Problem Set 7
Algebra III

Bennett Rennier
bennett@brennier.com

Ex 1. Show that the natural group homomorphism $\phi : R^* \to (R/J(R))^*$ is surjective.

Proof. Let $x + J(R)$ be an invertible element of $R/J(R)$. This means there is a $y + J(R)$ such that

$$(y + J(R))(x + J(R)) = yx + J(R) = 1 + J(R).$$

This proves that $1-yx \in J(R)$. Switching the order of multiplication, we also see that $1-xy \in J(R)$.

Since $1 - yx$ is in the Jacobson ideal, we know that $1 - (1 - yx) = yx$ invertible. Thus, there is a $r \in R$ such that $r(yx) = 1$. This means that x has a left inverse. We can apply the same reasoning to $1 - xy$ to conclude that x also has a right inverse. By uniqueness, these inverses must coincide and thus $x \in R^*$. This proves that $\phi(x) = x + J(R)$. Since $x + J(R)$ was arbitrary, we have that ϕ is surjective. □

Ex 2. Let k be a field of characteristic 3 and consider the standard representation $\rho : S_3 \to \mathrm{GL}_3(k)$. Prove that this representation is not completely reducible.

Proof. Suppose that ρ was completely reducible. This would mean that k^3 can be written as the direct sum of S_3-invariant subspaces. That is, $k^3 = U \oplus V$. Without loss of generality, we may assume that $\dim(U) = 1$ and $\dim(V) = 2$.

Let $(x, y, z) \in U$ be a non-zero vector. Since U is S_3-invariant, this means that all permutations of (x, y, z) also lie in U, and since U is 1-dimensional, it must be that all of these permutations are scalar multiples of (x, y, z). I claim that it must be that $x = y = z$ and so U must be the subspace $\{(x, x, x) : x \in k\}$. To prove this, assume without loss of generality that $x \neq 0$ (we can use a permutation to make this the case). Transposing the second and third coordinate, we have that $(x, y, z) = \alpha(x, z, y)$. Since $x = \alpha x$ and $x \neq 0$, it must be that $\alpha = 1$ and so $y = z$. Transposing the first and second coordinate we have that $(x, y, z) = \beta(y, x, z)$. As $z = \beta z$, either $\beta = 1$, giving us that $x = y = z$, or $z = 0$ in which case $y = 0$ as well and $x = \beta y = 0$, a contradiction. Thus, it must be that $x = y = z$ and so $U = \{(x, x, x) : x \in k\}$.

We note that the subspace $W = \{(x, y, z) : x + y + z = 0\}$ is also invariant under permutations and that for $(x, x, x) \in k^3$, $x + x + x = 3x = 0$ (as we are in characteristic zero), proving that $U \subseteq W$. Since W is of dimension 2, it must be that $W \cap V$ is not trivial. If $W \cap V$ is two-dimensional, then $W = V$. This is a contradiction, though, as $U \subseteq W$, so $U + W$ cannot be a direct sum. This means that $W \cap V$ must be 1-dimensional. Since W and V are both S_3-invariant, so is their intersection. By the previous paragrah though, U is the only 1-dimensional S_3-invariant space, so $W \cap V = U$. However, this implies that $V \cap U \neq \varnothing$, contradicting $U + V$ being a direct sum. Thus, we have that ρ cannot be completely reducible. □

Ex 3.

a) If G is a finite abelian group, show that any irreducible real representation of G is of degree 1 or 2.

b) If G is cyclic of finite order $n > 2$, construct an irreducible real representation of G of degree 2.

c) If G is of order 2, is there an irreducible real representation of G of degree 2?

Proof.

a) Finding a irreducible real representation of G is equivalent to finding a simple $\mathbb{R}[G]$-module. By Maschke's Theorem, since G is finite and $\mathbb{R}$ has characteristic zero, we know that $\mathbb{R}[G]$ is semisimple. Thus, from Artin-Weddenburn, we know that $\mathbb{R}[G]$ is isomorphic to $\oplus_{i \leq k} M_{n_i}(D_i)$ for some $\mathbb{R}$-division algebras D_i. Since $\mathbb{R}[G]$ is commutative, though, it must be that $n_i = 1$ for all $i \leq k$ and that the $\mathbb{R}$-division algebras D_i are really fields. Since the only fields over $\mathbb{R}$ are $\mathbb{R}$ and $\mathbb{C}$, we have that $\mathbb{R}[G]$ can be decomposed as the direct sum of copies of $\mathbb{R}$ and $\mathbb{C}$. Since every simple $\mathbb{R}[G]$-module appears in this decomposition, we have that every simple $\mathbb{R}[G]$-module is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. Since these have dimension 1 and 2 respectively (over the field $\mathbb{R}$), any irreducible real representation of G is of degree 1 or 2.

b) Consider the map $f : \mathbb{Z} \to \mathrm{GL}_2(\mathbb{R})$ given by sending 1 to the matrix

$$\begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}.$$

Since this matrix is a rotation by $2\pi/n$, it has order n. This means that $\ker(f) = n\mathbb{Z}$ and so we have a real representation $\rho : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_2(\mathbb{R})$. This representation is irreducible as we can see that there are no subspaces of $\mathbb{R}^2$ that are invariant under this action of rotation.

c) Let e and x be the elements of the cyclic group $\mathbb{Z}_2$. Consider the map $f : \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}[\mathbb{Z}_2]$ given by $f(a, b) = \frac{a+b}{2}e + \frac{a-b}{2}x$. Since $f(1, 0) = e + x$ and $f(0, 1) = e - x$ is a basis of $\mathbb{R}[\mathbb{Z}_2]$, we see that this map is a an isomorphism of vector spaces. Additionally, as

$$\begin{aligned} f(a,b)f(c,d) &= \left(\frac{a+b}{2}e + \frac{a-b}{2}x\right)\left(\frac{c+d}{2}e + \frac{c-d}{2}x\right) \\ &= \frac{(a+b)(c+d) + (a-b)(c-d)}{4}e + \frac{(a+b)(c-d) + (a-b)(c+d)}{4}x \\ &= \frac{ac+bd}{2}e + \frac{ac-bd}{2}x = f(ac, bd) = f((a,b)(c,d)), \end{aligned}$$

this proves that $\mathbb{R}[\mathbb{Z}_2] \simeq \mathbb{R} \oplus \mathbb{R}$ as rings. By the uniqueness of Artin-Weddenburn, we have that all simple $\mathbb{R}[\mathbb{Z}_2]$-modules are isomorphic to $\mathbb{R}$, meaning that all irreducible real representations of $\mathbb{Z}_2$ must be of degree 1. □

Ex 4. Let G be either D_8 or Q_8.

a) Show that for any field k with $\mathrm{char}(k) \neq 2$, G admits four inequivalent representations of degree 1.

b) Show that G admits a complex irreducible representation of degree 2.

c) Determine the structure of $\mathbb{C}[G]$.

Proof.

a) We note that if $G = D_8 = \langle r, s \mid r^4 = s^2 = (sr)^2 = 1\rangle$, then there are three index 2 subgroups: $\{1, r, r^2, r^3\}, \{1, r^2, s, sr^2\}$, and $\{1, r^2, sr, sr^3\}$. Similarly, if $G = Q_8 = \{-1, i, j, k : (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1\}$, then there are also three index 2 subgroups, namely $\{1, -1, i, -i\}, \{1, -1, j, -j\}$, and $\{1, -1, k, -k\}$.

Knowing this, let H_1, H_2, H_3 be the index two subgroups of G. Let $\rho_i : G \to M_1(k) \simeq k$ be the representation where ρ_i sends elements of H_i to 1 and sends elements not in H_i to -1 (this is well-defined because the subgroups are index two).

If ρ_1 and ρ_2 were equivalent, then there would be a vector space isomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f \circ \rho_1(g) = \rho_2(g) \circ f$ for all $g \in G$. But the only such f are $f(x) = \alpha x$ for some $\alpha \in \mathbb{R}^\times$. If we let $g \in H_1 \setminus H_2$, then $f \circ \rho_1(g) = f \circ \text{const}_1 = \text{const}_\alpha$ and $\rho_2(g) \circ f = \text{const}_0 \circ f = 0$. This proves that $\alpha = 0$, a contradiction. Thus, ρ_1 and ρ_2 are inequivalent. The same reasoning can be applied to show that ρ_1, ρ_2, ρ_3, and the trivial representation are all inequivalent represention of degree 1.

b) We can represent D_8 as $\phi : D_8 \to M_2(\mathbb{C})$ where

$$r \mapsto \begin{bmatrix} \cos(2\pi/4) & -\sin(2\pi/4) \\ \sin(2\pi/4) & \cos(2\pi/4) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad ; \quad s \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and we can represent Q_8 as $\psi : Q_8 \to M_2(\mathbb{C})$ where

$$1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad i \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad ; \quad j \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad ; \quad k \mapsto \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}.$$

We observe that these are indeed representations as $\phi(r)^4 = \phi(s)^2 = \phi(sr)^2 = 1$ and that $\psi(-1)^2 = \psi(1)$ and $\psi(i)^2 = \psi(j)^2 = \psi(k)^2 = \psi(ijk) = \psi(-1)$.

c) As $|G| = 8$ and $\mathbb{C}$ has characteristic zero, by Maschke's Theorem, $\mathbb{C}[G]$ is semi-simple. Thus, from Artin-Weddenburn, we get that $\mathbb{C}[G] \simeq \oplus_{i \leq k} M_{n_i}(D_i)$ where D_i is a $\mathbb{C}$-division algebra. Since $\mathbb{C}$ is algebraically closed, we know that $D_i = \mathbb{C}$ for every $i \leq k$. We also know that

$$8 = \dim(\mathbb{C}[G]) = \sum_{i \leq k} \dim(M_{n_i}(\mathbb{C})) = \sum_{i \leq k} n_i^2.$$

Additionally, each of these simple $\mathbb{C}[G]$ modules correspond to an irreducible representation. Since there are four representations of G of degree 1 and one representation of G of degree 2, it must be that

$$\mathbb{C}[G] = M_2(\mathbb{C}) \oplus \mathbb{C}^4.$$

□

Ex 5.

a) Show that D_8 admits a real irreducible representation of degree 2.

b) Show that Q_8 admits a real irreducible representation of degree 4.

c) Determine $\mathbb{R}[D_8]$ and $\mathbb{R}[Q_8]$ and show that they are not isomorphic as rings.

Proof.

a) The representation given in Exercise 4b works over $\mathbb{R}$ as well.

b) We can represent Q_8 as $\rho : Q_8 \to M_4(\mathbb{R})$ where

$$-1 \mapsto \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad ; \quad i \mapsto \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad ; \quad k \mapsto \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We can observe this is representation by seeing that $\phi(-1)^2 = \phi(1)$ and that $\phi(i)^2 = \phi(j)^2 = \phi(k)^2 = \phi(ijk) = \phi(-1)$.

c) By Exercise 4a, we know that both D_8 and Q_8 have four inequivalent real representations of degree 1. In the case of D_8, we also know that there is a real representation of degree 2, this means that the decomposition of $\mathbb{R}[D_8]$ has four copies of $\mathbb{R}$ and a copy of either $M_2(\mathbb{R})$ or $\mathbb{C}$. However, if $\mathbb{R}[D_8]$ contained a copy of $\mathbb{C}$, then it must be that $\mathbb{R}[D_8] \simeq \mathbb{R}^4 \oplus \mathbb{C} \oplus \mathbb{R}^2$ or $\mathbb{R}[D_8] \simeq \mathbb{R}^4 \oplus \mathbb{C} \oplus \mathbb{C}$. Neither of these can be the case, though, as $\mathbb{R}[D_8]$ is not commutative. Thus, it must be that $\mathbb{R}[D_8]$ contains a copy of $M_2(\mathbb{R})$ and so

$$\mathbb{R}[D_8] \simeq \mathbb{R}^4 \oplus M_2(\mathbb{R}).$$

Now, similarly, since Q_8 has a real representation of degree 4, this means that $\mathbb{R}[Q_8]$ contains a copy of either $M_2(\mathbb{C})$ or $\mathbb{H}$ in addition to four copies of $\mathbb{R}$. Since the dimension of $M_2(\mathbb{C})$ over reals is $4 \cdot 2 = 8$, it must be that $\mathbb{R}[Q_8]$ contains a copy of $\mathbb{H}$. This gives us that

$$\mathbb{R}[Q_8] \simeq \mathbb{R}^4 \oplus \mathbb{H}.$$

Since $\mathbb{H}$ is not isomorphic to $M_2(\mathbb{R})$ and these decompositions are unique up to permutation and isomorphisms by Artin-Weddenburn, we have that $\mathbb{R}[D_8]$ and $\mathbb{R}[Q_8]$ are not isomorphic. □

Ex 6.

a) $K \otimes_k k[x_1, \ldots, x_n] \simeq K[x_1, \ldots, x_n]$

b) $K \otimes_k k[x_1, \ldots, x_n]/(f) \simeq K[x_1, \ldots, x_n]/(f)$ for $f \in k[x_1, \ldots, x_n]$

Proof.

a) Let $f : K \to K[x_1, \ldots, x_n]$ and let $g : k[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]$ be the inclusion maps. Since the image of these maps commute, the map $\phi : K \otimes_k k[x_1, \cdot, x_n] \to K[x_1, \ldots, x_n]$ where $\phi(\lambda \otimes p(x)) = f(\lambda)g(p(x)) = \lambda p(x)$ is a well-defined k-algebra homomorphism.

If we let $v_1, \ldots, v_\ell$ be a basis of K as a k-vector space, then we see that $K \otimes_k k[x_1, \ldots, x_n]$ has basis $v_j \otimes \prod_{i \leq n} x_i^{e_i}$ and that $K[x_1, \ldots, x_n]$ has basis $v_j \prod_{i \leq n} x_i^{e_i}$. Since f is a bijection on these basis elements, we have that f is a bijection and thus that f is an isomorphism.

b) Let $f : K \to K[x_1, \ldots, x_n]/(f)$ and let $g : k[x_1, \ldots, x_n]\ (f) \to K[x_1, \ldots, x_n]\ (f)$ be the inclusion maps. Since the image of these maps commute, the map $\phi : K \otimes_k k[x_1, \cdot, x_n] \to$

$K[x_1, \ldots, x_n]$ where $\phi(\lambda \otimes (p(x) + (f))) = f(\lambda)g(p(x) + (f)) = \lambda p(x) + (f)$ is a well-defined k-algebra homomorphism.

[Proof that f is a bijection is incomplete.] $\square$

Ex 7.

a) $K \otimes_k M_n(k) \simeq M_n(K)$

b) $K \otimes_k \left(\frac{\alpha,\beta}{k}\right) \simeq \left(\frac{\alpha,\beta}{K}\right)$

Proof.

a) Let $f : K \to M_n(K)$ be the map $f(\lambda) = \lambda \operatorname{Id}_n$ and let $g : M_n(k) \to M_n(K)$ be the inclusion map. Since the image of these maps commute, the map $\phi : K \otimes_k M_n(k) \to M_n(K)$ where $\phi(\lambda \otimes A) = f(\lambda)g(A) = \lambda \operatorname{Id}_n A = \lambda A$ is a well-defined k-algebra homomorphism.

   If we let $v_1, \ldots, v_\ell$ be a basis of K as a k-vector space, then we see that $K \otimes_k M_n(k)$ has basis $v_m \otimes e_{ij}$ and that $M_n(K)$ has basis $v_j e_{ij}$. Since f is a bijection on these basis elements, we have that f is a bijection and thus that f is an isomorphism.

b) Let $f : K \to \left(\frac{\alpha,\beta}{K}\right)$ be the map $f(\lambda) = \lambda \cdot 1$ and let $g : \left(\frac{\alpha,\beta}{k}\right) \to \left(\frac{\alpha,\beta}{K}\right)$ be the inclusion map. Since the image of these maps commute, the map $\phi : K \otimes_k \left(\frac{\alpha,\beta}{k}\right) \to \left(\frac{\alpha,\beta}{K}\right)$ where $\phi(\lambda \otimes x) = f(\lambda)g(x) = \lambda x$ is a well-defined k-algebra homomorphism.

   If we let $v_1, \ldots, v_\ell$ be a basis of K as a k-vector space, then we see that $K \otimes_k \left(\frac{\alpha,\beta}{k}\right)$ has basis $v_m \otimes e_i$ (where $e_1 = 1$, $e_2 = i$, $e_3 = j$, and $e_4 = k$) and that $\left(\frac{\alpha,\beta}{K}\right)$ has basis $v_j e_{ij}$. Since f is a bijection on these basis elements, we have that f is a bijection and thus that f is an isomorphism. $\square$