1 Introduction

1.1 About me

- Some things I love:
  - Gardening
  - Walking
  - My job at Splunk
  - Backgammon and Ultimate Tic Tac Toe (Board games in general)
  - Avid proponent of Stoicism
- Some music I like:
  - Favorite artist: Marina (formerly Marina and the Diamonds)
  - Anima! (Blood, Aorta)
  - Regina Spektor
  - Paul Simon
  - My brightest diamond
  - Jazz music
  - A whole assortment of odd music.

1.2 Psychological Safety

The most important principle in this class, and one that I take VERY seriously, is that of psychological safety. This is a general practice used in many working environment, but is suited especially well for mathematics. What does psychological safety mean in the context of a math class?

- You should always feel comfortable asking a question. It does not matter if you've asked the question multiple times before or if the question is about pre-calculus. I don't care about what you don't know; I care about you learning and growing, and you asking questions is critical for me to be able to help you. You should feel safe saying "I don't know."

- Your opinion should feel validated and important, even in the light of incorrect work. Getting things wrong has been terribly stigmatized in much of the world (especially in math), but this has had a detrimental effect. Far more insight is gained from getting things wrong and understanding why than from writing down only correct work. Be brave, make suggestions. It is incredibly rare that someone will think negatively of it, and many people will thank you for asking or suggesting what you do. I've met no more than 1 or 2 students in my teaching career who scoffed at an other's question or comment and I did not and will not tolerate it. I promise you a class where suggestions are appreciated and validated.

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1.3 About my teaching style

- Communication, communication, communication!!!
  - Communicating with me, via email and video meetings. Please meet with me, I am very flexible. I prefer video meetings to email for many reasons, but either way, stay in contact.
  - Communication among each other. Collaboration is such a good thing on so many levels.
  - Communicating the ideas we learn. It is my opinion that the ability to tell a story is incredibly valuable, for learning and for career advancement.

  knowledge.

- Math is hard, and there is inevitable frustration that comes along with learning. In addition, this course is very fast paced. I promise to be inexhaustibly patient with each and everyone of you, but please be prepared for a challenging course. I'm fairly flexible with due dates, within reason; The sooner you ask, the better. I understand that you have jobs, families, and a life that is far more important than any math knowledge. However, if you are to get out of this class what you need, I ask that you put in a serious effort. I hope we can achieve a balance of seriousness and non-seriousness.

- `https://www.desmos.com`, `https://www.geogebra.org/3d?lang=en`, `https://www.wolframalpha.com`, and `https://www.symbolab.com`. All great tools that you should utilize.

- Some discussion to have about midterm and final structure. Timed test or longer term take-home project

1.4 Algebra

- $(xy)^n = x^n y^n$ Warning $(x+y)^n \neq x^n + y^n$
- $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$
- $\sqrt{xy} = \sqrt{x}\sqrt{y}$. Warning $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$
- $\dfrac{A}{C} + \dfrac{B}{C} = \dfrac{A+B}{C}$. Warning: You can't split fractions over plus or minus in the denominator.
- $\dfrac{a \pm b}{c+d} = \dfrac{a}{c+d} \pm \dfrac{b}{c+d}$

1.5 logs and exponential

Please be familiar with the basic properties of logs and exponents. Also be aware of the following equivalent notation for exponential functions:

$$e^x = \exp(x)$$

- $\ln(AB) = \ln(A) + \ln(B)$.
- $\ln(A^n) = n\ln(A)$. Be careful: $\ln(A^n) \neq \ln(A)^n$.
- $\ln\left(\dfrac{A}{B}\right) = \ln(A) - \ln(B)$
- Catch-all: $\ln\left(\dfrac{A^n B}{C^k}\right) = n\ln(A) + \ln(B) - k\ln(C)$
- $\exp(A+B) = \exp(A)\exp(B)$. Equivalently, $e^{x+y} = e^x e^y$
- $\exp(A)^n = \exp(An)$. Equivalently, $(e^x)^n = e^{xn}$
- $\exp(\ln(x)) = \ln(\exp(x)) = x$. We have to be careful here though - the first expression here is not defined for $x \leq 0$, whereas the second is, so my equal sign is technically false. The equality is true however for $x > 0$. What is really important here is understanding that these two functions are inverses of each other.

1.6 Sigma notation

Sums show up all over the place in math - there is no escape. The earlier you get used this notation the better. Let's start off with a simple example:

Example 1.1.

$$\sum_{k=1}^{5}(k^2-k) = (1^1-1)+(2^2-2)+(3^2-3)+(4^2-4)+(5^2-5) = 40$$

A little more verbose:

$$\sum_{k=1}^{5}(k^2-k) = \overbrace{(1^1-1)}^{k=1}+\overbrace{(2^2-2)}^{k=2}+\overbrace{(3^2-3)}^{k=3}+\overbrace{(4^2-4)}^{k=4}+\overbrace{(5^2-5)}^{k=5} = 40$$

Example 1.2. $\displaystyle\sum_{k=1}^{4}(x_k+\sqrt{k}) = (x_1+\sqrt{1})+(x_2+\sqrt{2})+(x_3+\sqrt{3})+(x_4+\sqrt{4})$

Example 1.3. There are are a class of identities that we will see occassionly. The HW refers to these as formulas for special sums:

- $\displaystyle\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{k=1}^{n} k^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{k=1}^{n} k^3 = 1^3+2^2+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$

We will work through these in class.

A critical skill when dealing with sums will be *re-indexing*. This is the act of rewriting a sum, without changing any of the terms of the sum, and thus the total as well. Let's look at a simple example:

Example 1.4. Notice the following two 'different' sums

- $\displaystyle\sum_{k=1}^{4} 2^{2k} = 2^2+2^4+2^6+2^8$

- $\displaystyle\sum_{k=3}^{6} 2^{2k-4} = 2^2 + 2^4 + 2^6 + 2^8$

1.7 Sets

A *set* is a collection of objects. In this class, most sets will be a collection of numbers. A set is written (most often) with a set of curly brackets. The general structure of a set is as follows:

$$\{\overbrace{\text{something}}^{\text{type of objects}} \mid \overbrace{\text{something else}}^{\text{Rule or restriction}}\}$$

Example 1.5. The following are examples of sets. The first few sets don't have braces because they are so fundamental.

1. $\mathbb{R}$. The set of all real numbers. I don't care much for the term 'real' numbers, but its standard. This is the collection of all non-complex (not imaginary) numbers. Think an infinitely long line.

2. $[0, 1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$. Let's break this down. The left side of the vertical bar tells us the type of object we're looking at – real numbers, which we denote with an x. The right side tells us which numbers we're keeping (or we could say which ones to throw out too).

3. $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. This is the plane, the set of all ordered pairs.

4. $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. This is just the set of integers. Note that there is no description on the right side in the braces, because no descriptions was needed.

5. $\mathbb{N} = \{0, 1, 2, \ldots\}$. This is the set of natural numbers. Some people don't include 0. It doesn't really matter, the lack of negative numbers is what is really important. Again, no description is needed.

6. $\mathrm{T} = \{x \in \mathbb{R} \mid x^2 = 1\}$. In this case, we are saying keep all real numbers x that square to 1. In this case, there are only 2 real numbers that this is true. Thus we can write the set T as follows:

$$T = \{-1, 1\} = \{x \in \mathbb{R} \mid x^2 = 1\}$$

Some context: We might have encountered this set if we had asked a question such as "What are the roots of the function $f(x) = x^2 - 1$?" They are the values of x that make $x^2 - 1 = 0$.

7. Jack $= \{\triangle \in \mathbb{R} \mid \triangle^2 = -1\}$. In this case, there are no numbers $\triangle$ that satisfy the relation on the right. Thus, this set is empty - it doesn't contain anything. Think an empty list in python.

$$\text{Jack} = \{\} = \emptyset$$

8. $L = \{(x, y) \in \mathbb{R}^2 \mid x = 1, y > 2\}$. Let's look at this is desmos. Note that this set forms a ? dimensional object.

9. $\zeta = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 4y^2 < 9\}$. Let's look at this is desmos. Note that this set forms a ? dimensional object.

10. $\square = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$. Let's look at this is desmos.

I don't want to overemphasize sets, but I also think a complete graduate understanding of calculus should include them.

1.8 Functions

There are 4 components of a function:

1. A <u>set</u> of input things, which are often represented by a variable.
2. A <u>set</u> of output things, represented by a different variable than the input.
3. A rule for taking input things to output things, with only one stipulation: that one input must yield *exactly* one output
4. A name for this rule.

Here is the format I write for functions when I want to be very clear:

$$\text{name} : \{\text{Input objects}\} \longrightarrow \{\text{Output objects}\}; \text{rule}$$

Example 1.6. A function in calculus you might probably have seen is $f(x) = x^2 + 1$. If I wanted to write this out fully, I would write

$$f : \mathbb{R} \to \mathbb{R}; x \mapsto x^2 + 1$$

This notation emphasizes that we start with a number (something in $\mathbb{R}$) and end up with a number (something else in $\mathbb{R}$). With this notation, we can restrict which inputs we consider or can describe the outputs in greater detail. For instance, this function could also be written as

$$f : \mathbb{R} \to [1, \infty); x \mapsto x^2 + 1$$

Example 1.7. Consider the function $f(x, y, z) = (xy, e^{yz})$. Note that the inputs for this function are ordered triples, and the outputs are pairs of number. Written formally:

$$f : \mathbb{R}^3 \to \mathbb{R}^2; (x, y, z) \mapsto (xy, e^{yz})$$

Example 1.8. Consider the rule $h : \{\text{days}\} \to \{\text{people}\}; d \mapsto \text{person whose birthday is } d$

Is this a function? What if we 'flip' the domain and codomain around?

Example 1.9. Here are few functions:

1. $h : \mathbb{R}^2 \to \mathbb{R}; (x, y) \mapsto x^2 + xy$. This would be written shorthand as $h(x, y) = x^2 + xy$
2. $D : \mathbb{R}^2 \to \mathbb{R}^3; (x, y) \mapsto (x + y, x - y, \ln(xy))$. Again, we could write this as $D(x, y) = (x + y, x - y, \ln(xy))$

One reason I emphasize all of this notation is to keep in mind when we can or cannot visualize things. When can we make a graph of function? What is the difference and/or relationship between a functions and a graph. There are many subtleties that don't show up for a while, but when they do can be a source of great frustration.

2 Derivatives

2.1 Difference Quotient

The fundamental pre-calculus object that leads to derivatives is the *difference quotient*. We cannot talk about a difference quotient unless we have a function at hand that we are analyzing. It doesn't make sense to talk about just 'a difference quotient'. The difference quotient is a function that is *derived* from another function. Further, it is a function that tells us about the function from which it is derived.

Definition 2.1 (Difference Quotient)**.** Given a function $f(x)$ and a fixed point a, the difference quotient of $f(x)$ at a is given by

$$v_a(h) = \frac{f(a+h) - f(a)}{h}$$

The difference quotient $v_a(h)$ represents the average rate of change of the function $f(x)$ starting at 'time' a over h units of time.When we look at an average rate of change, we must pick a time to start averaging and a length of time.

In calc 1 world, we don't think of a as a variable. We only think of a as a constant. As written above, the difference quotient is a function of a single variable, namely h. However, we could think of the difference quotient as a function of two variables:

$$v(a, h) = \frac{f(a+h) - f(a)}{h}$$

This is overkill at this point, but emphasizes the following important moral:

Moral 2.1. Not only do we need a function before we can talk about a difference quotient, but we also need a starting point.

We will work through this desmos link together: `https://www.desmos.com/calculator/cul5r7ojvn`

2.2 Derivatives the hard way: limit definition of the derivative

Let's just jump into right into the formal definition of the derivative.

Definition 2.2. The derivative of a function $f(x)$ is the function defined as a limit of the function's difference quotient:

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

A few things to note:

- Just as with the difference quotient, we need a function before we can meaningfully talk about a derivative. This is why we reuse the name, f, and only add a prime to distinguish the derivative as a separate function.

- The inputs to $f(x)$ and $f'(x)$ are the same - both input the same variable x. The outputs though measure different things. A function $f(x)$ tells us about some value at 'time' x, where $f'(x)$ tells us about how the function $f(x)$ is changing at that 'time' x.

2.3 NOTATION!!!

The following are some different ways to write the derivative of a function $y = f(x)$

$$\frac{d}{dx}[f(x)] = f'(x) = \frac{dy}{dx} = \frac{dy}{dx}(x) = \frac{df}{dx} = y'$$

I don't care much for the last one, especially when we move to multivariate functions, but nevertheless, it is still a valid notation you may use.