Question Two: Non-genericity of a pitchfork bifurcation

$f_a(x) = x^3 - \text{ax} + b$

```
syms a b x;
f = x^3 -a*x
```

f = $x^3 - a\,x$

```
fixedPoints = solve(f-x,x)
```

fixedPoints =

$$\begin{pmatrix} 0 \\ \sqrt{a+1} \\ -\sqrt{a+1} \end{pmatrix}$$

Part A: Demonstrate that $f_a(x)$ undergoes a pitchfork bifurcation when $b = 0$

Solution:

We proceed using the criteria established in Wiggins to show that $f_a(x)$ has a pitchfork bifurcation when b = 0:

1. $\frac{\partial f}{\partial a}(a, x) = 0$
2. $\frac{\partial f}{\partial x}(a, x) = 1$
3. $\frac{\partial^2 f}{\partial x^2}(a, x) = 0$
4. $\frac{\partial^2 f}{\partial x \partial a}(a, x) \neq 0$
5. $\frac{\partial^3 f}{\partial x^3}(a, x) \neq 0$

We begin by determining at what values of a condition 2 is verified:

```
pF_x = diff(f,x);

x = fixedPoints(1)
```

x = 0

```
pF_x1 = subs(pF_x)
```

pF_x1 = $-a$

```
a1 = solve(pF_x1-1,a)
```

a1 = -1

```
x = fixedPoints(2)
```

x = $\sqrt{a+1}$

```
pF_x2 = subs(pF_x)
```

pF_x2 = $2\,a+3$

```
a2 = solve(pF_x2-1,a)
```

a2 = -1

```
x = fixedPoints(3)
```

x = $-\sqrt{a+1}$

```
pF_x3 = subs(pF_x)
```

pF_x3 = $2\,a+3$

```
a2 = solve(pF_x3-1,a)
```

a2 = -1

Interpretation: Note that for all fixed points, $a_{1,2,3} = 1$. Plugging in a_2, a_3 into x_2, x_3 respectively returns the same fixed point $x = 0$. Thus, we expect the pitchfork bifurcation to happen at $(a, x) = (-1, 0)$.

To confirm, we calculate the derivatives outlined by Wiggins and evaluate them at $(a, x) = (-1, 0)$.

```
syms a x;
f = x^3 -a*x
```

f = $x^3 - a\,x$

```
pF_x = diff(f,x)
```

pF_x = $3\,x^2 - a$

```
pF_xx = diff(pF_x,x)
```

pF_xx = $6x$

```
pF_xxx = diff(pF_xx,x)
```

pF_xxx = 6

```
pF_a = diff(f,a)
```

pF_a = $-x$

```
pF_ax = diff(pF_a,x)
```

pF_ax = -1

The calculation above returns:

1. $\frac{\partial f}{\partial a}(a,x) = -x$
2. $\frac{\partial f}{\partial x}(a,x) = 3x^2 - a$
3. $\frac{\partial^2 f}{\partial x^2}(a,x) = 6x$
4. $\frac{\partial^2 f}{\partial x \partial a}(a,x) = -1$
5. $\frac{\partial^3 f}{\partial x^3}(a,x) = -6$

Substituting, we immeadiately note that:

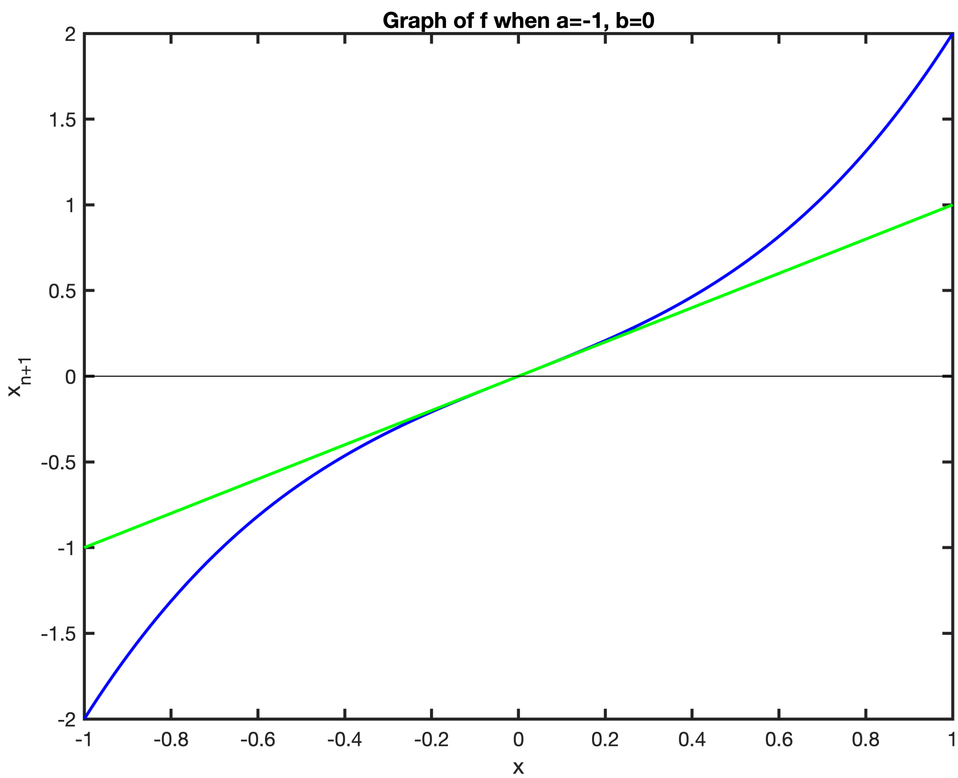
1. $\frac{\partial f}{\partial a}(-1,0) = 0$
2. $\frac{\partial f}{\partial x}(-1,0) = 1$
3. $\frac{\partial^2 f}{\partial x^2}(-1,0) = 0$
4. $\frac{\partial^2 f}{\partial x \partial a}(-1,0) = -1$
5. $\frac{\partial^3 f}{\partial x^3}(-1,0) \ = \ 6$

Thus all conditions for a pitchfork bifurcation are met.

Part B: Identify what bifurcation gives rise to the birth of two fixed points when $b \neq 0$

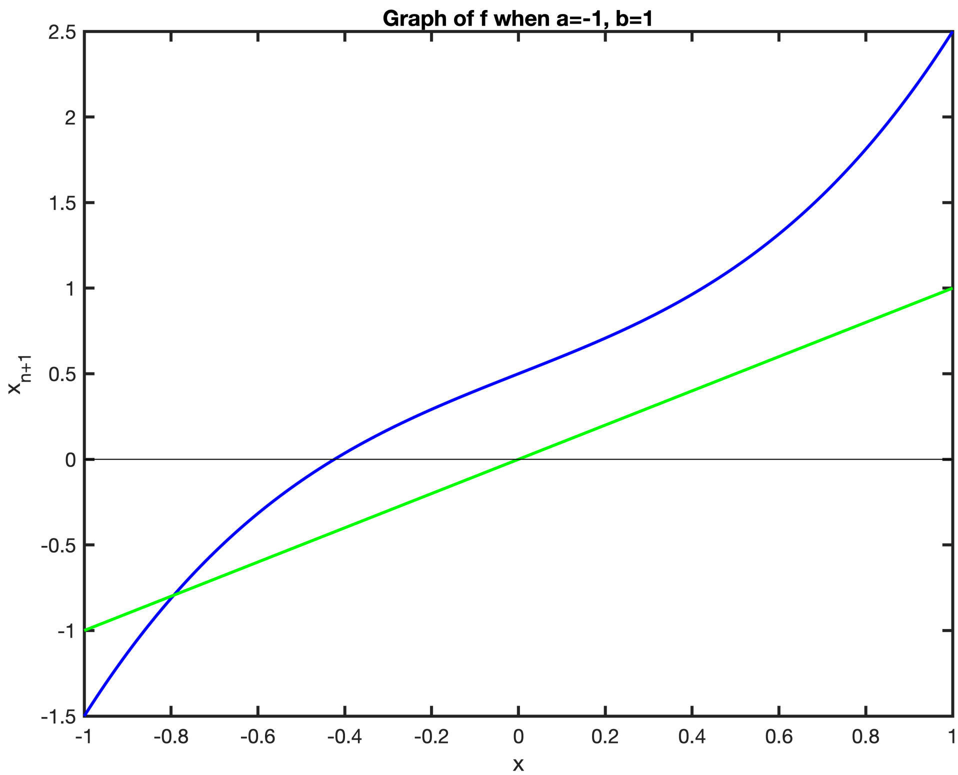
Intuition (and the book) point to a saddle node bifurcation. To show this, consider the graph of $f_a(x) = x^3 - \mathrm{ax} + b$ when $a = -1, b = 0$.

```
x=[-1:0.01:1];
y = x.^3 + x;
z=x;
w = x-x;
plot(x,y,'b','linewidth',1.5)
hold on
plot(x,z, 'g', 'linewidth',1.5)
plot(x, w, 'k')
set(gcf,'color','w');
set(gca,'linewidth',1.5)
xlabel('x');
ylabel('x_{n+1}');
title('Graph of f when a=-1, b=0');
hold off
```

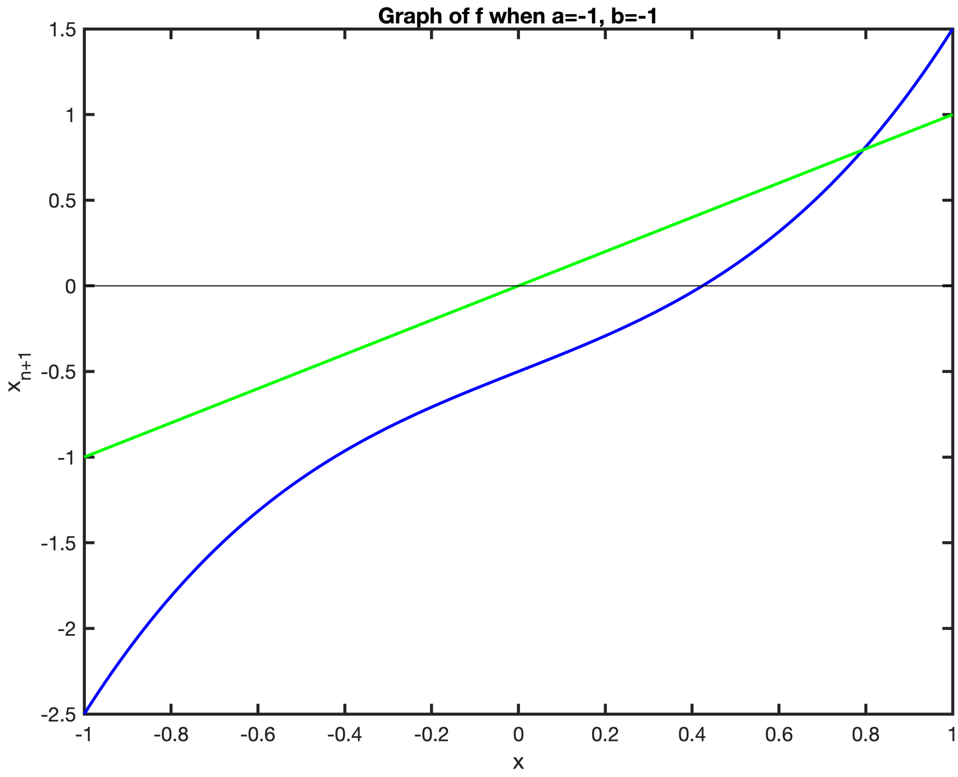



```
y = x.^3 + x +0.5;
plot(x,y,'b','linewidth',1.5)
hold on
plot(x,z, 'g', 'linewidth',1.5)
plot(x, w, 'k')
set(gcf,'color','w');
```

```
set(gca,'linewidth',1.5)
xlabel('x');
ylabel('x_{n+1}');
title('Graph of f when a=-1, b=1');
hold off
```




```
y = x.^3 + x -0.5;
plot(x,y,'b','linewidth',1.5)
hold on
plot(x,z, 'g', 'linewidth',1.5)
plot(x, w, 'k')
set(gcf,'color','w');
set(gca,'linewidth',1.5)
xlabel('x');
ylabel('x_{n+1}');
title('Graph of f when a=-1, b=-1');
hold off
```



Notice that as we "shift" b away from zero, we succeed in shifting the fixed point from zero by some $\epsilon \neq 0$. Thus the conditions established in Part A no longer hold - instead:

1. $\dfrac{\partial f}{\partial a}(a, \epsilon) = -\epsilon \neq 0$
2. $\dfrac{\partial f}{\partial x}(a, \epsilon) = 3\epsilon^2 - a = 1$
3. $\dfrac{\partial^2 f}{\partial x^2}(a, \epsilon) = 6\epsilon \neq 0$
4. $\dfrac{\partial^2 f}{\partial x \partial a}(a, \epsilon) = -1$
5. $\dfrac{\partial^3 f}{\partial x^3}(a, \epsilon) = -6$

and the conditions for a pitchfork bifurcation have been violated. Instead, we now satisfy the conditions for a saddle node bifurcation in which:

1. $\dfrac{\partial f}{\partial a}(-1, 0) \neq 0$
2. $\dfrac{\partial f}{\partial x}(-1, 0) = 1$

3. $\dfrac{\partial^2 f}{\partial x^2}(-1,0) \neq 0$