Problem Set 1
Homological Algebra

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Ex 1 Let $f : A \to B$ and $g : B \to C$ be morphisms in some category. Show that:

a) If $g \circ f$ is a monomorphism, then f is also a monomorphism.

b) If $g \circ f$ is an epimorphism, then g is also an epimorphism.

Proof.

a) To prove that f is a monomorphism, we must show that $f \circ h_1 = f \circ h_2 \implies h_1 = h_2$ for any morphisms $h_1, h_2 : D \to A$. We see though, if we start with $f \circ h_1 = f \circ h_2$, we can then compose on the left by g to get $g \circ f \circ h_1 = g \circ f \circ h_2$. Using the fact that $g \circ f$ is a monomorphism we deduce that $h_1 = h_2$, proving that f is a monomorphism.

b) To prove that g is a epimorphism, we must show that $h_1 \circ g = h_2 \circ g \implies h_1 = h_2$ for any morphisms $h_1, h_2 : C \to D$. We see though, if we start with $h_1 \circ g = h_2 \circ g$, we can then compose on the right by f to get $h_1 \circ g \circ f = h_2 \circ g \circ f$. Using the fact that $g \circ f$ is an epimorphism we deduce that $h_1 = h_2$, proving that g is an epimorphism. $\square$

Ex 2 In the category of (left) R-modules, where R is a ring, show that a morphism $f : A \to B$ is

a) a monomorphism if and only if f is injective;

b) an epimorphism if and only if f is surjective.

Proof.

a) monomorphism $\implies$ injective) If $f : A \to B$ is monomorphism, then consider the inclusion map $i : \ker(f) \to A$ and the zero map $0 : \ker(f) \to A$. We see that $f \circ i = 0 = f \circ 0$. As f is a monomorphism, this means that $i = 0$ and so $\ker(f) = \mathrm{Im}(i) = \mathrm{Im}(0) = \{0\}$. This proves that f is injective

injective $\implies$ monomorphism) Let $g_1, g_2 : X \to A$ be homomorphisms such that $f \circ g_1 = f \circ g_2$. This means for any $x \in X$, $f(g_1(x)) = f(g_2(x))$. By the definition of injectivity, it must be that $g_1(x) = g_2(x)$, proving that f is a monomorphism.

b) epimorphism $\implies$ surjective) If $f : A \to B$ is an epimorphism, then consider the projection map $\pi : B \to B/\mathrm{Im}(f)$ and the zero map $0 : B \to B/\mathrm{Im}(f)$. As $\pi \circ f = 0 \circ f$ and f is an epimorphism, we can conclude that $\pi = 0$ and so $\mathrm{Im}(f) = \ker(\pi) = \ker(0) = B$. This proves that f is surjective.

surjective $\implies$ epimorphism) Let $g_1, g_2 : B \to X$ be homomorphisms such that $g_1 \circ f = g_2 \circ f$. Now if $b \in B$, then that means $b = f(a)$ for some $a \in A$. We can use this to see that

$$g_1(b) = g_1(f(a)) = (g_1 \circ f)(a) = (g_2 \circ f)(a) = g_2(f(a)) = g_2(b).$$

Since this is true for any $b \in B$, we have that $g_1 = g_2$ and so f is an epimorphism. □

Ex 3 Show in detail that the τ defined in Example 1.2.7 indeed yields a natural transformation between the identity functor on the category of all K-vector spaces and the functor D^2.

Proof. Recall that for each vector space V, $\tau_V : V \to V^{**}$ is the function $\tau_V(v) = \text{eval}_v$. To prove that τ is a natural transformation, we need only to verify that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\tau_V"] \arrow[d, "\phi"] & V^{**} \arrow[d, "\phi^{**}"] \\ W \arrow[r, "\tau_W"] & W^{**} \end{tikzcd}$$

To verify this, we need to prove that $\phi^{**} \circ \tau_V = \tau_W \circ \phi$. So, if let $v \in V$, we see that

$$(\phi^{**} \circ \tau_V)(v) = \phi^{**}(\tau_V(v)) = \phi^{**}(\text{eval}_v) = \text{eval}_v \circ \phi^*$$

and that

$$(\tau_W \circ \phi)(v) = \tau_W(\phi(v)) = \text{eval}_{\phi(v)}.$$

Now to prove that these are the same elements of W^{**}, we note that for any $f \in W^*$, we have that

$$(\text{eval}_v \circ \phi^*)(f) = \text{eval}_v(f \circ \phi) = (f \circ \phi)(v) = f(\phi(v)) = \text{eval}_{\phi(v)}(f).$$

Thus, the diagram above commutes and τ is a natural transformation. □

Ex 4 Prove that the direct limit S (and the associated homomorphisms) as constructed in 1.3.5 satisfies the universal property of colimits.

Not attempted. □

Ex 5 Assume that for a poset I and a family of groups $\{G_i : i \in I\}$ with connecting homomorphisms f_{ij} as in 1.3.6 the direct limit G, together with associated homomorphisms $s_i : G_i \to G$, exists. Prove, *without* using the construction of G (only its universal property) that $G = \langle s_i(G_i) : i \in I \rangle$.

Proof. Let $s_i|^{\langle s_i(G_i) \rangle}$ the map s_i, but where we now restrict the target to $\langle s_i(G_i) \rangle$. We can then use the universal property of G to find a unique homomorphism $\phi : G \to \langle s_i(G_i) \rangle$ where $\phi \circ s_i = s_i|^{\langle s_i(G_i) \rangle}$.

$$\begin{tikzcd} G_i \arrow[dd, "f_{ij}"'] \arrow[dr, "s_i"] \arrow[drr, bend left, "s_i|^{\langle s_i(G_i) \rangle}"] & & \\ & G \arrow[r, dotted, "\phi"] & \langle s_i(G_i) \rangle \\ G_j \arrow[ur, "s_j"'] \arrow[urr, bend right, "s_j|^{\langle s_i(G_i) \rangle}"'] & & \end{tikzcd}$$

If $i : \langle s_i(G_i)\rangle \to G$ is the inclusion map, then we see that $i \circ \phi \circ s_i = i \circ s_i|^{\langle s_i(G_i)\rangle} = s_i$ and that $\phi \circ i \circ s_i|^{\langle s_i(G_i)\rangle} = \phi \circ s_i = s_i|^{\langle s_i(G_i)\rangle}$. This proves that i commutes with the rest of the maps in the above diagram, meaning $\phi \circ i$ and $i \circ \phi$ must be the identity maps. Thus, the inclusion map i is actually an isomorphism. This proves that $G = \langle s_i(G_i) : i \in I\rangle$. $\square$

Ex 6 With the notations of the "special case" in 1.3.6, set $G_1 = \langle a\rangle$ with $|a| = 9$, $G_2 = \langle b\rangle$ with $|b| = 3$, and $G_3 = A_9$. Let the homomorphisms f_{12} and f_{13} be defined by $f_{12}(a) = b$ and $f_{13}(a) = (123456789)$. Prove that the direct limit ("amalgam") of this set of groups and homomorphisms is trivial, again without using the construction of direct limits or amalgams.

Proof. Writing this as a diagram, we get that

$$\begin{tikzcd} A_9 \arrow[dr, "f_3"] & \\ \mathbb{Z}_9 \arrow[u, "f_{13}"] \arrow[r, "f_1"] \arrow[d, "f_{12}"'] & L \\ \mathbb{Z}_3 \arrow[ur, "f_2"'] & \end{tikzcd}$$

Since we know $f_2 \circ f_{12} = f_1$, we see that

$$f_1(a)^3 = f_1(a^3) = f_2(f_{12}(a^3)) = f_2(b^3) = f_2(1) = 1.$$

Furthermore, since $f_3 \circ f_{13} = f_1$, we also have that

$$1 = f_1(a)^3 = f_1(a^3) = f_3(f_{13}(a^3)) = f_3((147)(258)(369)).$$

This means that $(147)(259)(369) \in \ker(f_3)$. Since A_9 is simple, this means that f_3 must be the zero map. As $f_3 \circ f_{13} = f_1$, f_1 must also be the zero map and by similar reasoning f_2 is the zero map as well. Using Ex 5, we know that the direct limit equals $\langle f_i(G_i)\rangle = \langle 0\rangle = 0$. Thus, the direct limit and it's associated homomorphisms are all trivial. $\square$

Exercises from Atiyah-MacDonald:

Ex 7 (A&M Ch 2 Ex 15 pg 33) In the sitation of Ex 14, show that every element of M can be written in the form $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$.

Proof. Let $x \in M$. Since the projection $\mu : \oplus_i M_i \to M$ is surjective, there is some finite sum $\sum_{i\in S} x_i$ where $S \subseteq I$ and $\mu(\sum_{i\in S} x_i) = x$. Since S is finite and I is directed, there is some $k \in I$ such that $k \geq i$ for all $i \in S$ (just apply the directed system hypothesis to each element of S inductively). Since $M = \oplus_i M_i/D$ where D is the submodule generated by the elements $\{\mu_{ij}(x_i) - x_i : i \leq j\}$, we see that if we let $x_j = \sum_{i\in S} \mu_{ij}(x_i)$, then $x_j = \sum_{i\in S} x_i$ in M and

$$\mu_j(x_j) = \mu(x_j) = \mu\left(\sum_{i\in S} \mu_{ij}(x_i)\right) = \mu\left(\sum_{i\in S} x_i\right) = x.$$

Thus, any $x \in M$ can be written in the form $\mu_j(x_j)$ for some $j \in I$ and some $x_j \in M_j$. $\square$

Ex 8 (A&M Ch 2 Ex 18 pg 33) Let $\mathbf{M} = (M_i, \mu_{ij})$, $\mathbf{N} = (N_i, \nu_{ij})$ be direct systems of A-modules over the same directed set. Let M, N be the direct limits and $\mu_i : M_i \to M$, $\nu_i : N_i \to N$ the associated homomorphisms.

A *homomorphism* (read: natural transformation) $\Phi : \mathbf{M} \to \mathbf{N}$ is by definition a family of A-module homomorphisms $\phi_i : M_i \to N_i$ such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ whenever $i \leq j$. Show that Φ defines a unique homomorphism $\phi = \lim_{\to} \phi_i : M \to N$ such that $\phi \circ \mu_i = \nu_i \circ \phi_i$ for all $i \in I$.

Proof. We let $\alpha_i : M_i \to N$ be the map where $\alpha_i = \nu_i \circ \phi_i$. Now if $i \leq j$, we see that

$$\alpha_j \circ \mu_{ij} = \nu_j \circ \phi_j \circ \mu_{ij} = \nu_j \circ \nu_{ij} \circ \phi_i = \nu_i \circ \phi_i = \alpha_i.$$

This means we may use the universal property of direct limits to obtain a unique homomorphism $\phi : M \to N$ such that the following diagram is commutative:

$$\begin{tikzcd}
M_i \arrow[dr, "\mu_i"] \arrow[dd, "\mu_{ij}"'] \arrow[drr, bend left, "\alpha_i=\nu_i\circ\phi_i"] & & \\
& M \arrow[r, dotted, "\phi"] & N \\
M_j \arrow[ur, "\mu_j"'] \arrow[urr, bend right, "\alpha_j=\nu_i\circ\phi_i"'] & &
\end{tikzcd}$$

That is $\phi \circ \mu_i = \alpha_i = v_i \circ \phi_i$ for all $i \in I$, as we wanted. $\square$

Ex 9 (A&M Ch 2 Ex 17 pg 33) Let $(M_i)_{i \in I}$ be a family of submodules of an A-module, such that for each pair of indices i, j in I there exists $k \in I$ such that $M_i + M_j \subseteq M_k$. Define $i \leq j$ to mean $M_i \subseteq M_j$ and let $\mu_{ij} : M_i \to M_j$ be the embedding of M_i in M_j. Show that

$$\lim_{\to} M_i = \sum_i M_i = \bigcup_i M_i.$$

In particular, any A-module is the direct limit of its finitely-generated sub-modules.

Proof. We note that $M_j \in \sum_i M_i$ for every j and so $\cup_j M_j \subseteq \sum_i M_i$. Conversely, let $y = \sum_{i \in S} x_i \in \sum_i M_i$ where S is a finite subset of I. As this is a direct system, there is some $j \in I$ such that $j \geq i$ for all $i \in S$ (just apply the directed system hypothesis to each element of S inductively). This means $y = \sum_{i \in S} x_i \in \sum_{i \in S} M_i \subseteq M_j \subseteq \cup_j M_j$. Thus, $\sum_i M_i = \cup_i M_i$.

Now to prove the statement, we'll show that $\cup_i M_i$ satisfies the universal property of the direct limit. Let $\mu_i : M_i \to \cup_i M_i$ be the inclusion map. Since each μ_{ij} is also an inclusion map, we see that for all $i \leq j$ the following diagram commutes:

$$\begin{tikzcd}
M_i \arrow[dr, "\mu_i"] \arrow[dd, "\mu_{ij}"'] & \\
& \cup_i M_i \\
M_j \arrow[ur, "\mu_j"'] &
\end{tikzcd}$$

Now suppose that L where an A-module with homomorphisms $\alpha_i : M_i \to L$ such that $\alpha_i = \alpha_j \circ \mu_{ij}$ for $i \leq j$. Since μ_{ij} is the inclusion map, this says that $\alpha_i = \alpha_j|_{M_i}$. This means that the α_i's are consistent on their intersections and so there is a unique function $\alpha : \cup_i M_i \to L$. We note that α is an A-module homomorphism becuase it's restriction to each M_i is. This proves the second citerion for the direct limit, meaning $\lim_{\to} M_i = \cup_i M_i$ as we wanted.

We note that for an A-module M, we have that for each $x \in M$, Ax is a finitely-generated submodule, and so $M = \cup_{x \in M} Ax$. We also note that if $M_1, M_2 \subseteq M$ are finitely-generated, then they're both contained in $M_1 + M_2 \subseteq M$ which is also finitely-generated. Thus, using our previous work, every module is the direct limit of its finitely-generated sub-modules. $\square$