Sample problems with solutions for Homework 7

1. Evaluate the sum $\sum_{j=1}^{\infty} a_j$ (or determine that it diverges) given that the formula for its nth partial sum is $S_n = \dfrac{5+e^n}{n}$

2. Use a term-size comparison to determine whether $\sum_{n=1}^{\infty} \dfrac{1}{ne^n}$ converges.

3. Which comparison test is best used to determine if the series $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{2n+1}}$ converges?

4. Determining whether a series converges requires you to choose which tool best applies (divergence test, comparison test, limit comparison test, geometric series test, alternating series test). This website gives a strategy for choosing tests:
`http://tutorial.math.lamar.edu/Classes/CalcII/SeriesStrategy.aspx`

5. Find the sum of the series
$$\sum_{n=2}^{\infty} \frac{e^{2n+1}}{5\pi^{3n-2}}$$

6. Find a power series representation of the function $f(x) = \dfrac{1}{x+c}$

7. Find a power series representation of the function $f(x) = \dfrac{x}{(1-x)^2}$

8. Find the power series representation of $f(x) = \ln(1+\sqrt{x})$

Solutions

1. The point of this example is to keep straight these quantities:

   - a_j, which are the individual terms of the sequence,
   - S_n, which is the nth partial sum, meaning the sum of the first n terms
   - S, the sum of the infinite series.

   Note that
$$S = \sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} S_n$$

   Don't get the individual terms a_j mixed up with the partial sums S_n. For this problem, to find $S = \sum_{j=1}^{\infty} a_j$, just find the limit of S_n, since we've already been given a formula for the partial sums. We must find:
$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{5+e^n}{n}$$

   Informally, since the limit is of the form "$\dfrac{\infty}{\infty}$" and the numerator grows at a faster rate than the denominator, the value of this limit will be ∞. For a formal solution, using L'Hôpital's rule, we get $S = \lim_{n\to\infty} \dfrac{e^n}{1} = \infty$.

2. Let's define $a_n = \dfrac{1}{ne^n}$. There are two initial choices one might choose to compare the sequence a_n to. One is $b_n = \dfrac{1}{n}$ and the other $c_n \dfrac{1}{e^n}$. The following inequalities are both true:

$$a_n = \frac{1}{ne^n} < \frac{1}{n} = b_n$$

and

$$a_n = \frac{1}{ne^n} < \frac{1}{e^n} = c_n$$

However the, if we consider the corresponding series for b_n and c_n, we see that

$$\sum_{n=1}^{\infty} \frac{1}{n} \to \infty \qquad \text{Harmonic series diverges}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{1}{1-e^{-1}} \qquad \text{Geometric series with } r = \frac{1}{e}$$

A mantra I use is "smaller than small is good", meaning if the inequality we have is $<$, we want to compare to a finite series. Thus we can use direct comparison of a_n with c_n to conclude that $\sum_{n=1}^{\infty} a_n$ is convergent.

3. Note that we cannot use direct comparison to approach this series, as the inequality

$$\frac{1}{\sqrt{2n+1}} < \frac{1}{\sqrt{2n}}$$

is not a useful one, as the series of $\dfrac{1}{\sqrt{2n}}$ is divergent (smaller than big is a bad comparison). This tells us we might want to use the limit comparison test. Let's try it with the same comparison sequence. The limit of the the two sequences is

$$\lim_{n\to\infty} \frac{\sqrt{2n+1}}{\sqrt{2n}} = 1$$

Since the limit of the ratio of the two sequences is a finite number, we can conclude, using the limit comparison test, that $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{2n+1}}$ diverges.

5. We can confirm that this is a geometric series by taking the ratio of consecutive terms. Let $a_n = \dfrac{e^{2n+1}}{5\pi^{3n-2}}$. Then $\dfrac{a_{n+1}}{a_n} = \dfrac{e^{2(n+1)+1}}{5\pi^{3(n+1)-2}} \cdot \dfrac{5\pi^{3n-2}}{e^{2n+1}} = \dfrac{e^2}{\pi^3} < 1$. The fact that the ratio is a constant confirms it is a geometric series, and the ratio less than 1 means the series confirms. Note: if the functions of n in the exponents are all linear, then the series will be geometric. The sum of a convergent geometric series is $\dfrac{\text{first term}}{1-\text{ratio}}$. Here, noting that the first term is when $n = 2$ the first term is $\dfrac{e^5}{5\pi^4}$. The sum is thus $\dfrac{e^5}{5\pi^4 \cdot (1-\frac{e^2}{\pi^3})}$

6. One of the most fundamental series we encounter is

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k,$$

which holds for $|x| < 1$. If we do some factoring in the denominator, we can rewrite $f(x)$ as

$$f(x) = \frac{1}{c} \cdot \frac{1}{1-(-x)}$$

$$\frac{1}{c} \cdot \frac{1}{1-(-x)} = \frac{1}{c}\sum_{k=0}^{\infty}(-x)^k = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{c}$$

7. We know the power series representation $\dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k$ holds for $|x| < 1$ (shown using the geometric series test). While we may be tempted to just square both sides of the equation, squaring an infinite sum is sometimes doable, but not trivial. Another strategy is to take the derivative of both sides of the equation with respect to x. This gives

$$\left(\frac{1}{1-x}\right)^2 = \frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1}$$

Now multiply both sides of the equation by x:

$$\frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} kx^k$$

8. Among the series we should have memorized is the power series for $\ln(1+x)$.

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} \qquad \text{for } -1 < x \leq 1$$

Replacing x with $x^{\frac{1}{2}}$ we get

$$\ln(1+x^{\frac{1}{2}}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{\frac{1}{2}(k+1)}}{k+1}$$