Sample problems with solutions for Homework 4

1. Calculate $\displaystyle\int_3^7 x^2+\pi$ by evaluating it as the limit of a Riemann Sum.

2. For examples of finding critical values, for classifying critical values as local max/local min/neither, and for finding absolute extrema, see these pages:
`http://tutorial.math.lamar.edu/Classes/CalcI/CriticalPoints.aspx`
`http://tutorial.math.lamar.edu/Classes/CalcI/MinMaxValues.aspx`
`http://tutorial.math.lamar.edu/Classes/CalcI/AbsExtrema.aspx`

3. Use the following identity (which we will be able to verify soon) $\displaystyle\int_0^x t\sin(t^2)\,dt = -\frac{1}{2}\cos(x^2)+\frac{1}{2}$, evaluate the following:

   (a) $\displaystyle\int_0^{\sqrt{\pi}} t\sin(t^2)\ dt$

   (b) $\displaystyle\int_{\sqrt{\pi}}^{\sqrt{2\pi}} t\sin(t^2)\ dt$

   (c) $\displaystyle\int_{-\sqrt{3\pi}}^{\sqrt{3\pi}} t\sin(t^2)\ dt$

4. Graph the function $f(x)=2x+7$ on the interval $[-2,1]$, and use this to calculate $\displaystyle\int_{-2}^{1} f(x)\ dx$ **without** taking an antiderivative, but rather using basic geometry.

5. Write the following sum as a definite integral. Do not evaluate, (though soon we will learn a way to do this).

$$\lim_{k\to\infty}\sum_{i=0}^{k}\frac{1}{k}\cdot\frac{i}{k}\cdot e^{\frac{i}{k}}$$

Solutions

1. The sequence of right-hand end points of the interval is given by

$$x_i = 3 + i \cdot \frac{7-3}{n} = \frac{4}{n}, \quad i = 1, 2, \ldots, n$$

Note that the last point, when $i = n$ is

$$x_n = 3 + n \cdot \frac{4}{n} = 3 + 4 = 7 = b$$

Then we have

$$\begin{aligned}
A(n) &= \sum_{i=1}^{n} \overbrace{f\left(3 + \frac{4i}{n}\right)}^{\text{height}} \cdot \underbrace{\frac{4}{n}}_{\text{width}} \\
&= \sum_{i=1}^{n} \left(\left(3 + \frac{4i}{n}\right)^2 + \pi\right) \frac{4}{n} \\
&= \frac{4}{n} \sum_{i=1}^{n} \left(9 + 24\frac{i}{n} + 16\frac{i}{n^2} + \pi\right) \\
&= \frac{36}{n} \sum_{i=1}^{n} 1 + \frac{96}{n^2} \sum_{i=1}^{n} i + \frac{64}{n^3} \sum_{i=1}^{n} i^2 + \frac{4\pi}{n} \sum_{i=1}^{n} 1 \\
&= \frac{36}{n} \cdot n + \frac{96}{n^2} \cdot \frac{n(n+1)}{2} + \frac{64}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{4\pi}{n} n \\
A(n) &= 36 + 4\pi + 48\frac{n(n+1)}{n^2} + \frac{32}{3} \frac{n(n+1)(2n+1)}{n^3} \\
A(n) &= 36 + 4\pi + 48\frac{n^2 + O(n)}{n^2} + \frac{32}{3} \frac{2n^3 + O(n^2)}{n^3}
\end{aligned}$$

The actual area is equal to the limit of this function:

$$\int_3^7 x^2 + \pi \; dx = \lim_{n \to \infty} A(n) = 36 + 4\pi + 48 + \frac{64}{3}$$

2. Solutions to each example is given on the websites.

3. Let's do this:

(i) $\displaystyle\int_0^{\sqrt{\pi}} t \sin(t^2) \; dt = -\frac{1}{2} \cos((\sqrt{\pi})^2) + \frac{1}{2} = -\frac{1}{2}(-1) + \frac{1}{2} = 1$

(ii) $\displaystyle\int_{\sqrt{\pi}}^{\sqrt{2\pi}} t \sin(t^2) \; dt = \int_0^{\sqrt{2\pi}} t \sin(t^2) \; dt - \int_0^{\sqrt{\pi}} t \sin(t^2) \; dt = -1$

(iii) $\displaystyle\int_{-\sqrt{3\pi}}^{\sqrt{3\pi}} t \sin(t^2) \; dt = \int_{-\sqrt{3\pi}}^{0} t \sin(t^2) \; dt + \int_0^{\sqrt{3\pi}} t \sin(t^2) \; dt = 0$

4. Using this link, we can see the area is $9 + 9 = 18$, the area of a rectangle and a triangle. The integral is also computed in desmos to see this.

5. Writing the limit of a summation as a definite integral is definitely more tricky than the other direction, that is of writing an integral as a Riemann sum. This is made more difficult by the fact that there is more than one acceptable answer when writing the limit of a sum as an integral. However, there is a 'natural' answer.

   A tricky piece of this puzzle is the fact that the first term of this sum is 0 (when $i = 0$). Thus we can rewrite the sum as

   $$\sum_{i=1}^{k} \frac{1}{k} \cdot \frac{i}{k} \cdot e^{\frac{i}{k}}$$

   Looking at the definition of a Riemann sum, the first think we might identify is what part of this summation corresponds to $f(a + i\Delta x)$ and which piece corresponds to Δx. Δx, in this class, will always look like a constant over k (or whatever variable we're using), where the numerator is the length of the interval (why?). From this I gues $\Delta x = \frac{1}{k}$. Thus our integral will be over an interval of length 1.

   Again, there isn't one way to approach exercise, but since we don't see any + signs showing up, we might assume that $a = 0$, and since the length of the interval is 1, we deduce that $b = 1$.

   Now to find $f(x)$, we can replace $\frac{i}{k}$ with x everywhere in the summation (why?). This tells us $f(x) = xe^x$. This is everything we need. The limit of this sum is

   $$\lim_{k\to\infty} \sum_{i=1}^{k} \frac{1}{k} \cdot \frac{i}{k} \cdot e^{\frac{i}{k}} = \int_0^1 xe^x \; dx$$