Problem Set 2
Complex Analysis

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Ex 1 Find the power series expansion about $z = 1$ of

$$\frac{z+2i}{(z-2)(z^2+1)}$$

and find the radius of convergence of this power series.

Proof. Using partial fraction decomposition (and the trick for easily finding the numerators), we get that

$$\frac{z+2i}{(z-2)(z^2+1)} = \frac{A}{z-2} + \frac{B}{z+i} + \frac{C}{z-i} = \frac{\frac{2+2i}{5}}{z-2} + \frac{\frac{1}{2i+4}}{z+i} + \frac{\frac{3}{2i-4}}{z-i}$$

We note that

$$\frac{A}{z-2} = \frac{-A}{1-(z-1)} = -A\sum_{n=0}^{\infty}(z-1)^n = \sum_{n=0}^{\infty} -A\cdot(z-1)^n.$$

Similarly, we get that

$$\frac{B}{z+i} = \frac{-B}{(-1-i)-(z-1)} = \frac{\frac{-B}{(-1-i)}}{1-\frac{z-1}{-1-i}} = \frac{-B}{-1-i}\sum_{n=0}^{\infty}\left(\frac{z-1}{-1-i}\right)^n = \sum_{n=0}^{\infty}\frac{-B}{(-1-i)^{n+1}}(z-1)^n$$

and finally that

$$\frac{C}{z-i} = \frac{-C}{(i-1)-(z-1)} = \frac{\frac{-C}{(i-1)}}{1-\frac{z-1}{i-1}} = \frac{-C}{i-1}\sum_{n=0}^{\infty}\left(\frac{z-1}{i-1}\right)^n = \sum_{n=0}^{\infty}\frac{-C}{(i-1)^{n+1}}(z-1)^n.$$

Thus, combining everything togther we get that

$$\frac{z+2i}{(z-2)(z^2+1)} = \sum_{n=0}^{\infty}\left(-A - \frac{B}{(-1-i)^{n+1}} - \frac{C}{(i-1)^{n+1}}\right)(z-1)^n$$

is the power series expansion about $z = 1$ where A, B, C are the constants as determined above. We see that the radius of convergence for the respective terms are $|z-1| < 1$, $|z-1| < |-1-i| = \sqrt{2}$ and $|z-1| < |i-1| = \sqrt{2}$. Thus, the radius of convergence of the expression is at least 1. We see that it's also at most 1, as there is a pole at $z = 2$. This proves that the radius of convergence is exactly 1. □

Ex 2 Let $f(z) = |z|^2$ and $g(z) = \overline{z}$. Find the points at which f, g are differentiable.

Proof. We note that the following limit does not exist

$$\lim_{|z| \to 0} \frac{\overline{z}}{z}.$$

This is because if we take z to be approaching 0 along the real axis, we get that the limit is 1, but if we take z to be approaching along the imaginary axis, the limit becomes -1. Since a limit cannot converge to two different numbers, this limit cannot exist. For $f(z)$, we see that

$$\frac{|z+h|^2 - z^2}{h} = \frac{(z+h)\overline{(z+h)} - z\overline{z}}{h} = \frac{z\overline{h} + \overline{z}h + h\overline{h}}{h} = \overline{z} + \overline{h} + \frac{z\overline{h}}{h}.$$

So if we take the limit $|h| \to 0$ of both sides, we have that

$$f'(z) = \overline{z} + z \lim_{|h| \to 0} \frac{\overline{h}}{h}.$$

Since the limit on the right-hand side does not exist, the only point where $f(z) = |z|^2$ is differentiable is at $z = 0$. Now for $g(z) = \overline{z}$, we see that

$$g'(z) = \lim_{|h| \to 0} \frac{\overline{z+h} - \overline{z}}{h} = \lim_{|h| \to 0} \frac{\overline{h}}{h}$$

which we proved does not exist anywhere. Thus, $g(z)$ is differentiable nowhere. □

Ex 3 An hv path is a continuous function $\gamma : [a, b] \to \mathbb{C}$ so that there are numbers $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ with the property that $\gamma|_{[t_{i-1}, t_i]}$ is a straight line path which is either vertical or horizontal. Let $U \subseteq \mathbb{C}$ be a domain. Prove that if $z, w \in U$ then there is an hv path $\gamma : [a, b] \to U$ so that $\gamma(a) = z, \gamma(b) = w$.

Proof. Let $z \in U$ and let P be the set of all points $u \in U$ such that there exists an hv path from z to u. We note that z is trivially in P. We will prove that P is both open and closed, which means $P = U$ by connectedness.

Let $u \in P$. Since U is open, there exists an $r > 0$ such that $B_r(u) \subseteq U$. As u in P and $B_r(u)$ is convex, we can take an hv path from a to u, from u to $u + \mathrm{Re}(w - u)$, and then from $u + \mathrm{Re}(w-u)$ to $u + \mathrm{Re}(w-u) + \mathrm{Im}(w-u)$, which is simply w. This proves that $B_r(u) \subseteq P$ and thus that P is open.

Let $u \notin P$. Again, since U is open, there exists an $r > 0$ such that $B_r(u) \subseteq U$. Now suppose there were a $w \in B_r(u)$ such that $w \in P$. By the same reasoning as the last paragraph, we could extend the hv path from a to w into a hv path from a to u. This is a contradiction as $u \notin P$, which means that there is no $w \in B_r(u)$ such that $w \in P$. In other words, $B_r(u) \subseteq P^c$. Thus P^c is open and P is closed.

As P is non-empty, open, closed, and lies inside the connected space U, it must be that $P = U$. This proves that there exists a hv from the point $z \in U$ to any point $w \in W$. Note that our initial point $z \in U$ was arbitary; meaning there is a hv path between any two points in U. $\square$

Ex 4

a) For an integer n and an $a \in \mathbb{C}$, find all solutions to $z^n = a$.

b) Compute $\sum_{j=0}^{n} \cos(j\theta), \sum_{j=0}^{n} \sin(j\theta)$.

Proof.

a) Let $a = |a|e^{i\theta}$. We see that

$$(\sqrt[n]{|a|}e^{i(\theta+2\pi k)/n})^n = |a|e^{i\theta+2\pi ik} = |a|e^{i\theta} = a,$$

so $\sqrt[n]{|a|}e^{i(\theta+2\pi k)/n}$ are solutions to the given equation. We note that these solutions are distinct for k an integer such that $0 \leq k < n$. We also note that there can only be n roots to the polynomial $z^n - a$ as each root can be taken out as a factor which reduces the degree of the polynomial. Thus, these solutions represent all possible solutions to the given equation.

b) We note that $\sum_{j=0}^{n} e^{ij\theta} = \sum_{j=0}^{n} \cos(j\theta) + i\sum_{j=0}^{n} \sin(j\theta)$. We see that

$$\begin{aligned}
\sum_{j=0}^{n} e^{ij\theta} = \sum_{j=0}^{n} (e^{i\theta})^j &= \frac{1-(e^{i\theta})^{n+1}}{1-e^{i\theta}} = \frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} \cdot \frac{1-e^{-i\theta}}{1-e^{-i\theta}} \\
&= \frac{1-e^{-i\theta}-e^{i(n+1)\theta}+e^{i(n+1)\theta-i\theta}}{1-e^{i\theta}-e^{-i\theta}+e^{i\theta-i\theta}} = \frac{1-e^{-i\theta}-e^{i(n+1)\theta}+e^{in\theta}}{1-2\cos(\theta)+1} \\
&= \frac{1-(\cos(\theta)-i\sin(\theta))-(\cos((n+1)\theta)+i\sin((n+1)\theta))+(\cos(n\theta)+i\sin(n\theta))}{2-2\cos(\theta)} \\
&= \frac{1-\cos(\theta)-\cos((n+1)\theta)+\cos(n\theta)}{2-2\cos\theta} + i\frac{\sin(\theta)-\sin((n+1)\theta)+\sin(n\theta)}{2-2\cos(\theta)}.
\end{aligned}$$

This gives us the following identities

$$\begin{aligned}
\sum_{j=0}^{n} \cos(j\theta) &= \frac{1-\cos(\theta)-\cos((n+1)\theta)+\cos(n\theta)}{2-2\cos(\theta)} \\
\sum_{j=0}^{n} \sin(j\theta) &= \frac{\sin(\theta)-\sin((n+1)\theta)+\sin(n\theta)}{2-2\cos(\theta)}.
\end{aligned}$$

$\square$

Ex 5 Let U be a domain and let $V = \{\overline{z} : z \in U\}$.

a) Suppose that $f : U \to \mathbb{C}$ is holomorphic and define $g : V \to \mathbb{C}$ by $g(z) = \overline{f(\overline{z})}$. Prove that g is holomorphic.

b) Let $F : U \to \mathbb{C}$ be analytic, and let $g : V \to \mathbb{C}$ by $g(z) = \overline{f(\overline{z})}$. Prove that g is analytic.

Proof. We note that in these cases we also have that $g(z) = \overline{f(\overline{z})}$.

a) Using the limit definition of derivative, we see that

$$\begin{aligned}\frac{\partial}{\partial z} g(z) &= \lim_{|h|\to 0} \frac{g(z+h) - g(z)}{h} = \lim_{|h|\to 0} \frac{\overline{f(\overline{z_0 + h})} - \overline{f(\overline{z})}}{h} = \lim_{|h|\to 0} \overline{\frac{f(\overline{z} + \overline{h}) - f(\overline{z})}{\overline{h}}} \\ &= \lim_{|\overline{h}|\to 0} \overline{\frac{f(\overline{z} + \overline{h}) - f(\overline{z})}{\overline{h}}} = \lim_{|h'|\to 0} \overline{\frac{f(\overline{z} + h') - f(\overline{z})}{h'}} = \overline{\frac{\partial}{\partial z} f(\overline{z})}\end{aligned}$$

since f is holomorphic, this limit converges. Thus, g is differentiable. Now we recall that conjugation is continuous, as the inverse image of an open set is just its mirror across the real axis. Since the derivative of f is continuous as well, we get that

$$\lim_{z\to z_0} \frac{\partial}{\partial z} g(z) = \lim_{z\to z_0} \frac{\partial}{\partial z} \overline{f(\overline{z})} = \frac{\partial}{\partial z} \overline{f(\overline{\lim_{z\to z_0} z})} = \frac{\partial}{\partial z} \overline{f(\overline{z_0})} = \frac{\partial}{\partial z} g(z_0),$$

which proves that the derivative of g is continuous as well. Thus, g is holomorphic as desired.

b) Let $z_0 \in U$. As f is analytic, there is a power series $\sum_{n=0}^{\infty} a_n(z - z_0)$ which converges uniformly to f for radius less than

$$r = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Now, for $z_0 \in V$ and $z \in B_r(z_0)$, we have that

$$g(z) = \overline{f(\overline{z})} = \overline{\sum_{n=0}^{\infty} a_n(\overline{z} - \overline{z_0})} = \sum_{n=0}^{\infty} \overline{a_n}(z - z_0)$$

which is a power series for g at z_0 with radius of convergence

$$\frac{1}{\limsup_{n\to\infty} \sqrt[n]{|\overline{a_n}|}} = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} = r.$$

Thus, g is analytic, where the coefficients of the power series at z_0 are simply the conjugate of those in the power series of f at $\overline{z_0}$.

□

Ex 6 Let $(a_n)_{n=0}^{\infty}$ be complex numbers. Suppose that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. Show that the radius of convergence of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is 1. Compute $\lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta$.

Proof. We see that for $|z| < 1$,

$$\sum_{n=0}^{\infty} |a_n z^n|^2 - \sum_{n=0}^{N} |a_n z^n|^2 = \sum_{n=N}^{\infty} |a_n z^n|^2 = \sum_{n=N}^{\infty} |a_n|^2 |z^n|^2 \leq \sum_{n=N}^{\infty} |a_n|^2.$$

Since $\sum_{n=0}^{\infty} |a_n|^2$ converges, we can choose a large enough N to make the difference of the two sums as small as we want. Thus, $\lim_{N\to\infty} \sum_{n=0}^{N} a_n z^n$ converges absolutely and uniformly for $|z| < 1$. This says that the radius of convergence of $f(z)$ is at least 1 (there is no reason why the radius of convergence should be exactly 1, though).

We see that

$$\begin{aligned}
\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta &= \int_0^{2\pi} f(re^{i\theta}) \overline{f(re^{i\theta})} \, d\theta = \int_0^{2\pi} \left(\sum_{n=0}^{\infty} a_n (re^{i\theta})^n \right) \overline{\left(\sum_{m=0}^{\infty} a_m (re^{i\theta})^m \right)} \, d\theta \\
&= \int_0^{2\pi} \left(\sum_{n=0}^{\infty} a_n (re^{i\theta})^n \right) \overline{\left(\sum_{m=0}^{\infty} a_m (re^{i\theta})^m \right)} \, d\theta \\
&= \int_0^{2\pi} \left(\sum_{n=0}^{\infty} a_n (re^{i\theta})^n \right) \left(\sum_{m=0}^{\infty} \overline{a_m} r^m e^{-im\theta} \right) d\theta \\
&= \int_0^{2\pi} \sum_{n,m=0}^{\infty} a_n \overline{a_m} r^{n+m} e^{i(n-m)\theta} \, d\theta.
\end{aligned}$$

Now let $\varepsilon > 0$. Since $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly, there is some N such that for all $n_0 \geq N$, we have that $\sum_{n=n_0}^{\infty} |a_n z^n| < \sqrt{\varepsilon}$. Since $r < 1$ (I assume we're approaching from the inside of the unit circle, but this isn't specified),

$$\begin{aligned}
&\sum_{n,m=0}^{\infty} |a_n \overline{a_m} r^{n+m} e^{i(n-m)\theta}| - \sum_{n,m=0}^{N} |a_n \overline{a_m} r^{n+m} e^{i(n-m)\theta}| = \sum_{n,m=N}^{\infty} |a_n \overline{a_m} r^{n+m} e^{i(n-m)\theta}| \\
&= \sum_{n,m=N}^{\infty} |a_n r^n e^{in\theta}| |\overline{a_m r^m e^{im\theta}}| = \left(\sum_{n=N}^{\infty} |a_n z^n| \right) \left(\sum_{m=N}^{\infty} |a_m z^m| \right) < \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon.
\end{aligned}$$

This proves that our sum converges absolutely and uniformly and thus we can interchange the sum and the intergral in our previous equation and get that

$$\begin{aligned}
\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta &= \int_0^{2\pi} \sum_{n,m=0}^{\infty} a_n \overline{a_m} r^{n+m} e^{i(n-m)\theta} \, d\theta = \sum_{n,m=0}^{\infty} \int_0^{2\pi} a_n \overline{a_m} r^{n+m} e^{i(n-m)\theta} \, d\theta \\
&= \sum_{n,m=0}^{\infty} a_n \overline{a_m} r^{n+m} \int_0^{2\pi} e^{i(n-m)\theta} \, d\theta.
\end{aligned}$$

We note that if $n \neq m$, then $\int_0^{2\pi} e^{i(n-m)\theta} \, d\theta = \frac{e^{i(n-m)\theta}}{i(n-m)} \Big|_{\theta=0}^{2\pi} = \frac{1-1}{i(n-m)} = 0$. And if $n = m$, then $\int_0^{2\pi} e^{i(n-m)\theta} \, d\theta = \int_0^{2\pi} 1 \, d\theta = 2\pi$. Thus, we get that

$$\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \sum_{n,m=0}^{\infty} a_n \overline{a_m} r^{n+m} \int_0^{2\pi} e^{i(n-m)\theta} \, d\theta = 2\pi \sum_{n=0}^{\infty} a_n \overline{a_n} r^{n+n} = 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

This gives us that

$$\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}|f(r^{i\theta})|^2\,d\theta=\lim_{r\to 1}\sum_{n=0}^{\infty}|a_n|^2r^{2n}.$$

We can interchange the sum and the integral because for $r<1$

$$\sum_{n=0}^{\infty}|a_n|^2r^{2n}-\sum_{n=0}^{N}|a_n|^2r^{2n}=\sum_{n=N}^{\infty}|a_n|^2r^{2n}\le\sum_{n=N}^{\infty}|a_n|^2|r^{2n}|\le\sum_{n=N}^{\infty}|a_n|^2.$$

Since this expression does not that depend r and the last sum converges, we can make this as small as we want by choosing a sufficiently large N. This proves that the sum converges uniformly, so we can safely interchange the limit and the integral to get that

$$\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}|f(r^{i\theta})|^2\,d\theta=\lim_{r\to 1}\sum_{n=0}^{\infty}|a_n|^2r^{2n}=\sum_{n=0}^{\infty}\lim_{r\to 1}|a_n|^2r^{2n}=\sum_{n=0}^{\infty}|a_n|^2.$$

□

Ex 7

a) Let $U\subseteq\mathbb{C}$ be an open set. Suppose that we are given a sequence $f_n:U\to\mathbb{C}$, $n\in\mathbb{N}$ of continuous functions. Assume that f_n converges uniformly on compact subsets of U to a function $f:U\to\mathbb{C}$. Prove that f is continuous.

b) Suppose that $E\subseteq\mathbb{C}$ is given and that a sequence of functions $f_n:E\to\mathbb{C}$ for $n\in\mathbb{N}$. Suppose that for every $z\in E$, there is an open $U_z\subseteq\mathbb{C}$ with $z\in U_z\cap E$ and so that $f_n|_{U_x\cap E}$ converges uniformly. Prove that f_n converges uniformly on compact subsets of E to a function $f:E\to\mathbb{C}$.

Proof.

a) Let z_0 be an arbitary point in U. Since U is open, there is some $2r>0$ such that $B_{2r}(z_0)\subseteq U$. This means that $\overline{B_r(z_0)}\subseteq B_{2r}(z_0)\subseteq U$. We note that $\overline{B_r(z_0)}$ is closed and bounded, and thus compact. Now $(f_i)_{i\in\mathbb{N}}$ convergences uniformly to f on such compact sets. That means for $\varepsilon>0$, we have that there's some N such that $|f_n(z)-f(z)|<\frac{\varepsilon}{3}$ for any $n\ge N$ and any $z\in\overline{B_r(z_0)}$. Fix one such n. Since f_n is continuous, it must be continous at the point z_0; that means that there's a $\delta>0$ such that $|z_0-z|<\delta$ implies that $|f_n(z_z)-f_n(z)|<\frac{\varepsilon}{3}$. Without loss of generality, we may assume that $\delta<r$. With this, we see that if $|z_0-z|<\delta$ then

$$|f(z_0)-f(z)|\le|f(z_0)-f_n(z_0)|+|f_n(z_0)-f_n(z)|+|f_n(z)-f(z)|=\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

This proves that f is continuous at $z_0\in U$. Since our choice of z_0 was arbitrary, we have that f is continuous on U.

b) Let K be a compact subset of E. By the hypothesis, for each $z\in K$, there is an open set U_z such that $z\in U_z\cap E$ and $f_n|_{U_z\cap E}$ convergences uniformly. Since these U_z's form a cover of K and K is compact, there exists a finite subcover $\{U_{z_1},\ldots,U_{z_j}\}$. Unpacking the definition of uniform convergence, let $\varepsilon > 0$. We have that for each U_{z_i}, there is a function f_{z_i} and an $N \geq 0$ such that for all $n \geq N_{z_i}$ and $z \in U_{z_i} \cap E$, we have that $|f_n(z) - f_{z_i}(z)| < \varepsilon$. Let $N = \max_{i \leq j}(N_{z_i})$. Thus, for all $n \geq N$, we have that $|f_n(z) - f_{z_i}(z)| < \varepsilon$ for $z \in U_{z_i} \cap E$.

Let $f(z) = f_{z_i}(z)$ if $z \in U_{z_i} \cap E$. We note this function is well-defined as uniform convergence implies pointwise convergence and the sequence $(f_i(z))_{i \in \mathbb{N}}$ can only converge to one point. Thus, we have that f_i convergences uniformly to f on $\cup_i (U_{z_i} \cap E) = U \cap E$. Since $K \subseteq U \cap E$, we have that f_i convergences uniformly on K. This proves that for each compact subset K of E, there is a function f_K such that $f_i \to f_K$ uniformly. By a similar argument as before, the function $f(z) = f_K(z)$ for $z \in K$ is well-defined and has the property that $f_i \to f$ converges uniformly on compact subsets of E.

□

Ex 8 Recall that if $A \subseteq \mathbb{C}$ is given, then $z \in A$ is isolated in A if there is an $\varepsilon > 0$ so that $B_\varepsilon(z) \cap A = \{z\}$.

a) Let U be a nonempty open set and $f : U \to \mathbb{C}$, $g : U \to \mathbb{C}$ be analytic. Let $E = \{z \in U : f(z) = g(z)\}$. Let F be the set of accumulation points of E in U. Show that F is relatively open and closed in U.

b) Let U be a domain and $f : U \to \mathbb{C}$, $g : U \to \mathbb{C}$ be analytic. Let $E = \{z \in U : f(z) = g(z)\}$. If E has at least one accumulation point in U, then $f = g$ throughout U.

c) Let f be analytic in a domain U containing the point $z = 0$. Suppose that there is an integer $n_0 \geq 1$ with $|f(1/n)| < e^{-n}$ for $n \geq n_0$. Prove that $f(z) = 0$ throughout U.

Proof.

a) Without loss of generality, we can simply prove it for $E = \{z \in U : f(z) = 0\}$ (just take f to be $f - g$). To prove that F is closed, we will show that it contains all of its limit points. Let x be a limit point of F and let U be any open neighborhood of x. As x is a limit point, there is some $f \in F$ such that $f \in U$. Let $r > 0$ be such that $B_r(f) \subseteq U$ and $x \notin B_r(f)$. Since f is a limit point of E and $B_r(f)$ is an open neighborhood of f, $B_r(f)$ contains some element $e \in E$ as well. We note that $e \neq x$ as $x \notin B_r(f)$. Thus, our open set U contains an element of E distinct from x. Since this works for any open neighborhood of x, we have that x is a limit point of E, proving that $x \in F$. Thus, F contains all of its limit points and is closed.

   Now to prove that F is open, let $x \in F$. By what we proved in class, since x is an accumulation point of the zeros of f, the power series of f at x is 0. Thus, f is 0 on some ball $B_r(x)$. Since all the points of a ball are accumulation points of the ball, we have that $B_r(x) \subseteq F$. This proves that F is open as desired.

b) Since E contains at least one point, by the previous part and by the connectedness of U, we have that $E = U$. This means that every point $x \in U$ is an accumulation point of the set of zeros of $f - g$. This means that $f - g$ is zero for some open ball $B_{r_x}(x)$ where $r_x > 0$. As these balls cover U, we get that $f - g = 0$ on U. Thus, $f = g$ throughout U.

c) Since f is analytic on U and $0 \in U$, we know that $f(z) = \sum_{j=0}^{\infty} a_j z^j$ for $|z| < r$ for some $r > 0$. I claim that $a_k = 0$ for all $k \in \mathbb{N}$, which we will do by induction. For the $k = 0$ case, we know that for $n \geq n_0$, we have

$$|f(1/n)| < e^{-n}.$$

If we take the limit of both sides as $n \to \infty$, then as f is continuous, we get

$$\lim_{n\to\infty} |f(1/n)| = |f(\lim_{n\to\infty} 1/n)| = |f(0)| \leq \lim_{n\to\infty} e^{-n} = 0.$$

This proves that $f(0) = 0$, which means that $f(0) = \sum_{j=0}^{\infty} a_j 0^j = a_0$. This completes the base case. Now suppose $a_j = 0$ for $j < k$. If we let $h(z) = \sum_{j=0}^{\infty} a_{j+k} z^j$, then we see that $z^k \cdot h(z) = f(z)$. We see that for $n \geq n_0$

$$|(1/n)^k \cdot h(1/n)| = |f(1/n)| < e^{-n}.$$

Thus, using a similar trick as before, we can multiply by $|n|^k$ and take the limit as $n \to \infty$

$$\lim_{n\to\infty} |h(1/n)| = |h(\lim_{n\to\infty} 1/n)| = |h(0)| \leq \lim_{n\to\infty} |n|^k e^{-n} = \lim_{n\to\infty} \frac{|n|^k}{e^n} = 0.$$

We note that the last limit is zero as expontential functions grow faster than any polynomial. This means that $h(0) = 0$ and thus that $0 = h(0) = \sum_{j=0}^{\infty} a_{j+k} 0^j = a_k$. By induction, we have that $a_j = 0$ for all j; meaning f is 0 on some ball $B_r(0)$. This proves 0 is an accumulation point of the set of zeros of f, proving that $f = 0$ on U by part (b).

□

Ex 9

a) Suppose that $U \subseteq \mathbb{C}$ is connected and open and let $f : U \to \mathbb{C}$ be analytic. Suppose that $\mathrm{Re}(f)$ is constant; prove that f is constant.

b) Suppose that $U \subseteq \mathbb{C}$ is connected and open and let $f : U \to \mathbb{C}$ be analytic. Suppose that $|f|$ is constant; prove that f is constant.

Proof.

a) Let $z_0 \in U$. Since U is open, there is an open ball $B_r(z_0)$ contained in U. Define $\gamma_1 : (-r, r) \to \mathbb{C}$ as $\gamma_1(t) = z_0 + t$ and $\gamma_2 : (-r, r) \to \mathbb{C}$ as $\gamma_2 = z_0 + it$. We note that these two paths intersect at a right angle. Since $\mathrm{Re}(f) = \alpha$ is constant, that means $f(\gamma_1(t))$ and $f(\gamma_2(t))$ both lie on the line $\{z : \mathrm{Re}(z) = \alpha\}$. Thus, the angle between $f(\gamma_1(t))$ and $f(\gamma_2(t))$ at $f(z_0)$ is either 0 or π. However, since f is analytic, it is holomorphic and conformal. By conformality, as long as $f'(z_0) \neq 0$, then the angle between $f(\gamma_1(t))$ and $f(\gamma_2(t))$ should be preserved. Since this is not the case, it must be that $f'(z_0) = 0$. Since z_0 was arbitrary, we have that $f'(z) = 0$ for all $z \in U$. This proves that f is constant.

b) Let $z_0 \in U$. Since U is open, there is an open ball $B_r(z_0)$ contained in U. Define $\gamma_1 : (-r, r) \to \mathbb{C}$ as $\gamma_1(t) = z_0 + t$ and $\gamma_2 : (-r, r) \to \mathbb{C}$ as $\gamma_2 = z_0 + it$. We note that these two paths intersect at a right angle. Since $|f| = \alpha$ is constant, that means $f(\gamma_1(t))$ and $f(\gamma_2(t))$ both lie on the circle $\{z : |z| = \alpha\}$. Thus, the angle between $f(\gamma_1(t))$ and $f(\gamma_2(t))$ at $f(z_0)$ is either 0 or π. However, since f is analytic, it is holomorphic and conformal. By conformality, as long as $f'(z_0) \neq 0$, the angle between $f(\gamma_1(t))$ and $f(\gamma_2(t))$ should be preserved. Since this is not the case, it must be that $f'(z_0) = 0$. Since z_0 was arbitrary, we have that $f'(z) = 0$ for all $z \in U$. This proves that f is constant.

$\square$