Classification of Level Two Leibniz Algebras via Annhiliating Operators

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Abstract

In this paper, we give a classification of all Leibniz algebras, up to isomorphism, which degenerate directly to an algebra of level one. To achieve this, we make extensive use of the anticommutator and squaring operator, and the fact that for all $x, y \in L$, we have that $xy + yx, x^2 \in \mathrm{Ann}_R(L)$.

Keywords: Leibniz algebra, degenerations, level of algebra

Introduction

Deformation theory has been a topic with a rich history in mathematics and in physics. It originally arose in the context of geometry, in which objects were studied by viewing what they could "deform" into and whether these new objects were isomorphic to the original. Kodaira and Spencer gave the original idea of infinitesimal deformations for complex analytic manifolds [6]. Most notably, they proved that infinitesimal deformations can be parametrized by a related cohomology group.

For this paper, we consider the case of a finite dimensional algebra over a closed field k. Such a algebra can be considered as an element $\mu \in \mathrm{Hom}(V \otimes V, V)$, where V is an n-dimensional k-vector space. We then endow the space $\mathrm{Hom}(V \otimes V, V)$ with the Zariski topology. The following is the definition of a formal deformation for rings and algebras which was first given by Gerstenhaber [4].

Definition 1. *A formal deformation of an algebra μ_0 is a family of algebras $\mu(t) \in V \otimes k[[t]]$ over the formal power series ring $k[[t]]$:*

$$\mu(t) = \mu_0 + t\varphi_1 + t^2\varphi_2 + \dots$$

where $\varphi_i \in \mathrm{Hom}(V \otimes V, V)$.

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Degenerations are a closely related topic based on the concept of orbit closure. The degenerations of Lie algebras were originally a physics problem introduced by Inönü and Wigner [5]. First, we consider the action of the general linear group $g \in \mathrm{GL}_n(k)$ which acts on $\mu \in \mathrm{Hom}(V \otimes V, V)$ by:

$$(g * \mu)(x, y) = g(\mu(g^{-1}x, g^{-1}y)).$$

We notice that the orbit of this action is exactly the isomorphism class of μ. This leads us to the definition of a degeneration.

Definition 2. *We say that λ degenerates to μ, or simply that $\lambda \to \mu$. If there exists a matrix $g_t \in GL_n(k(t))$ such that:*

$$\mu(x, y) = \lim_{t\to 0}(g_t * \lambda)(x, y) = \lim_{t\to 0} g_t(\lambda(g_t^{-1}x, g_t^{-1}y)).$$

From this definition, we see that if $\lambda \to \mu$, then we have that $\mu \in \overline{\mathrm{Orb}(\lambda)}$, where $\mathrm{Orb}(-)$ is the orbit under the action of changing bases and closure is with respect to the Zariski topology.

There is a deep connection between deformations and orbit closures, which can be seen more clearly if we consider the deformation $\mu(t)$ not as a family of algebras, but as a algebra over the field $k((t))$. This viewpoint led to the following known result by Grunewald and O'Halloran [8].

Theorem 1. *If $\lambda \to \mu$ nontrivially (i.e. $\lambda \not\cong \mu$), then this degeneration defines a non-trivial formal deformation of μ.*

Thus, by studying degenerations, we can obtain deformations.

Definition 3. *We call a degeneration $\lambda \to \mu$ a direct degeneration if there is no chain of non-trivial degenerations of the form $\lambda \to \nu \to \mu$.*

Definition 4. *A level of an algebra λ is the maximum length of a chain of direct degenerations. We denote the level of an algebra by $lev_n(\lambda)$.*

We note that any n-dimensional algebra μ degenerates to the abelian algebra ab_n by

$$\lim_{t\to 0}(t^{-1}I_n * \mu)(x, y) = \lim_{t\to 0} t^{-1}\mu(tx, ty) = \lim_{t\to 0} t\mu(x, y) = 0 = \mathrm{ab}_n(x, y).$$

where I_n is the $n \times n$ identity matrix. Thus, we say that ab_n is the only algebra of level zero. Concerning algebras of level one, we have the following result proved by Khudoyberdiyev and Omirov [2].

Theorem 2. *Let A be an algebra of level one. Then A is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{array}{rllll}
p_n^- : & e_1e_i = e_i, & e_ie_1 = -e_i, & 2 \le i \le n & \\
n_3^- \oplus \mathrm{ab}_{n-3} : & e_1e_2 = e_3, & e_2e_1 = -e_3 & & \\
\lambda_2 \oplus \mathrm{ab}_{n-2} : & e_1e_1 = e_2 & & & \\
\nu_n(\alpha) : & e_1e_1 = e_1, & e_1e_i = \alpha e_i, & e_ie_1 = (1-\alpha)e_i, & 2 \le i \le n
\end{array}$$

We shall now restrict ourselves to the variety of Leibniz algebras.

Definition 5. *A (right) Leibniz algebra is an non-associative algebra such that for all $x, y, z \in L$, the following identity holds:*

$$x(yz) = (xy)z - (xz)y.$$

This paper is mainly a generalization of the results achieved by Khudoyberdiyev which concern finding a list of all non-isomorphic algebras of level two in a particular variety [3].

Main result

Firstly, we let L be a n-dimensional Leibniz algebra and let $\{e_1, e_2, \dots, e_n\}$ be a basis of L. We then define the products to be $e_j e_k = \sum_{s=1}^{n} \gamma_{jk}^{s} e_s$. Now we have the following lemma.

Lemma 1. *Suppose that L is not level one. Then for all j, k, s distinct, we have that either $L \to L_4(\alpha), L_5$ or that*

$$(\gamma_{jj}^{s}, \gamma_{jk}^{s}, \gamma_{kj}^{s}, \gamma_{kk}^{s}) \in \{(0, \alpha, -\alpha, 0), (\alpha, \alpha, \alpha, \beta), (\beta, \alpha, \alpha, \alpha)\}$$

for some constants α, β.

Proof. Without loss of generality, we let $j = 1$, $k = 2$, and $\ell = 3$. We see that if we take the degeneration

$$g_t(e_1) = t^{-1} e_1 \qquad g_t(e_2) = t^{-1} e_2 \qquad g_t(e_i) = t^{-2} e_i \qquad i \neq 1, 2$$

then we have that e_1e_1, e_1e_2, e_2e_1, and e_2e_2 as our only products. Furthermore, if we take the additional degeneration

$$g_t(e_3) = t^{-2} e_3 \qquad g_t(e_i) = t^{-1} e_i \qquad i \neq 3$$

Then we have that

$$\begin{aligned} e_1e_1 &= \kappa_3 e_3 & \quad e_2e_2 &= \omega_3 e_3 \\ e_1e_2 &= \alpha_3 e_3 & \quad e_2e_1 &= \beta_{3e_3} \end{aligned}$$

as our only products. Due to the classification of Nilpotent Leibniz algebras of dimension less than five, we can conclude that this algebra degenerates to either $L_4(\alpha)$ or L_5 as long as it's not isomorphic to ab_n, λ_2, or n_3^-. We see that this algebra is isomorphic to ab_n if and only if $\kappa_3 = \alpha_3 = \omega_3 = \beta_3 = 0$. We also see that this algebra is isomorphic to n_3^- if and only if $\kappa_\ell = \omega_\ell = 0$ and $\alpha_\ell = -\beta_\ell$. Finally, by **??**, we know that this algebra isomorphic to λ_2 if and only if $\kappa_\ell = \alpha_\ell = \beta_\ell$ and $\kappa_\ell = \omega_\ell^2$. This proves the statement. □

Theorem 3. *Let L be a n-dimensional non-Lie Leibniz algebra which is not of level one (i.e. $L \not\to \lambda_2$). Then L degenerates to one of the following three algebras:*

$$\begin{array}{rlll} L_4(\alpha): & e_1e_1 = e_3, & e_2e_1 = e_3, & e_2e_2 = \alpha e_3 \\ L_5: & e_1e_1 = e_3, & e_1e_2 = e_3, & e_2e_1 = e_3 \\ r_n: & e_ie_1 = e_i, & 2 \leq i \leq n & \end{array}$$

Proof. Since L is a non-Lie Leibniz algebra, we know that there exists an element $y \in L$ such that $yy \neq 0$. We use this to make the basis $\{e_1, e_2, \ldots, e_n\}$, where $e_1 = y$ and $e_2 = yy$. We consider the linear annhiliating operater $\varphi(x) = e_1x + xe_1$ and let (α_{ij}) be its matrix form. Thus, we have that

$$\varphi = \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \ldots & \alpha_{1n} \\ 2 & \alpha_{22} & \alpha_{23} & \alpha_{24} & \ldots & \alpha_{2n} \\ 0 & \alpha_{32} & \alpha_{33} & \alpha_{34} & \ldots & \alpha_{3n} \\ 0 & \alpha_{42} & \alpha_{43} & \alpha_{44} & \ldots & \alpha_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_{n2} & \alpha_{n3} & \alpha_{n4} & \ldots & \alpha_{nn} \end{bmatrix}$$

Suppose that $\alpha_{jk} \neq 0$ for some $1, 2, j, k$ distinct and $k \neq 2$. Then we have the following products

$$\begin{aligned} e_1e_1 &= 0 + e_2 & e_je_j &= \sum_i \omega_i e_i \\ e_1e_j &= \beta_k e_k + \sum_{i \neq k} \beta_i e_i & e_je_1 &= (\alpha_{jk} - \beta_k)e_k - \sum_{i \neq k}(\alpha_{ji} - \beta_i)e_i \end{aligned}$$

This means that by Lemma 1, either $L \to L_4(\alpha), L_5$ or $\beta_k = \alpha_{jk} - \beta_k = 0$ or $\beta_k = -\alpha_{jk} + \beta_k$. We thus assume that we're in one of the two latter cases. In both of these cases, we see that $\alpha_{jk} = 0$. This gives us the following matrix

$$\varphi = \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \ldots & \alpha_{1n} \\ 2 & \alpha_{22} & \alpha_{23} & \alpha_{24} & \ldots & \alpha_{2n} \\ 0 & 0 & \alpha_{33} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \alpha_{44} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \alpha_{nn} \end{bmatrix}$$

Now suppose that $\alpha_{2i} \neq 0$ for some $i \neq 1, 2$. Without loss of generality, let $i = 3$. We see that we can scale e_3 so that $\alpha_{23} = 1$. Then we have the following products

$$\begin{aligned} e_1e_1 &= e_2 & e_je_j &= \sum_i \omega_i e_i \\ e_1e_3 &= \beta_2 e_2 + \sum_{i \neq 2} \beta_i e_i & e_3e_1 &= (1 - \beta_2)e_2 - \sum_{i \neq 2}(\alpha_{i3} - \beta_i)e_i \end{aligned}$$

Since it is impossible that $\beta_2 = 1 - \beta_2 = 1$ or $\beta_2 = 1 - \beta_2 = 0$, we see that by Lemma 1, we have that $L \to L_4(\alpha), L_5$. Thus, we can assume that $\alpha_{2i} = 0$ for all $i \neq 1, 2$. This means that

$$\varphi = \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \ldots & \alpha_{1n} \\ 2 & \alpha_{22} & 0 & 0 & \ldots & 0 \\ 0 & 0 & \alpha_{33} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \alpha_{44} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \alpha_{nn} \end{bmatrix}$$

Now suppose there exists a $j \geq 3$ such that $\alpha_{22} \neq \alpha_{jj}$. Without loss of generality, let $j = 3$. Then we see that if we take the basis change $e_3 \mapsto e_2 + e_3$, we have that

$$\begin{aligned}\varphi(e_3') = \varphi(e_2) + \varphi(e_3) &= (\alpha_{12} + \alpha_{13})e_1 + \alpha_{22}e_2 + \alpha_{33}e_3 \\ &= (\alpha_{12} + \alpha_{13})e_1 + (\alpha_{22} - \alpha_{33})e_2 + \alpha_{33}e_3'\end{aligned}$$

By reapplying our previous argument, we see that it must be that either $L \to L_4(\alpha), L_5$ or $\alpha_{22} - \alpha_{33} = 0$. Thus, we assume that $\alpha_{22} = \alpha_{33}$. Since 3 was arbitrary, we have that $\alpha = \alpha_{22} = \alpha_{ii}$ for $i \geq 2$. This gives us the matrix

$$\varphi = \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \dots & \alpha_{1n} \\ 2 & \alpha & 0 & 0 & \dots & 0 \\ 0 & 0 & \alpha & 0 & \dots & 0 \\ 0 & 0 & 0 & \alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha \end{bmatrix}$$

Case 1: $\alpha \neq 0$.

If $\alpha \neq 0$, then we can take the basis change $e_i' = \varphi(e_i)$ for $i \neq 1$. This means that $e_2, \dots, e_n \in \mathrm{Ann}_R(L)$ and thus that $\varphi(e_i) = e_i e_1$ for $i \neq 1$. If we apply what we've done so far, we rearrive at the matrix

$$\varphi = \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \dots & \alpha_{1n} \\ 2 & \alpha & 0 & 0 & \dots & 0 \\ 0 & 0 & \alpha & 0 & \dots & 0 \\ 0 & 0 & 0 & \alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha \end{bmatrix}$$

If $\alpha_{1i} \neq 0$ for some $i \neq 1$, then we have that $\alpha_{1i}e_1 = \varphi(e_i) - \alpha e_i \in \mathrm{Ann}_R(L)$, which is a contradiction to our choice of e_1e_1. If we then take the degeneration

$$g_t(e_1) = e_1 \qquad g_t(e_i) = te_i \qquad i \neq 1$$

we have that the only products are $e_i e_1 = \alpha e_i$ for $i \neq 1$. After rescaling, we see that this is exactly r_n.

Case 2: $\alpha = 0$.

We see that if $\alpha_{1i} \neq 0$ for some $i \neq 1$, then we have that $\alpha_{1i}e_1 = \varphi(e_i) \in \mathrm{Ann}_R(L)$, which is a contradiction. This means that we have the following matrix

$$\varphi = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 2 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

We will now prove that either $L \to L_4(\alpha), L_5$ or $e_2 \in \mathrm{Ann}(L)$. Since $e_2 \in \mathrm{Ann}_R(L)$ and $\varphi(e_2) = e_1e_2 + e_2e_1 = e_2e_1 = 0$, we need only to prove that $e_2e_i = 0$ for $i \neq 1, 2$. Without loss of generality, let $i = 3$, let $e_3e_3 = \sum_i \omega_i e_i$, and let $e_2e_3 = \sum_i \beta_i e_i$. Then we see that for all $\ell \neq 2, 3$, we have that

$$\begin{aligned} e_2e_2 &= 0 \qquad & e_3e_3 &= \omega_\ell e_\ell + \sum_{i\neq\ell} \omega_i e_i \\ e_3e_2 &= 0 \qquad & e_2e_3 &= \beta_\ell e_\ell + \sum_{i\neq\ell} \beta_i e_i \end{aligned}$$

which by Lemma 1, we have that either $L \to L_4(\alpha), L_5$ or $\beta_\ell = -0 = 0$. Thus, we assume that $\beta_\ell = 0$ for $\ell \neq 2, 3$. This gives us that $e_2e_3 = \beta_2e_2 + \beta_3e_3$. If $\beta_3 \neq 0$, then we have that $\beta_3e_3 = e_2e_3 - \beta_2e_2 \in \mathrm{Ann}_R(L)$. This means that $e_2e_3 = 0$, a contradiction as $\beta_2 \neq 0$. This means that $\beta_2 = 0$, and thus that $e_2e_3 = \beta_2e_2$. Now suppose that $\beta_2 \neq 0$. Then we can scale e_3, so that $e_2e_3 = e_2$. If we take the change of basis $e_2' = e_1 + e_2$, then we have that

$$\begin{aligned} &e_2'e_2' = e_1e_1 = e_2 = e_2' - e_1 \\ &e_2'e_3 + e_3e_2' = (e_3e_1 + e_1e_3) + (e_3e_2 + e_2e_3) = e_2e_3 = e_2 = e_2' - e_1 \end{aligned}$$

Under this basis change, we let $e_3e_2 = \sum_i \alpha_i e_i$ and $e_3e_3 = \sum_i \omega_i e_i$. This means that we have that

$$\begin{aligned} e_2e_2 &= -e_1 + e_2 & e_3e_3 &= \omega_1 e_1 + \sum_{i\neq 1} \omega_i e_i \\ e_3e_2 &= \alpha_1e_1 + \alpha_2e_2 + \sum_{i\neq 1,1} \alpha_i e_i \qquad & e_2e_3 &= -(1 + \alpha_1 e_1) + (1 - \alpha_2)e_2 - \sum_{i\neq 1,2} \alpha_i e_i \end{aligned}$$

Since it is impossible for $\alpha_1 = -(\alpha_1 + 1) = -1$ and also impossible for $\alpha_1 = -(\alpha_1 + 1) = 0$, by Lemma 1, we have that $L \to L_4(\alpha), L_5$. Thus, we may assume that $\beta_2 = 0$, which means that $e_2e_3 = 0$. Since e_3 was arbitrary, we have that $e_2e_i = 0$ for all $i \neq 1, 2$. This proves that $e_2 \in \mathrm{Ann}(L)$.

Now we will prove that either $L \to L_4(\alpha), L_5$ or $e_ie_1 = e_1e_i = e_ie_i = 0$ for $i \neq 1, 2$. Without loss of generality, let $e_1e_3 = -e_3e_1 = \sum_i \alpha_i e_i$ and $e_3e_3 = \sum_i \beta_i e_i$. Now let j be arbitrary and take the basis change $e_j' = e_j + e_2$. This means that

$$\begin{aligned} e_1e_1 &= e_2 \qquad e_3e_3 = \sum_i \beta_i e_i = \beta_j e_j' - \beta_j e_2 + \sum_{i\neq j} \beta_i e_i \\ e_3e_1 &= -e_1e_3 = \sum_i \alpha_i e_i = \alpha_j e_j' - \alpha_j e_2 + \sum_{i\neq j} \alpha_i e_i \end{aligned}$$

We note that even if $j = 1$ or $j = 3$, the products on the left side stay the same as $e_2 \in \mathrm{Ann}(L)$. Since j was arbitrary, if we let $j = 2$, then by Lemma 1, we have that either $L \to L_4(\alpha), L_5$ or $2\alpha_2 = 2\beta_2 = 0$. Thus, we assume that $\alpha_2 = \beta_2 = 0$. Similarly, if $j \neq 2$, we have that $L \to L_4(\alpha), L_5$ or $\alpha_j = \beta_j = 0$. This means that we must also assume that $\alpha_j = \beta_j = 0$, which proves that $e_3e_3 = e_1e_3 = e_3e_1 = 0$. Since e_3 was arbitrary, we have that $e_ie_i = e_ie_1 = e_1e_i = 0$ for all $i \neq 1, 2$.

Similarly to the above paragraph, we can prove that either $L \to L_4(\alpha), L_5$ or $e_je_k = e_ke_j = 0$ for $1, j, k$ distinct. If we assume the latter, then we have that $e_1e_1 = e_2$ as our only product, which is isomorphic to λ_2. This is a contradiction, and thus any non-Lie Leibniz algebra not of level one degenerates into either $L_4(\alpha)$, L_5, or r_n. $\square$

Theorem 4. *Let L be an n-dimensional Leibniz algebra of level two. Then L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$L_4(\alpha), \quad L_5, \quad r_n, \quad \alpha \in \mathbb{C}$$

Proof. Due to Theorem 3, it is sufficient to prove that these algebras do not degenerate to each other. To falicitate this, we call upon the following table:

$$\begin{array}{ll}
\dim \mathrm{Comm}(L_5) = 1, & \dim \mathrm{Ann}_R(L_5) = n-2 \\
\dim \mathrm{Comm}(L_4(\alpha)) = 0, & \dim \mathrm{Ann}_R(L_4(\alpha)) = n-2, \quad \alpha \neq 0 \\
\dim \mathrm{Der}(L_4(0)) = n^2 - 3n + 4, & \dim \mathrm{Ann}_R(r_n) = n-1 \\
\dim \mathrm{Der}(r_n) = (n-1)^2 = n^2 - 2n + 1, &
\end{array}$$

We first note that $L_4(\alpha)$ and L_5 cannot degenerate to r_n, as $L_4(\alpha)$ and L_5 are nilpotent and r_n is not. We also see that r_n does not degenerate to $L_4(\alpha)$ $(\alpha \neq 0)$ or L_5, as r_n has more linearly independent elements in its right annhiliator. Additionally, since for $n \geq 4$, r_n has more derivations than $L_4(0)$, we have that $r_n \not\to L_4(0)$. Lastly, we see that $L_4(\alpha) \not\to L_5$ and that $L_5 \not\to L_4(\alpha)$ by the following paper [7]. $\square$

Remark 1. *We note that in the context of left Leibniz algebras, we have that the algebra*

$$\ell_n : \qquad e_1 e_i = e_i, \qquad 2 \leq i \leq n$$

replaces the algebra r_n as a Leibniz algebra of level two.

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