Problem Set 8
Graph Theory

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October 21, 2017

Ex 3.1.5 Prove that $\alpha(G) \geq \frac{n(G)}{\Delta(G)+1}$ for every graph G.

Proof. We will define a sequence of sets $V_i \subseteq V(G)$. First, we let $V_0 = \varnothing$ and then iteratively define V_{i+1} as V_i with an added vertex from G. Each time we add a vertex, though, we then delete that vertex and all of its neighbors from G. At each step of this process, we will have added 1 vertex and will have deleted at most $\Delta(G)+1$ vertices. Since we're only deleting at most $\Delta(G)+1$ vertices at a time, we can perform at least $\frac{n(G)}{\Delta(G)+1}$ steps. Thus, after this process stops, we will have obtained an independent set with at least $\frac{n(G)}{\Delta(G)+1}$ vertices in it. □

Ex 3.1.6 Let T be a tree with n vertices, and let k be the maximum size of an independent set in T. Determine $\alpha'(T)$ in terms of n and k.

Proof. Firstly, since trees are bipartite, we see by Theorem 3.1.16 that $\alpha'(T) = \beta(T)$. We also know by Lemma 3.1.21 that $\alpha(T) + \beta(T) = n$. Thus,

$$\alpha'(T) = \beta(T) = n - \alpha(T) = n - k.$$

□

Ex 3.1.9 Prove that every maximal matching in a graph G has at least $\alpha'(G)/2$ edges.

Proof. Let M be a maximal matching and let V be a set of vertices that are saturated by M. Suppose there were an edge with no endpoint in V. Then, by the definition of V, that edge could've been added to M to make a better matching. This means that every edge has an endpoint in V. Thus, V is actually a vertex cover of G. Since $\alpha'(G)$ is a lower bound of the size of any vertex cover, we have that $2|M| = |V| \geq \beta(G) \geq \alpha'(G)$. Thus, every maximal matching has at least $\alpha'(G)/2$ edges. □

Ex 3.1.22 Prove that a bipartite graph G has a perfect matching if and only if $|N(S)| \geq |S|$ for all $S \subseteq V(G)$, and present an infinite class of examples to prove that this characterization does not hold for all graphs.

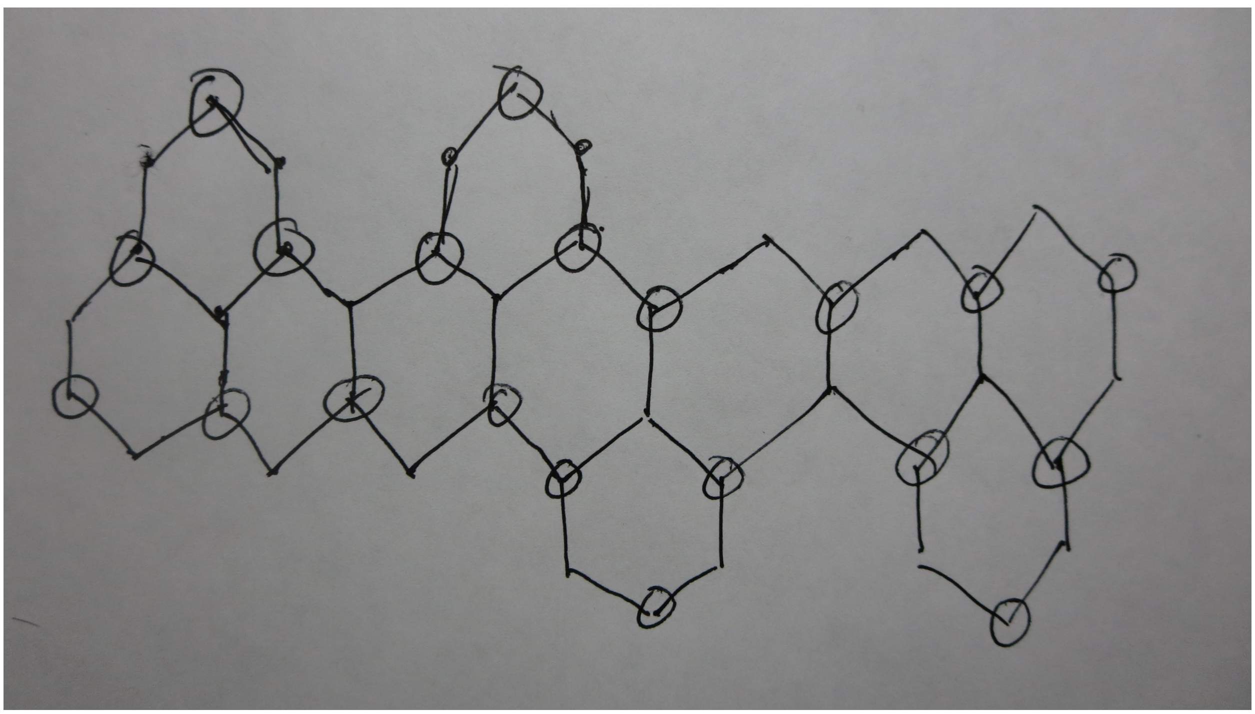
Proof. Let M be a perfect matching and let V be a set of vertices of G. We can map the vertices of V to their match to give a an injective map from V to $N(V)$. Thus, $|N(V)| \geq |V|$. Since V was arbitrary, we have that $|N(S)| \geq |S|$ for all $S \subseteq V(G)$.

Let X and Y be a bipartition of G. Suppose $|N(S)| \geq |S|$ for all $S \subseteq V(G)$. This means that, in particular, $|N(S)| \geq |S|$ for all $S \subseteq X$. By Hall's Theorem, the our graph has a matching that saturates X. This means that $|Y| \geq |X|$. Also, by our initial condition, we have that $|N(Y)| \geq |Y|$, which means that $|X| \geq |Y|$. This proves that $|X| = |Y|$, and thus the matching is actually perfect.

For an infinite class of counterexamples, consider the class of odd cycles: C_{2n+1} for $n \in \mathbb{N}$. Since $|C_{2n+1}|$ is odd, it cannot have a perfect match. However, if you take an set of vertices $V \subseteq V(C_{2n+1})$, then for each vertex in V, we can map it to a distinct neighbor by taking the next vertex in the cycle clockwise. This means that $N(S) \geq |S|$ for all $S \subseteq V(C_{2n+1})$. Thus, odd cycles are an infinite class of counterexamples to the claim. □

Ex 3.1.28 Exhibit a perfect matching in the graph below or give a short proof that it has none. [Graph drawn in the problem.]

Proof. We see that the graph below has 42 vertices, so a perfect match would require at least 21 edges. We also see that the vertices marked on the graph form a vertex cover of it. Since $\beta(G) \leq \alpha'(G)$, this means that the largest a match can be is 20. Thus, this graph has no perfect matching.



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