**Sample problems with solutions for Homework 2**

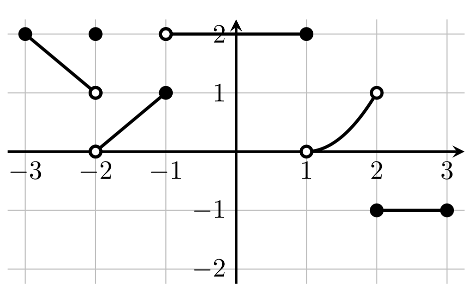
1. Rewrite the following expression using summation notation: $\ln\left(\prod_{k=1}^{4} x_k\right)$

2. Find explicit formulas for a sequence with the following recursive rules.
   (a) $a_1 = 9$ and $a_n = 8a_{n-1}$ for $n \geq 2$
   (b) $b_1 = 9$ and $b_n = 8nb_{n-1}$ for $n \geq 2$

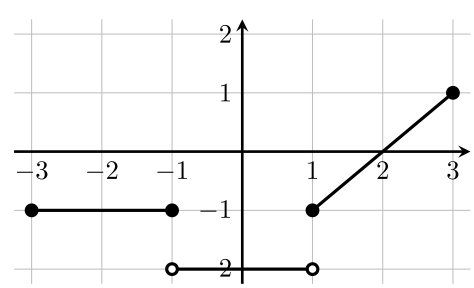
3. Each of the following sums is equal to exactly one of the others. Please find all of the pairs.
   (a) $\sum_{k=0}^{m+1}(3k+3)$
   (b) $\sum_{k=1}^{m}(k+2)$
   (c) $\sum_{k=2}^{m+3}(3k-5)$
   (d) $\sum_{k=2}^{m+2} k$
   (e) $\sum_{k=1}^{m+2} 3k$
   (f) $\sum_{k=0}^{m}(k+2)$
   (g) $\sum_{k=0}^{m+1}(3k+1)$
   (h) $\sum_{k=0}^{m-1}(k+3)$

4. Using the graphs below, find
   (a) $\lim_{x\to 1^-} g(f(x))$
   (b) $\lim_{x\to 1^+} g(f(x))$
   (c) $\lim_{x\to 1^-} f(g(x))$
   (d) $\lim_{x\to 1^+} f(g(x))$

**Graph of $f$**




**Graph of $g$**



# Solutions

1. Explicitly expanding the product notation to see the details fully fleshed out, we see:

$$\begin{aligned}
\ln\left(\prod_{k=1}^{4} x_k\right) &= \ln(x_1 \cdot x_2 \cdot x_3 \cdot x_4) \\
&= \ln(x_1) + \ln(x_2) + \ln(x_3) + \ln(x_4) \\
&= \sum_{k=1}^{4} \ln(x_k)
\end{aligned}$$

2. To find the pattern for the following, we write terms until we see a pattern arise.

   (a) (i) $a_1 = 9$
   (ii) $a_2 = 8 \cdot 9$
   (iii) $a_3 = 8 \cdot (8 \cdot 9)$
   (iv) $a_4 = 8 \cdot (8 \cdot 8 \cdot 9)$

   From this, we can see we have one more 8 multiplied each step, and the 9 is just along for the ride. Thus the general formula will be given by:

   $$a_n = 9 \cdot 8^{n-1}$$

   (b) This one is similar, just a little more work:

   i. $b_1 = 9$
   ii. $b_2 = 8 \cdot 2 \cdot 9$
   iii. $b_3 = 8 \cdot 3 \cdot (8 \cdot 2 \cdot 9)$
   iv. $b_4 = 8 \cdot 4 \cdot (8 \cdot 3 \cdot 8 \cdot 2 \cdot 9)$
   v. $b_5 = 8 \cdot 5 \cdot (8 \cdot 4 \cdot 8 \cdot 3 \cdot 8 \cdot 2 \cdot 9)$

   If we look carefully, we can see a factorial pattern arise, as well as an accumulation of 8s. There is also the extra 9 that is in each term. Putting this together, we get:

   $$b_n = 9 \cdot n! \cdot 8^{n-1}$$

3. To find which ones are equal, one could compute each sum (which would depend on $m$ or pick one and search through the others to find the match by comparing terms. I prefer the latter, at least for such a small sampling of summations.

   - $(a) = (e)$
   - $(b) = (h)$
   - $(c) = (g)$
   - $(d) = (f)$

4. (a) To compute $\lim_{x\to 1^-} g(f(x))$, we start by examining the behavior of $f(x)$ as $x \to 1^-$. We see that $f(x)$ has a constant value of 2. Substituting $x = 2$ into $g(x)$ gives 0, so $\lim_{x\to 1^-} g(f(x)) = 0$.

   (b) For $\lim_{x\to 1^+} g(f(x))$, we similarly examine the behavior of $f(x)$, but this time as $x \to 1^+$. We find $\lim_{x\to 1^+} f(x) = 0$. But notice that the $y$-values of $f(x)$ are approaching 0 from above, and thus decreasing towards 0. Since these are the values that are substituted into $g(x)$ we must find $\lim_{x\to 0^+} g(x)$, which gives $-2$.

   (c) This time, for $\lim_{x\to 1^-} f(g(x))$, we examine the behavior of $g(x)$ as $x \to 1^-$. We see $\lim_{x\to 1^-} g(x) = 2$. But notice that the $y$-values are a constant value of 2. These are the values that are being substituted into $f(x)$, so rather than taking a limit of $f(x)$, we just need the value of $f(-2)$, which is 2.

(d) For $\lim_{x\to 1^+} f(g(x))$, we first look at the behavior of $g(x)$ as $x \to 1^+$. We see this limit is $\lim_{x\to 1^+} g(x) = -1$. The $y$-values are approaching $-1$ from above, which means the values are decreasing towards $-1$. Substituting into $f(x)$, we see that we need $\lim_{x\to -1^+} f(x)$, which from the graph is 2.