Problem Set 7
Differential Topology

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Chapter 2, Section 1

Ex 7 Suppose that X is a manifold with boundary and $x \in \partial X$. Let $\varphi : U \to X$ be a local parametrization with $\phi(0) = x$, where U is an open subset of H^k. Then $d\phi_0 : \mathbb{R}^k \to T_x(X)$ is an isomorphism. Define the *upper half-space* $H_x(X)$ in $T_x(X)$ to be the image of H^k under $d\phi_0$, $H_x(X) = d\phi_0(H^k)$. Prove that $H_x(X)$ does not depend on the choice of local parametrization.

Proof. Let $\phi : U \to X$ and $\psi : V \to X$ be two local parametrizations of $x \in \partial X$ where U, V are open sets of H^k. Now without loss of generality, we can shrink U and V so that $\phi(U) = \psi(V)$. Let $h = \psi^{-1} \circ \phi$, which is a diffeomorphism from U to V. As a smooth map, it has a smooth extension $\tilde{h}$ which maps an open set $\tilde{U} \subseteq \mathbb{R}^k$ containing U to $\tilde{V}$. Keep in mind that $\tilde{h}$ maps H^k to H^k and things not in H^k to the things not in H^k. By definition, $dh_0 = d\tilde{h}_0$. Now let $v \in H^k$ and consider the path $\gamma : (-\varepsilon, \varepsilon) \to \tilde{U}$, where $\gamma(t) = vt$. Then we have that $\gamma(t) \notin H^k$ for $t < 0$ and $\gamma(0) = 0$. Thus

$$dh_0(v) = d\tilde{h}_0(v) = \frac{d}{dt} h(\gamma(t))\bigg|_{t=0} = \lim_{t \to 0} \frac{h(\gamma(t)) - h(\gamma(0))}{t - 0} = \lim_{t \to 0} \frac{h(vt)}{t} \in H^k.$$

Thus $dh_0(H^k) \subseteq H^k$, meaning $d\phi_0(H^k) \subseteq d\psi_0(H^k)$. Using a similar argument but with $g = \phi^{-1} \circ \psi$ instead of h, we see that $d\psi_0(H^k) \subseteq H^k$. Thus, $H_x(X)$ does not depend on the choice of local parametrization and is well-defined. □

Ex 8 Show that there are precisely two unit vectors in $T_x(X)$ that are prependicular to $T_x(\partial X)$ and that one lies inside $H_x(X)$ and the other outside. The one in $H_x(X)$ is called the *inward unit normal vector* to the boundary and the other is the *outward unit normal vector* to the boundary. Denote the outward unit normal by $\overrightarrow{n}$. Note that if X sits in $\mathbb{R}^N$, $\overrightarrow{n}$ may be considered to be a map of ∂X into $\mathbb{R}^N$. Prove that $\overrightarrow{n}$ is smooth. (Specifically, what is $\overrightarrow{n}$ for $X = H^k$?)

Proof. Let $x \in \partial X$. And let $\phi : U \to X$ be a local parametrization of x such that U is open in H^k. Since φ maps ∂U to ∂X, we have that $T_x(\partial X)$ is the image of $d(\phi|_{\mathbb{R}^{k-1}})_x$, that is $T_x(\partial X)$ is the image of $\mathbb{R}^{k-1}$ under $d\phi_x$. Since $T_x(\partial X)$ has dimension $k-1$ and lies in $T_x(X)$ which has dimension k, there exist exactly two unit vectors orthogonal to $T_x(\partial X)$, call them u and $-u$. As $T_x(\partial X)$ is the image of $\mathbb{R}^{k-1}$ under $d\phi_x$ and $H_x(X)$ is the image of H^k under $d\phi_x$ and $d\phi_x$ is a linear isomorphism so its conformal, we see that one of the vectors u or $-u$ lie in $H_x(X)$.

Let $x \in \partial X$. And let $\phi : U \to X$ be a local parametrization of x such that U is open in H^k. Now for any $z \in \partial U$, we have by the conformality of $d\psi_z$,

$$\overrightarrow{n}(z) = \frac{d\phi_z((0,\dots,-1))}{|d\phi_z((0,\dots,-1))|} = \frac{-d\phi_z((0,\dots,1))}{|d\phi_z((0,\dots,1))|} = \frac{-d\phi_z((0,\dots,1))}{|d\phi_z((0,\dots,1))|} = \frac{-D\phi_{(0,\dots,1)}(z)}{|D\phi_{(0,\dots,1)}(z)|}.$$

Since the directional derivative of a ϕ in the direction $(0,\dots,1)$ is just the partial derivative with respect to x_k, we see that $\overrightarrow{n}(z)$ is smooth on ∂U. Since smoothness is a local property and $x \in \partial X$ was arbitrary, we have that $\overrightarrow{n}(z)$ is smooth on ∂X. □

Ex 10 Let $x \in \partial X$ be a boundary point. Show that there exists a smooth non-negative function f on some open neighborhood U of x such that $f(z) = 0$ if and only if $z \in \partial U$, and if $z \in \partial U$ then $df_z(\overrightarrow{n}(z)) > 0$.

Proof. Let $x \in \partial X$ and let $\phi : U \to V$ be a chart of x such that V is open in H^k. We let $f : U \to \mathbb{R}$ be the map $\pi \circ \phi$, where $\pi : \mathbb{R}^k \to \mathbb{R}$ is the projection onto the last component. Since $\mathrm{Im}(\phi) = V \subseteq H_k$, we have that $f \geq 0$ and that $f(z) = 0$ if and only if $\phi(z) \in \partial H^k$, which happens if and only if $z \in \partial X$.

Now let $z \in \partial U$. Since $\overrightarrow{n}(z) \in T_z(U)$ is the outward unit normal vector, we can take a path $\gamma : (-\varepsilon, 0] \to U$ where $\gamma(0) = z$ and $\gamma'(0) = \overrightarrow{n}(z)$ and $\gamma(t) \notin \partial U$ for all $t \neq 0$. Since $\gamma \in U$ and $z \in \partial U$, we have that $\phi \circ \gamma$ is a map from $(-\varepsilon, 0]$ to H_k where $\phi(\gamma(0)) \in \partial H_k$ and $\phi(\gamma(t)) \notin \partial H_k$ for all $t \neq 0$. This means that $\pi(\phi(\gamma(t)))$ is a path in $\mathbb{R}$ such that $\pi(\phi(\gamma(t))) > 0$ for $t \neq 0$ and $\pi(\phi(\gamma(0))) = 0$, which says that $(\pi \circ \phi \circ \gamma)'(0) < 0$. Thus

$$df_z(\overrightarrow{n}(z)) = \frac{d}{dt} f(\gamma(t))\bigg|_{t=0} = \frac{d}{dt}\pi(\phi(\gamma(t)))\bigg|_{t=0} < 0,$$

which proves that the statement. □

Ex 11 Show that if X is any manifold with boundary, then there exists a smooth nonnegative function f on X, with regular value at 0, such that $\partial X = f^{-1}(0)$. [Use a partition of unity to glue together the local functions in Ex 10. What guarantees regularity?]

Proof. Since ∂X is a submanifold of X, there is a locally finite cover of ∂X, call this set $\{U_i\}_{i \in I}$. By 2.1.9, there are functions $f_i : U_i \to \mathbb{R}$ where $f_i \geq 0$, $f_i(x) = 0$ if and only if $x \in \partial U_i$, and $d(f_i)_z(\overrightarrow{n}(z)) < 0$ for all $z \in \partial U_i$. Since the sets are locally finite, we can define $f : X \to \mathbb{R}$ by $f = \sum_{i \in I} f_i$. We see then that $f = 0$ if and only if $x \in \cup_{i \in I} \partial U = \partial X$ and that $f_x > 0$ otherwise. This proves that $f^{-1}(0) = \partial X$. Now if $x \in \partial X$, then x lies in some of the open sets, say $U_1, \dots, U_n$. By construction, $d(f_i)_x(\overrightarrow{n}(x)) < 0$ for $1 \leq i \leq n$, so

$$df_x(\overrightarrow{n}(x)) = d(\sum_{i \in I} f_i)_x(\overrightarrow{n}(x)) = d(\sum_{i=1}^{n} f_i)_x(\overrightarrow{n}(x)) = \sum_{i=1}^{n} d(f_i)_x(\overrightarrow{n}(x)) < 0.$$

Since $df_x : T_x(X) \to \mathbb{R}$, we have that df_x is surjective. As x was arbitrary, df_x is surjective for all $x \in \partial X$. Since $f(\partial X) = 0$ and nothing else in X is mapped to 0, this proves that 0 is a regular value of f. Thus, $f^{-1}(0) = \partial X$ is a manifold. □

Chapter 2, Section 2

Ex 3 Find maps of the solid torus into itself having no fixed points. Where does the proof of the Brouwer theorem fail?

Proof. We proved in a previous homework that all tori are diffeomorphic to each other, so let's consider the torus parametrized by

$$f(\theta,\phi) = (\ (2+\cos(\theta))\cos(\phi),\ \ (2+\cos(\theta))\sin(\phi),\ \ \sin(\theta)\ )$$

then we see that the rotation sending ϕ to $\phi+1$ is map from the torus to itself that has no fixed points. Visually, this is equivalent to taking a torus lying on the xy-plane centered around the z-axis and then simply rotating it about the z-axis by some rotation (but not a full rotation). The reason why Brouwer's theorem fails is because the proof required the existence of a line from x to $f(x)$, but that line may not exist inside of a solid torus. That is, the theorem fails because the torus is not convex. □

Ex 4 Prove that the Brouwer theorem is false for the open ball $|x|^2 < a$.

Proof. From a previous homework we proved that the open ball $|x|^2 < a$ in $\mathbb{R}^k$ is diffeomorphic to all of $\mathbb{R}^k$ via the map

$$\phi(x) = \frac{ax}{a^2 - |x|^2}.$$

Let $f : \mathbb{R}^k \to \mathbb{R}^k$ where $f(x_1, \dots x_k) = (x_1, \dots, x_k + 1)$, which is a diffeomorphism of $\mathbb{R}^k$ that has no fixed points. We see then that the composition $\phi^{-1} \circ f \circ \phi$ is a diffeomorphism from our open ball to itself. If this map had a fixed point x, then $(\phi^{-1} \circ f \circ \phi)(x) = x$, which would mean that $f(\phi(x)) = \phi(x)$, which can't happen as f has no fixed points. Thus, $\phi^{-1} \circ f \circ \phi$ is a smooth map from the open ball $|x|^2 < a$ to itself with no fixed points. □

Ex 6 Prove the Brouwer theorem for continuous maps $f : B^n \to B^n$. Use the Weierstrass Approximation Theorem, which says that for any $\varepsilon > 0$ there exists a polynomial mapping $p : \mathbb{R}^n \to \mathbb{R}^n$ such that $|f - p| < \varepsilon$ on B^n.

Proof. Suppose that $f : B^n \to B^n$ were a continuous map with no fixed points. Since f has no fixed points, we know that $|f(x) - x| > c > 0$ on B^n for some $c \in \mathbb{R}$. By the Weierstrass Approximation Theorem, there's a polynomial $p : \mathbb{R}^n \to \mathbb{R}^n$ such that $|f - p| < \frac{c}{2}$ on B^n. If we take the function $\tilde{p}(x) = \frac{p(x)}{1+\frac{c}{2}}$, then we'd have that for $x \in B^n$

$$|\tilde{p}(x)| = \left|\frac{p(x)}{1+\frac{c}{2}}\right| = \frac{|p(x)|}{1+\frac{c}{2}} \leq \frac{|f(x)| + \frac{c}{2}}{1+\frac{c}{2}} \leq \frac{1+\frac{c}{2}}{1+\frac{c}{2}} = 1,$$

so $\tilde{p}(x)$ is a smooth function and can be restricted to be a map from B^n to B^n. We also see that for $x \in B^n$

$$\begin{aligned} |f(x) - \tilde{p}(x)| = \left|f(x) - \frac{p(x)}{1+\frac{c}{2}}\right| &= \frac{|f(x) + \frac{c}{2}f(x) - p(x)|}{1+\frac{c}{2}} \leq \frac{\frac{c}{2}|f(x)| + |f(x) - p(x)|}{1+\frac{c}{2}} \\ &< \frac{\frac{c}{2}|f(x)| + \frac{c}{2}}{1+\frac{c}{2}} \leq \frac{\frac{c}{2}+\frac{c}{2}}{1+\frac{c}{2}} = \frac{c}{1+\frac{c}{2}} \leq c. \end{aligned}$$

Thus, we have that for $x \in B^n$

$$c < |f(x) - x| = |f(x) - \tilde{p}(x) + \tilde{p}(x) - x| \leq |f(x) - \tilde{p}(x)| + |\tilde{p}(x) - x| < c + |\tilde{p}(x) - x|.$$

This proves that $|\tilde{p}(x) - x| > 0$ on B^n, which means that $\tilde{p}(x)$ has no fixed points in B^n. This is a contradiction to Brouwer's Fixed Point theorem for smooth maps. It must be then that that $f(x)$ has a fixed point, proving Brouwer's Fixed Point theorem for continuous maps. □

Ex 7 If the entries in an $n \times n$ real matrix A are all nonnegative, then A has a real nonnegative eigenvalue.

Proof. Suppose that A is an $n \times n$ real matrix with all nonnegative entries. If A is singular, then 0 is an eigenvalue of A. If A is invertible, then we can consider the map $\phi(v) = \frac{Av}{|Av|}$ from $S^{n-1} \to S^{n-1}$ where S^{n-1} is lying inside $\mathbb{R}^n$. Now, if v is a vector with all nonnegative entries, then we have that Av is also a vector with nonnegative entries (as all the entries in A are nonnegative). This means that ϕ restricts to a map from Q to Q where

$$Q = \{(x_1, \ldots, x_n) \in S^{n-1} : x_i \geq 0\}.$$

Since Q is homeomorpic to B^{n-1} and ϕ is continuous, by Ex 6, ϕ has a fixed point. This means that for some $v \in Q$, $v = \frac{Av}{|Av|}$. Thus, $Av = |Av|v$, proving that $|Av|$ is a nonnegative eigenvalue of A. □

Chapter 2, Section 3

Ex 1 Show that any neighborhood $\tilde{U}$ of Y in $\mathbb{R}^M$ contains some Y^ε. Moreover, if Y is compact, ε may be taken to be constant.

Proof. Since $\tilde{U}$ contains some open set around Y, for each $y \in Y$, there exists an $r_y > 0$ such that $B_{r_y}(y) \subseteq \tilde{U}$. Since Y is a manifold, there exists some partition of unity subordinate to $\{B_{r_y}(y)\}_{y \in Y}$. That is, there is a locally finite refinement, call it $\{U_i\}_{i \in I}$, such that each U_i is contained in some $B_{r_y}(y)$; we let ε_i be this r_y. Additionally, there are functions $\theta_i : \mathbb{R}^M \to \mathbb{R}$ such that $\sum_{i \in I} \theta_i = 1$ and for a fixed $y \in Y$, $\theta_i(y)$ is zero for all but finitely many i. If we take $\varepsilon = \sum \theta_i \varepsilon_i$, then ε is well-defined all but finitely many θ_i are zero for any given point, and ε is smooth as its the sum of smooth functions multiplied by a constant.

Let $w \in Y^\varepsilon$ and let y be the closest point of to w in Y. This means that $|w - y| < \varepsilon(y) = \sum \varepsilon_i \theta_i(y)$. Since this sum is finite, there's some ε' which is the maximum of the ε_i. This means that $|w - y| \leq \varepsilon' \sum \theta_i(y) = \varepsilon'$. This ε' correspond to some U_j, which is contained in one of the open balls $B_r(y)$, which was contained in $\tilde{U}$. Thus, $w \in \tilde{U}$, which proves that $Y^\varepsilon \subseteq \tilde{U}$.

If Y were compact, then we can assume that our locally finite refinement $\{U_i\}_{i \in I}$ is actually a finite set. In this case $\varepsilon = \sum_{i=1}^n \varepsilon_i \theta_i$. If we just take $\varepsilon' = \max(\varepsilon_1, \ldots, \varepsilon_n)$, then we see that our previous argument still works if we replace every ε_i with ε'. So we have that $Y^\varepsilon \subseteq \tilde{U}$ where $\varepsilon = \sum \varepsilon' \theta_i = \varepsilon' \sum \theta_i = \varepsilon'$, which is a constant. □

Ex 7 Suppose that X is a submanifold of $\mathbb{R}^N$. Show that "almost every" vector space V of any fixed dimensiona ℓ in $\mathbb{R}^N$ intersects X transversally.

Proof. Let $S \subseteq (\mathbb{R}^N)^\ell$ be the set consisting of all linearly independent ℓ-tuples of vectors in $\mathbb{R}^N$. This set is open as perturbing any vector slightly will still give a linearly independent set of vectors. Let $\phi : \mathbb{R}^\ell \times S \to \mathbb{R}^N$ be defined as

$$\phi(t_1, \ldots, t_\ell, v_1, \ldots, v_\ell) = \sum_{i=1}^{\ell} t_i v_i.$$

This means that

$$d\phi_{(t,v)} = \left[\ v_1 v_2 \cdots v_\ell \quad | \quad (t_1 \operatorname{Id}_\ell) \quad | \quad (t_2 \operatorname{Id}_\ell) \quad | \quad \cdots \quad | \quad (t_\ell \operatorname{Id}_\ell)\ \right].$$

This means that $d\phi$ has full rank. Therefore, ϕ is transverse to X. Thus, by the Transversality Theorem, for almost every choice of v, the map $\phi_v : \mathbb{R}^\ell \times S \to \mathbb{R}^N$ is transverse to X and its image is an ℓ-dimensional vector space. This means that almost every ℓ-dimensional vector space is transverse to X. $\square$