \langle F(\mu),\mu\rangle| \\
&= |\langle F(\mu_n) - F(\mu),\mu_n\rangle| + |\langle F(\mu),\mu_n - \mu\rangle| \\
&\leq \|F(\mu_n) - F(\mu)\|_\infty \|\mu_n\|_{\mathrm{TV}} + |\langle F(\mu),\mu_n - \mu\rangle| \\
&= \|F(\mu_n) - F(\mu)\|_\infty + |\langle F(\mu),\mu_n - \mu\rangle| \\
&\to 0
\end{aligned}$$

since again $F(\mu_n) \to F(\mu)$ by weak continuity of F, and since $\langle F(\mu),\mu_n - \mu\rangle \to 0$ by definition of weak convergence of μ_n to μ. Therefore, we conclude that

$$\theta_\nu(\mu_n) = E_F(\nu,\mu_n) - E_F(\mu_n,\mu_n) \to E_F(\nu,\mu) - E_F(\mu,\mu) = \theta_\nu(\mu),$$

which proves the claim. ☐

Proposition 7. *Let $\mu\colon [0,\infty) \to \mathcal{M}(S)$ be strongly differentiable. If $\mu([0,\infty)) \subseteq \mathcal{P}(S)$, then $\dot\mu(t) \in T\mathcal{P}(S)$ for all $t \in [0,\infty)$.*

Proof of Proposition 7. Suppose that $\mu([0,\infty)) \subseteq \mathcal{P}(S)$. Let $t \in (0,\infty)$. Since $\frac{\mu(t+\epsilon)-\mu(t)}{\epsilon}$ converges strongly to $\dot{\mu}(t)$ as $\epsilon \to 0$, it also converges weakly to $\dot{\mu}(t)$ as $\epsilon \to 0$, so

$$\lim_{\epsilon \to 0} \int_S f d\left(\frac{\mu(t+\epsilon)-\mu(t)}{\epsilon}\right) = \int_S f d\dot{\mu}(t)$$

for all $f \in C(S)$. In particular, taking f to be the function that is identically 1 on S yields that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(\mu(t+\epsilon)(S) - \mu(t)(S)\right) = \dot{\mu}(t)(S).$$

Since $\mu(t)$ and $\mu(t+\epsilon)$ are probability measures for all $\epsilon \in [-t,\infty)$, it holds that $\mu(t+\epsilon)(S) = \mu(t)(S) = 1$ for all such ϵ, and hence $\frac{1}{\epsilon}\left(\mu(t+\epsilon)(S) - \mu(t)(S)\right) = 0$ for all $\epsilon \in [-t,\infty) \setminus \{0\}$. Therefore, it must be that

$$\dot{\mu}(t)(S) = 0,$$

so indeed $\dot{\mu}(t) \in T\mathcal{P}(S)$. The case for $t = 0$ follows similarly. ☐

Proposition 8. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$ and let $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$. If v is Nash stationary, then the set of rest points of the EDM (5) equals* $\mathrm{NE}(F)$.

Proof of Proposition 8. Suppose that v is Nash stationary. Let $\mu \in \mathcal{P}(S)$ be a rest point of the EDM (5) with dynamics map v. Then $v(\mu, F(\mu)) = 0$. Since v is Nash stationary, this implies that $E_F(\nu,\mu) - E_F(\mu,\mu) = \langle F(\mu),\nu\rangle - \langle F(\mu),\mu\rangle \le 0$ for all $\nu \in \mathcal{P}(S)$. Thus, $\mu \in \mathrm{NE}(F)$. On the other hand, if $\mu \in \mathrm{NE}(F)$, then $\langle F(\mu),\nu\rangle - \langle F(\mu),\mu\rangle = E_F(\nu,\mu) - E_F(\mu,\mu) \le 0$, so $v(\mu, F(\mu)) = 0$ since v is Nash stationary. Thus, μ is a rest point of the EDM (5) with dynamics map v. ☐

Proposition 9 (Hofbauer et al., 2009; Cheung, 2014)**.** *If $v\colon \mathcal{P}(S) \times C(S) \to \mathcal{M}(S)$ is the dynamics map for either the BNN dynamics of Example 1 or the pairwise comparison dynamics of Example 2, then v is Nash stationary.*

Proof of Proposition 9. Let $\mu \in \mathcal{P}(S)$ and let $\rho \in C(S)$. First consider the BNN dynamics of Example 1. We have that

$$v(\mu,\rho)(B) = \int_B \max\{0, \langle\rho,\delta_s\rangle - \langle\rho,\mu\rangle\} d\lambda(s) - \mu(B)\int_S \max\{0, \langle\rho,\delta_s\rangle - \langle\rho,\mu\rangle\} d\lambda(s)$$

for all $B \in \mathcal{B}(S)$. If $\langle\rho,\nu\rangle \le \langle\rho,\mu\rangle$ for all $\nu \in \mathcal{P}(S)$, then it follows immediately that $v(\mu,\rho)(B) = 0$ for all $B \in \mathcal{B}(S)$, and hence $v(\mu,\rho) = 0$.

On the other hand, suppose that $v(\mu,\rho) = 0$. Suppose that $\int_S \max\{0, \langle\rho,\delta_s\rangle - \langle\rho,\mu\rangle\} d\lambda(s) = 0$. Then we find that

$$\int_B \max\{0, \langle\rho,\delta_s\rangle - \langle\rho,\mu\rangle\} d\lambda(s) = 0$$

for all $B \in \mathcal{B}(S)$. Hence, $\max\{0, \langle\rho,\delta_s\rangle - \langle\rho,\mu\rangle\} = 0$ for λ-almost every $s \in S$. Since λ has full support by assumption and $s \mapsto \max\{0, \langle\rho,\delta_s\rangle - \langle\rho,\mu\rangle\}$ is continuous, this shows that $\max\{0, \langle\rho,\delta_s\rangle - \langle\rho,\mu\rangle\} = 0$ for all $s \in S$. Hence, $\langle\rho,\delta_s\rangle \le \langle\rho,\mu\rangle$ for all $s \in S$. Since S is compact and ρ is continuous, the optimization $\sup_{s\in S}\rho(s)$ is attained by some $s' \in S$. Therefore, for all $\nu \in \mathcal{P}(S)$ it holds that $\langle\rho,\nu\rangle = \int_S \rho(s) d\nu(s) \le \int_S \rho(s') d\nu(s) = \rho(s') = \langle\rho,\delta_{s'}\rangle \le \langle\rho,\mu\rangle$. Now suppose that the other case holds, namely, that $\int_S \max\{0, \langle\rho,\delta_s\rangle - \langle\rho,\mu\rangle\} d\lambda(s) > 0$. Then it holds that

$$\mu(B) = \int_B \frac{\max\{0, \langle\rho,\delta_{s'}\rangle - \langle\rho,\mu\rangle\}}{\int_S \max\{0, \langle\rho,\delta_s\rangle - \langle\rho,\mu\rangle\} d\lambda(s)} d\lambda(s')$$

for all $B \in \mathcal{B}(S)$. Suppose for the sake of contradiction that there exists $\tilde{s} \in S$ such that $\langle\rho,\delta_{\tilde{s}}\rangle - \langle\rho,\mu\rangle > 0$. Then, by continuity of $s' \mapsto \langle\rho,\delta_{s'}\rangle - \langle\rho,\mu\rangle$, the preimage $\{s' \in S : \langle\rho,\delta_{s'}\rangle - \langle\rho,\mu\rangle > 0\}$ is open and contains $\tilde{s}$, and hence it must be the case that $\lambda(\{s' \in S : \langle\rho,\delta_{s'}\rangle - \langle\rho,\mu\rangle > 0\}) > 0$ by definition of $\mathrm{supp}(\lambda)$ and the fact that λ has full support. Therefore, since the Lebesgue integral of a positive function over a set of positive measure is positive, we find that

$$\begin{aligned}
\langle \rho, \mu \rangle &= \int_S \rho d\mu \\
&= \int_S \rho(s') \frac{\max\{0, \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle\}}{\int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s)} d\lambda(s') \\
&= \int_{\{s' \in S : \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle > 0\}} \langle \rho, \delta_{s'} \rangle \frac{\max\{0, \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle\}}{\int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s)} d\lambda(s') \\
&> \int_{\{s' \in S : \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle > 0\}} \langle \rho, \mu \rangle \frac{\max\{0, \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle\}}{\int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s)} d\lambda(s') \\
&= \langle \rho, \mu \rangle,
\end{aligned}$$

which is a contradiction. Hence, it must be that $\langle \rho, \delta_{\tilde{s}} \rangle \leq \langle \rho, \mu \rangle$ for all $\tilde{s} \in S$. Arguing as in the prior case, this yields that $\langle \rho, \nu \rangle \leq \langle \rho, \mu \rangle$ for all $\nu \in \mathcal{P}(S)$. Since this exhausts all cases to be considered, we conclude that indeed v is Nash stationary.

Now consider the pairwise comparison dynamics of Example 2. We have that

$$v(\mu, \rho)(B) = \int_S \int_B \gamma(s, s', \rho) d\lambda(s') d\mu(s) - \int_S \int_B \gamma(s', s, \rho) d\mu(s') d\lambda(s)$$

for all $B \in \mathcal{B}(S)$. By Lemma 3, which we prove after completing the current proof, it holds that $\langle \rho, \nu \rangle \leq \langle \rho, \mu \rangle$ for all $\nu \in \mathcal{P}(S)$ if and only if $\rho(s) \leq \rho(s')$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$. Thus, if $\langle \rho, \nu \rangle \leq \langle \rho, \mu \rangle$ for all $\nu \in \mathcal{P}(S)$, then $\max\{0, \rho(s) - \rho(s')\} = 0$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$, implying that $\operatorname{sign}(\max\{0, \rho(s) - \rho(s')\}) = 0$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$. Hence, since the conditional switch rate γ satisfies sign-preservation by assumption, we find that

$$\operatorname{sign}(\gamma(s', s, \rho)) = 0$$

for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$. This implies that $v(\mu, \rho)(B) = 0$ for all $B \in \mathcal{B}(S)$, and therefore that $v(\mu, \rho) = 0$.

On the other hand, suppose that $v(\mu, \rho) = 0$. Define the measures $v_1(\mu, \rho), v_2(\mu, \rho) \in \mathcal{M}(S)$ by

$$\begin{aligned}
v_1(\mu, \rho)(B) &\coloneqq \int_S \int_B \gamma(s, s', \rho) d\lambda(s') d\mu(s) = \int_B \int_S \gamma(s, s', \rho) d\mu(s) d\lambda(s'), \\
v_2(\mu, \rho)(B) &\coloneqq \int_S \int_B \gamma(s', s, \rho) d\mu(s') d\lambda(s) = \int_B \int_S \gamma(s', s, \rho) d\lambda(s) d\mu(s'),
\end{aligned}$$

so that $v(\mu, \rho) = v_1(\mu, \rho) - v_2(\mu, \rho)$. Since $v(\mu, \rho) = 0$, it holds that $v_1(\mu, \rho) = v_2(\mu, \rho)$, and hence $\langle \rho, v_1(\mu, \rho) \rangle = \langle \rho, v_2(\mu, \rho) \rangle$. Therefore,

$$\int_S \rho(s') \int_S \gamma(s, s', \rho) d\mu(s) d\lambda(s') = \int_S \rho(s') \int_S \gamma(s', s, \rho) d\lambda(s) d\mu(s').$$

Hence,

$$\int_S \int_S \rho(s') \gamma(s, s', \rho) d\mu(s) d\lambda(s') = \int_S \int_S \rho(s') \gamma(s', s, \rho) d\lambda(s) d\mu(s'),$$

implying that

$$\int_S \int_S (\rho(s) - \rho(s')) \gamma(s', s, \rho) d\mu(s') d\lambda(s) = 0.$$

By sign-preservation of the conditional switch rate γ, it holds that $\operatorname{sign}(\gamma(s', s, \rho)) = \operatorname{sign}(\max\{0, \rho(s) - \rho(s')\})$ for all $s, s' \in S$, and therefore, if $\rho(s) \geq \rho(s')$, we find that $\gamma(s', s, \rho) \geq 0$ so that $(\rho(s) - \rho(s'))\gamma(s', s, \rho) \geq 0$, and similarly if $\rho(s) \leq \rho(s')$, we find that $\gamma(s', s, \rho) = 0$ so that $(\rho(s) - \rho(s'))\gamma(s', s, \rho) = 0$. Hence, $(\rho(s) - \rho(s'))\gamma(s', s, \rho) \geq 0$ for all $s, s' \in S$, and also $\int_S (\rho(s) - \rho(s'))\gamma(s', s, \rho) d\mu(s') \geq 0$ for all $s \in S$.

Since $s \mapsto \int_S (\rho(s) - \rho(s'))\gamma(s', s, \rho) d\mu(s')$ is continuous (which follows from compactness of S and continuity of $s' \mapsto (\rho(s) - \rho(s'))\gamma(s', s, \rho)$, together with the dominated convergence theorem), the preimage $\{s \in S : \int_S (\rho(s) - \rho(s'))\gamma(s', s, \rho) d\mu(s') > 0\}$ is open and therefore must be empty, for otherwise $\int_S \int_S (\rho(s) - \rho(s'))\gamma(s', s, \rho) d\mu(s') d\lambda(s) > 0$ as λ has full support. Hence,

$$\int_S (\rho(s) - \rho(s'))\gamma(s', s, \rho) d\mu(s') = 0 \text{ for all } s \in S.$$

Similarly, since $s' \mapsto (\rho(s) - \rho(s'))\gamma(s', s, \rho)$ is continuous for all $s \in S$, the preimage $\{s' \in S : (\rho(s) - \rho(s'))\gamma(s', s, \rho) > 0\}$ is open for all $s \in S$, and hence for all $s' \in \operatorname{supp}(\mu)$ it must be the case that

$$(\rho(s) - \rho(s'))\gamma(s', s, \rho) = 0$$

for all $s \in S$. Thus, for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$, either $\rho(s) = \rho(s')$, or $\gamma(s', s, \rho) = 0$. In the latter case, we see by sign-preservation of the conditional switch rate that $\operatorname{sign}(\max\{0, \rho(s) - \rho(s')\}) = \operatorname{sign}(\gamma(s', s, \rho)) = 0$, and hence $\rho(s) \leq \rho(s')$. Therefore, we conclude that

$$\rho(s) \leq \rho(s') \text{ for all } s \in S \text{ and all } s' \in \operatorname{supp}(\mu).$$

By Lemma 3, this proves that $\langle \rho, \nu \rangle \leq \langle \rho, \mu \rangle$ for all $\nu \in \mathcal{P}(S)$, and consequently that v is Nash stationary. $\square$

Lemma 3. *It holds that $\langle \rho, \nu \rangle \leq \langle \rho, \mu \rangle$ for all $\nu \in \mathcal{P}(S)$ if and only if $\rho(s) \leq \rho(s')$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$.*

Proof of Lemma 3. Suppose first that $\rho(s) \leq \rho(s')$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$. Then, it holds that

$$\rho(s) = \int_S \rho(s) d\mu(s') \leq \int_S \rho(s') d\mu(s') = \langle \rho, \mu \rangle$$

for all $s \in S$. Therefore, for all $\nu \in \mathcal{P}(S)$, we conclude that

$$\langle \rho, \nu \rangle = \int_S \rho(s) d\nu(s) \leq \int_S \langle \rho, \mu \rangle \, d\nu(s) = \langle \rho, \mu \rangle \,,$$

which proves one direction of the lemma.

On the other hand, suppose that $\langle \rho, \nu \rangle \leq \langle \rho, \mu \rangle$ for all $\nu \in \mathcal{P}(S)$. Then we have that $\rho(s) = \langle \rho, \delta_s \rangle \leq \langle \rho, \mu \rangle$ for all $s \in S$. Furthermore,

$$\begin{aligned}
\sup_{s \in \operatorname{supp}(\mu)} \rho(s) &= \int_S \left(\sup_{s \in \operatorname{supp}(\mu)} \rho(s) \right) d\mu(s') \\
&= \int_{\operatorname{supp}(\mu)} \left(\sup_{s \in \operatorname{supp}(\mu)} \rho(s) \right) d\mu(s') \\
&\geq \int_{\operatorname{supp}(\mu)} \rho(s') d\mu(s') \\
&= \int_S \rho(s') d\mu(s') \\
&= \langle \rho, \mu \rangle \,.
\end{aligned}$$

Hence, $\sup_{s \in \operatorname{supp}(\mu)} \rho(s) = \langle \rho, \mu \rangle$. Suppose for the sake of contradiction that there exists $s' \in \operatorname{supp}(\mu)$ such that $\rho(s') < \sup_{s \in \operatorname{supp}(\mu)} \rho(s) = \langle \rho, \mu \rangle$. Since ρ is a continuous real-valued function on S, the preimage $U := \rho^{-1}((-\infty, \langle \rho, \mu \rangle)) = \{s \in S : \rho(s) < \langle \rho, \mu \rangle\}$ is open and contains s', and hence it must be the case that $\mu(U) > 0$ by definition of $\operatorname{supp}(\mu)$. Thus, since the Lebesgue integral of a positive function over a set of positive measure is positive, we find that

$$\begin{aligned}0 &= \langle \rho, \mu \rangle - \langle \rho, \mu \rangle \\ &= \int_S (\langle \rho, \mu \rangle - \rho(s)) d\mu(s) \\ &= \int_U (\langle \rho, \mu \rangle - \rho(s)) d\mu(s) + \int_{S \setminus U} (\langle \rho, \mu \rangle - \rho(s)) d\mu(s) \\ &\geq \int_U (\langle \rho, \mu \rangle - \rho(s)) d\mu(s) \\ &> 0,\end{aligned}$$

which is a contradiction. Hence, it must be the case that $\rho(s') = \sup_{s \in \operatorname{supp}(\mu)} \rho(s) = \langle \rho, \mu \rangle$ for all $s' \in \operatorname{supp}(\mu)$. Therefore, $\rho(s') = \langle \rho, \mu \rangle \geq \langle \rho, \nu \rangle$ for all $\nu \in \mathcal{P}(S)$ and all $s' \in \operatorname{supp}(\mu)$, and in particular, we find that $\rho(s') \geq \langle \rho, \delta_s \rangle = \rho(s)$ for all $s \in S$ and all $s' \in \operatorname{supp}(\mu)$. This concludes the proof. ☐

Theorem 2 (Main Result)**.** *Consider a game $F\colon \mathcal{P}(S) \to C(S)$, let $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$, and assume that Assumption 1 and Assumption 2 both hold. If v is Nash stationary and δ-dissipative with supply rate $w\colon \mathcal{M}(S) \times C(S) \to \mathbb{R}$ and it holds that*

$$w\left(\nu, D\overline{F}(\mu)\nu\right) \leq 0 \text{ for all } \mu \in \mathcal{P}(S) \text{ and all } \nu \in T\mathcal{P}(S), \tag{9}$$

then $\operatorname{NE}(F)$ *is weakly Lyapunov stable under the EDM* (5)*. If, additionally, Assumption 3 and Assumption 4 both hold and v is strictly δ-dissipative, then* $\operatorname{NE}(F)$ *is globally weakly attracting under the EDM* (5)*.*

Proof of Theorem 2. Since v is δ-dissipative with supply rate $w\colon \mathcal{M}(S) \times C(S) \to \mathbb{R}$, there exist $\sigma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}_+$ and $\Sigma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}_+$ with Σ having an appropriate extension $\overline{\Sigma}\colon U \times C(S) \to \mathbb{R}$ as in Definition 21. Define $V\colon \mathcal{P}(S) \to \mathbb{R}_+$ by $V(\mu) = \Sigma(\mu, F(\mu))$. By Proposition 5 and Proposition 6, $\operatorname{NE}(F)$ is weakly compact. Thus, by Lemma 7, it suffices to show that V is a global Lyapunov function for $\operatorname{NE}(F)$ under $\mu \mapsto v(\mu, F(\mu))$ (according to Definition 29). Let $\overline{V}\colon U \cap U' \to \mathbb{R}$ be defined by $\overline{V}(\mu) = \overline{\Sigma}(\mu, \overline{F}(\mu))$. Note that $U \cap U'$ is strongly open and contains $\mathcal{P}(S)$, and that $\overline{V}$ is weakly continuous and Fréchet differentiable since $\overline{\Sigma}$ is weak-∞-continuous and $\overline{F}$ is weakly continuous, and both $\overline{\Sigma}$ and $\overline{F}$ are Fréchet differentiable. Also note that $\overline{V}(\mu) = V(\mu)$ for all $\mu \in \mathcal{P}(S)$. Also, if $\mu \in \operatorname{NE}(F)$, then $v(\mu, F(\mu)) = 0$ by Proposition 8, and therefore $\overline{V}(\mu) = V(\mu) = \Sigma(\mu, F(\mu)) = 0$ by (7). Furthermore, if $\mu \in \mathcal{P}(S) \setminus \operatorname{NE}(F)$, then again by Proposition 8 we have that $v(\mu, F(\mu)) \neq 0$, so $\overline{V}(\mu) = V(\mu) = \Sigma(\mu, F(\mu)) > 0$ by (7). Therefore, the first two conditions from Definition 29 on V to be a global Lyapunov function for $\operatorname{NE}(F)$ under $\mu \mapsto v(\mu, F(\mu))$ are satisfied.

Next, since $\overline{\Sigma}$ and $\overline{F}$ are Fréchet differentiable,

$$D\overline{V}(\mu) = \partial_1 \overline{\Sigma}(\mu, \overline{F}(\mu)) + \partial_2 \overline{\Sigma}(\mu, \overline{F}(\mu)) \circ D\overline{F}(\mu)$$

for all $\mu \in U \cap U'$, and therefore, since $\overline{F}(\mu) = F(\mu)$ for all $\mu \in \mathcal{P}(S)$, it holds for all $\mu \in \mathcal{P}(S)$ that

$$\begin{aligned}D\overline{V}(\mu)v(\mu, F(\mu)) &= \partial_1 \overline{\Sigma}(\mu, F(\mu))v(\mu, F(\mu)) + \partial_2 \overline{\Sigma}(\mu, F(\mu)) D\overline{F}(\mu)v(\mu, F(\mu)) \\ &\leq -\sigma(\mu, F(\mu)) + w(v(\mu, F(\mu)), D\overline{F}(\mu)v(\mu, F(\mu))) \\ &\leq -\sigma(\mu, F(\mu)) \\ &\leq 0,\end{aligned} \tag{17}$$

where the first inequality follows from (6), and the second inequality follows from (9) together with the fact that $v(\mu, F(\mu)) \in T\mathcal{P}(S)$. Hence, V is indeed a global Lyapunov function for $\operatorname{NE}(F)$ under $\mu \mapsto v(\mu, F(\mu))$, so $\operatorname{NE}(F)$ is weakly Lyapunov stable under the EDM (5).

Now suppose that the δ-dissipativity of v is strict and that the additional hypotheses of Assumption 3 and Assumption 4 both hold. By Lemma 8, it suffices to show that V is a strict global Lyapunov function for $\operatorname{NE}(F)$ under $\mu \mapsto v(\mu, F(\mu))$ (according to Definition 29). This amounts to proving that $\mu \mapsto D\overline{V}(\mu)v(\mu, F(\mu))$ is weakly continuous and that $D\overline{V}(\mu)v(\mu, F(\mu)) < 0$ for all $\mu \in \mathcal{P}(S) \setminus \operatorname{NE}(F)$. Indeed, the continuity condition holds by Lemma 4, which we proof after completing the current proof.

Next, if $\mu \in \mathcal{P}(S) \setminus \mathrm{NE}(F)$, then Proposition 8 gives that $v(\mu, F(\mu)) \neq 0$ so $\sigma(\mu, F(\mu)) > 0$ by (8), implying that $D\overline{V}(\mu)v(\mu, F(\mu)) < 0$ for all such μ by (17). Hence, V is indeed a strict global Lyapunov function for $\mathrm{NE}(F)$ under $\mu \mapsto v(\mu, F(\mu))$, so $\mathrm{NE}(F)$ is globally weakly attracting under the EDM (5). $\square$

Lemma 4. *The map $\mu \mapsto D\overline{V}(\mu)v(\mu, F(\mu))$ is weakly continuous.*

Proof of Lemma 4. Since S is a metric space, the weak topology on $\mathcal{P}(S)$ is metrizable (Dudley, 2002, Theorem 11.3.3). Therefore, the weak topology on $\mathcal{P}(S)$ is first-countable and hence functions with domain $\mathcal{P}(S)$ are weakly continuous if they are weakly sequentially continuous. Thus, to prove the claim, it suffices to show that

$$D\overline{V}(\mu_n)v(\mu_n, F(\mu_n)) \to D\overline{V}(\mu)v(\mu, F(\mu))$$

whenever $\mu_n \to \mu$ weakly. To this end, let $\{\mu_n \in \mathcal{P}(S) : n \in \mathbb{N}\}$ be a sequence that converges weakly to $\mu \in \mathcal{P}(S)$. Then we have that

$$\begin{aligned} |D\overline{V}(\mu_n)v(\mu_n, F(\mu_n)) - D\overline{V}(\mu)v(\mu, F(\mu))| &= |D\overline{V}(\mu_n)v(\mu_n, \overline{F}(\mu_n)) - D\overline{V}(\mu)v(\mu, \overline{F}(\mu))| \\ &\leq |D\overline{V}(\mu)(v(\mu_n, \overline{F}(\mu_n)) - v(\mu, \overline{F}(\mu)))| \\ &\quad + |(D\overline{V}(\mu_n) - D\overline{V}(\mu))v(\mu, \overline{F}(\mu))| \\ &\quad + |(D\overline{V}(\mu_n) - D\overline{V}(\mu))(v(\mu_n, \overline{F}(\mu_n)) - v(\mu, \overline{F}(\mu)))| \end{aligned} \tag{18}$$

Assume for the time being that every Fréchet derivative $D\overline{V}(\nu)$ is weakly continuous, and that $D\overline{V}$ is weakly continuous on $\mathcal{P}(S)$. Then, under this assumption, it holds that

$$|D\overline{V}(\mu)(v(\mu_n, \overline{F}(\mu_n)) - v(\mu, \overline{F}(\mu)))| \to 0,$$

since $v(\mu_n, \overline{F}(\mu_n)) \to v(\mu, \overline{F}(\mu))$ weakly, as $\overline{F}$ is weakly continuous and v is continuous with respect to the weak-∞ topology on its domain and the weak topology on its codomain. Furthermore,

$$|(D\overline{V}(\mu_n) - D\overline{V}(\mu))v(\mu, \overline{F}(\mu))| \leq \|D\overline{V}(\mu_n) - D\overline{V}(\mu)\|_{\mathcal{M}(S)^*}\|v(\mu, \overline{F}(\mu))\|_{\mathrm{TV}} \to 0,$$

since $D\overline{V}(\mu_n) \to D\overline{V}(\mu)$ in the dual space $\mathcal{M}(S)^*$ with associated operator norm $\|\cdot\|_{\mathcal{M}(S)^*}$ induced by the total variation norm on $\mathcal{M}(S)$, as $D\overline{V} \colon U \cap U' \to \mathcal{M}(S)^*$ is weakly continuous on $\mathcal{P}(S)$ by our above assumption. Finally,

$$\begin{aligned} &|(D\overline{V}(\mu_n) - D\overline{V}(\mu))(v(\mu_n, \overline{F}(\mu_n)) - v(\mu, \overline{F}(\mu)))| \\ &\qquad \leq \|D\overline{V}(\mu_n) - D\overline{V}(\mu)\|_{\mathcal{M}(S)^*}\|v(\mu_n, \overline{F}(\mu_n)) - v(\mu, \overline{F}(\mu))\|_{\mathrm{TV}} \\ &\qquad \leq 2\|D\overline{V}(\mu_n) - D\overline{V}(\mu)\|_{\mathcal{M}(S)^*} \sup_{\nu \in \mathcal{P}(S)} \|v(\nu, \overline{F}(\nu))\|_{\mathrm{TV}} \\ &\qquad \to 0, \end{aligned}$$

since again $D\overline{V}(\mu_n) \to D\overline{V}(\mu)$ in $\mathcal{M}(S)^*$ by the weak continuity assumption on $D\overline{V}$, and also since $\sup_{\nu \in \mathcal{P}(S)} \|v(\nu, \overline{F}(\nu))\|_{\mathrm{TV}} \leq \sup_{(\nu, g) \in \mathcal{P}(S) \times \overline{F}(\mathcal{P}(S))} \|v(\nu, g)\|_{\mathrm{TV}} \leq M$ for some finite $M \in [0, \infty)$ by the $\|\cdot\|_{\mathrm{TV}}$-boundedness of v on weak-∞ compact subsets of $\mathcal{P}(S) \times C(S)$. Therefore, under the above assumptions, it must be that

$$D\overline{V}(\mu_n)v(\mu_n, F(\mu_n)) \to D\overline{V}(\mu)v(\mu, F(\mu)),$$

which is what was to be proven. Thus, it remains to prove the above assumptions, namely, that every Fréchet derivative $D\overline{V}(\nu)$ is weakly continuous, and that $D\overline{V}$ is weakly continuous on $\mathcal{P}(S)$.

Let us first prove that $D\overline{V}(\mu) \colon \mathcal{M}(S) \to \mathbb{R}$ is weakly continuous for all $\mu \in U \cap U'$. Let $\mu \in U \cap U'$. Since $\partial_2\overline{\Sigma}(\mu, \rho) \colon C(S) \to \mathbb{R}$ is continuous (with respect to the topology on $C(S)$ induced by $\|\cdot\|_\infty$) for all $\rho \in C(S)$ by definition of the Fréchet derivative, and since $D\overline{F}(\mu) \colon \mathcal{M}(S) \to C(S)$ is weakly continuous under the hypotheses of the theorem, it holds that the composition $\partial_2\overline{\Sigma}(\mu, \overline{F}(\mu)) \circ D\overline{F}(\mu) \colon \mathcal{M}(S) \to \mathbb{R}$ is weakly continuous. Since we also have that $\partial_1\overline{\Sigma}(\mu, \overline{F}(\mu)) \colon \mathcal{M}(S) \to \mathbb{R}$ is also weakly continuous under the hypotheses of the theorem, we conclude that

$$D\overline{V}(\mu) = \partial_1\overline{\Sigma}(\mu, \overline{F}(\mu)) + \partial_2\overline{\Sigma}(\mu, \overline{F}(\mu)) \circ D\overline{F}(\mu)$$

is weakly continuous, which proves the first assumption to be proven.

Finally, let us prove the remaining assumption, namely, that $D\overline{V}: U \cap U' \to \mathcal{M}(S)^*$ is weakly continuous on $\mathcal{P}(S)$ (that is, continuous with respect to the weak topology on its domain $U \cap U' \subseteq \mathcal{M}(S)$ and the topology on its codomain $\mathcal{M}(S)^*$ induced by the operator norm $\|\cdot\|_{\mathcal{M}(S)^*}$). Once again, since we are considering weak continuity of a function on $\mathcal{P}(S)$, where the weak topology is first-countable, it suffices to prove weak sequential continuity. Let $\{\mu_n \in \mathcal{P}(S) : n \in \mathbb{N}\}$ be a sequence that converges weakly to $\mu \in \mathcal{P}(S)$. Then

$$\begin{aligned}\|D\overline{V}(\mu_n) - D\overline{V}(\mu)\|_{\mathcal{M}(S)^*} \leq& \left\|\partial_1\overline{\Sigma}(\mu_n, \overline{F}(\mu_n)) - \partial_1\overline{\Sigma}(\mu, \overline{F}(\mu))\right\|_{\mathcal{M}(S)^*} \\ &+ \left\|\partial_2\overline{\Sigma}(\mu_n, \overline{F}(\mu_n)) \circ D\overline{F}(\mu_n) - \partial_2\overline{\Sigma}(\mu, \overline{F}(\mu)) \circ D\overline{F}(\mu)\right\|_{\mathcal{M}(S)^*}.\end{aligned}$$

It is clear that the first term in the above upper bound converges to 0 due to the weak-∞ continuity of $(\nu, \rho) \mapsto \partial_1\overline{\Sigma}(\nu, \rho)$ together with the weak continuity of $\overline{F}$. Further upper-bounding the second term in a similar manner to the bound (18) and appealing to the finiteness of $\|\varphi\|_{\mathrm{TV}}$ and $\|\psi\|_{\mathcal{M}(S)^*}$ for $\varphi \in C(S)^* = \mathcal{M}(S)$ and $\psi \in \mathcal{M}(S)^*$ together with the weak continuity of $\overline{F}$ and $D\overline{F}$ as well as the weak-∞ continuity of $(\nu, \rho) \mapsto \partial_2\overline{\Sigma}(\nu, \rho)$ yields that the second term converges to 0 as well. Thus, $D\overline{V}(\mu_n) \to D\overline{V}(\mu)$ in $\mathcal{M}(S)^*$, so $D\overline{V}$ is indeed weakly continuous on $\mathcal{P}(S)$. □

Proposition 10. *Suppose that the game $F: \mathcal{P}(S) \to C(S)$ is monotone. Then the following all hold:*

1. *Every Nash equilibrium of the game F is a GNSS of the game F.*
2. *Every strict Nash equilibrium of the game F is a GESS of the game F.*
3. *If F is strictly monotone, then every Nash equilibrium of the game F is a GESS of the game F.*

Proof of Proposition 10. Let $\mu \in \mathcal{P}(S)$ be a Nash equilibrium of the game F. Then $\int_S F_\mu d\nu \leq \int_S F_\mu d\mu$ for all $\nu \in \mathcal{P}(S)$, so by monotonicity it holds that

$$\begin{aligned}E_F(\nu, \nu) - E_F(\mu, \nu) &= \int_S F_\nu d\nu - \int_S F_\nu d\mu \\ &= \int_S F_\mu(\nu - \mu) + \int_S (F_\nu - F_\mu) d(\nu - \mu) \\ &\leq 0\end{aligned}$$

for all $\nu \in \mathcal{P}(S)$. Hence, μ is a GNSS of the game F. It is clear that if μ is a strict Nash equilibrium or if F is strictly monotone, then the above inequality becomes strict for $\nu \in \mathcal{P}(S) \setminus \{\mu\}$ and hence μ is a GESS of the game F in these cases. □

Corollary 1. *Every strict Nash equilibrium $\mu \in \mathcal{P}(S)$ of a monotone game $F: \mathcal{P}(S) \to C(S)$ with $h^F_{\nu:\mu}$ right-continuous at 0 for all $\nu \in \mathcal{P}(S)$ is necessarily the unique Nash equilibrium of the game F. Every Nash equilibrium $\mu \in \mathcal{P}(S)$ of a strictly monotone game $F: \mathcal{P}(S) \to C(S)$ with $h^F_{\nu:\mu}$ right-continuous at 0 for all $\nu \in \mathcal{P}(S)$ is necessarily unique.*

Proof of Corollary 1. This follows directly from Proposition 10 together with Proposition 4. □

Lemma 5. *Consider a game $F: \mathcal{P}(S) \to C(S)$ and let $N \subseteq \mathcal{P}(S)$ be an arbitrary set of population states. Let $\mathfrak{S}^F_N \subseteq \mathcal{P}(S)$ denote the set of all population states $\mu \in \mathcal{P}(S)$ such that, for all $\nu \in N$, it holds that*

$$E_F(\nu, \nu) \leq E_F(\mu, \nu).$$

Then, it holds that $\mathfrak{S}^F_N$ is a convex set.

Proof of Lemma 5. It holds that

$$\begin{aligned}\mathfrak{S}^F_N &= \{\mu \in \mathcal{P}(S) : E_F(\nu, \nu) \leq E_F(\mu, \nu) \text{ for all } \nu \in N\} \\ &= \bigcap_{\nu \in N} \{\mu \in \mathcal{P}(S) : E_F(\nu, \nu) \leq E_F(\mu, \nu)\}.\end{aligned}$$

Since E_F is linear in its first argument, the set $\{\mu \in \mathcal{P}(S) : E_F(\nu, \nu) \leq E_F(\mu, \nu)\}$ is convex for all $\nu \in N$, and therefore the set $\mathfrak{S}^F_N$, being the intersection of convex sets, is also a convex set. □

Proposition 11. *Suppose that the game $F\colon \mathcal{P}(S)\to C(S)$ is monotone. Then, if $h^F_{\nu:\mu}$ is right-continuous at 0 for every GNSS $\mu\in\mathcal{P}(S)$ of the game F and for all $\nu\in\mathcal{P}(S)$, then $\mathrm{NE}(F)$ is a convex set.*

Proof of Proposition 11. By Proposition 10, every Nash equilibrium of the game F is a GNSS of the game F, and by Proposition 2 every GNSS of the game F is a Nash equilibrium of the game F. Hence, the set of Nash equilibria of the game F equals the set of globally neutrally stable states of the game F, so $\mathrm{NE}(F)=\{\mu\in\mathcal{P}(S) : E_F(\nu,\nu)\le E_F(\mu,\nu) \text{ for all } \nu\in\mathcal{P}(S)\}$. Applying Lemma 5 with $N=\mathcal{P}(S)$ proves the claim. $\square$

Proposition 12. *If $v\colon \mathcal{P}(S)\times C(S)\to\mathcal{M}(S)$ is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2, then v is strictly δ-passive.*

Proof of Proposition 12. We prove the result for the two dynamics separately.

BNN dynamics. Consider the BNN dynamics of Example 1. We have that

$$v(\mu,\rho)(B)=\int_B \max\{0,\langle\rho,\delta_s\rangle-\langle\rho,\mu\rangle\}d\lambda(s)-\mu(B)\int_S \max\{0,\langle\rho,\delta_s\rangle-\langle\rho,\mu\rangle\}d\lambda(s)$$

for all $\mu\in\mathcal{P}(S)$, all $\rho\in C(S)$, and all $B\in\mathcal{B}(S)$. Define $\overline{\Sigma}\colon \mathcal{M}(S)\times C(S)\to\mathbb{R}$ and $\sigma\colon \mathcal{P}(S)\times C(S)\to\mathbb{R}$ by

$$\overline{\Sigma}(\mu,\rho)=\frac{1}{2}\int_S \max\{0,\langle\rho,\delta_s\rangle-\langle\rho,\mu\rangle\}^2 d\lambda(s),$$
$$\sigma(\mu,\rho)=\langle\rho,v(\mu,\rho)\rangle\int_S \max\{0,\langle\rho,\delta_s\rangle-\langle\rho,\mu\rangle\}d\lambda(s).$$

Notice that $\overline{\Sigma}(\mu,\rho)$ and $\sigma(\mu,\rho)$ are finite for all $\mu\in\mathcal{M}(S)$ and all $\rho\in C(S)$, since $s\mapsto\max\{0,\langle\rho,\delta_s\rangle-\langle\rho,\mu\rangle\}$ and $s\mapsto\max\{0,\langle\rho,\delta_s\rangle-\langle\rho,\mu\rangle\}^2$ are continuous and S is compact. Also notice that $\overline{\Sigma}(\mu,\rho)\ge 0$ for all $\mu\in\mathcal{M}(S)$ and all $\rho\in C(S)$. Thus, we may define $\Sigma\colon \mathcal{P}(S)\times C(S)\to\mathbb{R}_+$ by the restriction of $\overline{\Sigma}$ to the domain $\mathcal{P}(S)\times C(S)\subseteq\mathcal{M}(S)\times C(S)$. We claim that σ and Σ are appropriate maps to prove the strict δ-passivity of v.

To this end, first note that $\mathcal{M}(S)$ is strongly open, $\overline{\Sigma}$ is weak-∞-continuous, $\overline{\Sigma}$ is Fréchet differentiable, $(\mu,\rho)\mapsto\partial_1\overline{\Sigma}(\mu,\rho)$ and $(\mu,\rho)\mapsto\partial_2\overline{\Sigma}(\mu,\rho)$ are weak-∞-continuous, and every partial Fréchet derivative $\partial_1\overline{\Sigma}(\mu,\rho)$ is weakly continuous. All that remains to prove are (6) with $w\colon(\mu,\eta)\mapsto\langle\eta,\mu\rangle$, (7), (8), and that $\sigma\ge 0$.

Let $\mu\in\mathcal{P}(S)$ and $\rho\in C(S)$. It holds that $\Sigma(\mu,\rho)=0$ if and only if

$$\int_S \max\{0,\langle\rho,\delta_s\rangle-\langle\rho,\mu\rangle\}^2 d\lambda(s)=0. \tag{19}$$

Since $s\mapsto\max\{0,\langle\rho,\delta_s\rangle-\langle\rho,\mu\rangle\}^2$ is a continuous real-valued function on S, the preimage $U\coloneqq\{s\in S : \max\{0,\langle\rho,\delta_s\rangle-\langle\rho,\mu\rangle\}^2>0\}$ is open. Therefore, if U is nonempty, it contains some $s'\in S$, and hence since λ has full support, s' must be an element of $\mathrm{supp}(\lambda)$, implying that $\lambda(U)>0$. This in turn would imply that $\int_U \max\{0,\langle\rho,\delta_s\rangle-\langle\rho,\mu\rangle\}^2 d\lambda(s)>0$ as the Lebesgue integral of a positive function over a set of positive measure is positive. However, this would contradict (19). Thus, $\Sigma(\mu,\rho)=0$ if and only if

$$\max\{0,\langle\rho,\delta_s\rangle-\langle\rho,\mu\rangle\}^2=0 \text{ for all } s\in S,$$

which holds if and only if

$$\rho(s)\le\langle\rho,\mu\rangle \text{ for all } s\in S. \tag{20}$$

It is clear that, if $\langle\rho,\nu\rangle\le\langle\rho,\mu\rangle$ for all $\nu\in\mathcal{P}(S)$, then (20) holds. Conversely, if (20) holds, then $\langle\rho,\nu\rangle=\int_S\rho(s)d\nu(s)\le\int_S\langle\rho,\mu\rangle\,d\nu(s)=\langle\rho,\mu\rangle$ for all $\nu\in\mathcal{P}(S)$, and thus by Nash stationarity of v (Proposition 9) we conclude that $\Sigma(\mu,\rho)=0$ if and only if

$$v(\mu,\rho)=0,$$

which proves (7).

Again let $\mu \in \mathcal{P}(S)$ and $\rho \in C(S)$. If $v(\mu,\rho) = 0$, then certainly $\sigma(\mu,\rho) = 0$ due to linearity of $\langle \rho, \cdot \rangle$. Notice that

$$\begin{aligned}
\langle \rho, v(\mu,\rho) \rangle &= \int_S \rho(s') d(v(\mu,\rho))(s') \\
&= \int_S \rho(s') \max\{0, \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle\} d\lambda(s') - \int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s) \int_S \rho(s') d\mu(s') \\
&= \int_S \left(\rho(s') - \int_S \rho(\tilde{s}) d\mu(\tilde{s}) \right) \max\{0, \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle\} d\lambda(s') \\
&= \int_S (\langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle) \max\{0, \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle\} d\lambda(s').
\end{aligned}$$

Notice that $(\langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle) \max\{0, \langle \rho, \delta_{s'} \rangle - \langle \rho, \mu \rangle\} \geq 0$ for all $s' \in S$ and hence $\langle \rho, v(\mu,\rho) \rangle \geq 0$. Furthermore, by the usual arguments based on continuity and nonnegativity of the integrand together with full support of λ, we see that

$$\langle \rho, v(\mu,\rho) \rangle = 0$$

if and only if

$$\rho(s') = \langle \rho, \delta_{s'} \rangle \leq \langle \rho, \mu \rangle \text{ for all } s' \in S,$$

which, as shown above, holds true if and only if $v(\mu,\rho) = 0$. Furthermore, notice that by the same arguments,

$$\int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s) \geq 0,$$

with equality holding if and only if $v(\mu,\rho) = 0$. Thus,

$$\sigma(\mu,\rho) = \langle \rho, v(\mu,\rho) \rangle \int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s) \geq 0,$$

with equality holding if and only if $v(\mu,\rho) = 0$. This proves (8).

All that remains to be proven is (6) with $w\colon (\mu,\eta) \mapsto \langle \eta, \mu \rangle$. Let $\mu \in \mathcal{P}(S)$, $\rho \in C(S)$, and $\eta \in C(S)$. Define $\tau\colon \mathbb{R} \to \mathbb{R}_+$ by $\tau(r) = \max\{0,r\}^2$, so that $\tau'(r) = 2\max\{0,r\}$ and $\overline{\Sigma}(\mu,\rho) = \frac{1}{2}\int_S \tau(\langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle) d\lambda(s)$. Computing the first partial Fréchet derivative of $\overline{\Sigma}$ using the chain rule yields that

$$\begin{aligned}
\partial_1 \overline{\Sigma}(\mu,\rho) v(\mu,\rho) &= \frac{1}{2} \int_S \tau'(\langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle)(-\langle \rho, v(\mu,\rho) \rangle) d\lambda(s) \\
&= -\langle \rho, v(\mu,\rho) \rangle \int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s) \\
&= -\sigma(\mu,\rho).
\end{aligned}$$

Computing the second partial Fréchet derivative of $\overline{\Sigma}$ using the chain rule yields that

$$\begin{aligned}
\partial_2 \overline{\Sigma}(\mu,\rho)\eta &= \frac{1}{2} \int_S \tau'(\langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle)(\langle \eta, \delta_s \rangle - \langle \eta, \mu \rangle) d\lambda(s) \\
&= \int_S (\langle \eta, \delta_s \rangle - \langle \eta, \mu \rangle) \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s) \\
&= \int_S \left(\eta(s) - \int_S \eta(\tilde{s}) d\mu(\tilde{s}) \right) \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s) \\
&= \int_S \eta(s) \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s) - \int_S \eta(\tilde{s}) d\mu(\tilde{s}) \int_S \max\{0, \langle \rho, \delta_s \rangle - \langle \rho, \mu \rangle\} d\lambda(s) \\
&= \int_S \eta(s) d(v(\mu,\rho))(s) \\
&= \langle \eta, v(\mu,\rho) \rangle \\
&= w(v(\mu,\rho),\eta).
\end{aligned}$$

Thus, altogether we find that

$$\partial_1\overline{\Sigma}(\mu,\rho)v(\mu,\rho) + \partial_2\overline{\Sigma}(\mu,\rho)\eta = -\sigma(\mu,\rho) + w(v(\mu,\rho),\eta),$$

which shows that (6) holds and hence concludes the proof for the BNN dynamics.

Impartial pairwise comparison dynamics. Consider the impartial pairwise comparison dynamics of Example 2. We have that

$$v(\mu,\rho)(B) = \int_B\int_S \gamma(s,s',\rho)d\mu(s)d\lambda(s') - \int_B\int_S \gamma(s',s,\rho)d\lambda(s)d\mu(s')$$

for all $\mu \in \mathcal{P}(S)$, all $\rho \in C(S)$, and all $B \in \mathcal{B}(S)$. Since the pairwise comparison dynamics under consideration are impartial, it holds that for all $s' \in S$, there exists some continuous function $\phi_{s'}\colon \mathbb{R} \to \mathbb{R}_+$ such that

$$\gamma(s,s',\rho) = \phi_{s'}(\rho(s') - \rho(s))$$

for all $s \in S$ and all $\rho \in C(S)$. For all $s' \in S$, define $\tau_{s'}\colon \mathbb{R} \to \mathbb{R}_+$ by

$$\tau_{s'}(r) = \int_{[0,r]} \phi_{s'}(u)du,$$

where we see that $\tau_{s'}(r) = 0$ whenever $r < 0$, since we take $[0,r] = \emptyset$ in such cases. Notice that $\tau_{s'}$ is strictly increasing on $[0,\infty)$ since $\phi_{s'}(u) > 0$ for all $u > 0$: let $u > 0$, let $s, s' \in S$ be such that $s \neq s'$, and let $\rho \in C(S)$ be such that $\rho(s') - \rho(s) = u > 0$ (which exists by Urysohn's lemma and the fact that S is a metric space and hence normal), so that, by sign-preservation, we have that $\operatorname{sign}(\phi_{s'}(u)) = \operatorname{sign}(\phi_{s'}(\rho(s') - \rho(s))) = \operatorname{sign}(\gamma(s,s',\rho)) = \operatorname{sign}(\max\{0, \rho(s') - \rho(s)\}) = 1$. Define $\overline{\Sigma}\colon \mathcal{M}(S) \times C(S) \to \mathbb{R}$ and $\sigma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}$ by

$$\begin{aligned}
\overline{\Sigma}(\mu,\rho) &= \int_S\int_S \tau_s(\rho(s) - \rho(s'))d\lambda(s)d\mu(s'),\\
\sigma(\mu,\rho) &= -\overline{\Sigma}(v(\mu,\rho),\rho).
\end{aligned}$$

Notice that $\overline{\Sigma}(\mu,\rho)$ is finite for all $\mu \in \mathcal{M}(S)$ and all $\rho \in C(S)$ since $(s,s') \mapsto \tau_s(\rho(s) - \rho(s'))$ is continuous and S is compact. Also notice that $\overline{\Sigma}(\mu,\rho) \geq 0$ for all $\mu \in \mathcal{P}(S)$ and all $\rho \in C(S)$. Thus, we may define $\Sigma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}_+$ by the restriction of $\overline{\Sigma}$ to the domain $\mathcal{P}(S) \times C(S) \subseteq \mathcal{M}(S) \times C(S)$. We claim that σ and Σ are appropriate maps to prove the strict δ-passivity of v.

To this end, first note that $\mathcal{M}(S)$ is strongly open, $\overline{\Sigma}$ is weak-∞-continuous, $\overline{\Sigma}$ is Fréchet differentiable, $(\mu,\rho) \mapsto \partial_1\overline{\Sigma}(\mu,\rho)$ and $(\mu,\rho) \mapsto \partial_2\overline{\Sigma}(\mu,\rho)$ are weak-∞-continuous, and every partial Fréchet derivative $\partial_1\overline{\Sigma}(\mu,\rho)$ is weakly continuous. All that remains to prove are (6) with $w\colon (\mu,\eta) \mapsto \langle \eta, \mu\rangle$, (7), (8), and that $\sigma \geq 0$.

Let $\mu \in \mathcal{P}(S)$ and $\rho \in C(S)$. It holds that $\Sigma(\mu,\rho) = 0$ if and only if

$$\int_S\int_S \tau_s(\rho(s) - \rho(s'))d\lambda(s)d\mu(s') = 0,$$

which holds if and only if

$$\tau_s(\rho(s) - \rho(s')) = 0 \text{ for all } s \in S \text{ and all } s' \in \operatorname{supp}(\mu),$$

since λ has full support, $s \mapsto \tau_s(\rho(s) - \rho(s'))$ is nonnegative and continuous for all $s' \in S$, and $s' \mapsto \int_S \tau_s(\rho(s) - \rho(s'))d\lambda(s)$ is nonnegative and continuous (which follows from compactness of S together with the dominated convergence theorem). Since, for all $s \in S$, it holds that τ_s is strictly increasing on $[0,\infty)$ and $\tau_s(0) = 0$, it must be that $\Sigma(\mu,\rho) = 0$ if and only if

$$\rho(s) \leq \rho(s') \text{ for all } s \in S \text{ and all } s' \in \operatorname{supp}(\mu).$$

Therefore, by Lemma 3, it holds that $\langle \rho, \nu\rangle \leq \langle \rho, \mu\rangle$ for all $\nu \in \mathcal{P}(S)$. Hence, by Nash stationarity of v (Proposition 9), it holds that $\Sigma(\mu,\rho) = 0$ if and only if

$$v(\mu,\rho) = 0,$$

which proves (7).

Again let $\mu \in \mathcal{P}(S)$ and $\rho \in C(S)$. If $v(\mu,\rho) = 0$, then certainly $\sigma(\mu,\rho) = -\overline{\Sigma}(v(\mu,\rho),\rho) = 0$ due to linearity of $\overline{\Sigma}(\cdot,\rho)$. Writing out $\sigma(\mu,\rho)$, we find that

$$\begin{aligned}
\sigma(\mu,\rho) &= -\int_S\int_S \tau_s(\rho(s)-\rho(s'))d\lambda(s)d(v(\mu,\rho))(s') \\
&= -\int_S \left(\int_S \tau_s(\rho(s)-\rho(s'))d\lambda(s)\right)\left(\int_S \gamma(s,s',\rho)d\mu(s)\right)d\lambda(s') \\
&\quad + \int_S \left(\int_S \tau_s(\rho(s)-\rho(s'))d\lambda(s)\right)\left(\int_S \gamma(s',s,\rho)d\lambda(s)\right)d\mu(s') \\
&= \int_S\int_S \gamma(s',s,\rho)\int_S \left(\tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s')) - \tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s))\right)d\lambda(\tilde{s})d\lambda(s)d\mu(s').
\end{aligned}$$

For all $s,s' \in S$ such that $\rho(s) \leq \rho(s')$, it holds by sign-preservation that $\operatorname{sign}(\gamma(s',s,\rho)) = \operatorname{sign}(\max\{0,\rho(s)-\rho(s')\}) = 0$, and therefore $\gamma(s',s,\rho) = 0$ for all such s,s'. On the other hand, if $s,s' \in S$ are such that $\rho(s) > \rho(s')$, then $\operatorname{sign}(\gamma(s',s,\rho)) = \operatorname{sign}(\max\{0,\rho(s)-\rho(s')\}) = 1$, implying that $\gamma(s',s,\rho) > 0$. Furthermore, in this case with $\rho(s) > \rho(s')$, we see that $\rho(\tilde{s})-\rho(s) < \rho(\tilde{s})-\rho(s')$ for all $\tilde{s} \in S$, and therefore $\tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s')) \geq \tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s))$ for all $\tilde{s} \in S$ by the fact that every $\tau_{\tilde{s}}$ is nondecreasing. Thus, we immediately see that

$$\sigma(\mu,\rho) \geq 0.$$

We furthermore see that if $\sigma(\mu,\rho) = 0$, then

$$\gamma(s',s,\rho)\int_S \left(\tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s')) - \tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s))\right)d\lambda(\tilde{s}) = 0 \text{ for all } s \in S \text{ and all } s' \in \operatorname{supp}(\mu)$$

by the usual arguments based on continuity and nonnegativity of the integrand together with full support of λ. Thus, let $s \in S$ and $s' \in \operatorname{supp}(\mu)$. Either $\gamma(s',s,\rho) = 0$, or $\int_S \left(\tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s')) - \tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s))\right)d\lambda(\tilde{s}) = 0$. In the former case, it must be that $\rho(s) \leq \rho(s')$, for otherwise $\phi_s(\rho(s)-\rho(s')) > 0$, which would contradict the fact that $\phi_s(\rho(s)-\rho(s')) = \gamma(s',s,\rho) = 0$. Suppose that the latter case holds. Then either $\rho(s) \leq \rho(s')$ or $\rho(s) > \rho(s')$. If $\rho(s) > \rho(s')$, then, as argued above, we find that $\tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s')) - \tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s)) \geq 0$ for all $\tilde{s} \in S$, and hence by the usual arguments based on continuity and nonnegativity of the integrand together with the full support of λ, we conclude that $\tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s')) = \tau_{\tilde{s}}(\rho(\tilde{s})-\rho(s))$ for all $\tilde{s} \in S$. In this case, by the fact that every $\tau_{\tilde{s}}$ is strictly increasing on $[0,\infty)$ and $\rho(s') \neq \rho(s)$, it must be the case that, for all $\tilde{s} \in S$, we have that $\rho(\tilde{s})-\rho(s') \leq 0$ and $\rho(\tilde{s})-\rho(s) \leq 0$. But these two inequalities cannot hold simultaneously, as they would imply that $\rho(s) \leq \rho(s')$ and $\rho(s') \leq \rho(s)$, which contradicts the fact that $\rho(s) > \rho(s')$ in the case under consideration. Hence, we conclude that, when $\sigma(\mu,\rho) = 0$, it must hold that

$$\rho(s) \leq \rho(s') \text{ for all } s \in S \text{ and all } s' \in \operatorname{supp}(\mu).$$

Thus, by Lemma 3, we find that $\langle \rho,\nu\rangle \leq \langle \rho,\mu\rangle$ for all $\nu \in \mathcal{P}(S)$, and therefore by Nash stationarity of v (Proposition 9), it holds that $v(\mu,\rho) = 0$ whenever $\sigma(\mu,\rho) = 0$. This proves (8).

All that remains to be proven is (6) with $w\colon (\mu,\eta) \mapsto \langle \eta,\mu\rangle$. Let $\mu \in \mathcal{P}(S)$, $\rho \in C(S)$, and $\eta \in C(S)$. Since $\overline{\Sigma}(\cdot,\rho)$ is linear, it is immediate that $D(\overline{\Sigma}(\cdot,\rho))(\mu) = \overline{\Sigma}(\cdot,\rho)$, which implies that

$$\partial_1\overline{\Sigma}(\mu,\rho)v(\mu,\rho) = \overline{\Sigma}(v(\mu,\rho),\rho) = -\sigma(\mu,\rho).$$

Furthermore, computing the second partial Fréchet derivative of $\overline{\Sigma}$ using the chain rule yields that

$$\partial_2\overline{\Sigma}(\mu,\rho)\eta = \int_S\int_S \tau_s'(\rho(s)-\rho(s'))(\eta(s)-\eta(s'))d\lambda(s)d\mu(s'),$$

where the derivatives of the functions $\tau_s\colon \mathbb{R} \to \mathbb{R}_+$ are computed via the fundamental theorem of calculus:

$$\tau_s'(r) = \frac{d}{dr}\int_{[0,r]} \phi_s(u)du = \phi_s(r).$$

By impartiality of the pairwise comparison dynamics under consideration, we find that

$$
\begin{aligned}
\partial_2 \overline{\Sigma}(\mu, \rho)\eta &= \int_S \int_S \gamma(s', s, \rho)(\eta(s) - \eta(s'))d\lambda(s)d\mu(s') \\
&= \int_S \eta(s) \int_S \gamma(s', s, \rho)d\mu(s')d\lambda(s) - \int_S \eta(s') \int_S \gamma(s', s, \rho)d\lambda(s)d\mu(s') \\
&= \int_S \eta(s') \int_S \gamma(s, s', \rho)d\mu(s)d\lambda(s') - \int_S \eta(s') \int_S \gamma(s', s, \rho)d\lambda(s)d\mu(s') \\
&= \int_S \eta(s')d(v(\mu, \rho))(s') \\
&= \langle \eta, v(\mu, \rho) \rangle \\
&= w(v(\mu, \rho), \eta).
\end{aligned}
$$

Thus, altogether we find that

$$
\partial_1 \overline{\Sigma}(\mu, \rho)v(\mu, \rho) + \partial_2 \overline{\Sigma}(\mu, \rho)\eta = -\sigma(\mu, \rho) + w(v(\mu, \rho), \eta),
$$

which shows that (6) holds and hence concludes the proof. $\square$

Theorem 3. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$, let $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$, and assume that Assumption 1 holds. Furthermore, assume that Assumption 2 holds and that the extension $\overline{F}$ is continuously Fréchet differentiable. If v is Nash stationary, v is δ-passive, and F is monotone, then $\mathrm{NE}(F)$ is weakly Lyapunov stable under the EDM (5). If, additionally, Assumption 3 and Assumption 4 both hold and v is strictly δ-passive, then $\mathrm{NE}(F)$ is globally weakly attracting under the EDM (5).*

Proof of Theorem 3. Suppose that v is Nash stationary, v is δ-passive, and F is monotone. Let $w\colon \mathcal{M}(S) \times C(S) \to \mathbb{R}$ be defined by $w(\mu, \eta) = \langle \eta, \mu \rangle$. Then it holds that v is δ-dissipative with supply rate w. Furthermore, by Lemma 2, F satisfies self-defeating externalities, and therefore

$$
w(\nu, D\overline{F}(\mu)\nu) = \left\langle D\overline{F}(\mu)\nu, \nu \right\rangle \leq 0 \text{ for all } \mu \in \mathcal{P}(S) \text{ and all } \nu \in T\mathcal{P}(S).
$$

Hence, by Theorem 2, it holds that $\mathrm{NE}(F)$ is weakly Lyapunov stable under the EDM (5). The fact that $\mathrm{NE}(F)$ is globally weakly attracting under the EDM (5) given the additional hypotheses of Assumption 3 and Assumption 4 is immediate from Theorem 2. $\square$

Corollary 2. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$, let $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$, and assume Assumption 1 holds. Furthermore, assume that Assumption 2 holds and that the extension $\overline{F}$ is continuously Fréchet differentiable. If F is monotone and v is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2, then $\mathrm{NE}(F)$ is weakly Lyapunov stable under the EDM (5). If, additionally, Assumption 3 holds, then $\mathrm{NE}(F)$ is globally weakly attracting under the EDM (5).*

Proof of Corollary 2. Notice that weak Lyapunov stability of $\mathrm{NE}(F)$ follows immediately from Theorem 3 together with Proposition 9 and Proposition 12. Furthermore, global weak attraction of $\mathrm{NE}(F)$ under Assumption 3 follows by additionally noting that, for both the BNN dynamics and the impartial pairwise comparison dynamics, v satisfies the appropriate continuity conditions of Assumption 4 and v is $\|\cdot\|_{\mathrm{TV}}$-bounded on weak-∞ compact subsets of $\mathcal{P}(S) \times C(S)$ (the latter condition of which follows from the fact that $v(\mu, \rho)(B) \leq 4\|\rho\|_\infty$ for all $\mu \in \mathcal{P}(S)$ and all $\rho \in C(S)$ for the BNN dynamics and that the conditional switch rate γ is assumed bounded for the pairwise comparison dynamics). $\square$

Theorem 4. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$, let $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$, and let $u\colon \mathcal{P}(S) \times C(S) \to C(S)$. Assume that Assumption 5 holds and that Assumption 6 holds with some compact $K \subseteq C(S)$ containing $F(\mathrm{NE}(F))$. If v is Nash stationary and δ-dissipative with supply rate $w\colon \mathcal{M}(S) \times C(S) \to \mathbb{R}$ and u is F-payoff stationary and δ-antidissipative with supply rate $\tilde{w} \geq w$, then $P \coloneqq \{(\mu, \rho) \in \mathcal{P}(S) \times C(S) : v(\mu, \rho) = 0,\ u(\mu, \rho) = 0\}$ is a subset of $\mathrm{NE}(F) \times F(\mathrm{NE}(F))$ and is weak-∞-Lyapunov stable under the DPEDM (11). If, additionally, the δ-dissipativity of v and the δ-antidissipativity of u are both strict and v is $\|\cdot\|_{\mathrm{TV}}$-bounded on $\mathcal{P}(S) \times K$, then P is weak-∞-attracting under the DPEDM (11) from every $(\mu_0, \rho_0) \in \mathcal{P}(S) \times K$.*

Proof of Theorem 4. Since v is δ-dissipative with supply rate $w\colon \mathcal{M}(S) \times C(S) \to \mathbb{R}$, there exist $\sigma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}_+$ and $\Sigma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}_+$ with Σ having an appropriate extension $\overline{\Sigma}\colon U \times C(S) \to \mathbb{R}$ as in Definition 21. Furthermore, since u is δ-antidissipative with supply rate $\tilde{w}\colon \mathcal{M}(S) \times C(S) \to \mathbb{R}$, there exist $\gamma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}_+$ and $\Gamma\colon \mathcal{P}(S) \times C(S) \to \mathbb{R}_+$ with Γ having an appropriate extension $\overline{\Gamma}\colon \tilde{U} \times C(S) \to \mathbb{R}$ as in Definition 26. Define $V\colon \mathcal{P}(S) \times K \to \mathbb{R}_+$ by $V(\mu,\rho) = \Sigma(\mu,\rho) + \Gamma(\mu,\rho)$. Consider $P = \{(\mu,\rho) \in \mathcal{P}(S) \times C(S) : v(\mu,\rho) = 0,\ u(\mu,\rho) = 0\}$. If $(\mu,\rho) \in P$, then $u(\mu,\rho) = 0$, implying that $\rho = F(\mu)$ by F-payoff stationarity, and hence $v(\mu, F(\mu)) = 0$, so $\mu \in \mathrm{NE}(F)$ by Nash stationarity. Thus, $P \subseteq \mathrm{NE}(F) \times F(\mathrm{NE}(F)) \subseteq \mathcal{P}(S) \times K$. By Proposition 5 and Proposition 6, $\mathrm{NE}(F)$ is weakly compact, and hence $F(\mathrm{NE}(F))$ is compact as F is weakly continuous. Since v is continuous with respect to the weak-∞ topology on its domain and the weak topology on its codomain, and u is weak-∞-continuous, it holds that $P = v^{-1}(\{0\}) \cap u^{-1}(\{0\})$ is weak-∞-closed, and hence must be weak-∞-compact as well as $\mathrm{NE}(F) \times F(\mathrm{NE}(F))$ is. Thus, by Lemma 7, it suffices to show that V is a global Lyapunov function for P under $(\mu,\rho) \mapsto (v(\mu,\rho), u(\mu,\rho))$ (according to Definition 29). Let $\overline{V}\colon U \cap \tilde{U} \times C(S) \to \mathbb{R}$ be defined by $\overline{V}(\mu,\rho) = \overline{\Sigma}(\mu,\rho) + \overline{\Gamma}(\mu,\rho)$. Note that $U \cap \tilde{U}$ is strongly open and contains $\mathcal{P}(S)$, and that $\overline{V}$ is weak-∞-continuous and Fréchet differentiable since $\overline{\Sigma}$ and $\overline{\Gamma}$ are. Also note that $\overline{V}(\mu,\rho) = V(\mu,\rho)$ for all $(\mu,\rho) \in \mathcal{P}(S) \times K$. Also, if $(\mu,\rho) \in P$, then $v(\mu,\rho) = 0$ and $u(\mu,\rho) = 0$, so $\overline{V}(\mu,\rho) = 0$ by (7) and (13). Furthermore, if $(\mu,\rho) \in (\mathcal{P}(S) \times K) \setminus P$, then again by (7) and (13) we have that $\overline{V}(\mu,\rho) > 0$. Therefore, the first two conditions from Definition 29 on V to be a global Lyapunov function for P under (v,u) are satisfied.

Next, it holds for all $(\mu,\rho) \in \mathcal{P}(S) \times K$ that

$$
\begin{aligned}
D\overline{V}(\mu,\rho)(v(\mu,\rho),u(\mu,\rho)) &= \partial_1\overline{\Sigma}(\mu,\rho)v(\mu,\rho) + \partial_2\overline{\Sigma}(\mu,\rho)u(\mu,\rho) + \partial_1\overline{\Gamma}(\mu,\rho)v(\mu,\rho) + \partial_2\overline{\Gamma}(\mu,\rho)u(\mu,\rho) \\
&\le -\sigma(\mu,\rho) + w(v(\mu,\rho),u(\mu,\rho)) - \gamma(\mu,\rho) - \tilde{w}(v(\mu,\rho),u(\mu,\rho)) \\
&\le -\sigma(\mu,F(\mu)) - \gamma(\mu,\rho) \\
&\le 0,
\end{aligned} \tag{21}
$$

where the first inequality follows from (6) and (12), and the second inequality follows from the fact that $\tilde{w} \ge w$. Hence, V is indeed a global Lyapunov function for P under (v,u), so P is weak-∞-Lyapunov stable under the DPEDM (11).

Now suppose that the δ-dissipativity of v and the δ-antidissipativity of u are both strict, and that v is $\|\cdot\|_{\mathrm{TV}}$-bounded on $\mathcal{P}(S) \times K$. By Lemma 8, it suffices to show that V is a strict global Lyapunov function for P under $(\mu,\rho) \mapsto (v(\mu,\rho), u(\mu,\rho))$ (according to Definition 29). This amounts to proving that $(\mu,\rho) \mapsto D\overline{V}(\mu,\rho)(v(\mu,\rho),u(\mu,\rho))$ is weak-∞-continuous and that $D\overline{V}(\mu,\rho)(v(\mu,\rho),u(\mu,\rho)) < 0$ for all $(\mu,\rho) \in (\mathcal{P}(S) \times K) \setminus P$. Indeed, the continuity condition holds by Lemma 6, which we proof after completing the current proof.

Next, if $(\mu,\rho) \in (\mathcal{P}(S) \times K) \setminus P$, then $v(\mu,\rho) \neq 0$ or $u(\mu,\rho) \neq 0$, so $\sigma(\mu,\rho) > 0$ or $\gamma(\mu,\rho) > 0$ by (8) and (14), implying that $D\overline{V}(\mu,\rho)(v(\mu,\rho),u(\mu,\rho)) < 0$ for all such (μ,ρ) by (21). Hence, V is indeed a strict global Lyapunov function for P under (v,u), so P is globally weak-∞-attracting under the DPEDM (11) from K. ☐

Lemma 6. *The map $\mu \mapsto D\overline{V}(\mu)v(\mu, F(\mu))$ is weakly continuous.*

Proof of Lemma 6. The result follows from a nearly identical analysis as in the proof of Lemma 4 with minor changes. In particular, it follows from the $\|\cdot\|_{\mathrm{TV}}$-boundedness of v on $\mathcal{P}(S) \times K$, the weak-∞-to-weak continuity of v, the weak-∞ continuity of u, the weak continuity of every $\partial_1\overline{\Sigma}(\mu,\rho)$ and every $\partial_1\overline{\Gamma}(\mu,\rho)$, and the weak-$\infty$ continuity of the maps $(\mu,\rho) \mapsto \partial_1\overline{\Sigma}(\mu,\rho)$, $(\mu,\rho) \mapsto \partial_2\overline{\Sigma}(\mu,\rho)$, $(\mu,\rho) \mapsto \partial_1\overline{\Gamma}(\mu,\rho)$, and $(\mu,\rho) \mapsto \partial_2\overline{\Gamma}(\mu,\rho)$. ☐

Corollary 3. *Consider a game $F\colon \mathcal{P}(S) \to C(S)$, let $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$, and let $u\colon \mathcal{P}(S) \times C(S) \to C(S)$. Assume that Assumption 5 holds and that Assumption 6 holds with some compact $K \subseteq C(S)$ containing $F(\mathrm{NE}(F))$. If v is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2 and u is F-payoff stationary and strictly δ-antipassive, then $P \coloneqq \{(\mu,\rho) \in \mathcal{P}(S) \times C(S) : v(\mu,\rho) = 0,\ u(\mu,\rho) = 0\}$ is a subset of* $\mathrm{NE}(F) \times F(\mathrm{NE}(F))$ *and is weak-∞-Lyapunov stable under the DPEDM* (11) *and weak-∞-attracting under the DPEDM* (11) *from every $(\mu_0,\rho_0) \in \mathcal{P}(S) \times K$.*

Proof of Corollary 3. The proof follows analogously to that of Corollary 2. □

Proposition 13. *It holds that the continuous war of attrition game* $F\colon \mathcal{P}(S) \to C(S)$ *defined by* (16) *is monotone.*

Proof of Proposition 13. In this proof, we denote the indicator function on a set $A \subseteq \mathbb{R}$ by $\chi_A\colon \mathbb{R} \to \mathbb{R}$, where

$$\chi_A(t) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{if } t \notin A. \end{cases}$$

Let $\mu, \nu \in \mathcal{P}(S)$. It holds that

$$\begin{aligned}
&2\int_S \int_S \tilde{\Theta}(s-s')d(\mu-\nu)(s')d(\mu-\nu)(s) \\
&\quad = \int_S \int_S \tilde{\Theta}(s-s')d(\mu-\nu)(s')d(\mu-\nu)(s) + \int_S \int_S \tilde{\Theta}(s'-s)d(\mu-\nu)(s')d(\mu-\nu)(s) \\
&\quad = \int_S \int_S \left(\tilde{\Theta}(s-s') + \tilde{\Theta}(s'-s)\right) d(\mu-\nu)(s')d(\mu-\nu)(s) \\
&\quad = \int_S \int_S d(\mu-\nu)(s')d(\mu-\nu)(s) \\
&\quad = ((\mu-\nu)(S))^2 \\
&\quad = 0,
\end{aligned}$$

since $(\mu-\nu)(S) = \mu(S) - \nu(S) = 0$. Therefore,

$$\int_S \int_S \tilde{\Theta}(s-s')d(\mu-\nu)(s')d(\mu-\nu)(s) = 0.$$

Next, we note that

$$\begin{aligned}
\int_S \int_S \min\{s,s'\}d\mu(s')d\nu(s) &= \int_S \int_S \int_{[0,\min\{s,s'\}]} dt d\mu(s')d\nu(s) \\
&= \int_S \int_S \int_{[0,\infty)} \chi_{\{t'\in\mathbb{R}: t' \le \min\{s,s'\}\}}(t) dt d\mu(s')d\nu(s) \\
&= \int_S \int_S \int_{[0,\infty)} \chi_{\{t'\in\mathbb{R}: t' \le s\}}(t)\chi_{\{t'\in\mathbb{R}: t \le s'\}}(t) dt d\mu(s')d\nu(s) \\
&= \int_S \int_S \int_{[0,\infty)} \chi_{\{\tilde{s}\in S: \tilde{s} \ge t\}}(s)\chi_{\{\tilde{s}\in S: \tilde{s} \ge t\}}(s') dt d\mu(s')d\nu(s) \\
&= \int_{[0,\infty)} \int_S \chi_{\{\tilde{s}\in S: \tilde{s} \ge t\}}(s')d\mu(s') \int_S \chi_{\{\tilde{s}\in S: \tilde{s} \ge t\}}(s)d\nu(s)dt \\
&= \int_{[0,\infty)} \mu(S\cap[t,\infty))\nu(S\cap[t,\infty))dt.
\end{aligned}$$

Therefore, we find that

$$\begin{aligned}
&\int_S \int_S \min\{s,s'\}d(\mu-\nu)(s')d(\mu-\nu)(s) \\
&\quad = \int_{[0,\infty)} \left(\mu(S\cap[t,\infty))^2 - 2\mu(S\cap[t,\infty))\nu(S\cap[t,\infty)) + \nu(S\cap[t,\infty))^2\right) dt \\
&\quad = \int_{[0,\infty)} \left(\mu(S\cap[t,\infty)) - \nu(S\cap[t,\infty))\right)^2 dt.
\end{aligned}$$

Thus, overall, it holds that

$$\begin{aligned}
\langle F(\mu) - F(\nu), \mu - \nu\rangle &= \int_S (F_\mu(s) - F_\nu(s)) d(\mu-\nu)(s) \\
&= \int_S \int_S \tilde{f}(s,s') d(\mu-\nu)(s') d(\mu-\nu)(s) \\
&= V \int_S \int_S \tilde{\Theta}(s-s') d(\mu-\nu)(s') d(\mu-\nu)(s) - \int_S \int_S \min\{s,s'\} d(\mu-\nu)(s') d(\mu-\nu)(s) \\
&= -\int_{[0,\infty)} \left(\mu(S\cap[t,\infty)) - \nu(S\cap[t,\infty))\right)^2 dt \\
&\leq 0.
\end{aligned}$$

Hence, F is monotone. □

Corollary 4. *Consider the continuous war of attrition game $F\colon \mathcal{P}(S) \to C(S)$ defined by* (16)*. If $v\colon \mathcal{P}(S) \times C(S) \to T\mathcal{P}(S)$ is the dynamics map for either the BNN dynamics of Example 1 or the impartial pairwise comparison dynamics of Example 2 and if Assumption 1 holds, then* $\mathrm{NE}(F)$ *is weakly Lyapunov stable and globally weakly attracting under the EDM* (5)*.*

Proof of Corollary 4. This is immediate from Corollary 2 together with Proposition 13 and the fact that F satisfies all of the appropriate regularity conditions. □

B Supplementary Definitions and Results

Definition 29. Consider a Banach space X and a topology τ on X. Let $Y \subseteq X$, let $v\colon Y \to X$, and let $P \subseteq Y$ be τ-compact. A map $V\colon Y \to \mathbb{R}_+$ is a *global Lyapunov function for P under v* if it extends to a τ-continuous Fréchet differentiable map $\overline{V}\colon U \to \mathbb{R}$ defined on a norm-open set $U \subseteq X$ containing Y that satisfies the following conditions:

1. $\overline{V}(x) = 0$ for all $x \in P$.
2. $\overline{V}(x) > 0$ for all $x \in Y \setminus P$.
3. $D\overline{V}(x)v(x) \leq 0$ for all $x \in Y$.

If, additionally, the map $x \mapsto D\overline{V}(x)v(x)$ is τ-continuous and $D\overline{V}(x)v(x) < 0$ for all $x \in Y \setminus P$, then V is a *strict global Lyapunov function for P under v.*

Notice that the topology τ in Definition 29 need not coincide with the topology induced by the norm on X. Indeed, our dissipativity results for static feedback $\rho(t) = F(\mu(t))$ rely on taking $X = \mathcal{M}(S)$ with τ being the weak topology and $Y = \mathcal{P}(S)$.

Lemma 7. *Consider a Banach space X and a topology τ on X. Let $Y \subseteq X$, let $v\colon Y \to X$, and let $P \subseteq Y$ be τ-compact. If τ is weaker than the norm topology, Y is τ-compact, and there exists a global Lyapunov function for P under v, then P is τ-Lyapunov stable under v.*

Proof. Suppose that there exists a global Lyapunov function $V\colon Y \to \mathbb{R}_+$ for P under v, and let $\overline{V}\colon U \to \mathbb{R}$ be an appropriate extension as in Definition 29. Let $Q \subseteq Y$ be relatively τ-open and contain P. Then $Q = Y \cap O$ for some τ-open set $O \subseteq X$. Define $\partial_Y Q \coloneqq Y \cap \partial O$, where ∂O is the boundary of O in X with respect to τ. It holds that $\partial_Y Q$ is τ-compact since Y is τ-compact and ∂O is τ-closed. Therefore,

$$m \coloneqq \min_{x\in\partial_Y Q} \overline{V}(x)$$

exists, since $\overline{V}$ is τ-continuous. Notice that, since $\partial O \cap O = \emptyset$, it must be that $\partial_Y Q \cap Q = \emptyset$, and therefore $\partial_Y Q \cap P = \emptyset$. Hence, since $\overline{V}(x) > 0$ for all $x \in Y \setminus P$, it must be that $\overline{V}(x) > 0$ for all $x \in \partial_Y Q$ and thus $m > 0$.

Now, let

$$R = \{x \in Q : \overline{V}(x) \in (-\infty, m)\}.$$

Since $\overline{V}$ is τ-continuous and $(-\infty, m)$ is open, the preimage $\overline{V}^{-1}((-\infty, m))$ is τ-open, and hence $R = Q \cap \overline{V}^{-1}((-\infty, m)) = Y \cap O \cap \overline{V}^{-1}((0, m)) \subseteq Y$ is relatively τ-open. Furthermore, since $P \subseteq Q$ and $P \subseteq V^{-1}((-\infty, m))$ as $\overline{V}(x) = 0$ for all $x \in P$, it holds that $P \subseteq R \subseteq Y$. Let $x\colon [0, \infty) \to Y$ be a solution to the differential equation $\dot{x}(t) = v(x(t))$ with $x(0) = x_0 \in Y$. Suppose that $x_0 \in R$. Then, since the Fréchet derivative of real-valued functions on $\mathbb{R}$ recovers the usual derivative, we have that

$$\frac{d\overline{V} \circ x}{dt}(t)\epsilon = D(\overline{V} \circ x)(t)\epsilon = (D\overline{V}(x(t)) \circ Dx(t))\epsilon = D\overline{V}(x(t))(\epsilon\dot{x}(t)) = \epsilon D\overline{V}(x(t))v(x(t))$$

for all $t \in [0, \infty)$ and all $\epsilon \in \mathbb{R}$, where we have used the chain rule for Fréchet differentiation, linearity of Fréchet derivatives. Hence,

$$\frac{d\overline{V} \circ x}{dt}(t) = D\overline{V}(x(t))v(x(t)) \leq 0$$

for all $t \in [0, \infty)$. Since $\overline{V}$ is τ-continuous and x is τ-continuous since it is necessarily norm-continuous and τ is weaker than the norm topology, we may apply the mean value theorem to find that $\overline{V}(x(t)) \leq \overline{V}(x(0)) < m$ for all $t \in [0, \infty)$. Since $R \subseteq Q$, we conclude that $x(t) \in Q$ for all $t \in [0, \infty)$, so indeed P is τ-Lyapunov stable under v. $\square$

Lemma 8. *Consider a Banach space X and a topology τ on X. Let $Y \subseteq X$, let $v\colon Y \to X$, and let $P \subseteq Y$ be τ-compact. Suppose that, for every $x_0 \in Y$, there exists a unique solution $x\colon [0, \infty) \to Y$ to the differential equation $\dot{x}(t) = v(x(t))$ with $x(0) = x_0$. If τ is weaker than the norm topology, Y is τ-compact, and there exists a strict global Lyapunov function for P under v, then P is globally τ-attracting under v.*

Proof. In this proof, we denote the complement of a subset $M \subseteq X$ by M^c.

Suppose that there exists a strict global Lyapunov function $V\colon Y \to \mathbb{R}_+$ for P under v, and let $\overline{V}\colon U \to \mathbb{R}$ be an appropriate extension as in Definition 29. Let $x_0 \in Y$ be arbitrary. Let $Q \subseteq Y$ be relatively τ-open and contain P, and let $x\colon [0, \infty) \to Y$ be the unique solution to the differential equation $\dot{x}(t) = v(x(t))$ with $x(0) = x_0$. It suffices to show that there exists $T \in [0, \infty)$ such that

$$x(t) \in Q \text{ for all } t \in [T, \infty). \tag{22}$$

Since V is a global Lyapunov function, Lemma 7 gives that P is τ-Lyapunov stable under v, which implies that there exists a relatively τ-open set $R \subseteq Y$ containing P such that $x(t) \in Q$ for all $t \in [0, \infty)$ whenever $x(0) \in R$. By time-invariance of the ordinary differential equation $\dot{x}(t) = v(x(t))$ with $x(0) = x_0$ and uniqueness of its solutions, if there exists $T \in [0, \infty)$ such that $x(T) \in R$, this implies that $x(t) \in Q$ for all $t \in [T, \infty)$. Thus, to prove (22), it suffices to prove that there exists $T \in [0, \infty)$ such that $x(T) \in R$.

For the sake of contradiction, suppose that $x(t) \notin R$ for all $t \in [0, \infty)$. Since R is relatively τ-open, $R = Y \cap O$ for some τ-open set $O \subseteq X$, and therefore $Y \setminus R = Y \cap (Y \cap O)^c = Y \cap (Y^c \cup O^c) = Y \cap O^c$ is τ-compact since O^c is τ-closed and Y is τ-compact. Hence,

$$m \coloneqq \max_{y \in Y \setminus R} D\overline{V}(y)v(y)$$

exists, since $y \mapsto D\overline{V}(y)v(y)$ is τ-continuous. Since $Y \setminus R \subseteq Y \setminus P$, it must hold that $m < 0$ as V is a strict global Lyapunov function. Furthermore, since $x(t) \in Y \setminus R$ for all $t \in [0, \infty)$, it holds that

$$\frac{d\overline{V} \circ x}{dt}(t) = D\overline{V}(x(t))v(x(t)) \leq m$$

for all $t \in [0, \infty)$. Since $\overline{V}$ is τ-continuous and x is τ-continuous since it is necessarily norm-continuous and τ is weaker than the norm topology, we may apply the mean value theorem to conclude that, for all $\tau \in (0, \infty)$, there exists $t \in (0, \tau)$ such that

$$\frac{\overline{V}(x(\tau)) - \overline{V}(x(0))}{\tau} = \frac{d\overline{V} \circ x}{dt}(t) \leq m,$$

and hence

$$\overline{V}(x(\tau)) \leq m\tau + \overline{V}(x(0))$$

for all $\tau \in (0, \infty)$. Since $m < 0$, $m\tau + \overline{V}(x(0)) \to -\infty$ as $\tau \to \infty$, which implies that there exists $\tau \in (0, \infty)$ such that $\overline{V}(x(\tau)) < 0$. Since, for such τ, it holds that $x(\tau) \in Y \setminus R \subseteq Y \setminus P$, this contradicts the property of the global Lyapunov function V that $\overline{V}(y) > 0$ for all $y \in Y \setminus P$. Therefore, the supposition that $x(t) \notin R$ for all $t \in [0, \infty)$ is false, and we conclude that indeed there exists $T \in [0, \infty)$ such that $x(T) \in R$, which completes the proof. $\square$

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