Homework 4

Brennen Green

July 4, 2020

1. `Ch4.1 Exercise #4`

   (a) $P(1)$ is the basis statement

   (b) $P(1) = 1^3 = (\frac{1(1+1)}{2})^2 = (2/2)^2 = 1^2 = 1^3 \quad \square$

   (c) **Inductive Hypothesis:** Assume that $P(n)$ holds for $P(k)$ or that:
   $$1^3 + 2^3 + \cdots + k^3 = (\frac{k(k+1)}{2})^2$$

   (d) In the inductive step we prove that the inductive hypothesis holds for $P(k+1)$

   (e) **Inductive Step:**
   $$P(k+1) = 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = (\frac{(k+1)((k+1)+1)}{2})^2$$
   $$(\frac{k(k+1)}{2})^2 + (k+1)^3 = (\frac{(k+1)((k+1)+1)}{2})^2$$
   $$\frac{(k^2+k)^2}{4} + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$
   $$\frac{(k^2+k)^2 + 4(k+1)^3}{4} = \frac{(k+1)^2(k+2)^2}{4}$$
   $$(k^2+k)^2 + 4(k+1)^3 = (k+1)^2(k+2)^2$$
   $$k^4 + 6k^3 + 13k^2 + 12k + 4 = k^4 + 6k^3 + 13k^2 + 12k + 4$$
   $\therefore P(k+1)$ holds $\quad \square$

   (f) These steps put together demonstrate mathematical induction to prove that $P(n)$ is true for $n \in \mathbb{Z}^+$

2. `Ch4.1 Exercise #5`

Proof.

$$\textbf{Prove}: P(n) = 1^2 + 3^2 + \cdots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

$$\textbf{Basis}: (2(0)+1)^2 = \frac{(0+1)(2(0)+1)(2(0)+3)}{3} = (1)^3 = \frac{(1)(1)(3)}{3} = 1^3$$

$$\textbf{Assume}: 1^2 + 3^2 + \cdots + (2(k)+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$$

Inductive Step :

$$1^2 + 3^2 + \cdots + (2(k)+1)^2 + (2(k+1)+1)^2 = \frac{(k+2)(2(k+1)+1)(2(k+1)+3)}{3}$$

$$\frac{(k+1)(2k+1)(2k+3)}{3} + (2(k+1)+1)^2 = \frac{(k+2)(2(k+1)+1)(2(k+1)+3)}{3}$$

$$\frac{(k+1)(2k+1)(2k+3) + 3(2(k+1)+1)^2}{3} = \frac{(k+2)(2k+3)(2k+5)}{3}$$

$$(k+1)(2k+1)(2k+3) + 3(2k+3)^2 = (k+2)(2k+3)(2k+5)$$

$$(k+1)(2k+1) + 3(2k+3) = (k+2)(2k+5)$$

$$2k^2 + 9k + 10 = 2k^2 + 9k + 10$$

$\therefore P(n)$ holds for all nonnegative $n \in \mathbb{Z}^+$ by mathematical induction □

3. `Ch4.1 Exercise #6`

Proof.

$$\textbf{Prove}: P(n) = 1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$$

$$\textbf{Basis}: 1(1)! = (1+1)! - 1 = 2! - 2 = 2 * 1 - 2 = 1$$

$$\textbf{Assume}: 1(1!) + 2(2!) + \cdots + k(k!) = (k+1)! - 1$$

Inductive Step :

$$1(1!) + 2(2!) + \cdots + k(k!) + (k+1)((k+1)!) = ((k+1)+1)! - 1$$

$$(k+1)! - 1 + (k+1)(k+1)! = (k+2)! - 1$$

$$(k+1)! + (k+1)(k+1)! = (k+2)!$$

$$(k+1)!(k+2) = (k+2)!$$

$$(k+2)! = (k+2)!$$

$\therefore P(n)$ holds for $n \in \mathbb{Z}^+$ by mathematical induction □

4. `Ch4.1 Exercise #10`

   (a) $P(n) = \frac{1}{1(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

   (b) *Proof.*

$$\textbf{Basis}: \frac{1}{1(2)} = \frac{1}{1+1} = \frac{1}{2}$$

$$\textbf{Assume}: \frac{1}{1(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

Inductive Step :

$$\frac{1}{1(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

$$\frac{k(k+2)}{k+1} + \frac{1}{k+1} = k+1$$

$$\frac{k(k+2)+1}{k+1} = k+1$$

$$k^2 + 2k + 1 = k^2 + 2k + 1$$

$\therefore P(n)$ holds for all $n \in \mathbb{Z}^+$ by mathematical induction $\square$

5. `Ch4.1 Exercise #31`

Proof.

Prove : $\forall n \in \mathbb{Z}^+(2|n^2 + n)$

Basis : $2|1^2 + 1 = 2|2$

Assume : $2|k^2 + k$

Inductive Step : $2|(k+1)^2 + (k+1)$

$2|k^2 + 2k + 1 + k + 1$

$2|k^2 + k + 2k + 2$

$2|(k^2 + k) + 2(k+1)$ Divisble by 2 by our assumption & divisibility rules

$\therefore 2|n^2 + n$ holds for $n \in \mathbb{Z}^+$ by mathematical induction $\square$

6. `Ch4.2 Exercise #2`

Proof.

$P(n) =$ the nth domino in the set will fall
Basis : $P(1), P(2), P(3)$ are true because the first 3 dominoes fall
Inductive Step : $P(k+1)$'s true because we know that
$P(k-2)$ is true fall and that every time a domino falls the domino three places after will fall as well
$\therefore P(n)$ will hold infinitely by strong induction ◻

7. `Ch4.2 Exercise #4`

(a) $P(18) = 7 + 7 + 4 = 18$ $P(19) = 4 + 4 + 4 + 7 = 19$
$P(20) = 4 + 4 + 4 + 4 + 4 = 20$ $P(21) = 7 + 7 + 7 = 21$

(b) The inductive hypothesis is that $P(18), P(19), P(20), P(21)..P(k)$ is true

(c) In the inductive step we need to prove the hypothesis holds for $P(k+1)$ with $k \geq 21$

(d) Since $k \geq 21$ then we know that $P(k-3)$ is true by adding a 4-cent stamp then we arrive back at $P((k-3)+4) = P(k+1)$ thus $P(k)$ holds for $k \geq 21$

(e) These statements prove that $P(n)$ is true while $n \geq 18$ due to the definition of strong mathematical induction

8. `Ch4.2 Exercise  #5`

(a) All possible postages that can be formed from 4-cent and 11-cent stamps are any combinations of $4x + 11y$ with $x, y \in (\mathbb{Z} \geq 0)$

$$4, 8, 11, 12, 15, 16, 19, 20, 22, 23, 24, 26, 27, 28, 30, 31, 32, etc$$

Showing that any integer $n \geq 30$ can be written in the form $4x + 11y$

(b) *Proof.*

Basis : $P(n)n = 30 : 4(2) + 11(2) = 30$
Assume : $P(k) : 2x + 11y$ is true
Inductive Step :
If $P(k)$ is true and can be formed with 11-cent stamps then the $k+1$-cent stamp can also be formed by replacing the 11-cent steps with three 4-cent stamps

If $P(k)$ is true and can be formed with at least eight 4-cent stamps then you can replace those eight stamps with three 11-cent stamps to get a $k+1$-cent stamp

$\therefore P(k+1)$ is true
$\therefore P(n)$ is true for $n \geq 30$ by mathematical induction □

(c) .

Proof.

Basis : $P(30) = 4 * 2 + 11 * 2 = 30$
$P(31) = 4 * 5 + 11 * 1 = 31$ $P(32) = 4 * 8 + 11 * 0 = 32$
$P(33) = 4 * 0 + 11 * 3 = 33$
Assume : $P(k) = 4x + 11y = k$
Inductive Step :
$P(k+1) = P(k-3) + 4$
$\therefore P(k+1)$ is true
$\therefore P(n)$ is true for $n \geq 30$ by strong induction □

9. `Ch4.2 Exercise  #7`

   Any combinations of 2x + 5y can be made with 2 and 5 dollar bills

   $$2, 4, 5, 6, 7, 8, 9, 10, etc$$

   Therefore any $n \in \mathbb{Z} \geq 4$ can be made with a combination of \$2 and \$5 bills

   Proof.

   Basis : $P(4) = 2 * 2 = 4$ $P(5) = 5 * 1 = 5$ $P(6) = 2 * 3 = 6$
   Assume : $P(4), P(5), P(6), \ldots, P(k)$ is true
   Inductive Step :
   Since $P(k-1)$ is true and $k - 1 + 2 = k + 1$ then $P(k+1)$ is true
   $\therefore P(n)$ is true for $n \geq 4$ by strong induction $\square$

10. `Ch4.2 Exercise #10`

    $$P(n) = \text{If a chocolate bar has n squares then the result is}$$
    $$\text{how many breaks it takes to get n seperate squares}$$

    $P(1)$ takes zero breaks, $P(2)$ takes one, $P(3)$ takes two. Therefore it appears that P(n) takes n-1 breaks to get n individual squares

    Proof.

    Basis : $P(1) = 1 - 1 = 0$ $P(2) = 2 - 1 = 0$
    $P(3) = 3 - 1 = 2$
    Assume : $P(1), P(2), P(3), \ldots, P(k)$ *are true*
    Inductive Step : Since the bar with k squares can be broken with k-1 breaks. Then if the bar $k + 1 = xy$ (x rows, y columns) Is first broken into two separate bars, one with k squares, and one bar equivalent to P(1), then the number of breaks is
    $P(k+1) = P(k) + 1 = (k-1) + 1 = k$
    $\therefore$ Our hypothesis holds for $n \in \mathbb{Z}^+ P(n)$ by strong induction $\square$

11. `Ch4.3 Exercise #2`

   (a) $f(1) = -2(3) = -6$ $f(2) = -2(-6) = 12$ $f(3) = -2(12) = -24$ $f(4) = -2(-24) = 48$ $f(5) = -2(48) = 96$

   (b) $f(1) = 3(3)+7 = 16$ $f(2) = 3(16)+7 = 55$ $f(3) = 3(55)+7 = 172$ $f(4) = 3(172)+7 = 523$ $f(5) = 3(523)+7 = 1576$

   (c) $f(1) = (3)^2 - 2(3) = 3$ $f(2) = (3)^2 - 2(3) = 3$ $f(3) = 3$ $f(4) = 3$ $f(5) = 3$

   (d) $f(1) = 3^{\frac{3}{3}} = 3$ $f(2) = 3^{\frac{3}{3}} = 3$ $f(3) = 3$ $f(4) = 3$ $f(5) = 3$

12. `Ch4.3 Exercise #2`

   (a) $a_n = 6 + a_{n-1}$ $a_0 = 0$

   (b) $a_n = 2 + a_{n-1}$ $a_0 = 1$

   (c) $a_n = 10 * a_{n-1}$ $a_0 = 1$

   (d) $a_n = a_{n-1}$ $a_0 = 5$

13. `Ch4.3 Exercise #10`

$$S_m(n) = S_m(n-1) + 1 \text{ for } n \geq 1 \text{ and } S_m(0) = m$$

14. `Ch4.3 Exercise #18`

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

$$\textbf{Basis} : A^1 = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\textbf{Assume} : A^k = \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix}$$

$$\textbf{Inductive Step} : A^{k+1} = A^k * A = \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} f_{m+2} & f_{m+1} \\ f_{m+1} & f_m \end{bmatrix} \quad \therefore A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} \text{ by induction}$$

15. `Ch4.3 Exercise #43`

Proof.

Basis : $n(T) = 1$ $h(T) = 0$: $1 \geq 2(0) + 1$

Assume : $n(T_1) \geq 2h(T_1) + 1$ *and* $n(T_2) \geq 2h(T_2) + 1$

Inductive Step :

$n(T) = 1 + n(T_1) + n(T_2)$

$h(T) = 1 + max(h(T_1), h(T_2))$

$n(T) = 1 + n(T_1) + n(T_2) \geq 1 + 2h(T_1) + 1 + 2h(T_2) + 1$

$\geq 1 + 2max(h(T_1), h(T_2)) + 2$

$\geq 1 + 2(max(h(T_1), h(T_2)) + 1) \quad (h(T) = 1 + max(h(T_1), h(T_2)))$

$\therefore n(T) \geq 2h(T) + 1$ by structural induction $\square$

16. `Ch4.4 Exercise #8`

```
def sum(n: positive integer)
    if n==1 then
        return 1
    else
        return sum(n - 1) + n
```

17. `Ch4.4 Exercise #10`

```
def arrayMax(arr: positive integer array, size: integer)
    if size == 1 then
        return arr[0]
    else
        return max(arrayMax(arr[0:size-1], size-1), arr[size])
```

18. Ch4.4 Exercise #32

```
def nthTerm(n: positive integer)
    if n == 0 then
        return 1
    if n == 1 then
        return 2
    if n == 2 then
        return 3

    return nthTerm(n-1) + nthTerm(n-2) + nthTerm(n-3)
```

19. Ch4.4 Exercise #35

```
Recursive:
def nthTerm(n: positive integer)
    if n == 0 then
        return 1
    if n == 1 then
        return 3
    if n == 2 then
        return 5
    return nthTerm(n-1) + nthTerm(n-2)^2 + nthTerm(n-3)^3

Iterative: (Iterative of the more efficient algorithm)
def nthTerm(n: positive integer)
    if n == 0 then
        return 1
    if n == 1 then
        return 3
    if n == 2 then
        return 5
    for i := 1 to n - 2
        a = d * c^2 * b^3
        b = c
        c = d
        d = a
    return a
```

20. `Ch4.4 Exercise #44`

$$
\begin{aligned}
1 &: \quad 4, 3, 2, 5, 1, 8, 7, 6 \\
2 &: \quad 4, 3, 2, 5|1, 8, 7, 6 \\
3 &: \quad 4, 3|2, 5|1, 8|7, 6 \\
4 &: \quad 4|3|2|5|1|8|7|6 \quad \text{Divide Step (vertical bar splits groupings)} \\
5 &: \quad 3, 4|2, 5|1, 8|6, 7 \\
6 &: \quad 2, 3, 4, 5|1, 6, 7, 8 \\
7 &: \quad 1, 2, 3, 4, 5, 6, 7, 8
\end{aligned}
$$