Problem Set 9
Differential Topology

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Chapter 3 Section 2

Ex 13 Prove that every compact hypersurface in Euclidean space is orientable.

Proof. By the Jordan Brouwer Separation Theorem, a compact hypersurface divides Euclidean space into two open components. Furthermore, one of these components, call it X, has the property that $\overline{X}$ is a compact manifold and $\partial\overline{X}$ is our compact hypersurface. Since X is an open manifold inside Euclidean space, we can orient it by giving it the orientation of its ambient space. This orientation on the interior of $\overline{X}$ natually induces an orientation on its boundary. Since it's boundary is exactly our hypersurface, we have proven that any hypersurface is orientable. □

Ex 17 Compute the orientation of $X \cap Z$ in the following examples by exhibiting positively oriented bases at every point.

a) X is x-axis, Z is y-axis in $\mathbb{R}^2$

b) X is S^1, Z is y-axis in $\mathbb{R}^2$

c) X is xy-plane, Z is z-axis in $\mathbb{R}^3$

d) X is S^2, Z is yz-plane in $\mathbb{R}^3$

e) X is S^1 in xy-plane, Z is yz-plane in $\mathbb{R}^3$

f) X is xy-plane, Z is yz-plane in $\mathbb{R}^3$

g) X is hyperboloid $x^2 + y^2 - z^2 = a$ with preimage orientation $a > 0$, Z is xy-plane in $\mathbb{R}^3$

Proof.

a) $X \cap Z = \{(0,0)\}$. This point has positive orientation as $T_{(0,0)}X + T_{(0,0)}Z = T_{(0,0)}\mathbb{R}^2 = \mathbb{R}^2$ in an orientation-preserving way.

b) $X \cap Z = \{(0,1), (0,-1)\}$. The orientation is positive for $(0,1)$ and negative for $(0,-1)$.

c) $X \cap Z = \{(0,0,0)\}$. This point has positive orientation.

d) $X \cap Z = \{(0,y,z) : y^2 + z^2 = 1\}$, i.e. S^1 in the yz-plane. At the point $(0,y,z)$, the set $\{(0,-z,y)\}$ is a positively oriented basis.

e) $X \cap Z = \{(0,1,0),(0,-1,0)\}$. The orientation is positive for $(0,1,0)$ and negative for $(0,-1,0)$.

f) $X \cap Z = \{(0,y,0) : y \in \mathbb{R}\}$, i.e. the y-axis. At any point of this manifold, the set $\{(0,1,0)\}$ is a positively oriented basis.

g) $X \cap Z = \{(x,y,z) : z = 0, x^2 + y^2 = a\}$, i.e. the circle of radius $\sqrt{a}$ in the xy-plane. At the point $(x,y,0)$, the set $\{(-y,x,0)\}$ is a positively oriented basis. $\square$

Ex 24 Suppose that X is not orientable. Prove that $X \times Y$ is never orientable, no matter what manifold Y may be.

Proof. Suppose $X \times \mathbb{R}$ is orientable. We note that this induces an orientation on the open submanifold $X \times I$. But the boundary of $X \times I$ is two disjoint copies of X with opposite orientations. This proves that X is orientable. By the contrapositive, we see that if X is not orientable, then $X \times \mathbb{R}$ is not orientable. Using induction, we can conclude that $X \times \mathbb{R}^\ell$ is not orientable for any $\ell \geq 0$.

Suppose then that X is not orientable and that $X \times Y$ is orientable. Since Y is locally diffeomorphic to $\mathbb{R}^k$ for some k, we have that there's an open set V of Y that is diffeomorphic to an open set U of $\mathbb{R}^k$. We proved previously that U is actually diffeomorphic to all of $\mathbb{R}^k$. Thus, $X \times Y$ has an open submanifold (along with an induced orientation) $X \times V$ that is diffeomorphic to $X \times \mathbb{R}^k$. This proves that $X \times \mathbb{R}^k$ is orientable. This is in contradiction to the previous paragraph, though, so it must be that $X \times Y$ is never orientable, no matter what manifold Y may be. $\square$

Chapter 3 Section 3

Ex 2

a) Compute the degree of the antipodal map $S^k \to S^k$, $x \mapsto -x$.

b) Prove that the antipodal map is homotopic to the identity if and only if k is odd.

c) Prove that there exists a nonvanishing vector field on S^k if and only if k is odd.

d) Could mod 2 theory prove parts (b) and (c)?

Proof.

a) We note that the antipodal map from S^k to S^k is just the composition of $k+1$ reflections. Since each reflection is an orientation-reversing diffeomorphism, we have that

$$\deg(x \mapsto -x) = \deg(r_1 \circ r_2 \circ \cdots \circ r_{k+1}) = \deg(r_1)\deg(r_2)\ldots\deg(r_{k+1}) = (-1)^{k+1}.$$

b) If k is even, then the antipodal map would have degree -1 and thus couldn't be homotopic to the identity map as it has degree 1. We see that antipodal map is homotopic to the identity for S^1 as the map $h : I \times S^1 \to S^1$ where

$$h_t(v) = \begin{bmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{bmatrix} v$$

is a homotopy from the identity map to the antipodal map. This proves the two maps are homotopic for S^1. For S^k, k odd, we can perform the same trick for each pairs of coordinates.

Thus, explicity, the map

$$\begin{bmatrix} \cos(\pi t) & -\sin(\pi t) & 0 & 0 & \dots & 0 & 0 \\ \sin(\pi t) & \cos(\pi t) & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \cos(\pi t) & -\sin(\pi t) & \dots & 0 & 0 \\ 0 & 0 & \sin(\pi t) & \cos(\pi t) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \cos(\pi t) & -\sin(\pi t) \\ 0 & 0 & 0 & 0 & \dots & \sin(\pi t) & \cos(\pi t) \end{bmatrix}$$

is a homotopy from the identity map to the antipodal map for S^k.

c) For S^1, we see that $(x_1, x_2) \mapsto (-x_2, x_1)$ is a nonvanishing vector field. Similar to previous part, we can scale this up to S^k (k odd) using the vector field $(-x_2, x_1, -x_3, x_4, \dots, x_{-(k+1)}, x_k)$. Thus, for k odd, S^k has a nonvanishing vector field.

Now suppose that S^k has a nonvanishing vector field $\mathbf{v}(x)$. Without loss of generality, we can assume that $|\mathbf{v}(x)| = 1$ for any $x \in S^k$. Pushing each $x \in S^k$ in the direction of $\mathbf{v}(x)$ (each x being moved at the same speed as $|\mathbf{v}(x)| = 1$), we can smoothly send every point to its antipodal point. This means that the identity map is homotopic to the antipodal map. By part (b), this implies that k is odd as desired.

d) No, since $1 = -1 \bmod 2$, we would have no way of differentiating between these two cases. $\square$

Ex 6 Show that $z^2 = e^{-|z|^2}$ for some complex number z.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = e^{-|x|^2} - x^2$. We see that this function is continuous and that $f(0) = e^0 - 0 = 1$ and $f(1) = 1/e - 1 < 0$. By the Intermediate Value Theorem, there must be some $c \in \mathbb{R}$ such that $f(c) = 0$. That is, there exists a real number c such that $c^2 = e^{-|c|^2}$. Since real numbers are also complex numbers, we have proven the statement. $\square$

Ex 2.4.8(a) Let $f : S^1 \to S^1$ be any smooth map. Prove that there exists a smooth map $g : \mathbb{R} \to \mathbb{R}$ such that $f(\cos(t), \sin(t)) = (\cos(g(t)), \sin(g(t)))$ and satisfying $g(2\pi) = g(0) + 2\pi q$ for some integer q.

Proof. We note that $p : \mathbb{R} \to S^1$ where $\pi(x) = (\cos(x), \sin(x))$ is a local diffeomorphism. Let $g : [0, 2\pi] \to \mathbb{R}$ where $g = p^{-1} \circ f \circ p|_{[0,2\pi]}$. This means that for $t \in [0, 2\pi]$,

$$(\cos(g(t)), \sin(g(t))) = (p \circ g)(t) = (f \circ p|_{[0,2\pi]})(t) = f(\cos(t), \sin(t)).$$

We see that $g(2\pi) = (p^{-1} \circ f \circ p\,|_{[0,2\pi]})(2\pi) = p^{-1}(f(1,0))$ and $g(0) = (p^{-1} \circ f \circ p\,|_{[0,2\pi]})(0) = p^{-1}(f(1,0))$. Since the preimage of a point under p is a discrete set of points separated by 2π, we have that $g(2\pi) = g(0) + 2\pi q$ for some integer q. We can now extend g to all of $\mathbb{R}$ by defining $g(t + 2\pi) = g(t) + 2\pi q$. We see that this new function still satisfies $p \circ g = f \circ p$ as desired. $\square$

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Ex 8 For any map $f : S^1 \to S^1$, there exists a map $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(\cos(t), \sin(t)) = (\cos(g(t)), \sin(g(t))).$$

Morever, g satisifies $g(t + 2\pi) = g(t) + 2\pi q$ for some integer q. Show that $\deg(f) = q$.

Proof. Let $g_s(t) = (1-s)g(t) + sqt$. We see that g_s is a homotopy between g and st. Furthermore, we see that

$$\begin{aligned} g_s(t+2\pi) &= (1-s)g_1(t+2\pi) + s(t+2\pi)q = (1-s)s(g_1(t)+2\pi q) + s(tq+2\pi q) \\ &= (1-s)g_1(t) + sqt + 2\pi q = g_s(t) + 2\pi q, \end{aligned}$$

for each $s \in [0,1]$. This proves that $f_s(\cos(t), \sin(t)) = (\cos(g_s(t)), \sin(g_s(t)))$ is a well-defined homotopy from f to f_1 where $f_1(\cos(t), \sin(t)) = (\cos(qt), \sin(qt))$. Since f_1 simply traverses the circle q times and any point of S^1 is a regular value of f_1, we have that for any $x \in S^1$

$$\deg(f) = I(f, \{x\}) = I(f_1, \{x\}) = |f_1^{-1}(x)| = q$$

as desired. □

Ex 9 Prove that two maps of the circle S^1 into itself are homotopic if and only if they have the same degree.

Proof. Let $f_0, f_1 : S^1 \to S^1$ be two maps that are homotopic to each other. Then, by Ex 8, we have that

$$\deg(f_0) = I(f_0, \{x\}) = I(f_1, \{x\}) = \deg(f_1),$$

which proves that there two maps have the same degree.

Now let $f_0, f_1 : S^1 \to S^1$ be two maps of the same degree. By Ex 8, this means there exists $g_0, g_1 : \mathbb{R} \to \mathbb{R}$ such that $g_i(t+1) = g_i(t) + 2\pi q$ and $f_i(\cos(t), \sin(t)) = (\cos(g_i(t)), \sin(g_i(t)))$ for $i = 1, 2$. Let $g_s(t) = sg_1(t) + (1-s)g_0(t)$. We see that

$$\begin{aligned} g_s(t+1) &= sg_1(t+1) + (1-s)g_0(t+1) = s(g_1(t) + 2\pi q) + (1-s)(g_0(t) + 2\pi q) \\ &= sg_1(t) + (1-s)g_0(t) + 2\pi q = g_s(t) + 2\pi q. \end{aligned}$$

This property ensures that $f_s : S^1 \to S^1$ where $f_s(\cos(t), \sin(t)) = (\cos(g_s(t)), \sin(g_s(t)))$ is a well-defined homotopy between f_0 and f_1. □