1 A few more remarks on derivatives

- First, let's recap the first derivative test. This test to used to find where a function has a *local* maximum or minimum. The steps are:

  1. Find the critical numbers of $f(x)$. That is, find where the derivative is 0 or undefined.
  2. Make a number line, and mark the critical numbers you found. It may be a good idea to use solid dots to denote where $f(x)$ is continuous, and open dots otherwise. The first derivative test requires continuity.
  3. Look for sign changes at the critical points. You only need to check where $f(x)$ is continuous.

  WARNING! This test does not find *global* maximums and minumums. To find these, do the following:

  1. Find local max and mins using the first derivative test.
  2. If there is only one critical number, the argument is fairly short. We'll discuss this in class.
  3. Otherwise, further analysis is needed. A rough sketch is suggested.

- We will also look at some simple examples of what curve sketching with derivatives look like, but nothing too intense.

2 Jumping into integration

- Here is a link to something we will be looking at in class. `https://www.desmos.com/calculator/ywrjreihvx`
- $\int f(x)\ dx$. This is the antiderivative or indefinite integral. Both terms are interchangeable. It is the family of functions that have the property that the derivative is $f(x)$. Unlike differentiation, we are not guaranteed the existence of an antiderivative (well, a I'm leaving out a small something here which I'll mention in class).

  e^{-x^2} has no antiderivative..., well it does, but that is something for later.

  $\int x^2\ dx$ - this should be the function that we take the derivative of and get x^2. This is $\frac{x^3}{3}+C$
- $\int_a^b f(x)\ dx$. This is a definite integral. The geometric interpretation of this is the signed area between the x axis and the graph of the function on the interval $[a, b]$. This means two things:
  1. $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
  2. area above the curve is considered positive, the area below is considered negative.
- What is the difference between $\int f(x)\ dx$ and $\int_a^b f(x)\ dx$
- Let's clarify some notation that has a reputation for being confusing.
  - Question: what is the difference between $f(x) = x^2$ and $f(y) = y^2$ (or $g(\square) = \square^2$)?
  - Consider $A(x) = \int_a^x f(t)\ dt$.
  - Consider $A(t) = \int_a^t f(x)\ dx$
  - The former is the more commonly used in literature.

3 Riemann Sums

Definition 3.1. The Riemann sum of a function $f(x)$ on an interval $[a,b]$, using right hand end-points, is given given by

$$A(n) = \sum_{i=1}^{n} f(a + i\Delta x)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$.

- $\displaystyle\int_a^b f(x)\ dx = \lim_{n\to\infty} \overbrace{\sum_{i=1}^{n} f(a+i\Delta x)\Delta x}^{A(n)} = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(a+i\Delta x)\Delta x$, where $\Delta x = \dfrac{b-a}{n}$
- When $i = n$ (i.e the last endpoint of the righthand sum) $a + n \cdot \dfrac{b-a}{n} = b$
- Δx is the width of the bases of the rectangles.
- $a + i\Delta x$ gives us the sequenxe of end points. In words, this expression says "Start at a, and move i times to the right"

Example 3.1. Let's calculate the following integral by evaulating the limit of a Riemann sum. $\int_1^{10} 4x^2 - x + 1\ dx$

Let's break this down:

1. $\Delta x = \dfrac{9}{n}$

2. $a + i\Delta x = 1 + \dfrac{9i}{n}$

3. $f\left(1 + \dfrac{9i}{n}\right) = 4\left(1 + \dfrac{9i}{n}\right)^2 - \left(1 + \dfrac{9i}{n}\right) + 1$

4. $\displaystyle\int_1^{10} 4x^2 - x + 1\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{9i}{n}\right)\frac{9}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left[4\left(1 + \frac{9i}{n}\right)^2 - \left(1 + \frac{9i}{n}\right) + 1\right]\frac{9}{n}$

5. We will need some formulas:

   (i) $\displaystyle\sum_{i=1}^{n} 1 = 1 + 1 + \cdots + 1 = n$

   (ii) $\displaystyle\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

   (iii) $\displaystyle\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

6. Let's make sure that we understand what these formulas are. They are tools that let us calculate a whole class of sums without having to actually add up the sum. More specifically, they allow us to replace a summation with a formula. Let's do a quick example:

$$\sum_{i=1}^{9} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45 = 9(10)/2$$

Let's recall what we are calculating (note that we actually want the limit of this, but we'll thrown that in later).

$$\sum_{i=1}^{n} f\left(1 + \frac{9i}{n}\right)\frac{9}{n}$$

Let's do this:

$$
\begin{aligned}
\sum_{i=1}^{n}\left[4\left(1+\frac{9i}{n}\right)^2-\left(1+\frac{9i}{n}\right)+1\right]\frac{9}{n} &= \sum_{i=1}^{n}\left[4\left(1+\frac{18i}{n}+\frac{81i^2}{n^2}\right)-1-\frac{9i}{n}+1\right]\frac{9}{n}\\
&= \sum_{i=1}^{n}\left[4+63\frac{i}{n}+324\frac{i^2}{n^2}\right]\frac{9}{n}\\
&= \sum_{i=1}^{n}36\frac{1}{n}+9\cdot 63\frac{i}{n^2}+9\cdot 324\frac{i^2}{n^3}\\
&= \sum_{i=1}^{n}36\frac{1}{n}+\sum_{i=1}^{n}9\cdot 63\frac{i}{n^2}+\sum_{i=1}^{n}9\cdot 324\frac{i^2}{n^3}\\
&= \frac{36}{n}\sum_{i=1}^{n}1+\frac{567}{n^2}\sum_{i=1}^{n}i+\frac{2916}{n^3}\sum_{i=1}^{n}i^2\\
&= \frac{36}{n}\cdot n+\frac{567}{n^2}\frac{n(n+1)}{2}+\frac{2916}{n^3}\frac{n(n+1)(2n+1)}{6}
\end{aligned}
$$

Now taking limits, we will get

$$
\int_1^{10} f(x)\ dx = \lim_{n\to\infty}\left[\frac{36}{n}\cdot n+\frac{567}{n^2}\frac{n(n+1)}{2}+\frac{2916}{n^3}\frac{n(n+1)(2n+1)}{6}\right] = 36+\frac{567}{2}+\frac{2916\cdot 2}{6}
$$

4 Fundamental theorem of Calculus

Fundamental theorem of calculus

Suppose $f(x)$ is continuous on $[a, b]$. Then

(i) If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.

(ii) If $F' = f$ (that is F is any antiderivative of f), then $\int_a^b f(x)dx = F(b) - F(a)$.

(ii)* $\int_a^b f'(x)dx = \int_a^b \frac{d}{dx}[f(x)]dx = f(b) - f(a)$. If you integrate a derivative, you get the net change of the original function from a to b.

5 u-substitution

These two, u-sub and IBP are both, in some sense, the reverse process from chain rule and product rule respectively. To see this, recall one formulation of half of the fundamental theorem of calculus.

$$\int_a^b \frac{d}{dx}[f(x)]\ dx = f(x)\Big|_a^b = f(b) - f(a)$$

The only restriction on $f(x)$ is that it is continuous. Thus, we can replace $f(x)$ with $f(g(x))$ or $f(x)g(x)$, supposing $g(x)$ is also continuous. Let's look at $f(g(x))$ first.

$$\begin{aligned}\int_a^b \frac{d}{dx}[f(g(x))]\ dx &= f(g(x))\Big|_a^b \\ \int_a^b f'(g(x)) \cdot g'(x)\ dx &= f(g(x))\Big|_a^b\end{aligned}$$

Example 5.1. Suppose we are looking at the integral

$$\int_0^1 (2x+1)e^{x^2+x}\ dx$$

Then consider $f(x) = e^x$ and $g(x) = x^2 + x$. Then $f(g(x)) = e^{x^2+x}$ and $\frac{d}{dx}[f(g(x))] = \frac{d}{dx}[e^{x^2+x}] = f'(g(x) \cdot g'(x) = (2x+1)e^{x^2+x}$. Then we have

$$\begin{aligned}\int_0^1 (2x+1)e^{x^2+x}\ dx &= \int_0^1 \frac{d}{dx}\left[e^{x^2+x}\right] dx \\ &= e^{x^2+x}\Big|_0^1 \\ &= e^2 - e^0\end{aligned}$$

The last example is not what we would typically do to solve integrals of that type, but it does illustrate the fundamentals of why substitution.

- Pick a u. This will usually (not always) be inside of another function. That is, it will often be inside of a square root, or in an exponent or log.

- Compute the differential of u, $du = f'(x)dx$. For example, if $u = x^2+x$, then $du = (2x+1)dx$
- Look for a term that resembles du. You may be off by a constant. Make any replacements that you can.
- You may have extra non-u terms. Go back to your u sub and solve for x, and see if this replacement helps.

Example 5.2. $2\int_0^1 \frac{e^{\sqrt{x}}}{2\sqrt{x}}dx$

1. $u = \sqrt{x}$
2. $du = \frac{1}{2\sqrt{x}}dx$

$2\int_0^1 \frac{e^{\sqrt{x}}}{2\sqrt{x}}dx = 2\int_0^1 e^u du$

Example 5.3. $\int \sqrt{x-1} \cdot x \ dx = \int \sqrt{u} \cdot (u+1) \ du$

1. $u = x-1$; $x = u+1$
2. $du = dx$

5.1 Integration by Parts

Recall that $\int_a^b f'(x)dx = \int_a^b \frac{d}{dx}[f(x)]dx = f(b) - f(a)$. We used this fact along with the chain rule to show how and why u-sub works. We now apply this rule along side the product rule to show a result that is known as integration by parts. We start with this fact:

$$\int_a^b \frac{d}{dx}\left[f(x)g(x)\right] \ dx = f(x)g(x)\Big|_a^b = f(b)g(b) - f(a)g(a)$$

Further we can rewrite the left side of this:

$$\begin{aligned}
\int_a^b \frac{d}{dx}\left[f(x)g(x)\right] \ dx &= \int_a^b \left[f(x)g'(x) + g(x)f'(x)\right] dx \\
\int_a^b \frac{d}{dx}\left[f(x)g(x)\right] \ dx &= \int_a^b f(x)g'(x) \ dx + \int_a^b g(x)f'(x)dx
\end{aligned}$$

Putting this all together, we get

$$\underbrace{\int_a^b f(x)g'(x)\ dx}_{\text{integral we want to solve}} + \int_a^b g(x)f'(x)dx = f(x)g(x)\Big|_a^b$$

The goal for integration by parts is identify an integral we are given with the integral on the left and then hope that the second integral in the equation above is easy to solve. Usually we just write the equation above by solving for the integral on the left:

Integration by parts

$$\int_a^b f(x)g'(x)\ dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$$

Let $u = f(x)$ and $v = g(x)$, then $du = f'(x)\ dx$ and $dv = g'(x)\ dx$. Then IBP can be written:

$$\int \overbrace{f(x)}^{u}\overbrace{g'(x)\ dx}^{dv} = \overbrace{f(x)g(x)}^{uv} - \int \overbrace{g(x)}^{v}\overbrace{f'(x)dx}^{du} \iff \int u\ dv = uv - \int v\ du$$

Example 5.4 (Repeated application)**.** Evaluate $\displaystyle\int_1^5 x^2e^{2x}dx$

1. $u_1 = x^2$
2. $dv_1 = e^{2x}\ dx$
3. $du_1 = 2xdx$
4. $v_1 = \dfrac{1}{2}e^{2x}$
5. $$\int_1^5 x^2e^{2x}dx = \frac{1}{2}x^2e^{2x}\Big|_1^5 - \int_1^5 xe^{2x}\ dx$$
6. $u_2 = x$
7. $dv_2 = e^{2x}dx$

8. $du_2 = dx$

9. $v_2 = \frac{1}{2}e^{2x}$

10. $\int_1^5 xe^{2x}\ dx = \left(\frac{1}{2}xe^{2x}\Big|_1^5 - \int_1^5 \frac{1}{2}e^{2x}\ dx\right)$

11.
$$\int_1^5 x^2e^{2x}dx = \frac{1}{2}x^2e^{2x}\Big|_1^5 - \left(\frac{1}{2}xe^{2x}\Big|_1^5 - \int_1^5 \frac{1}{2}e^{2x}\ dx\right)$$

Example 5.5 (Boomerang)**.** Evalaute $\int \sin(x)e^x dx$

Example 5.6. Here is a list of integrals I present to you as a gift:

(i) $\displaystyle\int 2x^3 \cos\left(\frac{x}{3}\right) dx$

(ii) $\displaystyle\int (2x-3)^2 e^{\frac{x}{2}}\, dx$

(iii) $\int \ln(x)dx$

(a) $u = \ln(x)$; $du = \frac{1}{x}dx$

(b) $dv = dx$; $v = x$

(c) $\int u\ dv = \int \ln(x)dx = x\ln(x) - \int dx = x\ln(x) - x + C$

(iv) $\int 2x^4 \ln(x)dx$

(v) $\int \arcsin(x)dx$

(vi) $\int x^3 3^x dx$

(vii) $\int \frac{\ln(x)}{x^2} dx$

(a) $u = \ln(x)$; $du = \frac{1}{x}\ dx$

(b) $dv = \frac{1}{x^2}\ dx$; $v = -\frac{1}{x}$

(c) $\int \frac{\ln(x)}{x^2} dx = uv - \int -\frac{1}{x^2}\ dx$

(a) $dv = \frac{\ln(x)}{x}\ dx$

(b) $u = \frac{1}{x}$

(viii) $\int (\ln(x))^2 dx$

(ix) $\displaystyle\int \cos(2x)e^{3x}dx$

6 Taking the fundamental theorem a step further

By now, you can hopefully take the derivative of something that looks like the following:

$$\frac{d}{dx}\left[\int_a^x f(t)\ dt\right] = f(x)$$

$$A(x) = \int_a^x \sin(t) + t\ dt = -\cos(t) + \frac{t^2}{2}\bigg|_a^x = (-\cos(x) + \frac{x^2}{2}) - (-\cos(a) + \frac{a^2}{2})$$

$$A'(x) = \sin(x) + x$$

Okay, but what if we want to take the derivative of something that looks like the following:

$$G(x) = \int_a^{\sqrt{x}+\sin(x)} t^2 - t\ dt$$

1. $b(x) = \sqrt{x} + \sin(x)$; $b'(x) = \dfrac{1}{2\sqrt{x}} + \cos(x)$
2. $A(x) = \displaystyle\int_a^x t^2 - t\ dt$; $A'(x) = x^2 - x$ by FTC
3. $G(x) = A(b(x))$
4. $G'(x) = A'(b(x)) \cdot b'(x) = \left((\sqrt{x} + \sin(x))^2 - (\sqrt{x} + \sin(x))\right) \cdot \left(\dfrac{1}{2\sqrt{x}} + \cos(x)\right)$

Example 6.1. Find the derivative of the following function:

The general rule for the derivative of $G(x) = \displaystyle\int_{a(x)}^{b(x)} f(t)\ dt$ is thus:

$$G'(x) = f(b(x)) \cdot b'(x) - f(a(x)) \cdot a'(x)$$

When $b(x) = x$ and $a(x) = a$ (constant a)

$G'(x) = f(x) \cdot 1 - f(a) \cdot 0 = f(x)$

7 L'Hôpital's Rule

L'Hôpital's Rule

If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

The new limit may again be an indeterminate form as above. In that case, the rule may be applied again.

Use L'ôpital's rule to find the following limits.

Example 7.1. $\lim_{x\to 0} \frac{\sin(x)}{x}$

Example 7.2. $\lim_{x\to\infty} \frac{-6x^2}{e^x} = \lim_{x\to\infty} \frac{-12x}{e^x} = \lim_{x\to\infty} \frac{-12}{e^x} = 0$

Example 7.3. $\lim_{x\to\infty} \dfrac{2x^2 - 7x - 2}{-7x^2 + 4x - 1}$

Example 7.4. Evaluate $\lim_{x\to 0} x \cdot \ln(x)$

Example 7.5. $ab = \dfrac{a}{\frac{1}{b}} = \dfrac{b}{\frac{1}{a}}$

Evaluate $\lim_{x\to 0^+} x \cdot \ln(x) = \lim_{x\to 0^+} \dfrac{x}{\frac{1}{\ln(x)}} = \lim_{x\to 0^+} \dfrac{\ln(x)}{\frac{1}{x}}$

Example 7.6. $\lim_{x \to -\infty} -xe^x$

Example 7.7. $\lim_{x \to 1} \frac{x^{\frac{1}{3}} - 1}{x^{\frac{1}{2}} - 1}$

Example 7.8. $\lim_{x\to 2} \frac{x^3 - 8}{(2x)^{\frac{1}{2}} - 2}$

Example 7.9. $\lim_{x\to 1^+} x^{\frac{1}{x-1}} = 1^\infty = 1$

Example 7.10. $\lim_{x \to 0} x^x$

Example 7.11. $\lim_{x \to 0^+} x^x$