Problem Set 5
Graph Theory

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Ex 2.1.4 Prove or disprove: Every graph with fewer edges than vertices has a component that is a tree.

Proof. Let G be a simple graph and let $G_1, G_2, \ldots, G_n$ be the components of G. Since each G_i is connected (as it's a component), it must have at least $|V(G_i)| - 1$ edges. Suppose that G_i is not a tree for all $1 \leq i \leq n$. Then $|E(G_i)| \neq |V(G_i)| - 1$ as well, which proves that $|E(G_i)| \geq |V(G_i)|$. That means that

$$|E(G)| = \sum_{i=1}^{n} |E(G_i)| \geq \sum_{i=1}^{n} |V(G_i)| = |V(G)|$$

Thus, by the contrapositive, if $|E(G)| < V(G)$, then G has a component that is a tree. □

Ex 2.1.7 Prove that every n-vertex graph with m edges has at least $m - n + 1$ cycles.

Proof. Let ℓ be the number of components in such a graph G. If we choose a spanning tree for each of these compontents, then it will have $n - \ell$ edges (as each component has one fewer edge than vertices). By Corollary 2.1.5, we know that if we add any edge back into our spanning tree, it will contain exactly one cycle. Since each cycle obtained this way uses the edge that you add back into the graph, adding different edges back in will result in obtaining a different cycle. Since there are $m-n+\ell$ edges left in the graph, there are at least $m-n+\ell$ cycles. Since $\ell \geq 1$, this proves that G has at least $m - n + 1$ cycles. □

Ex 2.1.11 Let x and y be adjacent vertices in a graph G. For all $z \in V(G)$, prove $|d_G(x,z) - d_G(y,z)| \leq 1$.

Proof. Let P be the shortest path from x to z and Q be the shortest path from y to z. By the definition of d_G, $|E(P)| = d_G(x,z)$ and $|E(Q)| = d_G(y,z)$. We see that yP is a path from y to z and xQ is a path from x to z. Since P and Q are the shortest paths with their respective endpoints, we have that

$$d_G(x,z) + 1 = |yP| \geq |Q| = d_G(y,z)$$

and

$$d_G(y,z) + 1 = |xQ| \geq |P| = d_G(x,z).$$

This proves that $d_G(y,z) - 1 \leq d_G(x,z) \leq d_G(y,z) + 1$, which means that $|d_G(x,z) - d_G(y,z)| \leq 1$. □

Ex 2.1.12 Compute the diameter and radius of the biclique $K_{m,n}$.

Proof. We see that if $m, n \geq 2$, every vertex of $K_{m,n}$ has an eccentricity of 2, which means that it has a radius and diameter of 2 as well. The remaining cases are:

G	$\mathrm{Rad}\, G$	$\mathrm{Diam}\, G$
$K_{1,n}$ $(n > 1)$	1	2
$K_{0,n}$ $(n > 1)$	∞	∞
$K_{1,1}$	1	1
$K_{0,1}$	0	0

□

Ex 2.1.18 Prove that every tree with maximum degree $\Delta > 1$ has at least Δ vertices of degree 1. Show that this is the best possible by constructing an n-vertex tree with exactly Δ leaves, for each choice of n, Δ with $n > \Delta \geq 2$.

Proof. Let $v \in V(G)$ a vertex of maximum degree Δ. Then, in the graph $G - v$, there are exactly Δ components (otherwise there'd be a cycle). Since we've only removed edges, there are still no cycles in G, which means that each component is a tree. If a component is trivial, then that means that it's only edge was with v, and thus was a vertex of degree 1. For a nontrivial compontent, there are at least two vertices. Since each component is a tree, each one has at least 2 leaves. We see that it cannot be that both leaves were originally edges with v, as that would create a cycle. Thus, each component contains at least one vertex that has degree 1 in G. This proves that there are at least Δ vertices of degree 1.

To construct such a tree, we start with the star $K_{1,\Delta}$, which has Δ leaves and $\Delta + 1$ vertices. Then, we choose one of the leaves and its edge and extend it to a path of length $n - \Delta$. We see that this doesn't add any new leaves and that G is still a tree. Since our tree still has Δ leaves and now has $n - \Delta + \Delta = n$ vertices, it satisfies the required properties. □