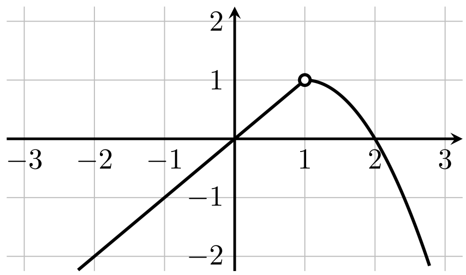
1 Recap

Many calculus classes will start discussions about limits with graphs that look like this:

Graph of f



If you are anything like me, this feels very contrived. What kind of graph is defined everywhere except one spot where there is a missing value? The difference quotient[1] is exactly the type of function whose graph looks like this. Further, the missing value in the difference quotient has a very important interprertation - it is the topic of calc 1.

[1]of a function $f(x)$ - we can only talk about a difference quotient of some existing function

2 Derivatives the less hard way: Rule-following game

Derivatives

- $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$
- $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) + g'(x)$
- $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
- $\frac{d}{dx}[g(f(x))] = g'(f(x)) \cdot f'(x)$
- $\frac{d}{dx}[a^x] = \ln(a)a^x$
- $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$
- $\frac{d}{dx}[\sin(x)] = \cos(x)$
- $\frac{d}{dx}[\cos(x)] = -\sin(x)$
- $\frac{d}{dx}[\tan(x)] = \sec^2(x)$
- $\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$
- $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$
- $\frac{d}{dx}[\cot(x)] = -\csc^2(x)$
- $\frac{d}{dx}[\log_a(x)] = \frac{1}{\ln(a)x}$
- $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

Being as sensitive as possible and using a word I rarely use, derivatives of polynomials are easy once you've done 10 of them. The biggest mistake people make is confusing polynomials with exponentials. $a^x \neq x^a$. Please remember the following:

$$\frac{d}{dx}[a^x] \neq x \cdot a^{x-1}$$

Example 2.1. Looking at some examples:

1. $\dfrac{d}{dx}\left[4x^3 - \pi x^\pi + 3\right] = 4\dfrac{d}{dx}\left[x^3\right] - \pi\dfrac{d}{dx}\left[x^\pi\right] + \dfrac{d}{dx}\left[4\right] = 4 \cdot 3x^2 = \pi \cdot \pi x^{\pi-1} + 0$

2. Watching for exponentials: $\dfrac{d}{dx}\left[x^{1.1} - (1.1)^x\right] = 1.1x^{0.1} - \ln(1.1) \cdot (1.1)^x$

I would like to jump right into chain rule, but first would like to do some practice together with function composition.

Example 2.2. We will do examples together as a class, but first, let's list out some 'elementary' functions.

- e^x
- $\ln(x)$
- x^n
- $\sqrt{x} = x^{\frac{1}{2}}$; $(\sqrt{x})^2 = (x^{\frac{1}{2}})^2 = x$. This is false. It is almost true.
- $\sin(x), \cos(x), \tan(x), \ldots$
- a^x (e^x is just a special case of this function).

Now let's talk through and construct some examples:

1. Consider $h(x) = \sqrt{\sin(x)}$ and consider $g(x) = \sin(x)$ and $j(x) = \sqrt{x}$. Then

$$h(x) = j(g(x))$$

2. $f(x) = x^2 + \ln(x)$ and $g(x) = \sin(x)$, then

$$p(x) = f(g(x)) = (\sin(x))^2 + \ln(\sin(x))$$

3. Consider $\square(x) = \ln(\sqrt{\arccos(x)})$. Then we can write this function as the composition of the following elementary functions.

   - $f_1(x) = \ln(x)$
   - $f_2(x) = \sqrt{x}$
   - $f_3(x) = \arccos(x)$

$$\square(x) = f_1(f_2(f_3(x)))$$

Example 2.3. Suppose we want to find the derivative of $q(x) = 2^{x^2+3}$. This function is not elementary (not a polynomial, e^x, $\ln(x)$, etc) so we can't use any of the explicit rules above. However, since this function is a composition, we will know to use chain rule. Chain rule is one that many people will eventually do on the fly, but at first I encourage you to be careful and explicit. Before we start, let's recall the chain rule:

$$\frac{d}{dx}[f(g(x)) = f'(g(x)) \cdot g'(x)$$

The key here is to rewrite the function $q(x)$ as the composition of two functions. Let's define $f(x) = 2^x$ and $g(x) = x^2 + 3$.

$$q(x) = f(g(x)) = 2^{x^2+3}$$

Note that we *can* take the derivatives of these functions using the rules above, so let's list these all now, along with one special term we will need:

- $f(x) = 2^x$
- $f'(x) = \ln(2) \cdot 2^x$
- $g(x) = x^2 + 3$
- $g'(x) = 2x$
- $f'(g(x)) = \ln(2) \cdot 2^{x^2+3}$

Then $q(x)$ can now be written as the composition of $f(x)$ and $g(x)$.

$$f(g(x)) = 2^{x^2+3}$$

We now have everything we need to compute the derivative of $q(x)$:

$$\frac{d}{dx}[q(x)] = \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) = \overbrace{\left(\ln(2) \cdot 2^{x^2+3}\right)}^{f'(g(x))} \cdot \overbrace{(2x)}^{g'(x)}$$

I like writing derivatives as $\frac{d}{dx}[f(x)]$ because I find it easier to play the derivative game. What I mean by this is that we often have to apply a sequence of rules, and the shorter notation can make things confusing. In particular, I like to think of the product rule as follows:

$$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)]\, g(x) + f(x)\frac{d}{dx}[g(x)]$$

rather than

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Let's do an example to illustrate what I mean by this.

Example 2.4. Compute the derivative of $f(x) = e^{\sin(x)} \cdot x^3$.

(I) First, I see a product, so my first thought is product rule. Writing as carefully as possible:

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[e^{\sin(x)}\right] \cdot x^3 + e^{\sin(x)} \cdot \frac{d}{dx}\left[x^3\right]$$

(II) Okay, I know how to handle the expression $\frac{d}{dx}\left[x^3\right]$ in the second term because x^3 is a fundamental function. However, the first term requires an application of the chain rule. Let's calculate this now.

(III) I'm trying to compute $\frac{d}{dx}\left[e^{\sin(x)}\right]$ and I know I need to use chain rule because I see function composition. I'll write this function as a composition first. I choose $j(x) = e^x$ and $k(x) = \sin(x)$. Then $e^{\sin(x)} = j(k(x))$. It is a good idea to write these down, as well as their derivatives:

- $j(x) = e^x$
- $j'(x) = \ln(e^1) \cdot e^x = e^x$
- $k(x) = \sin(x)$
- $k'(x) = \cos(x)$
- $j'(k(x)) = e^{\sin(x)}$

(IV) I can now write the derivative of this composition:

$$\frac{d}{dx}\left[e^{\sin(x)}\right] = \frac{d}{dx}\left[j(k(x))\right] = j'(k(x)) \cdot k'(x) = e^{\sin(x)} \cdot \cos(x)$$

(V) And now I can do the entire original derivative:

$$\frac{d}{dx}\left[f(x)\right] = \frac{d}{dx}\left[e^{\sin(x)}\right] \cdot x^3 + e^{\sin(x)} \cdot \frac{d}{dx}\left[x^3\right] = e^{\sin(x)} \cdot \cos(x) \cdot x^3 + e^{\sin(x)} \cdot 3x^2$$

People ask me all the time what they can do to get better in math. An acceptable answer is always "do as many examples as you can tolerate, and then do a few more." This is especially true of derivatives. Computing derivatives is just rule following and practice is the best method to see this. In theory, you should be able to come up with an infinite number of examples on your own.

Example 2.5. Needs a home: Please, please do not write $e^{-\infty} = 0$. Instead, just say, as $x \to \infty$, e^{-x} approaches 0.

3 Using derivatives

There are many questions we can ask about the graph of some function:

1. "Is the graph of this function is positive, zero, or negative?" There are two situations where we ask this question:

   (a) when we are looking at a function at a point.

   (b) when we are looking at a function on an interval.

2. "Is this function increasing, decreasing, or constant?" –We can definitely ask this questions about a function on an interval, but what do you think about at a point? Discuss.

3. "Does this function have a zero (root, x-int)?" and "What is the nature of these roots?" More specifically, does the function of the graph go through the x-axis, going from positive to negative (or the other way around) or does it 'bounce'?

4. "Is this graph concave up or concave down?"

What is going to make things difficult is that we will be using the same words (adjectives) in certain situations, but the significance and inference we draw from these words will be very different. This leads us to an important moral:

Moral 3.1. Make a strong effort to eradicate the word 'it' from your mathematical vocabulary. This will become more pressing later in the semester.

Let's now look at a table that you should always have ready to look at.

Useful table

$f(x)$	$\pm$	inc/dec	con. up/down
$f'(x)$	N/A	$\pm$	inc/dec
$f''(x)$	N/A	N/A	$\pm$

4 Recapping Derivatives

- A common error is for people to say "the derivative is the tangent line of a function." This is not quite correct. The actual statement should be "The derivative gives the slope of the tangent line to $f(x)$".
  - A line is function.
  - The slope of a line is a number.
  - The derivative is a function (that outputs a number, namely the slope of the tangent line).
- Fact - it is much easier, in general, to determine whether a function is positive/negative than it is to determine if the function is increasing or decreasing. This is especially true when we aren't looking at a graph. The great thing about derivatives is it allows us to make hard questions easier:

$$f(x) \text{ is increasing (decreasing)} \iff f'(x) \text{ is positive (negative)}$$

  We could make a similar statement about when $f(x)$ is concave up/down.
- Let's get some notation things out of the way. We have seen that the first and second derivative are denoted by $f'(x)$ and $f''(x)$ respectively. This is fine, but what if we want to talk about the 7th derivative. I personally do not want to write $f'''''''(x)$. If we want to talk about the nth derivative of a function, then we denote it by $f^{(n)}(x)$. So we could write $f'(x) = f^{(1)}(x)$. This notation will be indispensable when we get to Taylor Series.
- $\dfrac{d^n}{dx^n}[f(x)] = f^{(n)}(x)$
- First derivative test: If $f'(x)$ changes sign at a point where the derivative is 0 or undefined, and $f(x)$ is continuous, then $f(x)$ has a local max or min at that point.

Example 4.1. Consider the function $f(x) = 2x^3 + 3x^2 - 120x + 60$. We want to know when this function is increasing? And when is $f(x)$ decreasing?

First thing is take a derivative

$$f'(x) = 6x^2 + 6x - 120 = 6(x^2 + x - 20) = 6(x-4)(x+5)$$

This is zero when $x = 4$ and $x = -5$

Example 4.2. Suppose we took the derivative of some function $j(x)$, and the following was the result:

$$j'(x) = \frac{(x-3)(x+1)}{(x-1)(x-2)}$$

$\vec{x} = (x_1, x_2, \ldots, x_n)$

When is $j(x)$ increasing?