Problem Set 3
Complex Analysis I

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Ex 2.37 Calculate
$$\frac{1}{2\pi i}\oint_\gamma \frac{s^2+s}{(s-2i)(s+3)}\,ds,$$
where γ is the circle with center 1, radius 5, and counterclockwise orientation.

Proof. If we let γ_1 to be the counterclockwise circle defined by $|z-2i|=1$ and γ_2 to be the counterclockwise circle defined by $|z+3|=1$. By Cauchy Integral Formula that
$$\begin{aligned}\frac{1}{2\pi i}\oint_\gamma \frac{s^2+s}{(s-2i)(s+3)} &= \frac{1}{2\pi i}\oint_{\gamma_1} \frac{s^2+s}{(s-2i)(s+3)}\,ds + \frac{1}{2\pi i}\oint_{\gamma_2} \frac{s^2+s}{(s-2i)(s+3)}\,ds \\ &= \frac{1}{2\pi i}\oint_{\gamma_1} \frac{\frac{s^2+s}{s+3}}{s-2i}\,ds + \frac{1}{2\pi i}\oint_{\gamma_2} \frac{\frac{s^2+s}{s-2i}}{s+3}\,ds \\ &= \frac{(2i)^2+2i}{2i+3} + \frac{(-3)^2-3}{-3-2i} = -2+2i.\end{aligned}$$

□

Ex 2.40 Let γ_1 be the curve $\partial D(0,1)$ and let γ_2 be the curve $\partial D(0,3)$, both equipped with counterclockwise orientation. Note that the two curves taken together from the boundary of an annulus. Compute

a)
$$\frac{1}{2\pi i}\oint_{\gamma_2} \frac{s^2+5s}{s-2}\,ds - \frac{1}{2\pi i}\oint_{\gamma_1} \frac{s^2+5s}{s-2}\,ds$$

b)
$$\frac{1}{2\pi i}\oint_{\gamma_2} \frac{s^2-2}{s}\,ds - \frac{1}{2\pi i}\oint_{\gamma_1} \frac{s^2-2}{s}\,ds$$

c)
$$\frac{1}{2\pi i}\oint_{\gamma_2} \frac{s^3-3s-6}{s(s+2)(s+4)}\,ds - \frac{1}{2\pi i}\oint_{\gamma_1} \frac{s^3-3s-6}{s(s+2)(s+4)}\,ds$$

We think of these differences of integrals as the (oriented) complex line integral around the boundary of the annulus. Can you explain the answers you're getting in terms of the points at which the functions being integrated are not defined?

Proof. Using Cauchy's Integral Formula and Cauchy's Integral Theorem repeatedly, we get that

a)
$$\frac{1}{2\pi i}\oint_{\gamma_2}\frac{s^2+5s}{s-2}\,ds - \frac{1}{2\pi i}\oint_{\gamma_1}\frac{s^2+5s}{s-2}\,ds = f(2) - 0 = 2^2 + 5\cdot 2 - 0 = 14$$
where $f(z) = z^2 + 5z$.

b)
$$\frac{1}{2\pi i}\oint_{\gamma_2}\frac{s^2-2}{s}\,ds - \frac{1}{2\pi i}\oint_{\gamma_1}\frac{s^2-2}{s}\,ds = f(0) - f(0) = (0-2) - (0-2) = 0$$
where $f(z) = z^2 - 2$.

c)
$$\frac{1}{2\pi i}\oint_{\gamma_2}\frac{s^3-3s-6}{s(s+2)(s+4)}\,ds - \frac{1}{2\pi i}\oint_{\gamma_1}\frac{s^3-3s-6}{s(s+2)(s+4)}\,ds =$$
$$\frac{1}{2\pi i}\oint_{\gamma_3}\frac{s^3-3s-6}{s(s+2)(s+4)}\,ds + \frac{1}{2\pi i}\oint_{\gamma_1}\frac{s^3-3s-6}{s(s+2)(s+4)}\,ds - \frac{1}{2\pi i}\oint_{\gamma_1}\frac{s^3-3s-6}{s(s+2)(s+4)}\,ds =$$
$$f(-2) = 5 \qquad \blacksquare$$

where $f(z) = \frac{z^3-3z-6}{z(z+4)}$ and γ_3 is the counterclockwise curve along $|z-2| = 1$.

This integration along the boundary of an annulus is equivalent to doing Cauchy's Integral Formula, but only at poles that lie inside the annulus (poles inside the inner circle and outside the outer circle don't matter). $\square$

Ex 3.21 Prove that the function

$$f(z) = \sum_{j=0}^{\infty} 2^{-j} z^{2^j}$$

is holomorphic on $D(0,1)$ and continuous on $\overline{D}(0,1)$. Prove that if w is a $(2^N)^{\text{th}}$ root of unity, then $\lim_{r\to 1^-}|f'(rw)| = +\infty$. Deduce that f cannot be the restriction to $D(0,1)$ of a holomorphic function defined on a conected open set that is strictly larger than $D(0,1)$.

Proof. We see that the disc of convergence of the series is $\overline{D}(0,1) = \{z, |z| < 1\}$. On this disc, we have that

$$|\sum_{j=n+1}^{m} 2^{-j} z^{2^j}| \leq \sum_{j=n+1}^{m} 2^{-j} \leq \frac{1}{2^n}$$

which proves that the series converges uniformly on the disc. This proves that $f(z)$ is holomorphic on $D(0,1)$ and continuous on $\overline{D}(0,1)$. We also see that

$$f'(z) = \sum_{j=0}^{\infty} z^{2^j - 1}$$

which means that if w is a 2^Nth root of unity, then

$$\lim_{r \to 1^-} |f'(rw)| = +\infty. \qquad \text{[No proof of this]}$$

This means that $f'(z)$ is unbounded at the point w, which means that $f'(z)$ is not holomorphic at such points w. Since the set $\{x : x^{2n} = 1 \text{for some} n \in N$ is dense on the circle $|z| = 1$, $f(z)$ cannot be extended to a holomorphic function on a larger connected open set. $\square$

Ex 3.32 Suppose that f is bounded and holomorphic on $\mathbb{C} \setminus \{0\}$. Prove that f is constant. [Hint: Consider the function $g(z) = z^2 \cdot f(z)$ and endeavor to apply Theorem 3.4.4.]

Proof. Consider the function

$$g(z) = \begin{cases} z^2 f(z) & x \neq 0 \\ 0 & z = 0 \end{cases}$$

Since f is holomorphic on $\mathbb{C} \setminus \{0\}$, so is g. Since f is bounded on $\mathbb{C} \setminus \{0\}$, we see that

$$g'(0) = \lim_{z \to 0} \frac{g(z) - g(0)}{z - 0} = \lim_{z \to 0} \frac{z^2 f(z)}{z} = \lim_{z \to 0} z f(z) = 0.$$

Thus, we have that g is holomorphic on $\mathbb{C}$. Since $f(z)$ is bounded, there exists a M such that $|f| \leq M$ on $\mathbb{C} \setminus \{0\}$. This means that

$$|g(z)| \leq |z^2 f(z)| = |z^2| M$$

which implies that $g(z) = z^2 f(z) = a_0 + a_1 z + a_2 z^2$ on $\mathbb{C} \setminus \{0\}$ by Theorem 3.4.4. If we let $z \to 0$, since $f(z)$ is bounded, we get that $a_0 = 0$. Similarly, we see that $z f(z) = a_1 + a_2 z^2$, which means that if we let $z \to 0$ again, we have that $a_1 = 0$. Thus, $z^2 f(z) = a_2 z^2$, which means that $f(z) = a_2$. That is, f is constant. $\square$