Final Exam
Differential Topology

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Ex 1 Let $M \subseteq \mathbb{R}^n$ be an embedded smooth submanifold

a) Prove that there exists a $c \in \mathbb{R}$ such that the plane $x_1 = c$ instersects M tranversely.

b) Give an example that shows that the set of such c need not be open in $\mathbb{R}$.

c) In the case $M = \{(x, y, z) : x^2 + y^2 - z^2 = 1\} \subseteq \mathbb{R}^3$, find all $c \in \mathbb{R}$ such that the plane $x = c$ intersects M transversely.

Proof.

a) Consider the smooth projection map $p : \mathbb{R}^n \to \mathbb{R}$ where $p(x_1, \ldots x_n) = x_1$. Then, $\pi = p|_M$ is a smooth map from M to $\mathbb{R}$. I claim that $x_1 = c$ is not transverse to M if and only if c is a critical value of π. This is proved as follows:

$\Longrightarrow$) Let $c \in \mathbb{R}$ be such that $x_1 = c$ is not tranverse to M. Since $x_1 = c$ is a $n-1$ dimensional manifold, it must be that there is some $x \in M$ where $\pi(x) = c$ and T_xM is exactly the plane $x_1 = c$ (otherwise the sum of the tangent spaces would have full dimension n). As $d\pi$ is the projection map of tangent spaces, this means that $d\pi_x = 0$, which is not surjective. Thus, $\pi(x) = c$ is a critical value of π.

$\Longleftarrow$) Let c be a critical value of π. Then there's an $x \in M$ such that $\pi(x) = c$ and $d\pi_x : T_xM \to \mathbb{R}$ is not surjective. The only way that $d\pi_x$ can fail to be surjective is if $d\pi_x = 0$. Since $d\pi$ is the projection of tangent spaces, the tangent plane at x must be contained in the plane $x_1 = c$. This means that the plane $x_1 = c$ is not tranverse to M at x, and thus the plane is not transverse to M overall.

By Sard's Theorem, we know the critical values of a smooth map have measure zero. This means that we can always choose a regular value c of π and that for such c, the plane $x_1 = c$ will intersect M transversely.

b) From previous work, we know that there is a smooth bump function $b : \mathbb{R} \to [0, 1]$ such that $b'(0) = 0$, $b(x) = 1$ for $|x| < \frac{1}{4}$, and $b(x) = 0$ for $|x| > \frac{1}{3}$. Now, as $\mathbb{Q}$ is countable, we can enumerate all the rationals as $\{q_i : i \in \mathbb{N}\}$. We can then construct the function $f(x) = \sum_i q_i \cdot f(x + i)$. This function is well-defined because for any point in $\mathbb{R}$, only one bump function is non-zero. Additionally, the function is smooth as inbetween the bump functions there is an interval of length $1/3$ on which all the bump functions are identically zero. Since for any $q_i \in \mathbb{Q}$, we have that $f(-i) = q_i$ and $f'(-i) = 0$, we see that all of $\mathbb{Q}$ are critical values of f. This means that the plane $y = c$ does not intersect $\Gamma(f)$ transversally for any $c \in \mathbb{Q}$. In other words, the values of c that do intersect tranversally are contained in $\mathbb{R} \setminus \mathbb{Q}$, which has no open subsets.

c) Let $f(x, y, z) = x^2 + y^2 - z^2$ so that $M = f^{-1}(1)$. We have proved previously that $T_{(x,y,z)}M = \ker(df_{(x,y,z)}) = \ker((2x, 2y, -2z))$. By the Rank-Nullity Theorem, this means that $df_{(x,y,z)}$ is a normal vector to the point $(x, y, z) \in M$.

We see that the normal vectors at the points $(1, 0, 0)$ and $(-1, 0, 0)$ in M are $df_{(1,0,0)} = (2, 0, 0)$ and $df_{(-1,0,0)} = (-2, 0, 0)$ respectively. This means that the tangent planes $T_{(1,0,0)}M$ and $T_{(-1,0,0)}M$ coincide with the plane $x = 1$ and $x = -1$ respectively. Thus, the planes $x = 1$ and $x = -1$ do not intersect M transversely.

However, for any point $(x, y, z) \in M$ where $x \neq \pm 1$, it must be that either $y \neq 0$ or $z \neq 0$. In other words, the normal vector $df_{(x,y,z)}$ is not contained in $\langle (1, 0, 0) \rangle$. This means the tangent space at (x, y, z) does not coincide with $x = c$ for any $c \in \mathbb{R}$. Thus, the plane $x = c$ intersects M tranversally if and only if $c \neq \pm 1$. □

Ex 2 Prove or provide a counterexample: If Y is a smooth manifold and $M = f^{-1}(t)$ is the preimage of a regular value t of a smooth function $f : Y \to \mathbb{R}$, then M is orientable.

Proof. We showed in Ex 1(c) that the vector df_x is always normal to $x \in M$. Since df_x is smooth with respect to $x \in M$, is never the zero vector (as x is a critical point), and is defined uniquely for each $x \in M$, we can use it to define an orientation on M. That is, for the point $x \in M$, we can declare the sign of the ordered basis $v_1, \dots, v_n \in T_xM$ as the sign of $\det(df_x, v_1, \dots, v_{k-1})$. This proves that M is orientable. □

Ex 3 Let X and Y be smooth, closed, compact, oriented n-dimensional manifolds and $f : X \to Y$ a smooth map. Let $Z, W \subseteq Y$ be closed, oriented submanifolds of dimensions k and ℓ respectively, with $k + \ell = n$, $Z \pitchfork W$, and f tranverse to Z, W, and $Z \cap W$.

a) Prove that the inverse images $f^{-1}(Z)$ and $f^{-1}(W)$ are smooth submanifolds of X that intersect transversely and have dimensions k and ℓ respectively.

b) Prove that with the induced orientations, the intersection number $I(f^{-1}(Z), f^{-1}(W))$ satisfies

$$I(f^{-1}(Z), f^{-1}(W)) = \deg(f) I(Z, W).$$

c) Use the previous part to give a proof that any smooth map $f : S^2 \to T^2$ has degree zero.

Proof.

a) Since f is tranverse to Z and W, we know that $f^{-1}(Z)$ and $f^{-1}(W)$ are submanifolds of X. Furthermore, $\operatorname{codim}(f^{-1}(Z)) = \operatorname{codim}(Z)$ and $\operatorname{codim}(f^{-1}(W)) = \operatorname{codim}(W)$, which means $f^{-1}(Z)$ and $f^{-1}(W)$ have dimension k and ℓ respectively.

We will first prove that for $x \in W \cap Z$, df_x is an isomorphism. Now let $x \in W \cap Z$. Since $df_x(T_x f^{-1}(Z)) = T_{f(x)}Z$ and $df_x(T_x f^{-1}(W)) = T_{f(x)}W$ and $W \pitchfork Z$, we obtain that

$$\begin{aligned} T_{f(x)}Y &= T_{f(x)}Z + T_{f(x)}W \\ &= df_x(T_x f^{-1}(Z)) + df_x(T_x f^{-1}(Z)) \\ &= df_x(T_x f^{-1}(Z) + T_x f^{-1}(Z)) \subseteq df_x(T_x X), \end{aligned}$$

which means that df_x is a surjective map, and hence an isomorphism.

Now to prove that $f^{-1}(Z)$ and $f^{-1}(W)$ are tranverse. Let $v \in T_x f^{-1}(Z) \cap T_x f^{-1}(W)$. This means that $df_x(v) \in T_{f(x)}Z \cap T_{f(x)}W$. But $W \pitchfork Z$ and their dimensions add to n so it must be that

$$T_{f(x)}Z \oplus T_{f(x)}W = T_{f(x)}Y.$$

This means $df_x(v) \in T_{f(x)}Z \cap T_{f(x)}W = \{0\}$. As df_x is an isomorphism, it must be that $v = 0$. Thus,

$$\begin{aligned}\dim(T_x f^{-1}(Z) + T_y(F^{-1}(W))) &= \dim(T_x f^{-1}(Z) \oplus T_y(F^{-1}(W))) \\ &= \dim(T_x f^{-1}(Z)) + \dim(T_y(F^{-1}(W))) \\ &= k + \ell \\ &= \dim(T_{f(x)}Y)\end{aligned}$$

which proves that $f^{-1}(Z)$ and $f^{-1}(W)$ are transverse.

b) Let $x_1, \dots, x_n$ be the intersection points of Z and W and let $f^{-1}(\{x_1, \dots, x_n\} = \{y_1, \dots, y_m\}$ where $\varepsilon(x_i) = \pm 1$ and $\varepsilon(y_i) = \pm 1$ are the orientations. This means that

$$\begin{aligned}\deg(f) I(Z, W) = \deg(f) \sum_i \varepsilon(x_i) &= \sum_i I(f, \{x\}) \varepsilon(x_i) \\ = \sum_i I(f, \{x_i\}) &= \sum_i \varepsilon(y_i) = I(f^{-1}(Z), f^{-1}(W)).\end{aligned}$$

c) We will label the two primary circles of the torus so that $T^2 = S^1_a \times S^1_b$. We can easily see that S^1_a and S^1_b intersect at a single point tranversally, which means we can choose orientations on S^1_a and S^1_b so that $I(S^1_a, S^1_b) = 1$. Note that we can always homotope f to be simultaneously transverse to S^1_a, S^1_b, and $S^1_a \cap S^1_b$ (as homotoping f to be transversal is equivalent to finding a regular value).

Furthermore, we note that $f^{-1}(S^1_a)$ and $f^{-1}(S^1_b)$ are both 1-dimensional manifolds in S^2. Since any circle or line segment in S^2 is homotopic to any point in S^2, we can choose $x, y \in S^2$ where $x \neq y$ so that

$$\deg(f) = \deg(f) \cdot 1 = \deg(f) I(S^1_a, S^1_b) = I(f^{-1}(S^1_a), f^{-1}(S^1_b)) = I(\{x\}, \{y\}) = 0,$$

as we wanted to prove. □

Ex 4 Let C_1 and C_2 be disjoint, oriented submanifolds of $\mathbb{R}^3$, each diffeomorphic to the circle S^1. The *linking number* of C_1 and C_2 is defined as follows. Let $f : S^1 \to C_1$ and $g : S^1 \to C_2$ be orientation-preserving diffeomorphisms and define the map $L_{f,g} : T^2 \to S^2$ by

$$L_{f,g}(x, y) = \frac{f(x) - g(y)}{|f(x) - g(y)|},$$

where $(x, y) \in T^2 = S^1 \times S^1$. Then the linking number $\operatorname{lk}(C_1, C_2)$ is the integer given by

$$\operatorname{lk}(C_1, C_2) = \deg(L_{f,g}).$$

a) Prove that $\operatorname{lk}(C_1, C_2)$ does not depend on the choice of orientation-preserving diffeomorphisms f, g used in its definition, but reverses the sign if the orientation of C_1 or C_2 is reversed.

b) Prove that $\text{lk}(C_1, C_2) = \text{lk}(C_2, C_1)$.

c) Prove that if there exists a compact oriented submanifold $W \subseteq \mathbb{R}^3 \setminus C_2$ with $\partial W = C_1$, then $\text{lk}(C_1, C_2) = 0$.

d) Prove that if C_1 and C_1' are either smoothly homotopic in $\mathbb{R}^3 \setminus C_2$ or are cobordant in $\mathbb{R}^3 \setminus C_2$, then $\text{lk}(C_1, C_2) = \text{lk}(C_1', C_2)$.

Proof.

a) According to a previous homework problem (the last problem of homework 9), two smooths map from S^1 to S^1 are homotopic if and only if they have the same degree. Since f is a orientation-preserving diffeomorphism, it must be that $\deg(f) = 1$. This means that f is homotopic to the uniform speed parameterization $\phi : S^1 \to C_1$ (the one that preserves orientation). This homotopy extends to a homotopy from $L_{f,g}$ to $L_{\phi,g}$. Since the degree of a map is invariant under homotopy, we have that any orientation-preversing diffeomorphism gives the same linking number.

Now, if one reverses the sign of orientation of either C_1 or C_2 (but not both), then this reverses the orientation on T^2. Thus, $L_{f,g}$ reserves orientation, meaning the linking the number $\text{lk}(C_1, C_2)$ changes sign.

b) I don't think this is true. Swapping the two circles C_1 and C_2 is tantamount to swapping f and g. But we see that $L_{f,g} = -L_{g,f} = a \circ L_{g,f}$ where $a : S^2 \to S^2$ is the antipodal map. Since S^2 is an even sphere, the antipodal map has degree -1. This means that

$$\text{lk}(C_1, C_2) = \deg(L_{f,g}) = \deg(a \circ L_{g,f}) = \deg(a)\deg(L_{g,f}) = -\deg(L_{g,f}) = -\text{lk}(C_1, C_2).$$

c) Let $i : W \to \mathbb{R}^3 \setminus C_2$ be the inclusion map and let $L_{i,g} : W \times S^1 \to S^2$ be the map such that

$$L_{i,g}(x, y) = \frac{i(x) - g(y)}{|i(x) - g(y)|}.$$

Since this map is a smooth map such that $L_{i,g}|_{\partial(W \times S^1)} = L_{i|_{C_1}, g} = L_{f,g}$, we have by the Boundary Theorem that

$$\text{lk}(C_1, C_2) = \deg(L_{f,g}) = 0.$$

d) Now if C_1 and C_1' are smoothly homotopic in $\mathbb{R}^3 \setminus C_2$, then they are cobordant in $\mathbb{R}^3 \setminus C_2$ via the image of the homotopy map. Now if C_1 and C_1' are cobordant in $\mathbb{R}^3 \setminus C_2$, then this means there's a compact manifold $W \subseteq \mathbb{R}^3 \setminus C_2$ such that $\partial W = C_1 - C_2$. Let $i : W \to \mathbb{R}^3 \setminus C_2$ be the inclusion map and let $L_{i,g} : W \times S^1 \to S^2$ be defined as

$$L_{i,g}(x, y) = \frac{i(x) - g(y)}{|i(x) - g(y)|}.$$

Since this map is a smooth map such that $L_{i,g}|_{\partial(W \times S^1)} = L_{i|_{C_1 - C_1'}, g}$, we have by the Boundary Theorem that

$$0 = \deg(L_{i|_{C_1 - C_1'}, g}) = \deg(L_{i|_{C_1}, g}) + \deg(L_{i|_{-C_1'}, g}) = \text{lk}(C_1, C_2) - \text{lk}(C_1', C_2),$$

where the negative sign comes from (a), as $-C_1'$ has reversed orientation. This proves that $\text{lk}(C_1, C_2) = \text{lk}(C_1', C_2)$ as we wanted. $\square$