Graph Theory I
Problem Set 11

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Ex 4.1.1 Give a proof or counterexample for each statement below

a) Every graph with connectivity 4 is 2-connected.

b) Every 3-connected graph has connectivity 3.

c) Every k-connected graph is k-edge-connected.

d) Every k-edge-connected graph is k-connected.

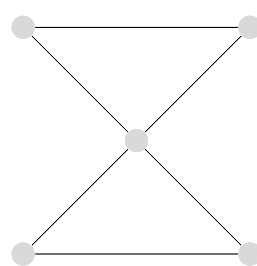
Proof.

a) True. A graph is 2-connected if its connectivity is at least 2. Since $4 > 2$, we see this is true.

b) False. The graph $K_{4,4}$ is 3-connected, but has connectivity greater than 3.

c) True. Since $\kappa'(G) \geq \kappa(G)$, every disconnecting set of edges in a k-connected graph has at least k edges.

d) False. Consider the graph below. It's fairly easy to see that $\kappa'(G) = 2$ and that $\kappa(G) = 1$. This means that G is 2-edge-connected is not 2-connected.



□

Ex 4.1.4 Prove that a graph G is k-connected if and only if $G \wedge K_r$ is $(k + r)$-connected.

Proof. In order to separate $G \wedge K_r$ it is necessary to delete all of the vertices in K_r as they are adjacent to all vertices. This leaves us with with only vertices in G. Thus, a set of vertices is separates $G \wedge K_r$ if and only if it contains the r vertices of K_r and a separating set of vertices of G. If we apply this to minimal separating sets, we see that G is k-connected if and only if $G \wedge K_r$ is $(k+r)$-connected. □

Ex 4.1.5 Let G be a connected graph with at least three vertices. Form G' from G by adding an edge with endpoints x, y whenever $d_G(x, y) = 2$. Prove that G' is 2-connected.

Proof. Firstly, since G' is simply G with more edges, it must be that G' is connected as well. Suppose that G' has a cut-vertex v. This means that v is also a cut vertex of G, as $G - v$ has even fewer edges than $G' - v$. However, we see that the neighbors of v in G are adjacent in G', which means that they cannot be in different components. Thus, $G' - v$ is actually connected. This is a contradiction and proves that G' has no cut-vertex. Since it takes at least 2 vertices to disconnect G', we see that G' is 2-connected. □

Ex 4.1.11 Prove that $\kappa'(G) = \kappa(G)$ when G is a simple graph with $\Delta(G) \leq 3$.

Proof. Let S be a minimum vertex cut set. Since it is always the case that $\kappa(G) \leq \kappa'(G)$, we need only to construct a edge cut of size $|S|$. Let H_1, H_2 be two components of $G - S$. As S is a minimum vertex cut set, we see that for each $v \in S$, it must be that v has a neighbor in H_1 and a neighbor in H_2. Additionally, since $\Delta(G) \leq 3$, it is impossible for v to have two neighbors in both H_1 and H_2. So, for each $v \in S$, we find the vertex which is the sole neighbor in H_i and delete the edge connecting them. If it happens to be that v has a sole neighbor in each H_i and has a third edge connecting to $v' \in S$ of the same form, then we simply delete the two edges going to H_1. This will ensure that G is disconnected. Since we delete one edge for each $v \in S$, we have an edge cut of size $|S|$ as desired. □

Ex 4.1.15 Use Proposition 4.1.12 and Theorem 4.1.11 to prove that the Petersen graph is 3-connected.

Proof. Since we know that the Petersen graph P is 3-regular, by Theorem 4.1.11, we need only to prove that P is 3-edge-connected. We let $[S, \overline{S}]$ be a minimum edge cut. Since $|S| + |S'| = n(P) = 10$, we may assume without loss of generality that $|S| \leq 5 \leq |\overline{S}|$. If we have that $|[S, \overline{S}]| < 3$, then by Proposition 4.1.12 we see that

$$\sum_{v \in S} d(v) - 2e(G[S]) = 3|S| - 2e(G[S]) \leq 2.$$

Since P has a girth of 5, if $|S| < 5$, then we have that $e(G[S]) \leq |S| - 1$. This means that $3|S| - 2(|S| - 1) \leq 2$, which gives that $|S| \leq 0$. This is obviously impossible as S can't be empty. For $|S| = 5$, we have that $3|S| - 2|S| \leq 2$, which simplifies to $|S| \leq 2$. This is again obviously false. Thus, there is no edge cut of size less than 3, which proves that the Petersen graph is 3-connected. □