Prime Numbers and Cicada Cycles

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Introduction:

In our project, we will be looking into the life cycles of insect Cicada, with the explanation of lcm (least common multiple) and gcd (greatest common divisor) theorem. The subspecies that is only found in Northern America is the Magicicada. What is so special about these specific insects is they tend to live their life mainly underground and only emerge to the surface between 13 to 17 years. There are three subspecies with a 13-year life cycle and three subspecies with a 17-year life cycle. The first known mention of these insects was by the Native Americans and was spoken about them for generations. However, the first literature piece that they were mentioned in was by William Bradford, the governor of Plymouth Colony in 1633. Studies dove more deeply into these insects because of their interesting pattern of emerging to the surface, and rarely being seen because of their infrequent surface. The conclusion came to be that these species only come to the surface on these prime number years 13 or 17, because it was a system used to help defend themselves against predators. Once they come to the surface they cling onto houses and trees while they break out of their shells and become winged adults. “In each species the nymphal stage remains in the soil for a lengthy period, then the adult cicada emerges after either 13 years or 17 years depending on the geographical area. “Even more strikingly, this emergence is synchronized among all members of a cicada species in any given area. The adults all emerge within the same few days, they mate, die a few weeks later and then the cycle repeats itself.” (Baker, Alan) Now that we have a general idea of what these bugs are and what they do, we will now further deep drive into their biological traits and help better understand them using math.

Problem Description:

It is tough for one scientist to study these species because they appear so infrequently, with our work we are hoping we can help scientists better understand the breach to the surface by these creatures, and a better understanding of the advantages of having this particular trait in their life cycle. In our analysis we will see if we can somehow see how these insects and their internal clocks pattern for coming to the surface and why they can benefit from applying this pattern. Then we will use the Greatest common divisor and Euclid's algorithm to better understand these insects that appear so infrequently. We will be looking at past times it has been recorded these insects were seen, the years passed between these sightings, and the years it was recorded. We will look at the first recorded sighting until now, the time this paper was written November 16th, 2021, this will give us our range and dates of when these insects have been sighted. “Biologists have long found features of the life-cycle of periodical cicadas mysterious, and this is reflected both in the substantial literature devoted to this topic and in biologists' specific remarks. There are at least five distinct features of this life-cycle for which explanations have been sought by biologists;” (Baker, Alan) Even though there are five distinct features of the cicada’s life cycles, we will specifically look at the prime-numbered-year in relation to cicada life cycle.

Importance of project:

We believe this is important to further research because it will show how beneficial the Greatest common divisor and Euclid's algorithm can be in not just math but in other fields of study. We hope by the end of this project, the reader will have a good understanding of what the Cicada insect is and its patterns for coming to the surface but also understanding how we can further use the Greatest common divisor and Euclid's algorithm to better understand these insects and their habits. We believe that the lcm and gcd have a lot of real-life applications, we hope to show in this report real life applications of these and how it can be used outside of just standard math classes. We also believe looking at the adaptation that the cicadas have made to make their emergence prime number years, will help biologists better understand how some species evolve to help avoid predators and keep their species from extinction.

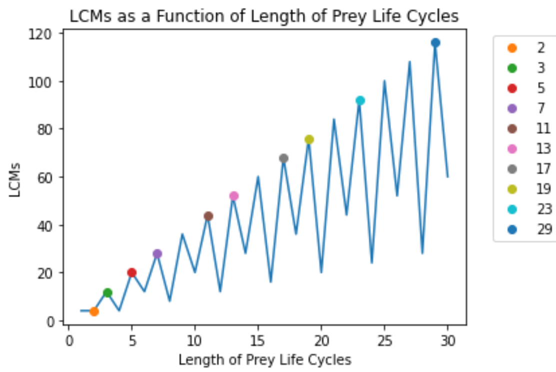
Theory and simulation:

Given that predators have higher nutritional demands at the end of their reproductive cycles, and that cicadas are only exposed to predatory threats when they emerge (at the end of their life cycles), it is an optimal survival strategy for cicadas to minimize the concurrence of these two events. LCM gives us a good way of understanding this problem mathematically since when LCM is maximized, the frequency of the concurrence of the ends of these cycles will be minimized. Two cycles of low LCM will have higher frequency of concurrent ends and two cycles of high LCM will have lower frequency of concurrent ends.

To understand why periodical cicadas have 13 and 17 year cycles exclusively, we simulated data that would help us observe LCM levels for varying cycle lengths of both predators and cicadas. This will help demonstrate the crucial importance of LCM, GCD, coprimality, and other important integer properties.

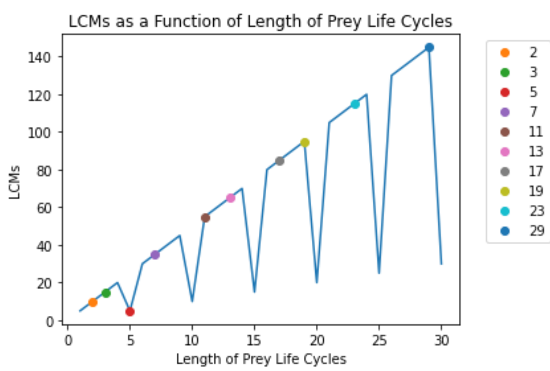
First we want to look at varying cicada life cycles and observe the LCMs when predator reproductive cycles are four, five, and six years. Any notes will be bullet pointed below the respective visualizations.

Predator Simulation: Predator with 4 Year Cycle



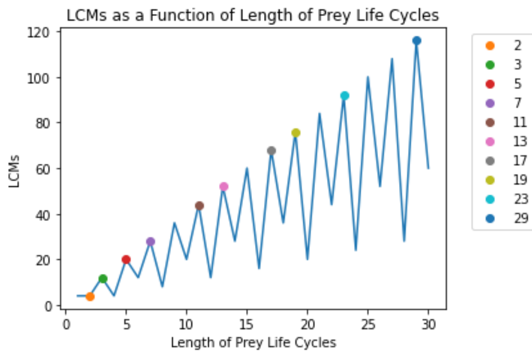
- We can see that the relationship between cicada life cycles and LCM with a predator of 4 year reproductive style show the uniqueness of prime numbered life cycles almost exclusively being the peaks of the function

Predator Simulation: Predator with 5 Year Cycle



- Again we see it for when predator reproductive cycles equal 5 years.

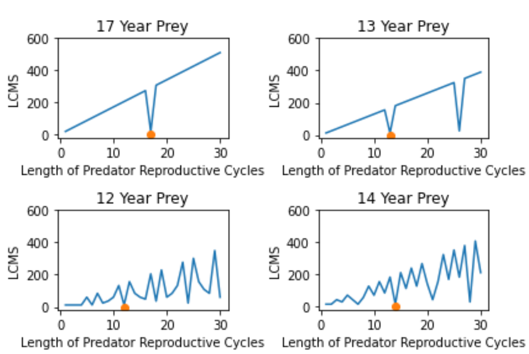
Predator Simulation: Predator with 6 Year Cycle



- Again we see it for when predator reproductive cycles equal 6 years.
- Not only are LCMs typically maximized when the cicada's life cycle falls on a prime number, we also note that it is minimized when the predator reproductive cycle equals the cicada life cycle. This makes sense since the concurrence of events would be every cicada life cycle which would be the worst survival outcome.

Now we will fix cicada life cycles at two prime numbers 13 and 17, and two non-prime numbers 12 and 14 to see the varying levels of LCM to varying levels of predator reproductive cycle lengths. This will give us a good sense of the "robustness" of prime numbered LCM performance relative to non-prime numbered LCM performance.

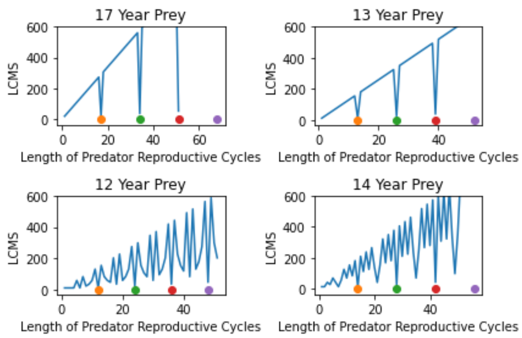
Predator Simulation: Different Life Cycles



- Here we vary the predator reproductive cycles and fixed the cicada life cycles to 17, 13, 12, and 14 to make a comparison between the LCMs at varying reproductive cycle lengths of prime and non-prime cicada cycles.

- Orange markers indicate when predator reproductive cycle = prey life cycle
- We can see that for almost all ranges, prime numbered prey life cycles provide **higher** LCMs across varying lengths of predator reproductive cycles compared to the non-prime numbered life cycles.
- Again we see the only exception being when the two cycles are of equal length

Coprime Cycle Lengths



- Same visualization but the range of predator reproductive cycles is 52
- Not only are the lowest LCMs when the two cycles lengths are equal, we notice that the most severe troughs in the function occur at every multiple of the cicada's life cycle. When a prime number, the life cycle of cicadas are co-prime with every number except the multiples of the prime number. When two numbers are co-prime, their GCD = 1 and for the prime numbers, GCD =/= 1 only when the predator reproductive cycle is a multiple of the prime numbered cicada reproductive cycle. So we can see that not only is LCM maximization ideal as a survival strategy, GCD minimization is also ideal.

So from our simulations, we demonstrate the optimal survival strategy of LCM maximization while also observing the implications of coprimality and GCD minimization that results from it. While these visualizations give us a strong demonstration of LCM, GCD, and coprimalitys' role in the survival strategy of cicada life cycle lengths, we dive further into the modelling and mathematical nature of cicada life cycles below.

Solution Description:

To find the relation between cicadas and Greatest common divisor and Euclid's algorithm, we will first find out the years they emerge. According to https://www.cicadamania.com, species M. septendecim, M. cassini, and M. septendecula often live in states of DE, GA, IL, IN, KY, MD, MI, NC, NJ, NY, OH, PA, TN, VA, WV, Washington.

The life cycle length of these species is 17 years. The years they emerge from 1950 till now were 1953, 1970, 1987, 2004, 2021, which follows the prime number feature as we assumed.

Early in the class, we discussed the prime number theorem. A good application of the theorem is that the probability that the selected number is prime from the year 1950 to 2021 is roughly 1/ln(70) = 23.54%. However, this result is still higher than what we would expect for the cicada species to emerge in the real world. As mentioned above, species M. septendecim, M. cassini, and M. septendecula only appear in the years 1953, 1970, 1987, 2004, 2021 – five times from 1950 to 2021. The probability of emerging is 5 / 71 = 7.04%.

So, why do cicadas have this internal clock pattern? Another answer lies in the stability of prime prey cycles. For the next step, we will compare the results of using a composite number 6 and a prime number 17 to have a clear look at the different survival rates.

To offer an intuition, let us think of a cicada species that has a mass-reproduction cycle of six years (not a prime number). A predator with a cycle of three years would not have enough food (the cicada species) in its first episode of mass-reproduction at the end of three years. In the second episode however (at the end of six years), there will be a coincidence of mass-reproduction between the predator and the prey, and the predator-numbers will flourish because of greater food-availability, making the mass-extinction of such cicadas more probable.

With prime numbered years, it is easy to see that the chance of this coincidence is significantly reduced - for instance, the first coincidence between our imagined predator and the cicada with a 17-year cycle would happen at the end of the fifty-first year, or after sixteen cycles of mass-reproduction without a corresponding rise in food-availability. Less food for predators means fewer of them will survive. (*Go.galegroup.com.ezproxy.fiu.edu*)

With this concept, we will next apply the Greatest common divisor and Euclid's algorithm to illustrate the process above. We assume a fungus (predator) of period F interacting with a cicada of period C. A momentary fitness $\phi_c(t)$ of the cicadas in a year t as follows: it is zero if cicadas are not present it is -1 if both fungi and cicadas are present, and it is +1 if the cicadas are present but the fungi are not. We do this because emergence uses up metabolic resources due to metamorphosis, mating and death; these resources are lost if the cicada eggs are eaten up by the fungi, while they are preserved if the cicadas stay as larvae below the ground. The fitness F_f, resp. F_c is defined by the sum over the $\phi_f(t)$, resp. $\phi_c(t)$, t = 0, …, FC, divided by the number of fungus, resp. cicada generations (Markus, M., & Goles, E., 2002).

$$F_f = \sum_{t=0}^{FC} \frac{\phi f(t)}{number\ of\ fungi}$$

Let Icm(F, C): least common multiple, gcd(F, C): greatest common divisor of F and C. In FC years, the fungi appear C times, both cicadas and fungi appear $FC/lcm(F,\ C)$ times; thus fungi without prey appear $C\ -\ \ FC/Icm(F,\ C)$ times. Considering that gcd(F, C) * lcm(F, C) =

FC, we thus obtain the fungus fitness $F_f(F, C) = 2gcd(F, C)/C - 1$, and cicada fitness $F_c(F, C) = 1 - 2gcd(F, C)/F$ (Markus, M., & Goles, E., 2002).

$$Ff(F,\ C) = \frac{(+1)\bullet\frac{FC}{lcm(f,C)} - (-1)\bullet\left(C - \frac{FC}{lcm(f,C)}\right)}{C}$$

$$= \frac{gcd(F,C) - C + gcd(F,C)}{C} = \frac{2gcd(F,C)}{C} - 1$$

Assume C = Cp is a prime; by virtue of condition K, any F is relatively prime to Cp; therefore gcd(F, Cp) = 1, so that starting from (F, Cp), there exists no fungus mutant that is fitter than a resident fungus. On the other hand, for any F, gcd(F, C') >= 1, where C ' is a cicada mutant, as compared to god(F, Cp) = 1, so that Fc(F, C') <= Fc(F, Cp), no prey mutant is fitter than a resident. In conclusion, any initial random choice of (F, C) and mutations fulfilling condition K will lead and lock to a prime C after a sufficiently large number of mutations (Markus, M., & Goles, E., 2002).

Those big, prime numbers might also minimize unfortunate hybridization between cicadas timed to breed on different cycles. When such cicadas' reproductive years coincide, any cross-breeding could doom offspring. Their half-brood genes could lead them to reproduce in some intermediate years between mom's and dad's regular cycle. Without the company of millions of pure-broods, hybrids would be easy pickings for predators and reproductive dead-ends for their family lineages. But with life spans of 13 and 17 years, the simultaneous emergence of broods on different schedules happens only once every 221 years (Milius, S., 2013).

Conclusion:

Based on the analysis, we concluded that the prime number patterns are indeed applied in the real world. For example, cicadas have a special internal clock pattern that an adult cicada emerges after either 13 years or 17 years depending on the geographical area. Our study covered some scientific explanations based on the Prime Number Theorem and further applications of the greatest common divisor and Euclid’s Theorem. However, our study does have weaknesses and limitations. This study only touched on the benefits of having this periodical pattern and comparisons to having a normal emergence. Further extensions may include more sophisticated algorithms when the years reach a possible limitation. Studies can also dig deeper into the reasons why a 13-year and a 17-year are the optimal survival strategies for cicadas with the knowledge of their lives and habits.

Yaqun: Through this project, I found the real-life applications of prime numbers and the greatest common divisors. To get to the reasons for one problem, we need to practice multiple possible theorems and combine them with natural patterns. This project also leads me to explore the connections of principles I learned from class and real-life, and how to apply the methodologies and algorithms to real-life problems.

David: This project has shown me that we can apply what we learn in our classes to real life problems that interest us or use for work we will be doing after we graduate. I also got to learn about cicadas, a species I had never heard of before and how they have such a unique trait that can only be explained by math and biologists. It's cool to see and look into how math can be used to describe and understand biological traits that these bugs have adapted over the course to ten of thousands of years. This project has definitely widened my view of the importance of math and all its real life applications.

Brendon: One of the biggest takeaways I had from this project is how number theory can be shockingly explicit in the real world. Many of the results of our analysis of periodical cicadas are due to their unique life cycle characteristics. These characteristics also allow number theory to be almost directly applied to understanding their behavior mathematically. LCM maximization as a survival strategy demonstrates a clear uniqueness to prime numbers. Conversely, we can see that GCD minimization also leads to more optimal outcomes and these seemingly theoretical ideas have clear and direct application in the studies of periodical cicadas.

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