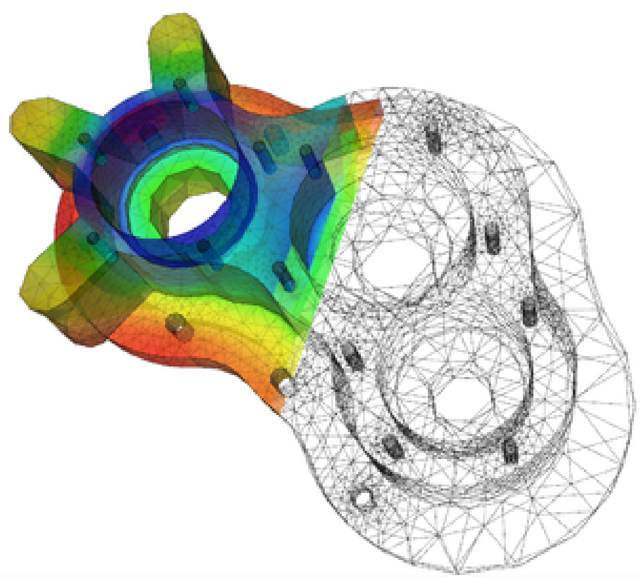
Mastering The Fundamentals of Partial Differential Equations



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1 Numerical Methods of Solving Partial Differential Equations

1.1 Order of Error

$$\frac{df}{dx} = \lim_{x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \Longrightarrow \qquad \frac{df}{dx} \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Taylor's Theorem: $\quad f(x+\Delta x) \approx f(x) + f'(x)\Delta x + \frac{f''(x)(\Delta x)^2}{2} + \frac{f'''(x)(\Delta x)^3}{3!} + \frac{f^{(4)}(x)(\Delta x)^4}{4!} + ...$

Taylor's Theorem can be simplified to contain less terms by using the comprehensive error term $[\frac{f''(c)(\Delta x)^2}{2!}]$.

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + ... + [\tfrac{f''(c)(\Delta x)^2}{2!}] \qquad , \qquad \text{Where } c \in (x, x+\Delta x)$$

$$\text{Error}(\Delta x) = \tfrac{M(\Delta x)^2}{2}$$

The equation can be rearranged to give us

$$\boxed{f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{\text{Error}(\Delta x)}{\Delta x}}$$

$$\tfrac{\text{Error}(\Delta x)}{\Delta x} \to \tfrac{M\Delta x}{2} \to O(\Delta x)$$

In order to find the order of error, $O(\Delta x)$, of a given equation $f(x)$, you must multiply $f'(x)$ by the given Δx.

Mastery Check:
Approximate $u'(x)$ using $u(x-h)$, $u(x)$, and $u(x+3h)$.

Rewrite $u(x-h)$, $u(x)$, and $u(x+3h)$ as their Taylor Series approximations:

$$\begin{aligned} u(x-h) &= u(x) - u'(x)h + \frac{u''(x)h^2}{2} - \frac{u'''(x)h^3}{6} + \dots \\ u(x+3h) &= u(x) + u'(x)3h + \frac{u''(x)(3h)^2}{2} + \frac{u'''(x)(3h)^3}{6} + \dots \\ u(x) &= u(x) \end{aligned}$$

Figure out the coefficients:

$$\begin{aligned} A(\frac{u''(x)h^2}{2}) + B(\frac{9u''(x)h^2}{2}) &= 0 \longrightarrow A = -9B \longrightarrow A = -9, B = 1 \\ C(u(x)) + B(u(x)) + A(u(x)) &= 0 \longrightarrow C + 1 - 9 = 0 \longrightarrow C = 8 \\ A(\frac{u'''(x)h^3}{6}) + B(\frac{u'''(x)27h^3}{6}) &= \frac{18u'''(x)h^3}{6} = 3u'''(x)h^3 \end{aligned}$$

Combine the Taylor Series using the coefficients:

$$\begin{aligned} 8u(x) + u(x+3h) - 9u(x-h) &= 0 + 3u'(x)h + 9u'(x)h + 0 + ... \\ u'(x) &= \frac{8u(x) + u(x+3h) - 9u(x-h)}{12h} + \frac{O(h^3)}{12h} \end{aligned}$$

Solving for $u'(x)$ and simplifying gives the final equation:

$$\boxed{u'(x) = \frac{8u(x) + u(x+3h) - 9u(x-h)}{12h} + O(h^2)}$$

1.1.1 Error

The error is of order h^2. Order of error is directly proportional to h^2 (ie. $O(h^2) \propto h^2$). In the limit as $h^2 \to 0$, the error in the definition of a derivative also goes to 0. You cannot let h^2 become too small as your computer will be unable to calculate such small numbers. The *machine epsilon* is the smallest number your computer can store and limits how small your calculations can reach.

1.2 Neumann Stability

Neumann Stability is a method of checking the stability of finite difference schemes. In order for the given scheme to be stable, the following condition must be met

$$|\lambda| \leq 1.$$

The problems generally begin by considering one Fourier mode of the solution:

$$e^{ikx_m} = u(t_j, x_m).$$

The following substitutions may then be made:

$$u_{j+1,m} = \lambda e^{ikx_m}, \quad u_{j,m+1} = e^{ik(x_m+\Delta x)}, \quad u_{j,m} = e^{ikx_m}, \quad u_{j,m-1} = e^{ik(x_m-\Delta x)}.$$

The given scheme may then be solved for the variable λ and simplified to find if and when $|\lambda| \leq 1$.
If $|\lambda|$ is never ≤ 1, then the scheme is unstable. If $|\lambda| \leq 1$ for some $a \leq \sigma = \frac{c\Delta t}{\Delta x} \leq b$, then the scheme is conditionally stable. If $|\lambda|$ is always ≤ 1, then the scheme is unconditionally stable.

Mastery Check:
Find the Neumann Stability for the following scheme.

$$\begin{aligned}
u_{j+1,m} &= \frac{1}{2}\sigma(\sigma-1)u_{j,m+1} - (\sigma^1 - 1)u_{j,m} + \frac{1}{2}\sigma(\sigma+1)u_{j,m-1} \\
\sigma &= \frac{c\Delta t}{\Delta x} \\
u_t + cu_x &= 0
\end{aligned}$$

Begin by setting the following substitutions:

$$u_{j+1,m} = \lambda e^{ikx_m}, \quad u_{j,m+1} = e^{ik(x_m+\Delta x)}, \quad u_{j,m} = e^{ikx_m}, \quad u_{j,m-1} = e^{ik(x_m-\Delta x)}.$$

Plug the substitutions into the scheme and solve for λ

$$\begin{aligned}
\lambda e^{ikx_m} &= \frac{1}{2}\sigma(\sigma-1)e^{ik(x_m+\Delta x)} - (\sigma^1 - 1)e^{ikx_m} + \frac{1}{2}\sigma(\sigma+1)e^{ik(x_m-\Delta x)} \\
\lambda &= \sigma^2 \cos(k\Delta x) - i\sigma \sin(k\Delta x) + 1 - \sigma^2 \\
|\lambda| \leq 1 &\implies |\lambda|^2 \leq 1 \\
|\lambda|^2 &= (1 - \sigma^2(1 - \cos(k\Delta x)))^2 + \sigma^2 \sin^2(k\Delta x) \\
|\lambda|^2 &= (\sigma^2 - 1)\sigma^2 \cos^2(k\Delta x) + 2(1-\sigma^2)\sigma^2 \cos(k\Delta x) + (1-\sigma^2)^2 + \sigma^2
\end{aligned}$$

Set

$$g = |\lambda|^2 \quad , \quad \alpha = \cos(k\Delta x) \to \alpha \in [-1, 1]$$

to make $g(\alpha)$ a quadratic function

$$g(\alpha) \quad = \quad (\sigma^2-1)\sigma^2\alpha^2 + 2(1-\sigma^2)\sigma^2\alpha + (1-\sigma^2)^2 + \sigma^2$$

which implies that the max occurs at

$$g'(\alpha) = 0 \qquad \forall \qquad \alpha \in [-1,1]$$
$$\text{or at}$$
$$\alpha \quad = \quad \pm 1.$$

We need to find: $|g| \leq 1$ with $\alpha \in [-1,1]$.
First, $g(1)$ and $g(-1)$ can be checked

$$\begin{aligned} g(1) &= 1 \\ g(-1) &= \left(1-2\sigma^2\right)^2 \end{aligned}$$

Then, each case for the value of $|\sigma|$ can be checked.
Case 1: $|\sigma| > 1$

$$g(-1) \quad = \quad \left(1-2\left(|\sigma|>1\right)^2\right)^2 \implies |g|>1 \quad \forall \quad |\sigma|>1$$

Unstable when $|\sigma| > 1$.
Case 2: $|\sigma| = 1$

$$g(\alpha) = 1 \quad \forall \quad \sigma = \pm 1$$

Stable when $|\sigma| = 1$.
Case 3: $|\sigma| < 1$

$$g'(\alpha) = 0 = 2\left(\sigma^2-1\right)\sigma^2\alpha + 2\left(1-\sigma^2\right)\sigma^2 \quad \implies \quad \text{Max occurs at } \alpha = 1 \rightarrow g(1) = 1$$

Stable when $|\sigma| < 1$.
Summarizing the outcomes yields the solution:

$$\boxed{\text{Conditional Stability: } |\sigma| \leq 1}$$

1.3 The CFL Condition

The CFL condition is a condition for convergence used when solving partial differential equations numerically using explicit time-dependent schemes.

Constant Wave Speed c:
The CFL condition typically has the following form

$$\sigma = \frac{c\Delta t}{\Delta x} \leq \sigma_{\max}$$

where c is the magnitude of the velocity of the function, Δt is the time step, and Δx is the spatial step. If an explicit scheme is used then typically $\sigma_{\max} = 1$. Using an implicit scheme, larger values of $\sigma_{\max}$ may be tolerated.

Upwind Scheme (Nonconstant Wave Speed c):
In order to overcome the sign restriction on a nonconstant wave speed two different schemes must be used,

the forward difference scheme when the wave speed is negative and the backwards scheme when it is positive. The resulting scheme for nonconstant wave speeds is

$$u_{j+1,m} \quad = \quad \begin{cases} -\sigma_{j,m}u_{j,m+1} + (\sigma_{j,m}+1)u_{j,m}, & c_{j,m} \leq 0 \\ -(\sigma_{j,m}-1)u_{j,m} + \sigma_{j,m}u_{j,m-1}, & c_{j,m} > 0 \end{cases},$$

where

$$\sigma_{j,m} = c_{j,m}\frac{\Delta t}{\Delta x}, \qquad c_{j,m} = c(t_j, x_m).$$

In order to remain stable, the step size must remain small and the CFL condition,

$$\frac{\Delta x}{\Delta t} \leq |c_{j,m}|,$$

must be satisfied at each node.

Mastery Check:
Find the CFL condition of the following scheme.

$$\begin{aligned} u_t + cu_x &= 0 \\ u_{j+1,m} &= Au_{j,m+3} + Bu_{j,m+2} + Cu_{j,m+1} + Du_{j,m} + Eu_{j,m-1} \end{aligned}$$

The numerical domain of dependence can be graphed by observing where the variables of u would land in the graph. For example, $u_{j,m+2}$ would be put at the point $(j, m+2)$ on the graph. The points can all be connected then to create the domain of dependence for the scheme.

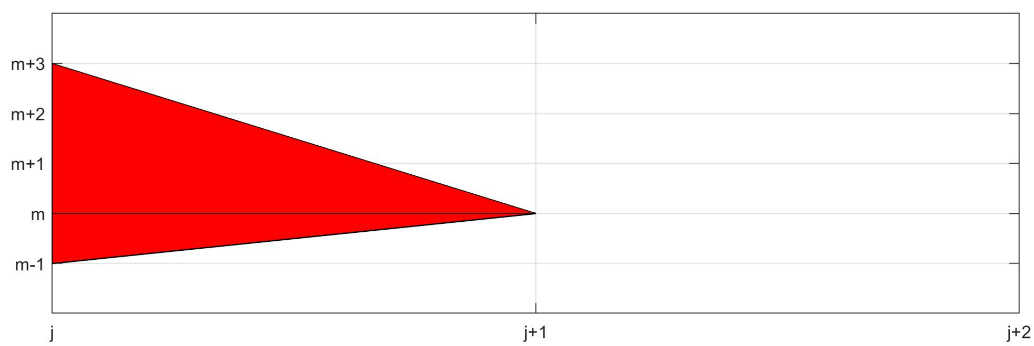


Figure 1: Numerical Domain of Dependence

The CFL condtion states that the value must fall within the domain of dependence, shown above in Figure 1.

$$\begin{aligned} x_{m-1} \leq & \quad x_m - ct_{j+1} & \leq x_{m+3} \\ -\Delta x \leq & \quad -c\Delta t & \leq 3\Delta x \end{aligned}$$

$$\boxed{-1 \leq \sigma = \frac{-c\Delta t}{\Delta x} \leq 3}$$

1.4 Implementing an Explicit Difference Method

The explicit difference method uses the data from previous time steps to calculate the value at the current time step. A boundary value problem is a set of differential equations that have additional constraints, called boundary conditions. The boundary conditions used in the explicit difference method are often referred to as α and β. The boundary conditions are used to determine the values $u(t_j, x_0)$ and $u(t_j, x_n)$ on the boundary nodes. The remaining values in the solution matrix u are determined using the equation $u^{(j+1)} = Au^{(j)} + b^{(j)}$, where

$$A = \begin{pmatrix} 1-2\mu & \mu & & & \\ \mu & 1-2\mu & \mu & & \\ & \mu & \ddots & \ddots & \\ & & \ddots & \ddots & \mu \\ & & & \mu & 1-2\mu \end{pmatrix}, \quad b^{(j)} = \begin{pmatrix} \mu\alpha_j \\ 0 \\ 0 \\ \vdots \\ 0 \\ \mu\beta_j \end{pmatrix}.$$

Note: Where $n =$ number of space steps $= \frac{l}{\Delta x}$: A is an (n-1) x (n-1) matrix. b is an (n-1) column matrix.

In order to avoid potential instability while using the explicit difference method, the following equation must be used

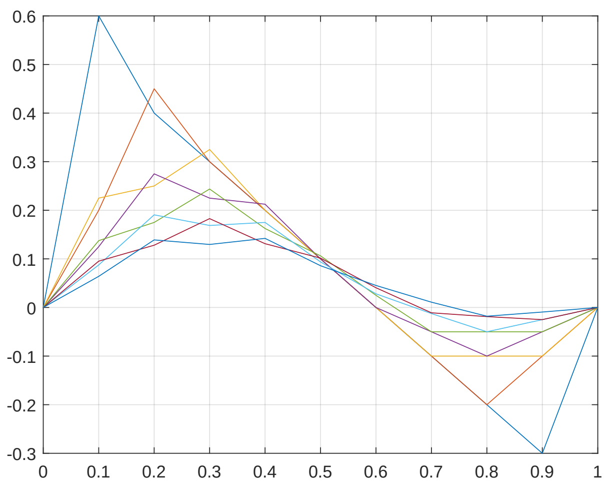
$$\Delta t \leq \frac{(\Delta x)^2}{2\gamma}\ .$$

This condition makes the explicit difference method *conditionally stable* because it must be met to maintain stability in the solution.

Example 5.4.

$$\gamma = 1, \quad l = 1, \quad \Delta x = 0.1$$

$$u(0,x) = f(x) = \begin{cases} -x, & \text{if } 0 \leq x \leq \frac{1}{5} \\ x - \frac{2}{5}, & \text{if } \frac{1}{5} \leq x \leq \frac{7}{10} \\ 1 - x, & \text{if } \frac{7}{10} \leq x \leq 1 \end{cases}$$



Mastery Problem:

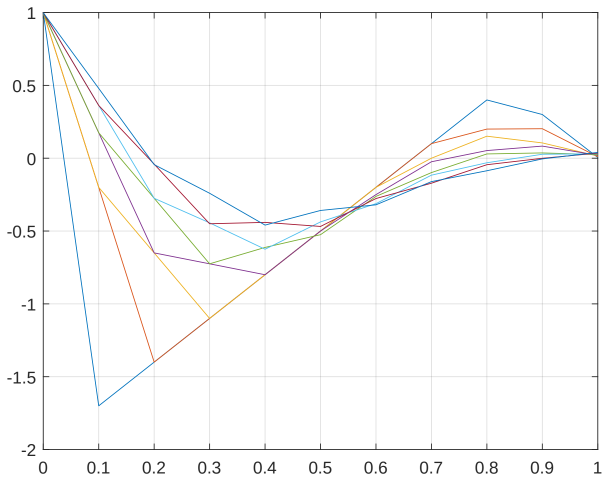
$$\begin{aligned} u(t,0) &= \cos(t) \\ u(t,1) &= \sin(t) \\ \Delta x &= 0.1 \\ u_t &= u_{xx} \end{aligned}$$

$$u(t,0) = \begin{cases} 2|x-\frac{1}{6}|, & \text{if } 0 \leq x \leq \frac{1}{3} \\ 0, & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3} \\ \frac{1}{2} - 3|x-\frac{5}{6}|, & \text{if } \frac{2}{3} \leq x \leq 1 \end{cases}$$

Code:

```
%Inputs
L = 1;
gamma = 1;  %diff(u,t) = gamma*diff(u,x,2);
delta_x = .1;
delta_t = delta_x^2/(2*gamma);  % needs to be <= delta_x^2/2*gamma
t_end = 0.02;
mu = (gamma*delta_t)/(delta_x^2);
n = L/delta_x;  % number of space steps
T = round(t_end/delta_t);  % number of time steps
% Creates Matrix A
A = full(gallery('tridiag', n-1, mu, 1-2*mu, mu));  % dimensions, diag: bottom, middle, upper
B = zeros(n-1,1);  % Creates Matrix B
u = zeros(n+1, T);  % Creates u-matrix
for z = 1:n-1
    x = z*delta_x;
    u(z+1, 1) = (0<=x<=(1/3)).*2*abs(x-(1/6)) + ((1/3<=x)&(x<=2/3)).*0 +
((2/3<=x)&(x<=1)).*(1/2)-3*abs(x-(5/6));
end
for j = 1:T
    B(1) = mu*alpha(j*delta_t);
    B(end) = mu*beta(j*delta_t);
    u(2:n,j+1) = (A*u(2:n,j))+B;
    u(1,j) = alpha(j*delta_t);  % Adds in alpha to u-matrix
    u(end,j) = beta(j*delta_t);  % Adds in beta to u-matrix
end
u(:,T+1) = [];  % Removes the T+1'th column
figure
plot(linspace(0,L,n+1),u)
grid
function [a] = alpha(t)  % Defines u(t,0)
    a = cos(t);  % u(t,0)
end
function [b] = beta(t)  % Defines u(t,L)
    b = sin(t);  % u(t,L)
end
```

Final Solution:



1.5 Implementing an Implicit Difference Method

The Implicit Difference Method uses data from the current time step to calculate it's value. This method also has the boundary conditions α and β, which represent the solution function, u, at the designated time steps $t_0 = 0$ and $t_f = L$. $\hat{A}$ is found from replacing μ with $-\mu$ in the Matrix A. The boundary conditions are used to determine the values $u(t_j, x_0)$ and $u(t_j, x_n)$ on the boundary nodes. The remaining values in the solution matrix u are determined using the equation $u^{(j+1)} = \hat{A}^{-1}(u^{(j)} + b^{(j+1)})$, where

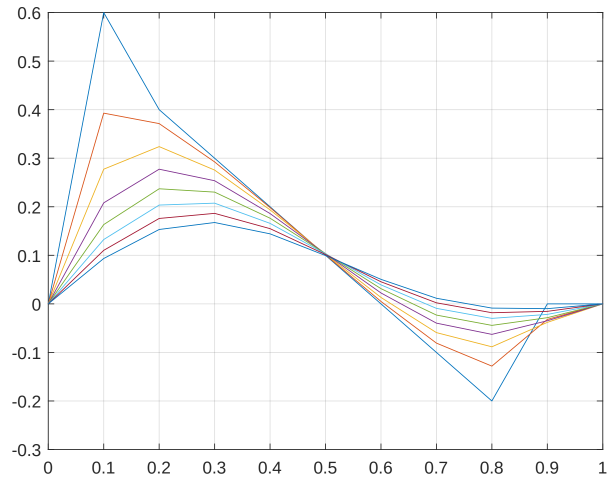
$$\hat{A} = \begin{pmatrix} 1+2\mu & -\mu & & & \\ -\mu & 1+2\mu & -\mu & & \\ & -\mu & \ddots & \ddots & \\ & & \ddots & \ddots & -\mu \\ & & & -\mu & 1+2\mu \end{pmatrix}, \quad b^{(j+1)} = \begin{pmatrix} \mu\alpha_{(j+1)} \\ 0 \\ 0 \\ \vdots \\ 0 \\ \mu\beta_{(j+1)} \end{pmatrix}.$$

Note: Where $n =$ number of space steps $= \frac{l}{\Delta x}$: $\hat{A}$ is an (n-1) x (n-1) matrix. b is an (n-1) column matrix.

While the explicit difference method was conditionally stable, the implicit difference method is *unconditionally stable* as it does not require the restriction of the time step to maintain stability. This unconditional stability comes at the cost of having to invert the matrix $\hat{A}$.

Example 5.5.

$$\begin{aligned} \Delta x &= 0.1 \\ \Delta t &= 0.01 \\ t &= 0.04 \end{aligned}$$



Mastery Problem:

$$\begin{aligned} u(0,x) &= \cos(t) \\ u(1,x) &= \sin(t) \\ \Delta x &= 0.1 \\ u_t &= u_{xx} \end{aligned}$$

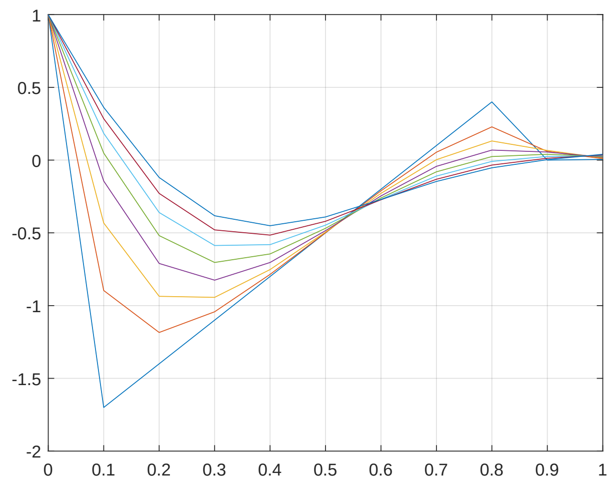
$$u(t,0) = \begin{cases} 2|x-\frac{1}{6}|, & \text{if } 0 \leq x \leq \frac{1}{3} \\ 0, & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3} \\ \frac{1}{2} - 3|x-\frac{5}{6}|, & \text{if } \frac{2}{3} \leq x \leq 1 \end{cases}$$

Code:

```
%Inputs
L = 1;
gamma = 1; %diff(u,t) = gamma*diff(u,x,2);
delta_x = .1;
delta_t = delta_x^2/(2*gamma); % needs to be <= delta_x^2/2*gamma
t_end = 0.02;
mu = (gamma*delta_t)/(delta_x^2);
n = L/delta_x; % number of space steps
T = round(t_end/delta_t); % number of time steps
% Creates Matrix A
A_hat = full(gallery('tridiag', n-1, -mu, 1+2*mu, -mu)); % dimensions, diag: bottom, middle, upper
B = zeros(n-1,1);% Creates Matrix B
u = zeros(n+1, T); % Creates u-matrix
for z = 1:n-2
    x = z*delta_x;
    u(z+1, 1) = (0<=x<=(1/3)).*2*abs(x-(1/6)) + ((1/3<=x)&(x<=2/3)).*0 + ((2/3<=x)&(x<=1)).*(1/2)-3*abs
end
for j = 1:T
    B(1) = mu*alpha((j+1)*delta_t);
    B(end) = mu*beta((j+1)*delta_t);
    u(2:n,j+1) = (inv(A_hat)*(u(2:n,j)+B));
```

```
    u(1,j) = alpha(j*delta_t); % Adds in alpha to u-matrix
    u(end,j) = beta(j*delta_t); % Adds in beta to u-matrix
end
u(:,T+1) = []; % Removes the T+1'th column
figure
plot(linspace(0,L,n+1),u)
grid
function [a] = alpha(t) % Defines u(0,x)
    a = cos(t);
end
function [b] = beta(t) % Defines u(L,x)
    b = sin(t);
end
```

Final Solution:



2 Method of Characteristics

The Method of Characteristics is a technique for solving partial differential equations. The method uses the equation $\frac{du}{dt} = \frac{\partial u}{\partial t}\frac{dt}{dt} + \frac{\partial u}{\partial x}\frac{dx}{dt}$ to turn the partial differential equation into a series of ordinary differential equations. The ordinary differential equations can then be solved using the characteristic curves to find the solution to the given partial differential equation.

2.1 Solving $u_t + cu_x + du = g(x)$ with $u(0,x) = f(x)$

$$\begin{aligned} u(0,x) &= f(x) \\ u_t + {\color{red}c}u_x + {\color{blue}d}u &= g(x) \end{aligned}$$

Rearrange equation to have all derivatives on one side.

$$u_t + {\color{red}c}u_x = g(x) - {\color{blue}d}u$$

Differentiate the function u with respect to t to create an equation that looks similar to what was given.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial t}\frac{dt}{dt} + \frac{\partial u}{\partial x}\frac{dx}{dt} \\
\frac{\partial u}{\partial t} &= u_t + \frac{dx}{dt}u_x
\end{aligned}$$

Solve for the coefficient c and integrate to find the constant ξ.

$$\begin{aligned}
\frac{dx}{dt} &= c \\
dx &= c dt \\
x &= ct + \xi \\
\xi &= x - ct
\end{aligned}$$

Solve to find the equation h(t).

$$\begin{aligned}
\frac{du}{dt} &= u_t + \frac{dx}{dt}u_x \\
\frac{du}{dt} &= u_t + cu_x \\
u_t + cu_x &= g(x) - du \\
\frac{du}{dt} &= g(x) - du
\end{aligned}$$

Integrate then find the constant of integration $k(\xi)$ using the initial condition $u(0, x)$. Plug $k(\xi)$ back into the equation $h(t)$ to find the final solution.

Mastery Problem:

$$\begin{aligned}
u_t - 3u_x &= 12 \\
u(0, x) &= \frac{\arctan(x)}{x^2 + 1}
\end{aligned}$$

Differentiate the function u with respect to t to create an equation that looks similar to what was given.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial t}\frac{dt}{dt} + \frac{\partial u}{\partial x}\frac{dx}{dt} \\
\frac{\partial u}{\partial t} &= u_t + \frac{dx}{dt}u_x
\end{aligned}$$

Solve for the coefficients and integrate to find the constant ξ.

$$\begin{aligned}
\frac{dx}{dt} &= -3 \\
dx &= -3dt \\
x &= -3t + \xi \\
\xi &= x + 3t
\end{aligned}$$

Solve to find the equation h(t).

$$\begin{aligned}
\frac{\partial u}{\partial t} &= u_t + \frac{dx}{dt}u_x \\
\frac{\partial u}{\partial t} &= u_t - 3u_x \\
u_t - 3u_x &= 12 \\
\frac{du}{dt} &= 12 \\
\int du &= \int 12dt \\
h(t) &= 12t + k(\xi)
\end{aligned}$$

Find $k(\xi)$ using the initial condition $u(0, x)$.

$$k(\xi) = u(0, \xi) = \frac{\arctan(x+3t)}{(x+3t)^2+1}$$

Plug $k(\xi)$ back into the equation $h(t)$ to find the final solution.

$$\boxed{h = 12t + \frac{\arctan(x+3t)}{(x+3t)^2+1}}$$

2.2 Solving the linear transport equation with polynomial coefficients

$$\begin{aligned}
u_t + c(x)u_x &= 0 \\
u(0, x) &= f(x)
\end{aligned}$$

$$\begin{aligned}
\frac{dx}{dt} &= c(x) \\
\xi &= x(0) \\
\int_\xi^x \frac{1}{c(x)}\,dx &= \int_0^t 1\,dt
\end{aligned}$$

Solve for ξ in terms of x and t. Then plug into initial function.

$$\begin{aligned}
u(0, x) &= f(x) \\
u(t, x) &= f(\xi)
\end{aligned}$$

Check by plugging 0 in for t in the equation $f(\xi)$. It should equal the original function $f(x)$.

$$f(x(0)) = u(0, x)$$

Mastery Check:

$$\begin{aligned}
u_t + (x-5)(x-1)(t+6)u_x &= 0 \\
u(0, x) &= \frac{3\cos(x)}{x^2+6}
\end{aligned}$$

Determine $\frac{dx}{dt}$ and integrate to solve for ξ.

$$\begin{aligned}
\frac{dx}{dt} &= (x-5)(x-1)(t+6) \\
\int_{\xi}^{x} \frac{1}{(x-5)(x-1)} dx &= \int_{0}^{t} t + 6dt \\
\frac{1}{4}\left(\ln\left|\frac{x-5}{x-1}\right| - \ln\left|\frac{\xi-5}{\xi-1}\right|\right) &= \frac{1}{2}t^2 + 6t \\
\left(\frac{x-5}{x-1}\right)\left(\frac{\xi-1}{\xi-5}\right) &= e^{2t^2+24t} \\
\xi &= \frac{1-5\left(\frac{x-1}{x-5}\right)\left(e^{2t^2+24t}\right)}{1-\left(\frac{x-1}{x-5}\right)\left(e^{2t^2+24t}\right)}
\end{aligned}$$

Plug ξ in for x in $u(0,x)$ to find the solution equation $u(t,x)$.

$$u(t,\xi) = \frac{3\cos(\xi)}{\xi^2+6}$$

$$\boxed{u(t,x) = \frac{3\cos(\frac{1-5\frac{x-1}{x-5}e^{2t^2+24t}}{1-\frac{x-1}{x-5}e^{2t^2+24t}})}{(\frac{1-5\frac{x-1}{x-5}e^{2t^2+24t}}{1-\frac{x-1}{x-5}e^{2t^2+24t}})^2+6}}$$

2.3 Solving first order nonlinear PDEs using the method of characteristics

$$\begin{aligned}
u_t + g(u)u_x &= 0 \\
u(0,x) &= f(x)
\end{aligned}$$

Solve for x by plugging $u(0,\xi)$ into $g(u)$ because u is constant along all time t, and integrating.

$$\begin{aligned}
\frac{dx}{dt} &= g(u) \\
\frac{dx}{dt} &= g(u(0,\xi)) = g(f(\xi)) \\
\int_{\xi}^{x} dx &= \int_{0}^{t} g(f(\xi))\, dt \\
x &= g(f(\xi))t + \xi
\end{aligned}$$

Check for a rarefaction or a shock in the characteristic curves and resolve them if there is one using the methods listed in the next sections. A suitable conservation law (ie. Conservation of mass, energy, etc.) in terms of $g(u)$ would be needed if there is a shock. A conservation law ensures that the solution follows the laws of physics and provides us with a realistic solution.

Mastery Check:

$$\begin{aligned} u_t + (u-3)^3 u_x &= 0 \\ u(0,x) &= \begin{cases} 2, & x < 1 \\ 4, & x > 1 \end{cases} \end{aligned}$$

$$\begin{aligned} \frac{dx}{dt} &= (u-3)^3 \\ x &= (u(0,\xi)-3)^3 t + \xi \\ x &= \begin{cases} \xi - t, & \xi < 1 \\ \xi + t & \xi > 1 \end{cases} \end{aligned}$$

The rarefaction begins at $\xi = 1$, or $(0,1)$.

$$\begin{aligned} m &= \frac{x-1}{t-0} = \frac{x-1}{t} \\ (u-3)^3 &= \frac{x-1}{t} \\ u &= \sqrt[3]{\frac{x-1}{t}} + 3 \end{aligned}$$

Final Solution:

$$\boxed{u(t,x) = \begin{cases} 2, & x < 1-t \\ \sqrt[3]{\frac{x-1}{t}} + 3, & 1-t < x < 1+t \\ 4, & x > 1+t \end{cases}}$$

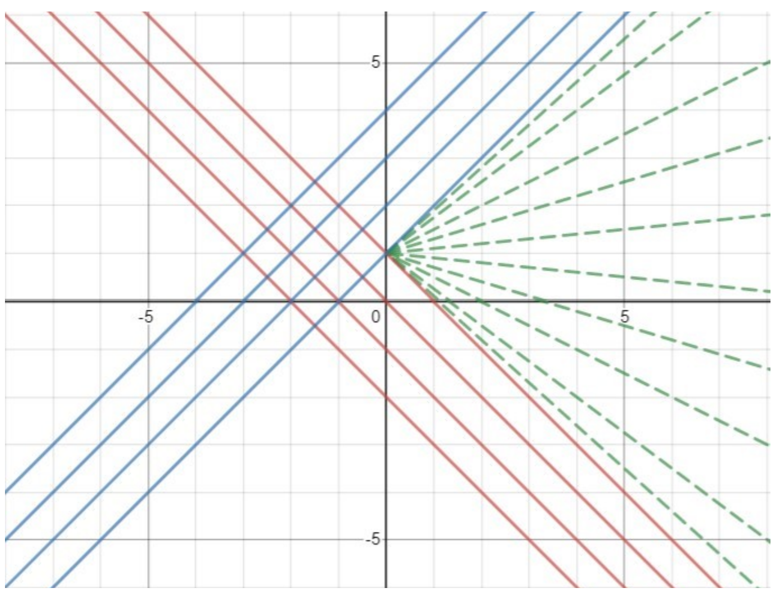


Figure 2: Graph showing the rarefaction region.

2.4 Shocks

A shock occurs when the characteristic curves intersect with each other. The time a shock begins can be found using

$$t_* \quad := \quad \min\left\{-\frac{1}{f'(x)}\middle| f'(x) < 0\right\}.$$

Shocks need to be resolved because a shock means that there are multiple values for the function at the points in the shock region. Resolving the shock using the method below creates a line $\sigma(t)$ that separates the conflicting characteristic curves and resolves the problem of having multiple values for each point in the region.

$$\begin{aligned} u_t + g(u)u_x &= 0 \\ u(0,x) &= f(x) \\ \text{Conservation of Mass:} & \\ T = u \quad , & \quad X = \frac{u^2}{2} \end{aligned}$$

Solve for the Shock line $\sigma(t)$ by using the conservation law given.

$$\begin{aligned} \frac{dx}{dt} &= g(u(0,x)) \\ x &\longrightarrow \sigma \\ \frac{d\sigma}{dt} &= \frac{X^+ - X^-}{T^+ - T^-} = \frac{\frac{(g(u^+))^2}{2} - \frac{(g(u^-))^2}{2}}{g(u^+) - g(u^-)} \end{aligned}$$

Solve and integrate to find $\sigma(t)$ using the initial conditions and the characteristic curves of $f(x)$ to figure out the constant of integration.

Mastery Check:

$$\begin{aligned} u_t + uu_x &= 0 \\ u(0,x) &= \begin{cases} 0, & x < -1 \\ x, & -1 < x < 0 \\ 3, & 0 < x \end{cases} \\ \text{Conservation of Mass:} & \\ T = u \quad , & \quad X = \frac{u^2}{2} \end{aligned}$$

Shock:
Solve for the Shock line $\sigma(t)$ by using the conservation law given.

$$\begin{aligned}
\frac{dx}{dt} &= u(0,x) \\
\frac{d\sigma}{dt} &= \frac{\frac{(u^+)^2}{2} - \frac{(u^-)^2}{2}}{u^+ - u^-} \\
u^+ &= \xi = \frac{\sigma}{t+1} \\
\frac{d\sigma}{dt} &= \frac{\frac{\left(\frac{\sigma}{t+1}\right)^2}{2} - 0}{\sigma - 0} = \frac{\sigma}{2(t+1)^2}
\end{aligned}$$

The Shock begins at $(0,-1)$ based off of the intitial conditions and by looking at the graph (Figure 3).

$$\begin{aligned}
\sigma(0) &= -1 \\
\sigma(t) &= -e^{-\frac{1}{2(t+1)}}
\end{aligned}$$

Rarefaction:
An equation needs to be found that can reflect the slope of each line in a rarefaction fan to fill in the empty space in the graph.
The rarefaction begins at $(0,0)$.

$$m = \frac{x-0}{t-0} = \frac{x}{t}$$

Final Solution:

$$\boxed{u(t,x) = \begin{cases} 0, & x < -e^{-\frac{1}{2(t+1)}} \\ x, & -e^{-\frac{1}{2(t+1)}} < x < 0 \\ \frac{x}{t}, & 0 < x < 3t \\ 3, & x > 3t \end{cases}}$$

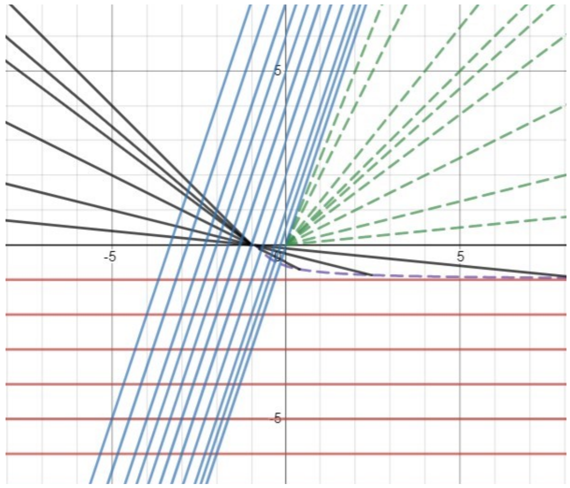


Figure 3: Graph showing the shock and rarefaction regions.

2.5 Rarefaction Waves

A rarefaction occurs whenever there are no values for the function in a region of points. On a graph of characteristic curves, this looks like a region of blank space between characteristic curves. To resolve a characteristic curve, an equation must be found to reflect the slope of each line in a rarefaction fan to fill in the region of empty space. The green lines in Figure 3 represent the lines of the rarefaction fan filling in the rarefaction region between the blue characteristic curves and the black characteristic curves.

$$\begin{aligned} u_t + g(u)u_x &= 0 \\ u(0,x) &= f(x) \end{aligned}$$

Solve for the slope of the rarefaction lines, where (t_0, x_0) is the initial point where the rarefaction begins, by knowing that any point in the rarefaction region (t, x) will have a slope of:

$$m = \frac{x - x_0}{t - t_0}$$

Set

$$g(u) = m$$

And solve for u to find the function $h(t, x)$ that will be the value of $u(t, x)$ inside the bounds of the rarefaction region.

$$u(t,x) = h(t,x)$$

Mastery Check:

$$\begin{aligned} u_t + uu_x &= 0 \\ u(0,x) &= \begin{cases} 0, & x < -1 \\ x, & -1 < x < 0 \\ 3, & 0 < x \end{cases} \end{aligned}$$

Conservation of Mass:

$$T = u \quad , \quad X = \frac{u^2}{2}$$

Shock:
Solve for the Shock line $\sigma(t)$ by using the conservation law given.

$$\begin{aligned} \frac{dx}{dt} &= u(0,x) \\ \frac{d\sigma}{dt} &= \frac{\frac{(u^+)^2}{2} - \frac{(u^-)^2}{2}}{u^+ - u^-} \\ u^+ &= \xi = \frac{\sigma}{t+1} \\ \frac{d\sigma}{dt} &= \frac{\frac{\left(\frac{\sigma}{t+1}\right)^2}{2} - 0}{\sigma - 0} = \frac{\sigma}{2(t+1)^2} \end{aligned}$$

The Shock begins at $(0, -1)$ based off of the intitial conditions and by looking at the graph (Figure 3).

$$\begin{aligned} \sigma(0) &= -1 \\ \sigma(t) &= -e^{-\frac{1}{2(t+1)}} \end{aligned}$$

Rarefaction:
An equation needs to be found that can reflect the slope of each line in a rarefaction fan to fill in the empty space in the graph.
The rarefaction begins at $(0, 0)$.

$$m = \frac{x - 0}{t - 0} = \frac{x}{t}$$

Final Solution:

$$\boxed{u(t,x) = \begin{cases} 0, & x < -e^{-\frac{1}{2(t+1)}} \\ x, & -e^{-\frac{1}{2(t+1)}} < x < 0 \\ \frac{x}{t}, & 0 < x < 3t \\ 3, & x > 3t \end{cases}}$$

Refer to Figure 3 for a graph of the characteristic curves of the final solution.

3 Fourier Series

Fourier Series are able to represent any function as an infinite series of sines and cosines. Many modern technologies, such as phones, televisions, and computers, were made using Fourier Series. The Gibbs Phenomenon is an overshoot that occurs at the point of a jump discontinuity in the Fourier Series representation of a function. The region of overshoot becomes narrower as the number of terms increases, but the amount of overshoot relative to the rest of the approximation remains the same.

3.1 Real Fourier Series

The Fourier Series for a function $f(x)$ on an interval $[-L, L]$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \tag{1}$$

The variables a_0, a_k, and b_k can be found using the following equations

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx \tag{2}$$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)dx \tag{3}$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)dx \tag{4}$$

The variables solved for using the above equations can then be inserted into the Fourier Series equation to find the Fourier Series for the function $f(x)$.

Mastery Check:

$$f(x) = |x-2|$$

on the interval $[-5, 5]$.

Calculate the variable a_0 using equation (2) and simplify.

$$a_0 = \frac{1}{5}\int_{-5}^{5} |x-2|dx$$
$$a_0 = \frac{29}{5}$$

Calculate the variable a_k using equation (3) and simplify.

$$a_k = \frac{1}{5}\int_{-5}^{5} |x-2|\cos\left(\frac{k\pi x}{5}\right)dx$$
$$a_k = \frac{10(-1)^k - 10\cos\left(\frac{2\pi k}{5}\right)}{\pi^2 k^2}$$

Calculate the variable b_k using equation (4) and simplify.

$$\begin{aligned}
b_k &= \frac{1}{5}\int_{-5}^{5} |x-2| \sin\left(\frac{k\pi x}{5}\right) dx \\
b_k &= \frac{4\pi k(-1)^k - 10\sin\left(\frac{2\pi k}{5}\right)}{\pi^2 k^2}
\end{aligned}$$

Plug the values for the variables a_0, a_k, and b_k back into equation (1) to find the solution.

$$\boxed{f(x) \sim 2.9 + \sum_{k=1}^{\infty} \frac{10(-1)^k - 10\cos\left(\frac{2\pi k}{5}\right)}{\pi^2 k^2} \cos\left(\frac{k\pi x}{5}\right) + \frac{4\pi k(-1)^k - 10\sin\left(\frac{2\pi k}{5}\right)}{\pi^2 k^2} \sin\left(\frac{k\pi x}{5}\right)}$$

3.2 Complex Fourier Series

The complex Fourier Series for a function $f(x)$ is

$$f(x) \sim C_0 + \sum_{k=-\infty}^{\infty} C_k e^{ikx}$$

The following formulas are often used to simplify f(x) before calculating C_k to make the math simpler.

$$\begin{aligned}
e^{i\pi} &= -1 \\
e^{ikx} &= \cos(kx) + i\sin(kx) \\
e^{-ikx} &= \cos(kx) - i\sin(kx) \\
\cos(kx) &= \frac{e^{ikx} + e^{-ikx}}{2} \\
\sin(kx) &= \frac{e^{ikx} - e^{-ikx}}{2i} \\
a_k &= C_k + C_{-k} \\
b_k &= i\left(C_k - C_{-k}\right) \\
C_k &= \frac{1}{2}\left(a_k - ib_k\right) \\
C_{-k} &= \frac{1}{2}\left(a_k + ib_k\right) \\
\sin(\pi k) &= 0 \text{ when } k \in \mathbb{Z}
\end{aligned}$$

The variable C_k can be found using the following equation

$$C_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx} dx$$

Do not forget to check for when $k = 0$ by using

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) dx$$

The variables solved for using the above equations can then be inserted into the Complex Fourier Series equation to find the Complex Fourier Series for the function $f(x)$.

Mastery Check:

$$f(x) = \sin^4(x)$$

on the interval $[-\pi, \pi]$.

$$\begin{aligned} C_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^4(x) e^{-ikx} dx \\ C_k &= \frac{48\sin(\pi k)}{2\pi(k^5 - 20k^3 + 64k)} \end{aligned}$$

$\sin(\pi k) = 0$ because $k \in \mathbb{Z}$.
When denominator equals 0:

$$\begin{aligned} k^5 - 20k^3 + 64k &= 0 \\ k_1 = -4, 4 &\longrightarrow C_{k_1} = \frac{\pi}{8} \\ \frac{C_{k_1}}{2\pi} &= \frac{1}{16} \\ k_2 = -2, 2 &\longrightarrow C_{k_2} = \frac{-\pi}{2} \\ \frac{C_{k_2}}{2\pi} &= \frac{-1}{4} \\ k_3 = 0 &\longrightarrow C_{k_3} = \frac{3\pi}{4} \\ \frac{C_{k_3}}{2\pi} &= \frac{3}{8} \\ \sin^4(x) &\sim \frac{3}{8} - \frac{1}{4}e^{-2ix} - \frac{1}{4}e^{2ix} + \frac{1}{16}e^{-4ix} + \frac{1}{16}e^{4ix} \end{aligned}$$

$$\boxed{\sin^4(x) \sim \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)}$$

3.3 Convergence of Fourier Series

If the Fourier coefficients of

$$\sum_{k=1}^{\infty} C_k e^{ikx}$$

satisfy

$$\begin{aligned} |C_k| &\leq \frac{M}{|k|^\alpha} \quad \forall \quad k \gg 0, \ \ \alpha > 1 \\ \alpha &> n+1 \\ \sum_{k=-\infty}^{\infty} |k|^n |C_k| &< \infty \end{aligned}$$

Where $M > 0$ is some constant and $n \in \mathbb{Z}^+$, then, by the Weierstrass M-test, the Fourier Series converges uniformly to an n-times differentiable 2π-periodic function.

Common Comparison tests to determine convergence:
Harmonic series $\longrightarrow$ diverges

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

Alternating harmonic series $\longrightarrow$ converges

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

P-series $\longrightarrow$ converges for $p > 1$

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

Alternating p-series $\longrightarrow$ converges for $p > 1$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$$

Mastery Check:
Discuss the convergence of

$$\sum_{k=1}^{\infty} \frac{1}{k^3 + k} \sin(kx).$$

According to $\sum C_k e^{ikx}$,

$$C_k = \frac{1}{k^3 + k}.$$

The condition stated below must be met in order for the Fourier Series to be convergent to an n-times differentiable 2π-periodic function.

$$\sum_{k=1}^{\infty} |k|^n |C_k| < \infty \quad \longrightarrow \quad \sum_{k=1}^{\infty} \left| \frac{k^n}{k^3 + k} \right| < \infty$$

Find the maximum value of n to find how many times the function is differentiable.
$n = 1$:

$$\sum_{k=1}^{\infty} \left| \frac{k}{k^3 + k} \right|$$

The series converges when $n = 1$ by the comparison test using a p-series where $p = 2$.
$n = 2$:

$$\sum_{k=1}^{\infty} \left| \frac{k^2}{k^3 + k} \right|$$

The series diverges when $n = 2$ by the integral test where $\int_1^{\infty} \frac{x^2}{x^3+x} dx = \infty$.
As the series diverges when $n = 2$, we can conclude that

$$\max\{n\} = 1.$$

$\boxed{\text{The Fourier Series is convergent to a 1-times differentiable } 2\pi\text{-periodic function.}}$

Checking our work:
The first derivative of the Fourier Series,

$$\frac{d}{dx}\left(\sum_{k=1}^{\infty}\frac{1}{k^3+k}\sin(kx)\right) \quad \rightarrow \quad \sum_{k=1}^{\infty}\frac{1}{k^2+1},$$

converges by the comparison test using a p-series where $p = 2$.
The second derivative of the Fourier Series,

$$\frac{d^2}{dx^2}\left(\sum_{k=1}^{\infty}\frac{1}{k^3+k}\sin(kx)\right) \quad \rightarrow \quad \sum_{k=1}^{\infty}\frac{-k}{k^2+1},$$

diverges by the series integral test where $\int_1^\infty \frac{-x}{x^2+1}dx = -\infty$.
The third derivative of the Fourier Series,

$$\frac{d^3}{dx^3}\left(\sum_{k=1}^{\infty}\frac{1}{k^3+k}\sin(kx)\right) \quad \rightarrow \quad \sum_{k=1}^{\infty}\frac{-k^2}{k^2+1},$$

diverges by the series integral test where $\int_1^\infty \frac{-x^2}{x^2+1}dx = -\infty$.

3.4 Integrability and Differentiability of Fourier Series

Integrability:
The mean of a function $f(x)$ follows the formula

$$m \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$$

If a function f is piecewise continuous and has a mean of zero on the interval $[-\pi, \pi]$, then its Fourier Series

$$f(x) \quad \sim \quad \sum_{k=1}^{\infty} a_k\cos(kx) + b_k\sin(kx)$$

can be integrated term-by-term to produce

$$\int_0^x f(y)dy \quad \sim \quad m + \sum_{k=1}^{\infty} -\frac{b_k}{k}\cos(kx) + \frac{a_k}{k}\sin(kx).$$

If a function $f(x)$ has a nonzero mean m, then integrating the function would produce

$$\int_0^x f(y)dy \quad \sim \quad \frac{a_0}{2}x + m + \sum_{k=1}^{\infty} -\frac{b_k}{k}\cos(kx) + \frac{a_k}{k}\sin(kx)$$

where x can be replaced by its Fourier Series and the result will be the Fourier Series for the integral of the function $f(x)$

$$\int_0^x f(y)dy \quad \sim \quad \frac{a_0}{2}\left(2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin(kx)\right) + m + \sum_{k=1}^{\infty} -\frac{b_k}{k}\cos(kx) + \frac{a_k}{k}\sin(kx).$$

Differentiability:
The function $f(x)$ must have a piecewise C^2 and continuous 2π-periodic extension in order for the derivative of its Fourier Series to converge to the derivative of the function. If the aforementioned criteria are met, then the Fourier Series of the function $f(x)$ can be differentiated term-by-term to produce the Fourier Series

$$f'(x) \quad \sim \quad \sum_{k=1}^{\infty} kb_k \cos(kx) - ka_k \sin(kx)$$

Or for Complex Fourier Series

$$f'(x) \quad \sim \quad \sum_{k=-\infty}^{\infty} ikc_k e^{ikx}.$$

Mastery Check:
Find the Fourier Series for x^3, then differentiate the series and discuss the results. Start with:

$$x \quad \sim \quad 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx)$$

The Fourier Series of x is integrable because the function x is piecewise continuous and has mean zero on the interval $[-\pi, \pi]$.

$$\begin{aligned}
\frac{x^2}{2} \quad &\sim \quad m + 2\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kx) \\
m \quad &= \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{x^2}{2} dx = \frac{\pi^2}{6} \\
\frac{x^2}{2} \quad &\sim \quad \frac{\pi^2}{6} + 2\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kx)
\end{aligned}$$

Integrate again to find the Fourier Series for $\frac{x^3}{3}$.

$$\begin{aligned}
\frac{x^3}{3} \quad &\sim \quad m + 2\frac{\pi^2}{6}x + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \sin(kx) \\
m \quad &= \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{x^3}{3} dx = 0
\end{aligned}$$

Replace x with the Fourier Series for x, then simplify and solve for the Fourier series of x^3.

$$\begin{aligned}
\frac{\pi^2}{6}x \quad &= \quad \frac{\pi^2}{6}\left(4\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx)\right) \\
\frac{x^3}{3} \quad &\sim \quad \frac{\pi^2}{6}\left(4\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx)\right) + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \sin(kx) \\
x^3 \quad &\sim \quad 12\left(\sum_{k=1}^{\infty} \frac{\pi^2(-1)^k + 1}{6k} \sin(kx) + \frac{(-1)^k}{k^3} \sin(kx)\right)
\end{aligned}$$

$$\boxed{x^3 \sim 2\sum_{k=1}^{\infty} \sin(kx)\left(\frac{(-1)^k(6-\pi^2k^2)}{k^3}\right)}$$

Derivative of the Fourier Series of x^3:

$$3x^2 \quad \nsim \quad 2\sum_{k=1}^{\infty}\left(\frac{(-1)^k(6-\pi^2k^2)}{k^2}\right)\cos(kx)$$

The Fourier Series of x^3 cannot be differentiated term-by-term because the derivative of the Fourier Series does not converge to the derivative of the function. The 2π-periodic extension of x^3 is discontinuous. A function must be continuous 2π-periodic in order for the derivative of the Fourier Series to converge to the derivative of the function.

3.5 Boundary Conditions

Homogeneous means the boundary conditions are equal to zero.
Nonhomogeneous means the boundary conditions are not equal to zero, but instead equal to a function.

- Dirichlet Boundary Conditions
  Prescribes the value on the endpoints/boundary of the region.

$$\begin{aligned} u(t,0) &= \alpha(t) \\ u(t,L) &= \beta(t) \end{aligned}$$

  Note: Homogeneous Dirichlet boundary conditions produce a Fourier Sine series when used with 2nd order PDEs.

- Neumann Boundary Conditions
  Prescribes the value of the normal derivative on the boundary.

$$\begin{aligned} u_x(t,0) &= \alpha(t) \\ u_x(t,L) &= \beta(t) \end{aligned}$$

  Note: Homogeneous Neumann boundary conditions produce a Fourier Cosine series when used with 2nd order PDEs.

- Periodic Boundary Conditions
  The value at the endpoints remain the same and the value of the normal derivatives at the endpoints remain the same.

$$\begin{aligned} u(t,0) &= u(0,L) \\ u_x(t,0) &= u_x(t,L) \end{aligned}$$

- Robin Boundary Conditions
  A combination of Dirichlet and Neumann boundary conditions with a constant coefficient β.

$$u(t,0) + \beta u_x(t,o) \quad = \quad \mu(t)$$

Mastery Check:
Heat Equation: $u_t = \gamma u_{xx}$
Give:

- Homogeneous Dirichlet Boundary Conditions on $[0, 6]$.
- Nonhomogeneous Neumann Boundary Conditions on $[0, 3]$.

Homogeneous Dirichlet Boundary Conditions:

$$\begin{aligned} u(t,0) &= 0 \\ u(t,6) &= 0 \end{aligned}$$

Produces a Fourier Sine series when homogeneous.

Nonhomogenous Neumann Boundary Conditions:

$$\begin{aligned} u_x(t,0) &= \sin(t) \\ u_x(t,3) &= \cos(t) \end{aligned}$$

Produces a Fourier Cosine series when homogeneous.

4 Separation of Variables

The function $u(t, x)$ can be separated using the following assumptions:

$$\begin{aligned} u(t,x) &= w(t)v(x) \\ w(t) &\not\equiv 0 \\ v(x) &\not\equiv 0 \end{aligned}$$

The single variable functions $w(t)$ and $v(x)$ can be plugged into the given equation and then each can be solved for. A heat equation example would be:

$$\begin{aligned} w'(t)v(x) &= \gamma w(t)v''(x) \\ \frac{w'(t)}{w(t)} &= \gamma\frac{v''(x)}{v(x)} = -\lambda \\ w(t) &= e^{-\lambda t} \end{aligned}$$

The constant λ has three possibilities of it's value, $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$. This is used along with the given boundary conditions to find the values of the single variable functions that are dependent on λ. The final solution can be found using

$$u(t,x) = a_0 + \sum_{k=1}^{\infty} a_k w(t)v(x).$$

4.1 Equilibrium behavior of a solution

A system is in equilibrium when it does not change with respect to time. Equilibrium solutions are found by removing any derivatives with respect to time and then solving for the solution of that system.

$$\begin{aligned} u_t &= \gamma u_{xx} \\ u(t,0) &= \alpha \\ u(t,L) &= \beta \end{aligned}$$

The first step to solving an equilibrium problem is to set all derivatives with respect to time equal to 0.

$$0 = \gamma u_{xx}$$

Solving the differential equation gives the general solution

$$u^*(x) = Ax + B$$

The given boundary conditions can then be plugged into the general solution to find the unknown coefficients A and B. Because the general solution is a straight line between the boundary values, the final solution can be written as

$$u^*(x) = \frac{\beta - \alpha}{L}x + \alpha,$$

where α and β are constants. If $\alpha(t)$ and $\beta(t)$ are not constants, then

$$u^*(x) = \lim_{t\to\infty} u(t,x).$$

$u^*(x)$ can be found by solving for

$$\lim_{t\to\infty}\alpha(t) = \alpha^* \quad \text{and} \quad \lim_{t\to\infty}\beta(t) = \beta^*$$

then plugging them into

$$u^*(x) = \frac{\beta^* - \alpha^*}{L}x - \alpha^*.$$

Mastery Check:
Solve

$$\begin{aligned} u_t &= .04u_{xx} \\ u(0,x) &= \frac{11(x+5)+3(x+2)}{7} + \frac{(x^2+3x+10)\sin^2(x)}{x^2+1} \\ u(t,-5) &= -3 \\ u(t,2) &= 11 \end{aligned}$$

Set all derivatives with respect to time equal to 0 because a system is in equlibrium when it does not change with respect to time.

$$u^*_{xx} = 0$$

Then integrate to find a general solution to u^*.

$$u^* = Ax + B$$

Plug the given boundary conditions into the formula for u^*.

$$\begin{aligned} u^*(-5) &= -5A + B = -3 \\ u^*(2) &= 2A + B = 11 \end{aligned}$$

Solve the system of equations to find the variables A and B.

$$A = 2 \quad , \quad B = 7$$

Plug the values for A and B back into the formula for u^* to find the final equilibrium solution.

$$\boxed{u^*(x) = 2x + 7}$$

This can be related to the general solution of

$$u^*(x) = \alpha + \frac{\beta - \alpha}{L}x,$$

where $\alpha = u(t,0) = 7$ and $\beta = u(t,L) = 11$.

4.2 Solving the 1D heat equation

The 1D heat equation is given by

$$u_t = \gamma u_{xx}$$

The initial and boundary conditions will be provided in the problem. They will be Dirichlet, Neumann, Periodic, Robin, or a mix of boundary conditions.
Assume

$$\begin{aligned} u(t,x) &= w(t)v(x) \\ w(t) &\not\equiv 0 \\ v(x) &\not\equiv 0 \end{aligned}$$

Then

$$\begin{aligned} w'(t)v(x) &= \gamma w(t)v''(x) \\ \frac{w'(t)}{w(t)} &= \gamma\frac{v''(x)}{v(x)} = -\lambda \\ w(t) &= e^{-\lambda t} \end{aligned}$$

The equation for $v(x)$ is dependent on the value of λ. There are three different possible cases for the value of λ:

$$\begin{array}{llll} \text{Case 1:} & \lambda = 0 & \Longrightarrow & v(x) = Ax + B \\ \text{Case 2:} & \lambda < 0 & \Longrightarrow & v(x) = A\cos(\omega x) + B\sin(\omega x), \text{ where } \omega = \sqrt{\frac{\lambda}{\gamma}} \\ \text{Case 3:} & \lambda > 0 & \Longrightarrow & v(x) = A\cosh(\omega x) + B\sinh(\omega x), \text{ where } \omega = \sqrt{\frac{-\lambda}{\gamma}} \end{array}$$

Each case must be solved using the boundary conditions given for the problem. If the resultant equation cancels out with both of the unknown coefficients, A and B, equaling 0, then the solution to that case is trivial and the next case must be attempted.

After ω is found, it can be solved to find λ. $u(t,x)$ can then be found using

$$\begin{aligned} u(t,x) &= w(t)v(x) \\ u(t,x) &= \sum_{k=1}^{\infty} a_k e^{-\lambda t} v(x) \\ a_k &= \frac{2}{L}\int_0^L |f(x)| \sin\left(\frac{k\pi x}{L}\right) dx \end{aligned}$$

If $v(x)$ is linear as in Case 1, an a_0 term is necessary, which can be found using

$$a_0 = \frac{1}{L}\int_0^L |f(x)| dx$$

This results in a final solution equation of

$$u(t,x) = a_0 + \sum_{k=1}^{\infty} a_k e^{-\lambda t} v(x)$$

Mastery Check:
Solve:

$$\begin{aligned} u_t &= .5u_{xx} \\ u(t,0) &= 0 \\ u_x(t,4) &= 0 \\ u(0,x) &= x(x-4)^2 \end{aligned}$$

Assume

$$\begin{aligned} u(t,x) &= w(t)v(x) \\ w(t) &\not\equiv 0 \\ v(x) &\not\equiv 0 \end{aligned}$$

Then

$$\begin{aligned} w'(t)v(x) &= .5w(t)v''(x) \\ \frac{w'(t)}{w(t)} &= .5\frac{v''(x)}{v(x)} = -\lambda \\ w(t) &= e^{-\lambda t} \end{aligned}$$

Plug in the given boundary conditions for each case of the value of λ.
Case 1: $\lambda = 0$

$$\begin{aligned} v(x) &= Ax + B \\ v(0) &= B = 0 \\ v'(x) &= A \\ v'(4) &= A = 0 \end{aligned}$$

Case 1 results in a trivial solution.

Case 2: $\lambda < 0$

$$\begin{aligned} v(x) &= A\cos(\omega x) + B\sin(\omega x) \\ v(0) &= A = 0 \\ v'(x) &= B\omega\cos(\omega x) - A\omega\sin(\omega x) \\ v'(4) &= B\omega\cos(4\omega) = 0 \end{aligned}$$

Results in:

$$B = 0 \quad \text{OR} \quad \cos(4\omega) = 0$$

Case 2 has a nontrivial solution as $\cos(4\omega) = 0$ is true for $\omega = \frac{\pi}{8} + \frac{k\pi}{4}$.

Case 3: $\lambda > 0$

$$\begin{aligned} v(x) &= A\cosh(\omega x) + B\sinh(\omega x) \\ v(0) &= A = 0 \\ v'(x) &= \omega Ae^{\omega x} - \omega Be^{-\omega x} \\ v'(4) &= -\omega Be^{-4\omega} = 0 \\ B &= 0 \end{aligned}$$

Case 3 results in a trivial solution as both coefficients are equal to 0.

Case 2 was the only nontrivial solution, so we must move forward using the value of ω from that case.

$$\begin{aligned}
\sqrt{2\lambda} &= \omega = \frac{\pi}{8} + \frac{k\pi}{4} \\
\lambda &= \frac{1}{2}\left(\frac{\pi}{8} + \frac{k\pi}{4}\right)^2 \\
w(t) &= Ce^{-\lambda t} = Ce^{-\frac{1}{2}\left(\frac{\pi}{8}+\frac{k\pi}{4}\right)^2 t} \\
v(x) &= B\sin(\omega x) = B\sin\left(\left(\frac{\pi}{8} + \frac{k\pi}{4}\right)x\right)
\end{aligned}$$

Putting it all together now

$$\begin{aligned}
u(t,x) &= w(t)v(x) \\
u(t,x) &= \sum_{k=1}^{\infty} a_k e^{-\frac{1}{2}\left(\frac{\pi}{8}+\frac{k\pi}{4}\right)^2 t} \sin\left(\left(\frac{\pi}{8} + \frac{k\pi}{4}\right)x\right)
\end{aligned}$$

The variable a_k is found using the integral of a Fourier Sine Series.

$$\begin{aligned}
a_k &= \frac{1}{2}\int_0^4 x(x-4)^2 \sin\left(\left(\frac{\pi}{8} + \frac{k\pi}{4}\right)x\right)dx \\
a_k &= \frac{-2048(\pi(2k+1)(\sin(\pi k)-2) + 6\cos(\pi k))}{\pi^4(2k+1)^4}
\end{aligned}$$

Plugging a_k back into the equation gives the final solution

$$\boxed{u(t,x) = \sum_{k=1}^{\infty} \frac{-2048(\pi(2k+1)(\sin(\pi k)-2) + 6\cos(\pi k))}{\pi^4(2k+1)^4} e^{-\frac{1}{2}\left(\frac{\pi}{8}+\frac{k\pi}{4}\right)^2 t} \sin\left(\left(\frac{\pi}{8} + \frac{k\pi}{4}\right)x\right)}$$

4.3 Solving the Laplace equation

Solve:

$$\begin{aligned}
u_{xx} + u_{yy} &= 0 \\
0 \leq\ & x \ \leq L \\
0 \leq\ & y \ \leq H \\
u(x,0) &= f_1(x) \\
u(x,H) &= f_2(x) \\
u(0,y) &= g_1(y) \\
u(L,y) &= g_2(y)
\end{aligned}$$

Separable solutions to Laplace's Equation:

λ	$v(x)$	$w(y)$
$\lambda = -\omega^2 < 0$	$\cos(\omega x), \sin(\omega x)$	$e^{-\omega y}, e^{\omega y}$
$\lambda = 0$	$1, x$	$1, y$
$\lambda = \omega^2 > 0$	$e^{-\omega x}, e^{\omega x}$	$\cos(\omega y), \sin(\omega y)$

1) Begin by setting all boundary conditions equal to 0 except for one.

$$\begin{aligned} u(x,0) &= f_1(x) \\ u(x,H) &= 0 \\ u(0,y) &= 0 \\ u(L,y) &= 0 \end{aligned}$$

Assume

$$\begin{aligned} u_1(x,y) &= w(x)v(y) \\ w(t), v(y) &\not\equiv 0 \end{aligned}$$

Plug in the homogenous boundary conditions

$$\begin{aligned} w(0)v(y) &= w(1)v(y) = 0 \\ w(0) &= w(1) = 0 \end{aligned}$$

Cases: $\lambda < 0 \to \lambda = -\omega^2$

$$\begin{aligned} w(x) &= A\cos(\omega x) + B\sin(\omega x) \\ w(0) &= A = 0 \\ w(L) &= B\sin(\omega) = 0 \\ \omega &= \frac{k\pi}{L} \end{aligned}$$

Recombining gives

$$\begin{aligned} w(x) &= B\sin\left(\frac{k\pi x}{L}\right) \\ v(y) &= C\cosh\left(\frac{k\pi y}{L}\right) + D\sinh\left(\frac{k\pi y}{L}\right) \\ v(H) &= C\cosh\left(\frac{k\pi H}{L}\right) + D\sinh\left(\frac{k\pi H}{L}\right) = 0 \longrightarrow e^{\frac{2k\pi H}{L}} = -\frac{D}{C} \end{aligned}$$

A shift is required: Set $C = 1$, then $D = -e^{\frac{2k\pi H}{L}}$. The following conditions must be satisfied by the shift

$$v_k(H) = C_k \sinh(0) = 0,$$

yielding the resultant equation

$$v_k(y) = C_k \sinh\left(\frac{k\pi(H-y)}{L}\right).$$

Combine the terms found.

$$u_1(x,y) = \sum_{k=1}^{\infty} a_k \sinh\left(\frac{k\pi(H-y)}{L}\right)\sin\left(\frac{k\pi x}{L}\right)$$

Plug in the inhomogenous boundary condition.

$$u_1(x,0) = f_1(x) = \sum_{k=1}^{\infty} a_k \sinh\left(\frac{k\pi H}{L}\right)\sin\left(\frac{k\pi x}{L}\right)$$

Solve for a_k the same way as solving for the coefficients in a Fourier Sine Series.

$$
\begin{aligned}
a_k &= \frac{2}{L\sinh\left(\frac{k\pi H}{L}\right)} \int_0^L f_1(x)\sin\left(\frac{k\pi x}{L}\right)dx \\
u_1(x,y) &= \sum_{k=1}^{\infty} \frac{2}{L\sinh\left(\frac{k\pi H}{L}\right)} \left(\int_0^L f_1(x)\sin\left(\frac{k\pi x}{L}\right)dx\right)\sinh\left(\frac{k\pi(H-y)}{L}\right)\sin\left(\frac{k\pi x}{L}\right)
\end{aligned}
$$

2) Repeat the previous steps but with different boundary conditions.

$$
\begin{aligned}
u(x,0) &= 0 \\
u(x,H) &= f_2(x) \\
u(0,y) &= 0 \\
u(L,y) &= 0
\end{aligned}
$$

The same information holds true for $w(x)$, but $v(y)$ must change to account for the new boundary conditions.

$$
\begin{aligned}
w(x) &= B\sin\left(\frac{k\pi x}{L}\right) \\
v_k(0) &= C_k\sinh(0) = 0 \\
v_k(y) &= C_k\sinh\left(\frac{k\pi y}{L}\right)
\end{aligned}
$$

Recombining $w(x)$ and $v(y)$ gives

$$
u_2(x,y) = \sum_{k=1}^{\infty} b_k\sinh\left(\frac{k\pi y}{L}\right)\sin\left(\frac{k\pi x}{L}\right).
$$

The boundary condition must then be plugged in to solve for the coefficient b_k.

$$
\begin{aligned}
u_2(x,H) &= f_2(x) = \sum_{k=1}^{\infty} b_k\sinh\left(\frac{k\pi H}{L}\right)\sin\left(\frac{k\pi x}{L}\right) \\
b_k &= \frac{2}{L\sinh\left(\frac{k\pi H}{L}\right)} \int_0^L f_2(x)\sin\left(\frac{k\pi x}{L}\right)dx
\end{aligned}
$$

Plugging b_k back into $u_2(x,y)$ yields

$$
u_2(x,y) = \sum_{k=1}^{\infty} \frac{2}{L\sinh\left(\frac{k\pi H}{L}\right)} \left(\int_0^L f_2(x)\sin\left(\frac{k\pi x}{L}\right)dx\right)\sinh\left(\frac{k\pi y}{L}\right)\sin\left(\frac{k\pi x}{L}\right).
$$

3) Repeat the previous steps but with different boundary conditions.

$$
\begin{aligned}
u(x,0) &= 0 \\
u(x,H) &= 0 \\
u(0,y) &= g_1(y) \\
u(L,y) &= 0
\end{aligned}
$$

The work done previously can be used as a template of what to do for the rest of the cases.

$$\begin{aligned}
u_3(x,y) &= \sum_{k=1}^{\infty} c_k \sinh\left(\frac{k\pi(L-x)}{H}\right)\sin\left(\frac{k\pi y}{H}\right)\\
u_3(0,y) &= g_1(y) = \sum_{k=1}^{\infty} c_k \sinh\left(\frac{k\pi L}{H}\right)\sin\left(\frac{k\pi y}{H}\right)\\
c_k &= \frac{2}{H\sinh\left(\frac{k\pi L}{H}\right)}\int_0^H g_1(y)\sin\left(\frac{k\pi y}{H}\right)dy\\
u_3(x,y) &= \sum_{k=1}^{\infty}\frac{2}{H\sinh\left(\frac{k\pi L}{H}\right)}\left(\int_0^H g_1(y)\sin\left(\frac{k\pi y}{H}\right)dy\right)\sinh\left(\frac{k\pi(L-x)}{H}\right)\sin\left(\frac{k\pi y}{H}\right)
\end{aligned}$$

4) Repeat the previous steps but with different boundary conditions.

$$\begin{aligned}
u(x,0) &= 0\\
u(x,H) &= 0\\
u(0,y) &= 0\\
u(L,y) &= g_2(y)
\end{aligned}$$

The work done previously can be used as a template of what to do for the rest of the cases.

$$\begin{aligned}
u_4(x,y) &= \sum_{k=1}^{\infty} d_k \sinh\left(\frac{k\pi x}{H}\right)\sin\left(\frac{k\pi y}{H}\right)\\
u_4(L,y) &= g_2(y) = \sum_{k=1}^{\infty} d_k \sinh\left(\frac{k\pi L}{H}\right)\sin\left(\frac{k\pi y}{H}\right)\\
d_k &= \frac{2}{H\sinh\left(\frac{k\pi L}{H}\right)}\int_0^H g_2(y)\sin\left(\frac{k\pi y}{H}\right)dy\\
u_4(x,y) &= \sum_{k=1}^{\infty}\frac{2}{H\sinh\left(\frac{k\pi L}{H}\right)}\left(\int_0^H g_2(y)\sin\left(\frac{k\pi y}{H}\right)dy\right)\sinh\left(\frac{k\pi x}{H}\right)\sin\left(\frac{k\pi y}{H}\right)
\end{aligned}$$

Recombining all of the solutions gives

$$\begin{aligned}
u(t,x) &= u_1(x,y)+u_2(x,y)+u_3(x,y)+u_4(x,y)\\
u(t,x) &= \sum_{k=1}^{\infty}\left[\frac{2}{L\sinh\left(\frac{k\pi H}{L}\right)}\left(\int_0^L f_1(x)\sin\left(\frac{k\pi x}{L}\right)dx\right)\sinh\left(\frac{k\pi(H-y)}{L}\right)\sin\left(\frac{k\pi x}{L}\right)\right.\\
&\quad+\frac{2}{L\sinh\left(\frac{k\pi H}{L}\right)}\left(\int_0^L f_2(x)\sin\left(\frac{k\pi x}{L}\right)dx\right)\sinh\left(\frac{k\pi y}{L}\right)\sin\left(\frac{k\pi x}{L}\right)\\
&\quad+\frac{2}{H\sinh\left(\frac{k\pi L}{H}\right)}\left(\int_0^H g_1(y)\sin\left(\frac{k\pi y}{H}\right)dy\right)\sinh\left(\frac{k\pi(L-x)}{H}\right)\sin\left(\frac{k\pi y}{H}\right)\\
&\quad\left.+\frac{2}{H\sinh\left(\frac{k\pi L}{H}\right)}\left(\int_0^H g_2(y)\sin\left(\frac{k\pi y}{H}\right)dy\right)\sinh\left(\frac{k\pi x}{H}\right)\sin\left(\frac{k\pi y}{H}\right)\right].
\end{aligned}$$

Mastery Check

$$\begin{array}{rcl} u_{xx}+u_{yy} & = & 0 \\ 0 \leq & x,y & \leq 1 \\ u(x,0) & = & x \\ u(x,1) & = & 1-x \\ u(0,y) & = & y \\ u(1,y) & = & 1-y \end{array}$$

1) First set all but one condition to 0:

$$\begin{array}{rcl} u(x,0) & = & x \\ u(x,1) & = & 0 \\ u(0,y) & = & 0 \\ u(1,y) & = & 0. \end{array}$$

Cases: $\lambda < 0 \rightarrow \lambda = -\omega^2$

$$\begin{array}{rcl} w(x) & = & A\cos(\omega x)+B\sin(\omega x) \\ w(0) & = & A=0 \\ w(1) & = & B\sin(\omega)=0 \\ \omega & = & k\pi \end{array}$$

Recombining gives

$$\begin{array}{rcl} w(x) & = & B\sin(k\pi x) \\ v(y) & = & C\cosh(k\pi y)+D\sinh(k\pi y) \\ v(1) & = & C\cosh(k\pi)+D\sinh(k\pi)=0 \longrightarrow e^{2k\pi} = -\dfrac{D}{C} \\ v_k(y) & = & C_k\sinh(0)=0 \\ v_k(y) & = & C_k\sinh(k\pi(1-y)). \end{array}$$

Plugging in $w(x)$ and $v(y)$ yields

$$\begin{array}{rcl} u_1(x,y) & = & \displaystyle\sum_{k=1}^{\infty} a_k \sinh(k\pi(1-y))\sin(k\pi x) \\ u_1(x,0) & = & x = \displaystyle\sum_{k=1}^{\infty} a_k\sinh(k\pi)\sin(k\pi x) \\ a_k & = & \dfrac{2}{\sinh(k\pi)}\displaystyle\int_0^1 x\sin(k\pi x)dx = \dfrac{2}{\sinh(k\pi)}\left(\dfrac{-\cos(k\pi)}{k\pi}\right) \\ u_1(x,y) & = & \displaystyle\sum_{k=1}^{\infty}\dfrac{2}{\sinh(k\pi)}\left(\dfrac{-\cos(k\pi)}{k\pi}\right)\sinh(k\pi(1-y))\sin(k\pi x). \end{array}$$

2) Repeat the previous steps for the second boundary condition.

$$\begin{array}{rcl} u(x,1) & = & 1-x \\ u(x,0) & = & 0 \\ u(0,y) & = & 0 \\ u(1,y) & = & 0 \end{array}$$

$$\begin{aligned}
u_2(x,y) &= \sum_{k=1}^{\infty} b_k \sinh(k\pi y)\sin(k\pi x) \\
u_2(x,1) &= \sum_{k=1}^{\infty} b_k \sinh(k\pi)\sin(k\pi x) \\
b_k &= \frac{2}{\sinh(k\pi)}\int_0^1 (1-x)\sin(k\pi x)dx = \frac{2}{k\pi\sinh(k\pi)} \\
u_2(x,y) &= \sum_{k=1}^{\infty} \frac{2}{k\pi\sinh(k\pi)}\sinh(k\pi y)\sin(k\pi x)
\end{aligned}$$

3) Repeat the previous steps for the third boundary condition.

$$\begin{aligned}
u(0,y) &= y \\
u(1,y) &= 0 \\
u(x,1) &= 0 \\
u(x,0) &= 0
\end{aligned}$$

$$\begin{aligned}
u_3(x,y) &= \sum_{k=1}^{\infty} c_k \sinh(k\pi(1-x))\sin(k\pi y) \\
u_3(0,y) &= y = \sum_{k=1}^{\infty} c_k \sinh(k\pi)\sin(k\pi y) \\
c_k &= \frac{2}{\sinh(k\pi)}\int_0^1 y\sin(k\pi y)dy = \frac{2}{\sinh(k\pi)}\left(\frac{-\cos(k\pi)}{k\pi}\right) \\
u_3(x,y) &= \sum_{k=1}^{\infty} \frac{2}{\sinh(k\pi)}\left(\frac{-\cos(k\pi)}{k\pi}\right)\sinh(k\pi(1-x))\sin(k\pi y)
\end{aligned}$$

4) Repeat the previous steps for the fourth boundary condition.

$$\begin{aligned}
u(1,y) &= 1-y \\
u(0,y) &= 0 \\
u(x,1) &= 0 \\
u(x,0) &= 0
\end{aligned}$$

$$\begin{aligned}
u_4(x,y) &= \sum_{k=1}^{\infty} d_k \sinh(k\pi x)\sin(k\pi y) \\
u_4(1,y) &= \sum_{k=1}^{\infty} d_k \sinh(k\pi)\sin(k\pi y) \\
d_k &= \frac{2}{\sinh(k\pi)}\int_0^1 (1-y)\sin(k\pi y)dy = \frac{2}{k\pi\sinh(k\pi)} \\
u_4(x,y) &= \sum_{k=1}^{\infty} \frac{2}{k\pi\sinh(k\pi)}\sinh(k\pi x)\sin(k\pi y)
\end{aligned}$$

Putting it all together: $u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)$

$$\begin{aligned} u(x,y) &= \sum_{k=1}^{\infty} \left[\frac{2}{\sinh(k\pi)} \left(\frac{-\cos(k\pi)}{k\pi} \right) \sinh(k\pi(1-y)) \sin(k\pi x) \right. \\ &\quad + \frac{2}{k\pi \sinh(k\pi)} \sinh(k\pi y) \sin(k\pi x) \\ &\quad + \frac{2}{\sinh(k\pi)} \left(\frac{-\cos(k\pi)}{k\pi} \right) \sinh(k\pi(1-x)) \sin(k\pi y) \\ &\quad \left. + \frac{2}{k\pi \sinh(k\pi)} \sinh(k\pi x) \sin(k\pi y) \right]. \end{aligned}$$

Simplifying:

$$\begin{aligned} u(x,y) &= \sum_{k=1}^{\infty} \left[\frac{2}{k\pi \sinh(\pi k)} \left(\sin(k\pi x)((-1)^{k+1} \sinh(k\pi(1-y)) + \sinh(k\pi y)) \right. \right. \\ &\quad \left. \left. + \sin(k\pi y) \left((-1)^{k+1} \sinh(k\pi(1-x)) + \sinh(k\pi x) \right) \right) \right]. \end{aligned}$$

4.4 Solving the 1D wave equation

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(t,0) &= u(t,L) = 0 \\ u(0,x) &= f(x) \\ u_t(0,x) &= g(x) \end{aligned}$$

Assume $u(t,x) = w(t)v(x)$.

λ	$w(t)$	$v(x)$
$\lambda = -\omega^2 < 0$	$A\cos(\omega t) + B\sin(\omega t)$	$R\cos\left(\frac{\omega x}{c}\right) + S\sin\left(\frac{\omega x}{c}\right)$
$\lambda = 0$	$At + B$	$Rx + S$
$\lambda = \omega^2 > 0$	$Ae^{-\omega t} + Be^{\omega t}$	$Re^{\frac{-\omega x}{c}} + Se^{\frac{\omega x}{c}}$

Use the above table to solve the cases for a nontrivial solution to ω.
Cases: $\lambda < 0$

$$\begin{aligned} v(x) &= R\cos\left(\frac{\omega x}{c}\right) + S\sin\left(\frac{\omega x}{c}\right) \\ v(0) &= R = 0 \\ v(L) &= S\sin\left(\frac{\omega L}{c}\right) = 0 \\ S = 0 \quad &\text{OR} \quad \sin\left(\frac{\omega L}{c}\right) = 0 \\ \frac{\omega L}{c} &= k\pi \quad \forall \quad k \in \mathbb{Z}^+ \\ \omega &= \frac{k\pi c}{L} \end{aligned}$$

Plug in ω to find the solution equations for $w(t)$ and $v(x)$.

$$\begin{aligned} w(t) &= A\cos\left(\frac{k\pi ct}{L}\right) + B\sin\left(\frac{k\pi ct}{L}\right) \\ v(x) &= S\sin\left(\frac{k\pi x}{L}\right) \end{aligned}$$

Plug in the found solution equations from the cases to the assumption $u(t,x) = w(t)v(x)$.

$$u(t,x) = a_0 + b_0 t + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi ct}{L}\right)\sin\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi ct}{L}\right)\sin\left(\frac{k\pi x}{L}\right)$$

Solve for the coefficients a_k, a_0, b_k, and b_0 using the initial conditions.

$$\begin{aligned} u(0,x) = f(x) &= \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right) \\ a_k &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)dx \\ a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ u_t(0,x) = g(x) &= \sum_{k=1}^{\infty} b_k \frac{k\pi}{L}\sin\left(\frac{k\pi x}{L}\right) \\ b_k &= \frac{2}{k\pi c}\int_0^L g(x)\sin\left(\frac{k\pi x}{L}\right)dx \\ b_0 &= \frac{2}{k\pi c}\int_0^L g(x)dx \end{aligned}$$

Plug the values for the variables back into the following equation to give the solution.

$$u(t,x) = a_0 + b_0 t + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi ct}{L}\right)\sin\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi ct}{L}\right)\sin\left(\frac{k\pi x}{L}\right)$$

Mastery Check:

$$\begin{aligned} u_{tt} &= u_{xx} \\ u_x(t,0) &= u_x(t,1) = 0 \\ u(0,x) &= x(1-x) \\ u_t(0,x) &= 0 \end{aligned}$$

Assume

$$\begin{aligned} u(t,x) &= w(t)v(x) \\ \frac{w''(t)}{w(t)} &= \frac{v''(x)}{v(x)} = \lambda \\ w''(t) &= \lambda w(t) \\ v''(x) &= \lambda v(x) \end{aligned}$$

Case 1: $\lambda = 0$

$$\begin{aligned} w(t) &= At + B \\ w'(0) &= A = 0 \\ v(x) &= Ct + D \\ v'(0) = v'(1) &= C = 0 \end{aligned}$$

B and D are nontrivial.
Case 2: $\lambda < 0$

$$\begin{aligned} w(t) &= A\cos(\omega t) + B\sin(\omega t) \\ w'(0) &= B\omega = 0 \\ B &= 0 \\ v(x) &= C\cos(\omega x) + D\sin(\omega x) \\ v'(0) &= D\omega = 0 \\ D &= 0 \\ v'(1) &= -C\omega\sin(\omega) = 0 \\ C &= 0 \end{aligned}$$

A is nontrivial.
Case 3: $\lambda > 0$

$$\begin{aligned} w(t) &= A\cosh(\omega t) + B\sinh(\omega t) \\ w'(t) &= \omega A e^{\omega t} - \omega B e^{-\omega t} \\ w'(0) &= \omega(A - B) = 0 \\ A &= B \\ v(x) &= C\cosh(\omega x) + D\sinh(\omega x) \\ v'(x) &= \omega C e^{\omega x} - \omega D e^{-\omega x} \\ v'(0) &= \omega(C - D) = 0 \\ C &= D \\ v'(1) &= \omega C(e^{\omega} - e^{-\omega}) = 0 \\ C &= 0 = D \end{aligned}$$

The general equation for the solution can be given by

$$u(t,x) = a_0 + b_0 t + \sum_{k=1}^{\infty} [a_k\cos(k\pi t)\cos(k\pi x) + b_k\sin(k\pi t)\cos(k\pi x)].$$

Calculate the coefficients a_k, b_k, a_0, and b_0.

$$\begin{aligned} a_k &= 2\int_0^1 x(1-x)\cos(k\pi x)dx = \frac{2\left((-1)^{k+1} - 1\right)}{\pi^2 k^2} \\ b_k &= \frac{2}{k\pi}\int_0^1 0dx = 0 \\ a_0 &= \int_0^1 x(1-x)dx = \frac{1}{6} \\ b_0 &= \int_0^1 0dx = 0 \end{aligned}$$

Plugging the values back in yields the final solution

$$\boxed{u(t,x) = \frac{1}{6} + \sum_{k=1}^{\infty} \frac{2\left((-1)^{k+1} - 1\right)}{\pi^2 k^2} \cos(k\pi t)\cos(k\pi x).}$$

4.5 Solving the 1D wave equation using d'Alembert's formula

Starting on a bounded interval with periodic boundary conditions, the initial data can carefully be extended in a way designed to match the boundary conditions and then d'Alembert's formula can be applied

$$u(t,x) \quad = \quad \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2}\int_{x-ct}^{x+ct} g(z)dz.$$

Note: the constant c denotes the wave speed.

Homogeneous Dirichlet Boundary Conditions: Take odd periodic extensions of both $u(0,x) = f(x)$ and $u_t(0,x) = g(x)$ to $f(-x) = -f(x), g(-x) = -g(x)$ on $-L \le x \le 0$ then take a $2L$-periodic extension of the result from $-L \le x \le L$ and copy periodically.

Homogeneous Neumann Boundary Conditions: Take an even periodic extension on $-L \le x \le 0$ then take a $2L$-periodic extension of the result from $-L \le x \le L$ and copy periodically.

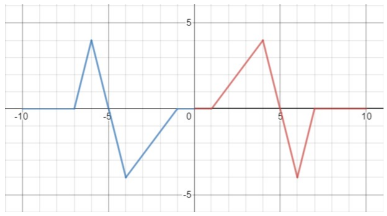
Mastery Check:
Sketch at least 3 solutions at later times of the following wave equation.

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(t,0) = u(t,10) &= 0 \\ u(0,x) &= f(x) \\ u_t(0,x) &= 0 \end{aligned}$$

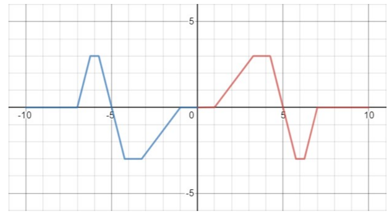
Applying d'Alembert's formula to the given equation would yield

$$u(t,x) \quad = \quad \frac{f(x-t)+f(x+t)}{2}.$$

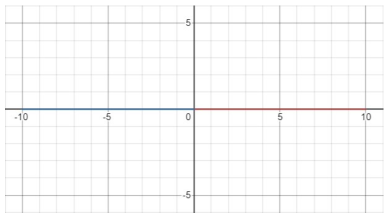
Using the given boundary conditions and the d'Alembert's formula from above, the following graphs can be created.



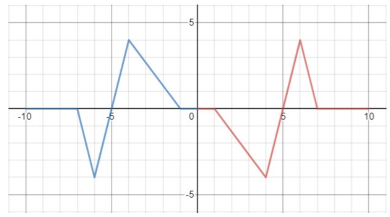
(a) $t = 0$



(b) $t = 1$



(c) $t = 5$



(d) $t = 10$

Figure 4: Solutions to d'Alembert's formula for the given wave equation

References

[1] P. Olver, *Introduction to partial differential equations*, Springer, May 2016.