Sample problems with solutions for Homework 10

1. Approximate $\sqrt{2}$ to within three decimal places using Newton's method.
2. Evaluate $\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3)\ dy\ dx$.
3. A uniform probability density function is defined on the region in the first quadrant bounded by $y = \sqrt{x}$ and $x = 4$. Find the probability that a pair (x, y) satisfies $y < 1 - \frac{1}{4}x$.
4. If you were converting a solid region to polar coordinates and its base is defined by $x^2 + y^2 \leq 0.64$, what should be the range for r and θ?
5. Find the volume of the solid bounded between the surfaces $f(x, y) = 4 - x^2 - 3y^2$ and $g(x, y) = 3x^2 + y^2$. You can see a visualization of this solid here `https://www.geogebra.org/3d/skmebp5c`

Solutions

1. To find the requested value, we need to think out side of the box a a little. Consider the function $f(x) = x^2 - 2$. This function has a root when $x = \sqrt{2}$. Thus, if we use Newton's method to find the zero of $f(x)$, then we will really have found an approximation for $\sqrt{2}$. Let's start with an initial guess of $x_0 = 2$. Newton's method will give us the next term as follows:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{4}{4} = 1$$

If we continue this process we will get the following sequence of approximations $\{1, 1.4166, 1.41421, 1.4142136\}$

2. If we try to integrate the function with the order of integration that is given, we will quickly see that this isn't possible as $\cos(y^3)$ is not integrable. Thus we must change the order of integration. One should get the following integral after changing the order:

$$\int_0^1 \int_0^{y^2} \cos(y^3)\ dx\ dy$$

Let's integrate!

$$\begin{aligned}
\int_0^1 \int_0^{y^2} \cos(y^3)\ dx\ dy &= \int_0^1 \left(x\cos(y^3)\Big|_{x=0}^{x=y^2} \right) dy \\
&= \int_0^1 y^2\cos(y^3) dx \\
&= \frac{1}{3}\sin(u)\Big|_0^1 \\
&= \frac{1}{3}\sin(1)
\end{aligned}$$

3. The region in the first quadrant bounded by $y = \sqrt{x}$ and $y = 4$ is shown here XXXBrendt insert url for Desmos page here. The probability density function is a constant c defined over this 2D region. So we need $\int_0^4 \int_0^{\sqrt{x}} c\, dydx = c \int_0^4 \int_0^{\sqrt{x}} dydx = 1$. Since the integrand is 1, the double integral actually represents the area of the region, which is $\int_0^4 \sqrt{x}\, dx = \frac{16}{3}$. This tells us that $c = \frac{3}{16}$. The region constrainted by $y < 1 - \frac{1}{4}x$ is shown here XXXBrendt insert link for second region here. This probabiity of this region can be defined either by integrating with respect to y on the inside integral, or vice versa. If the

4. The region is a disk of radius $r = 0.8$. So the region is covered by $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 0.8$.

5. The function $f(x, y)$ defines a downward-facing elliptical paraboloid and $g(x, y)$ defines an upward-facing elliptical paraboloid. As seen in the image, the intersection of these two surfaces looks like the edge of a saddle. To find the "top-view" formula of the region bounded by these two surfaces, we find their intersection, where $f(x, y) = g(x, y)$. This gives $4 - x^2 - 3y^2 = 3x^2 + y^2$. Solving gives $x^2 + y^2 = 1$, a disk of radius 1. This disk defines the region of integration in polar coordinates. The volume is given by the following integral, where R is the region of integration

$$\iint_R f(x, y) - g(x, y)\, dA = \iint_R 4 - x^2 - 3y^2 - (3x^2 + y^2)\, dA = \iint_R 4 - 4x^2 - 4y^2\, dA$$

$$= \int_0^{2\pi} \int_0^1 (4 - 4r^2)\, r dr d\theta = \int_0^{2\pi} \int_0^1 (4r - 4r^3)\, dr d\theta = 2\pi$$