Problem Set 4
Differential Topology

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Chapter 1, Section 2

Ex 6 The tangent space to S^1 at a point (a,b) is a one-dimensional subspace of $\mathbb{R}^2$. Explicitly calculate the subspace in terms of a and b.

Proof. Let $(a,b) = (\cos(x_0), \sin(x_0))$ for some $x_0 \in [0, 2\pi)$. Then the map $\phi : (x_0 - \pi, x_0 + \pi) \to S^1$ where $\phi(x) = (\cos(x), \sin(x))$ is chart for the point (a,b). Then we have that $d_{x_0}\phi : \mathbb{R} \to \mathbb{R}^2$ is a linear map such that for an arbitrary vector $[t] \in \mathbb{R}$

$$d_{x_0}\phi([t]) = \begin{bmatrix} -\sin(x_0) \\ \cos(x_0) \end{bmatrix} [t] = \begin{bmatrix} -\sin(x_0)t \\ \cos(x_0)t \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} t.$$

Since $T_{(a,b)}S^1$ is the image of this linear map, we have that $T_{(a,b)}S^1$ is the span of the vector $(-b, a)$. $\square$

Ex 8 What is the tangent space to the paraboloid defined by $x^2 + y^2 - z^2 = a$ at $(\sqrt{a}, 0, 0)$ where $a > 0$?

Proof. Let $\gamma_1(t) = (\sqrt{a}\cos(t), \sqrt{a}\sin(t), 0)$ and $\gamma_2 = (\sqrt{a}, t, t)$ be paths where $t \in (-\pi, \pi)$. We see that these paths are bijections onto their image and thus one-dimensional manifolds (their charts at any point are simply themselves). Since $(\sqrt{a}\cos(t))^2 + (\sqrt{a}\sin(t))^2 - 0^2 = a(\cos(t)^2 + \sin(t)^2) = a$ and $\sqrt{a}^2 + t^2 - t^2 = a$, we have that $\operatorname{Im}(\gamma_1)$ and $\operatorname{Im}(\gamma_2)$ are both submanifolds of the given paraboloid. Since $\gamma_1(0) = \gamma_2(0) = (\sqrt{a}, 0, 0)$, their respective tangent spaces at $(\sqrt{a}, 0, 0)$ are $\gamma_1'(0) = (-\sqrt{a}\sin(0), \sqrt{a}\cos(0), 0) = (0, \sqrt{a}, 0)$ and $\gamma_2'(0) = (0, 1, 1)$. As submanifolds, these tangent spaces are subspaces of the tangent space of the paraboloid at $(\sqrt{a}, 0, 0)$. As these vectors are linearly independent and the tangent space of the paraboloid is two-dimensional, we have that the tangent space is simply the span of the vectors $(0, \sqrt{a}, 0)$ and $(0, 1, 1)$ (which is just the subspace $(0, \alpha, \beta)$ for $\alpha, \beta \in \mathbb{R}$). $\square$

Ex 9

a) Show that for any manifolds X and Y,

$$T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y).$$

b) Let $f : X \times Y \to X$ be the projection map $(x, y) \mapsto x$. Show that

$$df_{(x,y)} : T_x(X) \times T_y(Y) \to T_x(X)$$

is the analogous projection $(v, w) \mapsto v$.

c) Fixing any $y \in Y$ gives an injection mapping $f : X \to X \times Y$ by $f(x) = (x, y)$. Show that $df_x(v) = (v, 0)$.

d) Let $f : X \to X'$, $g : Y \to Y'$ be any smooth maps. Prove that

$$d(f \times g)_{(x,y)} = df_x \times dg_y.$$

Proof.

a) Let $f : \mathbb{R}^n \to X$ and $g : \mathbb{R}^m \to Y$ be local charts that contain x and y respectively. Then $f \times g : \mathbb{R}^{n+m} \to X \times Y$ is a chart of $X \times Y$ containing (x, y). We see that for $v \in \mathbb{R}^n$ and $v' \in \mathbb{R}^m$,

$$d(f \times g)_{(x,y)}(v, v') = \begin{bmatrix} \frac{\partial (f\times g)_1}{\partial x_1} & \cdots & \frac{\partial (f\times g)_1}{\partial x_{n+m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial (f\times g)_{n+m}}{\partial x_1} & \cdots & \frac{\partial (f\times g)_{n+m}}{\partial x_{n+m}} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_{n+m} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{n+m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_{n+m}} \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_{n+m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_{n+m}} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_{n+m} \end{bmatrix}$$

$$= \begin{bmatrix} df_x & 0 \\ 0 & dg_y \end{bmatrix} \begin{bmatrix} v \\ v' \end{bmatrix} = \begin{bmatrix} df_x(v) \\ dg_y(v') \end{bmatrix}.$$

This proves that

$$T_{(x,y)} = \operatorname{Im}(d(f \times y)_{(x,y)}) = \operatorname{Im}(df_x \times dg_y) = \operatorname{Im}(df_x) \times \operatorname{Im}(dg_y) = T_x(X) \times T_y(Y)$$

as we wanted.

b) Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$. If $(v, w) \in \mathbb{R}^n \times \mathbb{R}^m$, then we see that

$$df_{(x,y)}(v, w) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial x_{n+1}} & \cdots & \frac{\partial f_1}{\partial x_{n+m}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial x_{n+1}} & \cdots & \frac{\partial f_n}{\partial x_{n+m}} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \\ w_1 \\ \vdots \\ w_m \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \cdots & \frac{\partial x_1}{\partial x_n} & \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x_1} & \cdots & \frac{\partial x_n}{\partial x_n} & \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_m} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \\ w_1 \\ \vdots \\ w_m \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{Id}_n & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = v$$

which proves what we want.

c) Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, then for $v \in \mathbb{R}^n$, we see that

$$\begin{aligned} df_x(v) &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \\ \frac{\partial f_{n+1}}{\partial x_1} & \cdots & \frac{\partial f_{n+1}}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{n+m}}{\partial x_1} & \cdots & \frac{\partial f_{n+m}}{\partial x_n} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \cdots & \frac{\partial x_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x_1} & \cdots & \frac{\partial x_n}{\partial x_n} \\ \frac{\partial y}{\partial x_1} & \cdots & \frac{\partial y}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y}{\partial x_1} & \cdots & \frac{\partial y}{\partial x_n} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Id}_n \\ 0 \end{bmatrix} \begin{bmatrix} v \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix} \end{aligned}$$

as we wanted.

d) If we let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, then I don't really know how this is different from the first part of part (a). We still have that

$$d(f \times g)_{(x,y)} = df_x \times dg_y,$$

it's just in this case we don't necessarily have tangent spaces as $f \times g$ is not necessarily a diffeomorphism. □

Ex 10

a) Let $f : X \to X \times X$ be the mapping $f(x) = (x, x)$. Check that $df_x(v) = (v, v)$.

b) If Δ is the diagonal of $X \times X$, show that its tangent space $T_{(x,x)}(\Delta)$ is the diagonal of $T_x(X) \times T_x(X)$.

Proof.

a) Let $X \subseteq \mathbb{R}^n$. Then $f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, x_1, \ldots, x_n)$ and $df_x : \mathbb{R}^n \to \mathbb{R}^{2n}$ is a linear map such for a vector $v \in \mathbb{R}^n$

$$df_x(v) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{x_2} & \cdots & \frac{\partial f_1}{x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{x_2} & \cdots & \frac{\partial f_2}{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{2n}}{\partial x_1} & \frac{\partial f_{2n}}{x_2} & \cdots & \frac{\partial f_{2n}}{x_n} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \mathbb{1}_n \\ \mathbb{1}_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \\ v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v \\ v \end{bmatrix}.$$

This proves that $df_x(v) = (v, v)$ as desired.

b) We note that the image of df_x described in the last problem is precisely the tangent space $T_{(x,x)}(\Delta)$. Similarly, the tangent space $T_x(X)$ is the image of $d\mathbb{1}_x = \mathbb{1}_n$ where $\mathbb{1}$ is the identity from X to X and $\mathbb{1}_n$ is the identity matrix from $\mathbb{R}^n$ to $\mathbb{R}^n$. Thus, for $v \in \mathbb{R}^n$ we see that

$$df_x(v) = (v, v) = d\mathbb{1}_x(v) \times d\mathbb{1}_x(v),$$

which proves that $T_{(x,x)}(\Delta) = \text{Im}(df_x)$ is the diagonal of $T_x(X) \times T_x(X) = \text{Im}(d\mathbb{1}_x \times d\mathbb{1}_x)$. $\square$

Ex 11

a) Suppose that $f : X \to Y$ is a smooth map and let $F : X \to X \times Y$ be $F(x) = (x, f(x))$. Show that

$$dF_x(v) = (v, df_x(v)).$$

b) Prove that the tangent space to the graph of f at the point $(x, f(x))$ is the graph of $df_x : T_x(X) \to T_{f(x)}(Y)$.

Proof.

a) Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$, $f : X \to X$, and $F : X \to X \times Y$ where $F(x_1, \dots, x_n) = (x_1, \dots, x_n, f(x_1, \dots, x_n))$. Then we see that for $v \in \mathbb{R}^n$

$$dF_x(v) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \\ \frac{\partial F_{n+1}}{\partial x_1} & \cdots & \frac{\partial F_{n+1}}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_{n+m}}{\partial x_1} & \cdots & \frac{\partial F_{n+m}}{\partial x_n} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \text{Id}_n \\ df_x \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v \\ df_x(v) \end{bmatrix}$$

as desired.

b) We note that F is the map from X to the graph of f. Thus, the image of dF_x is precisely the tangent space of the graph of f at $(x, f(x))$. Since for an arbitrary $v \in T_x(X)$ we have that $dF_x(v) = (v, df_x(v))$, the tangent space of the graph of f at $(x, f(x))$ is the graph of df_x. $\square$

Chapter 1, Section 3

Ex 2 Suppose that Z is an ℓ-dimensional submanifold of X and that $z \in Z$. Show that there exists a local coordinate system $\{x_1, \dots, x_k\}$ defined in a neighborhood U of z in X such that $Z \cap U$ is defined by the equations $x_{i+1} = 0, \dots, x_k = 0$.

Proof. Let $z \in X$ and let U be an open neighborhood of z such that $\phi : U \to \mathbb{R}^k$ is a local coordinate map. We see then that we have the following diagram

$$\begin{tikzcd} z \in U \subseteq X^k \arrow[r, "\phi"] & \mathbb{R}^k \arrow[d, "\pi"] \\ z \in Z^\ell \arrow[u, hook, "i"] \arrow[r, dotted, "\pi \circ \phi \circ i"] & \mathbb{R}^\ell \end{tikzcd}$$

If we choose $\{e_i\}_{i\leq \ell}$ be a basis of the image of $\pi \circ \phi \circ i$, then we can extend this to a basis $\{e_i\}_{i\leq k}$ of $\mathbb{R}^k$. With respect to this basis, ϕ is a local coordinate system defined on U, a neighborhood of z, such that $Z \cap U$ is defined with the local coordinates $\{x_1, \ldots, x_k, 0, \ldots, 0\}$. $\square$

Ex 5 Prove that a local diffeomorphsim $f : X \to Y$ is actually a diffeomorphism of X onto an open subset of Y, provided that f is one-to-one.

Proof. Without loss of generality, we may assume that f is surjective (just take Y as $\mathrm{Im}(f)$). This means that f is a bijection, and thus has an inverse f^{-1}. Let $y \in Y$. Since f is a bijection, there exists exactly one $x \in X$ such that $f(x) = y$. As f is a local diffeomorphism, there exists an open set U_x containing x such that $f|_{U_x} : U_x \to f(U_x)$ is a diffeomorphism onto its image. We note that $y \in f(U_x)$, so we let $V_y = f(U_x)$. Since $f|_{U_x}$ is a diffeomorphism, its inverse also is. Since inverses are unique and $f^{-1}|_{V_y}$ is such an inverse, this proves that $f^{-1}|_{V_y}$ is a diffeomorphism. Thus, for any $y \in Y$, there's an open set V_y such that $f^{-1}|_{V_y}$ is smooth. This implies that f^{-1} is smooth everywhere, proving that f is a "global" diffeomorphism. $\square$

Ex 10 Let $f : X \to Y$ be a smooth map that is one-to-one on a compact submanifold Z of X. Suppose that for all $x \in Z$,

$$df_x : T_x(X) \to T_{f(x)}(Y)$$

is an isomorphism. Then f maps Z diffeomorphically onto $f(Z)$ (why?). Prove that f, in fact, maps an open neighborhood of Z in X diffeomorphically onto an open neighborhood of $f(Z)$ in Y. Note that when Z is a single point, this specializes to the Inverse Function Theorem.

Proof. Since df_x is an isomorphism for all $x \in Z$, by the Inverse Function Theorem, f is a local diffeomorphism on Z. As we know that f is injective on Z, by Ex 5, f maps Z diffeomorphically onto $f(Z)$. Since f is locally diffeomorphic on Z, it's locally diffeomorphic on some open set V containing Z.

Suppose that f is not injective for any open neighborhood of Z. Let U_i be the $1/i$ open neighborhood of Z in X for $i \in \mathbb{N}$. By assumption, this means that $f|_{U_i}$ is not injective, so there are some $a_i, b_i \in U_i$ such that $f(a_i) = f(b_i)$. This allows us to construct the sequence $\{(a_i, b_i)\}_{i\in\mathbb{N}}$ such that $f(a_i) = f(b_i)$. Since Z is compact, it is bounded, which means U_1 is bounded distance away from a bounded set. Thus, $\overline{U_1}$ is compact. By Tychonoff's Theorem, $\overline{U_1} \times \overline{U_1}$ is compact (I understand this complete overkill). Since $\{(a_i, b_i)\}_{i\in\mathbb{N}}$ is a sequence in a compact set, we have that there is a subsequence $\{(a_{k_i}, b_{k_i})\}_{i\in\mathbb{N}}$ converging to some point (a, b). By construction, the a_{k_i}'s and the b_{k_i}'s are sequences approaching Z and so a and b must be in Z. Since f is continuous, we have that

$$f(a) = f(\lim_{i\to\infty} a_{k_i}) = \lim_{i\to\infty} f(a_{k_i}) = \lim_{i\to\infty} f(b_{k_i}) = f(\lim_{i\to} b_{k_i}) = f(b).$$

But $a, b \in Z$ and f is injective on Z, so we actually have that $a = b$. This means that $\{a_{k_i}\}_{i\in\mathbb{N}}$ and $\{b_{k_i}\}_{i\in\mathbb{N}}$ are sequences both approaching $a = b \in Z$ such that $f(a_{k_i}) = f(b_{k_i})$. Since f is a local diffeomorphism, there is some open neighborhood W of $a = b \in Z$ such that f restricted to W is a diffeomorphism onto its image. However, by the topological definition of convergence, there is some N such that for all $i \geq N$, a_{k_i} and b_{k_i} are both in W. In particular, we have that a_{k_N} and b_{k_N} are both in W and that $f(a_{k_N}) = f(b_{k_N})$. This is a contradiction to f being a diffeomorphism when restricted to W. Thus, we have that f is injective for some open neighborhood of Z, call this set U.

This means we have an open set V containing Z such that f is locally diffeomorphic on V and an open set U containing Z such that f is injective on U. Their intersection $U \cap V$ is an open set containing Z such that $f|_{U\cap V}$ is both injective and locally diffeomorphic. By Ex 5, this proves that $f|_{U\cap V}$ is a diffeomorphism onto its image. $\square$

Assigned Problems

Ex 3 A smooth map $f : M \to N$ between n-manifolds is a *local diffeomorphism* if for every $x \in M$ there is a neighborhood U of x such that $f|_U$ is a diffeomorphism $U \to f(U)$. Prove that the function $f(x, y) = (e^x \cos(y), e^x \sin(y))$ is a surjective local diffeomorphism $\mathbb{R}^2 \to \mathbb{R}^2 \setminus \{(0, 0)\}$, but not a diffeomorphism.

Proof. Let $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. We can rewrite this point in polar coordinates

$$(r, \theta) = (\sqrt{x^2 + y^2}, \arg(x, y))$$

where the function $\arg : \mathbb{R}^2 \to (0, 2\pi]$ is the angle between (x, y) and the positive x-axis. Then we see that

$$f(\ln(r), \theta) = (e^{\ln(r)} \cos(\theta), e^{\ln(r)} \sin(\theta)) = (r \cos(\theta), r \sin(\theta)) = (x, y)$$

which proves that f is surjective. We note that f is not injective as $f(0, 0) = (1, 0) = f(0, 2\pi)$, which also proves that f is not a diffeomorphism. Since

$$\begin{aligned}\det(df_{(x,y)}) &= \det \begin{bmatrix} \frac{\partial}{\partial x} e^x \cos(y) & \frac{\partial}{\partial y} e^x \cos(y) \\ \frac{\partial}{\partial x} e^x \sin(y) & \frac{\partial}{\partial y} e^x \sin(y) \end{bmatrix} = \det \begin{bmatrix} e^x \cos(y) & -e^x \sin(y) \\ e^x \sin(y) & e^x \cos(y) \end{bmatrix} \\ &= e^x(\cos(y)\cos(y) + \sin(y)\sin(y)) = e^x(\cos(y)^2 + \sin(y)^2) = e^x\end{aligned}$$

is never zero, we have that $df_{(x,y)}$ is never singular at any point $(x, y) \in \mathbb{R}^2$. By the Inverse Function Theorem, f is a local diffeomorphism. $\square$

Ex 4 Let $x = (x_1, \ldots, x_p)$ and $y = (y_1, \ldots, y_q)$ be the standard coordinates on $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, and let $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{p+q} \to \mathbb{R}^k$ be a smooth map. Let $(x_0, y_0) \in \mathbb{R}^p \times \mathbb{R}^q$ and let $w_0 = f(x_0, y_0)$. We say "the equation $f(x, y) = w_0$ determines y a function of x near (x_0, y_0) " if there are neighborhoods U of x_0 and V of y_0 and a (unique) smooth function $g : U \to V$ such that

$$g(x_0) = y_0 \text{ and } f(x, g(x)) = w_0$$

for all $x \in U$. Prove that if this is the case, then $d_x f = -d_y f \circ dg$ where $d_x f$ is the $k \times p$ matrix of partial derivatives of f with respect to the x_i and d_j is the $k \times q$ matrix of partials of f with respect to y_j.

Proof. Let $h(x) = (x, g(x))$ from $\mathbb{R}^p$ to $\mathbb{R}^k$. This means that $f(x, g(x)) = f(h(x))$. Since $f \circ h$ is constant on some neighborhood of U of x_0, then for all $x \in U$ $d(f \circ h)_x$ is 0. By the chain rule, we have that $d(f\circ h)_x = df_{h(x)}\circ h_x = df_{(x,y)}\circ h_x$. Thus, we get that

$$0 = d(f\circ h)_x = df_{(x,y)}\circ h_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_p} & \frac{\partial f_1}{\partial x_{p+1}} & \cdots & \frac{\partial f_1}{\partial x_{p+q}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_p} & \frac{\partial f_k}{\partial x_{p+1}} & \cdots & \frac{\partial f_k}{\partial x_{p+q}} \end{bmatrix} \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_p} \\ \frac{\partial h_{p+1}}{\partial x_1} & \cdots & \frac{\partial h_{p+1}}{\partial x_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_{p+q}}{\partial x_1} & \cdots & \frac{\partial h_{p+q}}{\partial x_p} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_p} & \frac{\partial f_1}{\partial x_{p+1}} & \cdots & \frac{\partial f_1}{\partial x_{p+q}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_p} & \frac{\partial f_k}{\partial x_{p+1}} & \cdots & \frac{\partial f_k}{\partial x_{p+q}} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \cdots & \frac{\partial x_1}{\partial x_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_p}{\partial x_1} & \cdots & \frac{\partial x_p}{\partial x_p} \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_q}{\partial x_1} & \cdots & \frac{\partial g_q}{\partial x_p} \end{bmatrix}$$

$$= \begin{bmatrix} df_x & df_y \end{bmatrix} \begin{bmatrix} \mathbb{1}_p \\ dg_x \end{bmatrix} = df_x \circ \mathbb{1}_p + df_y \circ dg_x = df_x + df_y \circ dg_x$$

which proves that $df_x = -df_y \circ dg_x$ as desired. $\square$