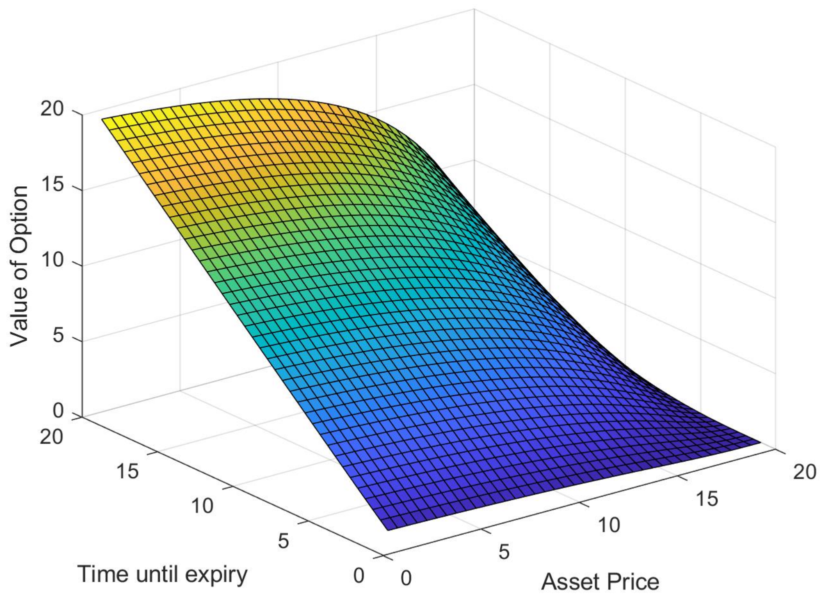
Mastering
The Black-Scholes Equation



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1 The Black-Scholes Equation

1.1 Introduction

The Black-Scholes Equation is very useful in economics. It is able to provide the value of a single option based on the volatility of the asset, the fixed interest rate, the exercise price, and the time until expiration. The Black-Scholes Equation assumes the following to be true, [2]:

- The asset price follows geometric Brownian motion.
- The risk-free interest rate r is known.
- The asset volatility σ is known.
- The asset does not pay dividends.
- There are no transaction costs.
- The option cannot be sold before its expiry (ie. it is a European option).

1.2 Derivation

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0$$

The dependent variable $u(t,x)$ represents the value of a single option, a contract to either buy or sell at an exercise price p at a future time t_*. The constant $\sigma > 0$ represents the asset's volatility. r is the fixed interest rate for bank deposits, where a return is guaranteed instead of buying the option. Borrowed money is indicated by a negative value of r.

The Black-Scholes equation is a backwards diffusion process as the diffusion term u_{xx} in the following equation has a negative coefficient, [3].

$$u_t = -\frac{\sigma^2}{2} x^2 u_{xx} - rxu_x + ru$$

In order to evaluate the equation, we need to determine the option's value $u(t,x)$ at the current time t and asset value x. For a European call option, the asset is bought at the exercise price $p > 0$ at the specified time and the final condition may be written as

$$u(t_*, x) = \max\{x - p, 0\},$$

representing a profit when $x > p$, or the option not being exercised at $x \leq p$ to avoid a loss. For a European put option, the asset is sold and the final condition may be written as

$$u(t_*, x) = \max\{p - x, 0\}.$$

The solution $u(t,x)$ is defined for all $t < t_*$ and $x > 0$ subject to the boundary conditions

$$u(t,0) = 0, \quad u(t,x) \sim x \quad \text{as } x \to \infty.$$

The ratio $\frac{u(t,x)}{x}$ tends to a constant as $x \to \infty$.

1.3 Solving the Black Scholes Equation

The Black-Scholes equation may be solved explicitly by transforming it into the heat equation, [3]. The first step is to convert it to a forward diffusion process:

$$\tau = \frac{1}{2}\sigma^2(t_* - t), \quad v(\tau, x) = u(t_* - 2\tfrac{\tau}{\sigma^2}, x)$$

where τ runs forward from 0 as the actual time t runs backwards from t_*. This converts the final condition into an initial condition $v(0, x) = f(x)$. $v(\tau, x)$ satisfies

$$\frac{\partial v}{\partial \tau} = x^2 \frac{\partial^2 v}{\partial x^2} + \kappa x \frac{\partial v}{\partial x} - \kappa v, \quad \text{where} \quad \kappa = \frac{2r}{\sigma^2}.$$

A change of variables using $x = e^y$ can then take place to remove the explicit dependence on the independent variable x.

$$\omega(\tau, y) = \quad v(\tau, e^y) \quad = v(\tau, x)$$

The following derivatives can be computed from above.

$$\begin{aligned} \frac{\partial \omega}{\partial \tau} &= \frac{\partial v}{\partial \tau} \\ \frac{\partial \omega}{\partial y} &= e^y \frac{\partial v}{\partial x} = x \frac{\partial v}{\partial x} \\ \frac{\partial^2 \omega}{\partial y^2} &= e^{2y} \frac{\partial^2 v}{\partial x^2} + e^y \frac{\partial v}{\partial x} = x^2 \frac{\partial^2 v}{\partial x^2} + x \frac{\partial v}{\partial x} \end{aligned}$$

We can use ω to solve the partial differential equation

$$\frac{\partial \omega}{\partial \tau} = \frac{\partial^2 \omega}{\partial y^2} + (\kappa - 1)\frac{\partial \omega}{\partial y} - \kappa \omega.$$

The heat equation can be found by setting

$$\omega(\tau, y) = e^{\alpha \tau + \beta y} z(\tau, y),$$

where α and β are constants. Substituting into the previous equation gives us

$$\frac{\partial z}{\partial \tau} + \alpha z = \frac{\partial^2 z}{\partial y^2} + 2\beta \frac{\partial z}{\partial y} + \beta^2 z + (\kappa - 1)\left(\frac{\partial z}{\partial y} + \beta z\right) - \kappa z.$$

The terms involving z and $\frac{\partial z}{\partial y}$ can be eliminated by setting

$$\alpha = -\frac{1}{4}(\kappa + 1)^2, \quad \beta = -\tfrac{1}{2}(\kappa - 1).$$

In conclusion, the heat equation,

$$\frac{\partial z}{\partial \tau} = \frac{\partial^2 z}{\partial y^2},$$

can be satisfied by the function

$$z(\tau, y) = e^{\frac{1}{4}(\kappa+1)^2 \tau + \frac{1}{2}(\kappa - 1)y} \omega(\tau, y).$$

So far, we have discovered that for $\tau > 0, -\infty < y < \infty$,

$$u(t,x) \quad = \quad x^{-\frac{1}{2}(\kappa-1)} e^{-\frac{1}{8}(\kappa+1)^2\sigma^2(t_*-t)} z\left(\frac{1}{2}\sigma^2(t_*-t), \log x\right) \tag{1}$$

solves the final value problem for the Black-Scholes Equation for $t < t_*$ and $0 < x < \infty$.

The solution to the initial value problem can be written as

$$z(\tau,y) \quad = \quad \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(y-\eta)^2}{4\tau} + \frac{(\kappa-1)\eta}{2}} f(e^\eta) d\eta.$$

This can be combined with (1) to produce an explicit solution formula for the general final value problem for the equation.

1.4 European Call Option

The initial condition,

$$z(0,y) = h(y) \quad = \quad e^{\frac{(\kappa-1)y}{2}} \max\{e^y - p, 0\},$$

results in the integral

$$z(\tau,y) \quad = \quad \frac{1}{2\sqrt{\pi\tau}} \int_{\log p}^{\infty} e^{-\frac{(y-\eta)^2}{4\tau} + \frac{(\kappa-1)\eta}{2}} (e^\eta - p) d\eta.$$

The integral can be evaluated to produce

$$z(\tau,y) \quad = \quad \frac{1}{2}\left[e^{\frac{(\kappa+1)^2\tau}{4} + \frac{(\kappa+1)y}{2}} \operatorname{erfc}\left(\frac{\log p - (\kappa+1)\tau - y}{2\sqrt{\tau}}\right) - pe^{\frac{(\kappa-1)^2\tau}{4} + \frac{(\kappa-1)y}{2}} \operatorname{erfc}\left(\frac{\log p - (\kappa-1)\tau - y}{2\sqrt{\tau}}\right)\right],$$

where $\operatorname{erfc}(x)$ is the complementary error function, or

$$\operatorname{erfc}(x) \quad = \quad \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2} dz = 1 - \operatorname{erf}(x).$$

Plugging this back into (1) results in the Black-Scholes Formula for a European call option:

$$\boxed{u(t,x) = \frac{1}{2}\left[x \operatorname{erfc}\left(-\frac{(r+\frac{1}{2}\sigma^2)(t_*-t) + \log(\frac{x}{p})}{\sqrt{2\sigma^2(t_*-t)}}\right) - pe^{-r(t_*-t)} \operatorname{erfc}\left(-\frac{(r-\frac{1}{2}\sigma^2)(t_*-t) + \log(\frac{x}{p})}{\sqrt{2\sigma^2(t_*-t)}}\right)\right].}$$

An example of this in use is shown in Figure 1 with the input values of $t_* = 10$, $r = .08$, $\sigma = .82$, and $p = 25$.

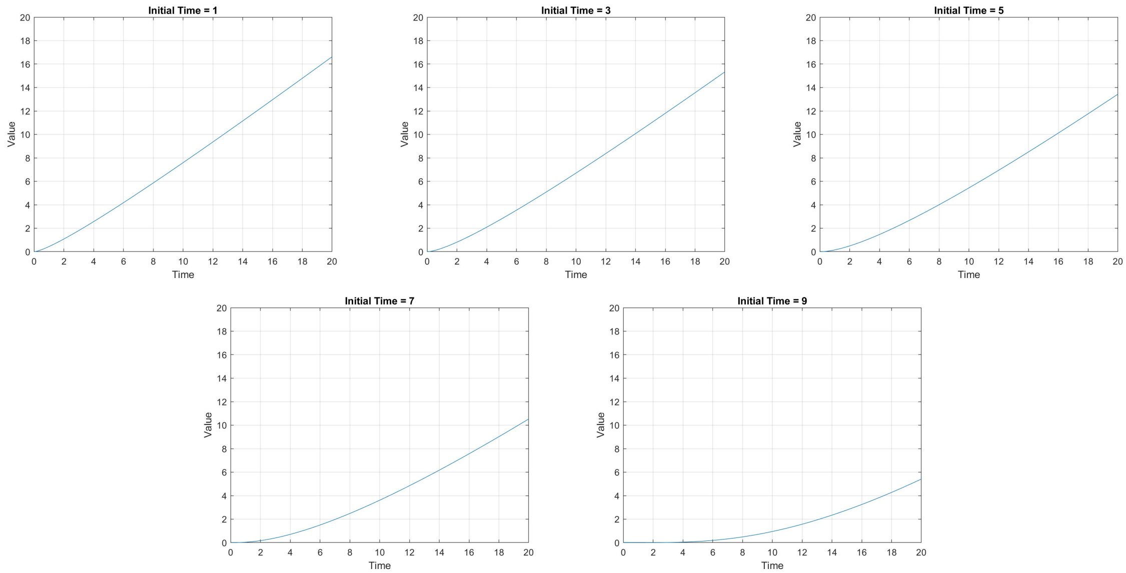

Figure 1: Graphs to the Solution of the Black-Scholes equation

The option's value decreases as the time gets closer to the exercise time t_*, lessening the chances of profit from the asset's remaining volatility.

2 Matlab Implementation

The Black-Scholes Equation can be implemented using Matlab for ease of calculation. Below are two Matlab programs, one solves the Black-Scholes Equation numerically and the other creates a graph of the solution.

2.1 Numerical Solution

The following code outputs the value of the asset according to the Black-Scholes Equation depending on the inputs t_*, σ, r, p, t, and x.

```
function u = BlackScholesEq(t_final, sigma, r, p, t, x)
% Computes the value of a European call option at expiration

%Inputs:
%t_final = Time until expiry
%sigma = Volatility of asset
%r = Fixed Interest Rate
%p = Exercise Price of asset
%t = initial time (usually 0)
%x = asset price at initial time

u = (1/2)*(x*erfc(-((r+0.5*sigma^2)*(t_final-t)+log(x/p))/(sqrt(2*sigma^2*(t_final-t))))-p*exp(-r*(t_fi
```

2.2 Graphical Solution

The second way to implement the equation using Matlab consists of graphing the output of the Black-Scholes Equation with respect to time until expiration. Examples of this program's output can be seen in Figure 1.

It can be seen that as initial time T progresses, the value of the asset decreases as time gets closer to the exercise time t_{final}. The function is dependent on the inputs t_*, σ, r, p, and T.

```
function u = BSGrph(t_final, sigma, r, p, T)
% Graphs the value of a European call option in respect to time until expiration

%Inputs:
%t_final = Time until expiry
%sigma = Volatility of asset
%r = Fixed Interest Rate
%p = Exercise Price of asset
%T = Initial Time (Requirement: T < t_final)

syms x t
u(t,x) = (1/2)*(x*erfc(-((r+0.5*sigma^2)*(t_final-t)+log(x/p))/(sqrt(2*sigma^2*(t_final-t))))-p*exp(-r*
fplot(u(T,x))
grid
xlim([0 20])
ylim([0 20])
title(['Initial Time = ', num2str(T)])
xlabel('Time')
ylabel('Value')
```

3 Downfalls of the Black-Scholes Equation

While being a popular equation used to model stock options and allowing one to trade without risk, the Black-Scholes Equation has its downfalls, [1]. There are numerous criticisms on its ability to accurately reflect the market:

- Returns do not absolutely follow a normal distribution. They tend to be leptokurtic, or more concentrated about the mean with fat tails.
- A risk-free interest rate is assumed in the calculations of the solution, but it is not possible to calculate an exactly risk-free rate. Investors tend to use the long-term yield of bonds as risk-free interest rates, but they are not truly risk-free. They are simply the least risky investment according to the market.
- The Black-Scholes model assumes a market using European options, where the asset cannot be sold until its expiry. Most options traded today are American options, where the asset can be sold at any point.

References

[1] Brilliant.org, *Black-scholes-merton* (29 Nov 2020), available at https://brilliant.org/wiki/black-scholes-merton/.

[2] R. Chan, *Math4210 notes: Black-scholes equations* (29 Nov 2020), available at https://www.math.cuhk.edu.hk/~rchan/teaching/math4210/chap08.pdf.

[3] P. Olver, *Introduction to partial differential equations*, Springer, May 2016.