Problem Set 3
Homological Algebra

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Ex 1. Prove the exactness of the long exact sequence at $H_n(\mathbf{C}'')$.

Proof. To help us keep track of things, let us recall the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r, "\partial'_{n+2}"] & C'_{n+1} \arrow[r, "\partial'_{n+1}"] \arrow[d, "f_{n+1}"] & C'_n \arrow[r, "\partial'_n"] \arrow[d, "f_n"] & C'_{n-1} \arrow[r, "\partial'_{n-1}"] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r, "\partial_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "g_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "g_{n-1}"] & \dots \\
\dots \arrow[r, "\partial''_{n+2}"] & C''_{n+1} \arrow[r, "\partial''_{n+1}"] \arrow[d] & C''_n \arrow[r, "\partial''_n"] \arrow[d] & C''_{n-1} \arrow[r, "\partial''_{n-1}"] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

which induces the following long exact sequence of relative homology groups:

$$\dots \longrightarrow H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(C'') \xrightarrow{\delta} H_{n-1}(C') \xrightarrow{f_*} H_{n-1}(C) \xrightarrow{g_*} \dots$$

where δ is the connecting homomorphism. Finally, we also recall that δ is defined by taking $c \in H_n(C'')$ on the following path through the chain complexes

$$\begin{tikzcd}
C'_{n+1} & C'_n & a \\
C_{n+1} & b \arrow[r, mapsto] & \partial_n b \arrow[u, mapsto] \\
C''_{n+1} & c \arrow[u, mapsto] & C''_{n-1}
\end{tikzcd}$$

Now, to actually start with the proof, let $b \in H_n(C) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$. We can use the path of δ to see that $\delta(g_*(b))$ maps b down to c, then back up to b, then to ∂b, which is 0 as $b \in \ker(\partial_n)$, then to a, which must also be 0 as f_{n-1} is injective. Thus, $\delta(g_*(b)) = 0$. Since b was arbitrary, this proves that $\operatorname{im}(g_*) \subseteq \ker(\delta)$.

Now to prove the reverse inclusion, let $c \in \ker(\delta) \subseteq H_n(C'') = \ker(\partial''_n)/\operatorname{im}(\partial''_{n+1})$. We see then that $\delta(c) = a$ must be the zero element in $H_{n-1}(C')$, meaning $a \in \operatorname{im}(\partial'_{n+1})$. That means that $a = \partial'_n a'$ for some $a' \in C'_n$. We see that the element $b - f_n(a')$ is a cycle as

$$\partial_n(b - f_n(a')) = \partial_n b - \partial_n f_n(a') = \partial_n b - f_{n-1}\partial'_n(a') = \partial_n b - f_{n-1}(a) = \partial_n b - \partial_n b = 0.$$

We also see that

$$g_n(b - f_n(a')) = g_n(b) - g_n(f_n(a')) = g_n(b) = c.$$

Thus, g_* sends the cycle $b - f_n(a')$ to c. This proves that $\ker(g_*) \subseteq \operatorname{im}(\delta)$, so we can conclude that the long exact sequence is exact at $H_n(C'')$. □

Ex 2. Incomplete.

Ex 3. Incomplete.

Ex 4. Let $\mathcal{C}$ be the category of (left) R-modules and let $F : \mathcal{C} \to \mathbf{Ab}$ be the contravariant functor $F = \operatorname{Hom}_R(-, M)$ for some fixed $M \in \mathcal{C}$. Prove that F is left exact, i.e. if $A \to B \to C \to 0$ is exact, then $0 \to F(C) \to F(B) \to F(A)$ is exact.

Proof. Let the following be an exact sequence

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

To prove that F is left is exact, we need to prove that the following is also an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(C, M) \xrightarrow{\psi_*} \operatorname{Hom}_R(B, M) \xrightarrow{\phi_*} \operatorname{Hom}_R(A, M)$$

To do this, we will first prove that ψ_* is injective. We recall that for any $f : C \to M$, we have that $\psi_*(f) = f \circ \psi$. This means that if $f, g : C \to M$ and $\psi_*(f) = \psi_*(g)$, then $f \circ \psi = g \circ \psi$. As ψ is surjective, it is a epimorphism, meaning we can cancel to obtain that $f = g$. Thus, we see that ψ_* is injective.

We will now proveness exactness at $\operatorname{Hom}_R(B, M)$. Since F is a contravariant functor, we see that $\phi_* \circ \psi_* = (\psi \circ \phi)_* = 0_* = 0$. This proves that $\operatorname{im}(\psi_*) \subseteq \ker(\phi_*)$. Now to prove the reverse inclusion, let $f \in \ker(\phi_*)$. We see then that $f \circ \phi = \phi_*(f) = 0$, which means $\operatorname{im}(\phi) \subseteq \ker(f)$. We know that $\operatorname{im}(\phi) = \ker(\psi)$, which means $\ker(\psi) \subseteq \ker(f)$. Thus, f can be factored through ψ, i.e. there is some function $g : C \to M$ such that $g \circ \psi = f$. This means that $\psi_*(g) = g \circ \psi = f$. Thus, $f \in \operatorname{im}(\psi_*)$. This proves exactness at $\operatorname{Hom}_R(B, M)$ as we wanted. □

Ex 5. Let D be an injective $\mathbb{Z}$-module. Provide $D^\# = \operatorname{Hom}_{\mathbb{Z}}(R, D)$ with the structure of a left R-module as in 1.4.2(b) and show that this R-module is also injective.

Proof. Let X, Y be R-modules such that we have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "g"] & Y \\ & \operatorname{Hom}_{\mathbb{Z}}(R, D) & \end{tikzcd}$$

We see that we can then define a map $\tilde{g} : X \to D$ where $\tilde{g}(x) = g(x)(1)$. We see that $\tilde{g}(x+y) = g(x+y)(1) = g(x)(1) + g(y)(1) = \tilde{g}(x) + \tilde{g}(y)$ since g is an R-module homomorphism. This proves that $\tilde{g}$ is a $\mathbb{Z}$-module homomorphism. Thus, if we forget about the R-module structure of X and Y, we can invoke the property of D being injective to obtain the following diagram of $\mathbb{Z}$-modules.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\tilde{g}"] & Y \arrow[dl, dotted, "\tilde{h}"] \\ & D & \end{tikzcd}$$

As D is injective, this implies there's a $\mathbb{Z}$-module homomorphism $\tilde{h} : Y \to D$ such that $\tilde{g} = \tilde{h} \circ f$. We use this $\tilde{h}$ to define an R-module homomorphism $h : Y \to \mathrm{Hom}_{\mathbb{Z}}(R, D)$ where $h(y) = \phi_y$ and $\phi_y(r) = \tilde{h}(ry)$. We see that

$$h(f(x))(r) = \phi_{f(x)}(r) = \tilde{h}(rf(x)) = \tilde{h}(f(rx)) = \tilde{g}(rx) = g(rx)(1) = g(x)(r).$$

This proves that $h \circ f = g$. Thus, we have the diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[dl, dotted, "h"] \\ & \mathrm{Hom}_{\mathbb{Z}}(R, D) & \end{tikzcd}$$

This diagram proves that $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is injective as an R-module. □

Ex 6. Let A be an R-module, let I be any nonempty index set and for each $i \in I$, let B_i be an R-module. Prove that following isomorphisms of abelian groups; when R is commutative prove that these are R-module isomorphisms.

a) $\mathrm{Hom}_R(\oplus_{i\in I} B_i, A) \simeq \prod_{i\in I} \mathrm{Hom}_R(B_i, A)$

b) $\mathrm{Hom}_R(A, \prod_{i\in I} B_i) \simeq \prod_{i\in I} \mathrm{Hom}_R(A, B_i)$.

Proof.

a) We note that we have canonical injections $\iota_i : B_i \to \oplus_i B_i$. Now, we can define a maps $\phi_i : \mathrm{Hom}_R(\oplus_i B_i) \to \mathrm{Hom}_R(B_i, A)$, where $\phi_i(f) = f \circ \iota_i$. By the universal property of direct products, this gives a homomorphism $\Phi : \mathrm{Hom}_R(\oplus_i B_i, A) \to \prod_i \mathrm{Hom}_R(B_i, A)$ such that $\pi_i \circ \Phi = \phi_i$, where π_i is the canonical projection from $\prod_i \mathrm{Hom}_R(B_i, A)$ to $\mathrm{Hom}_R(B_i, A)$. Now, we wish to prove that Φ is an isomorphism.

   We will first prove that Φ is injective. Let $f : \oplus_i B_i \to A$ be in $\ker(\Phi)$. This means that

   $$f \circ \iota_i = \phi_i(f) = (\pi_i \circ \Phi)(f) = \pi_i(\Phi(f)) = \pi_i(0) = 0.$$

   Since f is the zero map on every component of $\oplus_i B_i$, it must be that $f = 0$. Thus, $\ker(\Phi)$ is trivial, proving that Φ is injective.

   Now we will prove that Φ is surjective. Let $g = (g_i) \in \prod_i \mathrm{Hom}_R(B_i, A)$. We define $f \in \mathrm{Hom}_R(\oplus_i B_i, A)$ as $f(\oplus_i b_i) = \sum_i g_i(b_i)$. We see that if we let $b_i \in B_i$, then

   $$(\pi_i \circ \Phi(f))(b_i) = \phi_i(f)(b_i) = (f \circ \iota_i)(b_i) = f(\iota_i(b_i)) = g_i(b_i) = (\pi_i \circ g)(b_i)$$

Thus, $\Phi(f)$ and g are equal on every component. This proves that $\Phi(f) = g$, meaning Φ is surjective.

If R is commutative, then $\mathrm{Hom}_R(\oplus_i B_i, A)$ and $\prod_i \mathrm{Hom}_R(B_i, A)$ are left R-modules. We see that for any $r \in R$ and any $f \in \mathrm{Hom}_R(\oplus_i B_i, A)$,

$$\Phi(rf) = ((rf) \circ \iota_i)_{i \in I} = (r(f \circ \iota_i))_{i \in I} = r(f \circ \iota_i)_{i \in I} = r\Phi(f).$$

Thus, we can say that Φ is an R-module isomorphism.

b) We note that we have canonical projections $\pi_i : \prod_i B_i \to B_i$. Now, we can define a maps $\phi_i : \mathrm{Hom}_R(A, \prod_i B_i) \to \mathrm{Hom}_R(A, B_i)$ where $\phi_i(f) = \pi_i \circ f$. Using the universal property of direct products, this gives us a unique homomorphism $\Phi : \mathrm{Hom}_R(A, \prod_i B_i) \to \prod_i \mathrm{Hom}_R(A, B_i)$ such that $\pi_i \circ \Phi = \phi_i$. Now, we wish to prove that Φ is an isomorphism.

We will first prove that Φ is injective. Let $f : A \to \prod_i B_i$ be in $\ker(\Phi)$. This means that

$$\pi_i \circ f = \phi_i(f) = (\pi_i \circ \Phi)(f) = \pi_i(\Phi(f)) = \pi_i(0) = 0.$$

Since the projection of f onto each B_i is zero, it must be that f is the zero map. Thus, $\ker(\Phi)$ is trivial, proving that Φ is injective.

Now we will prove that Φ is surjective. Let $g = (g_i) \in \prod_i \mathrm{Hom}_R(A, B_i)$. We define $f \in \mathrm{Hom}_R(A, \prod_i B_i)$ as $f(a) = \prod_i g_i(a)$. We see then that

$$\pi_i(\Phi(f)(a)) = (\pi_i \circ \Phi)(f)(a) = \phi_i(f)(a) = (\pi_i \circ f)(a) = g_i(a).$$

Thus, $\Phi(f)$ and g agree on every projection, meaning $\Phi(f) = g$. This proves that Φ is surjective and thus, Φ is an isomorphism.

If R is commutative, then $\mathrm{Hom}_R(A, \prod_i B_i)$ and $\prod_i \mathrm{Hom}_R(A, B_i)$ are (left) R-modules. We see that for any $r \in R$ and any $f \in \mathrm{Hom}_R(A, \prod_i B_i)$,

$$\begin{aligned}(\pi_i \circ \Phi)(rf)(a) &= \phi_i(rf)(a) = (\pi_i \circ rf)(a) = \pi_i(rf(a)) = r\pi_i(f(a)) = r(\pi_i \circ f)(a) \\ &= r\phi_i(f)(a) = r(\pi_i \circ \Phi)(f)(a) = r\pi_i(\Phi)(f)(a) = \pi_i(r\Phi)(f)(a) = (\pi_i \circ r\Phi)(f)(a)\end{aligned}$$

Since $\Phi(rf)$ and $r\Phi(f)$ agree on every projection, we have that $\Phi(rf) = r\Phi(f)$ as desired. Thus, we can say that Φ is an R-module isomorphism. □

Ex 7. [Continuation of Exercise 6] If S is the the direct sum of the B_i, show that there is always a (canonical) embedding of the direct sum of the $\mathrm{Hom}_R(A, B_i)$ into $\mathrm{Hom}_R(A, S)$, but that this embedding needn't be surjective.

Proof. Let $\iota_i : B_i \to \oplus_i B_i$ and $j_i : \mathrm{Hom}_R(A, B_i) \to \oplus_i \mathrm{Hom}_R(A, B_i)$ be the canonical injections. We see that we can then define a map $\phi_i : \mathrm{Hom}_R(A, B_i) \to \mathrm{Hom}(A, \oplus_i B_i)$ where $\phi_i(f) = \iota_i \circ f$. By the universal property of the direct sum, this means there's a unique map $\Phi : \oplus_i \mathrm{Hom}_R(A, B_i) \to \mathrm{Hom}(A, \oplus_i B_i)$ where $\Phi \circ j_i = \phi_i$.

Now we would like to show that Φ is injective. Let $f \in \oplus_i \mathrm{Hom}_R(A, B_i)$ be in $\ker(\Phi)$. This means that $f = \sum_i j_i(f_i)$ where $f_i \in \mathrm{Hom}_R(A, B_i)$ and all but finitely many terms are nonzero. We see then that

$$0 = \Phi(f) = \Phi(\sum_i j_i(f_i)) = \sum_i \Phi(j_i(f_i)) = \sum_i \phi_i(f_i) = \sum_i (\iota_i \circ f_i)$$

Now, if we evaluate this at any $a \in A$, we get $0 = \sum_i (\iota_i \circ f_i)(a)$ which is an element of $\oplus_i B_i$. Since this is a direct sum, it must be that $(\iota_i \circ f_i)(a) = 0$ for all $a \in A$. Thus, $\iota_i \circ f_i = 0$ for all $i \in I$. Since ι_i are inclusion maps, they must be injective, proving that each f_i must be the zero map. Thus, $f = \sum_i j_i(0) = \sum_i 0 = 0$, so we can conclude that Φ is injective.

However, Φ need not be surjective. Consider the example of $\mathbb{R}$-vector spaces where $A = V$ is an infinite dimensional vector space with countable basis $\{e_1, e_2, \dots\}$, and $B_i = \mathbb{R}x_i$. Then we can define a linear transformation $f : V \to \oplus_i \mathbb{R}x_i$ by letting $f(e_i) = x_i$ for each $i \in \mathbb{N}$. However, f cannot be written as a finite sum of $f_i : V \to \mathbb{R}x_i$. This is because $\operatorname{rank}(f) = \infty$, but $\operatorname{rank}(\sum_{i \leq n} f_i) \leq \sum_{i \leq n} \operatorname{rank}(f_i) \leq \sum_{i \leq n} 1 = n < \infty$. Thus, this is an example where the map $\Phi : \oplus_i \operatorname{Hom}_R(A, B_i) \to \operatorname{Hom}_R(A, \oplus_i B_i)$ is not surjective. $\square$