Problem Set 7
Real Analysis I

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Ex 5.1 Suppose $(X, \mathcal{A})$ is a measurable space, f is a real-valued function, and $\{x \mid f(x) > r\} \in \mathcal{A}$ for each rational number r. Prove that f is measurable.

Proof. For any $a \in \mathbb{R}$, we see that from the denseness of $\mathbb{Q}$ in $\mathbb{R}$,

$$\{x \in X \mid f(x) > a\} = \bigcap_{q \in (a,\infty)\cap\mathbb{Q}} \{x \in X \mid f(x) > q\}$$

Since $\mathbb{Q}$ is countable, the intersection is a countable intersection of elements of the σ-algebra $\mathcal{A}$. Thus, the intersection is in $\mathcal{A}$, proving that f is measurable. $\square$

Ex 5.2 Let $f : (0, 1) \to \mathbb{R}$ be such that for every $x \in (0, 1)$ there exists $r > 0$ and a Borel measurable function g, both depending on x, such that f and g agree on $(x-r, x+r) \cap (0, 1)$. Prove that f is Borel measurable.

Proof. We see that for any $n \geq 2$

$$\left[\frac{1}{n}, 1 - \frac{1}{n}\right] \subseteq \bigcup_{x \in (0,1)} (x - r_x, x + r_x)$$

where r_x is the r in the question that depends in x. By compactness, there must be a finite subcovering. Let's denote it by $\{(x_i - r_i, x_i + r_i) \mid 0 < i \leq m\}$ for some $m \in \mathbb{N}$. Let g_i be the Borel measurable set that agrees with f on the interval $(x_i - r_i, x_i + r_i)$. We see then that for any $a \in \mathbb{R}$

$$B_i = \{x \in (x_i - r_i, x_i + r_i) \mid f(x) > a\} = \{x \in (x_i - r_i, x_i + r_i) \mid g_i(x) > a\}$$

is a Borel set. Thus,

$$C_n = \{x \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \mid f(x) > a\} = \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \cap \bigcup_{i=1}^{m} B_i$$

is a Borel set for $n = 1, 2, \ldots$. Finally, we see that

$$\{x \in (0, 1) \mid f(x) > a\} = \bigcup_{n=1}^{\infty} C_n$$

is a Borel set for each $a \in \mathbb{R}$. This proves that f is Borel measurable. $\square$

Ex 5.3 Suppose f is measurable and $f(x) > 0$ for all x. Let $g(x) = 1/f(x)$. Prove that g is a measurable function.

Proof. Since f is positive, then it's clear that g is also positive. Thus, if $a \leq 0$, then $\{x \in X \mid g(x) > a\} = X \in \mathcal{A}$. If $a > 0$, we see that

$$\{x \in X \mid g(x) > a\} = \{x \in X \mid f(x) < \frac{1}{a}\}$$

which is in $\mathcal{A}$, by Proposition 5.5. This proves that g is measurable. □

Ex 5.4 Suppose f_n are measurable functions. Prove that a

$$A = \{x \mid \lim_{n\to\infty} f_n(x) \text{ exists } \}$$

is a measurable set.

Proof. By Proposition 5.8, we know that $\limsup f_n$ and $\liminf f_n$ are both measurable functions, if they are finite. Also, if they are both finite, then by Proposition 5.7, $\limsup f_n - \liminf f_n$ is measurable as well. It follows that

$$A_1 = \{x \mid \lim_{n\to\infty} f_n(x) \text{ exists and is finite } \} = \{x \mid \limsup f_n(x) - \liminf f_n(x) = 0\}$$

$$A_2 = \{x \mid \lim_{n\to\infty} f_n(x) = \infty\} = \bigcap_{i=1}^{\infty} \{x \mid \liminf f_n(x) > i\}$$

$$A_2 = \{x \mid \lim_{n\to\infty} f_n(x) = -\infty\} = \bigcap_{i=1}^{\infty} \{x \mid \liminf f_n(x) < -i\}$$

are measurable sets. Thus, $A = A_1 \cup A_2 \cup A_3$ is also measurable. □

Ex 5.8 Give an example of a collection of measurable non-negative functions $\{f_\alpha\}_{\alpha \in A}$ such that if g is defined by $g(x) = \sup_{\alpha \in A} f_\alpha(x)$, then g is finite for all x but g is non-measurable. (A can be uncountable.)

Proof. Consider $(\mathbb{R}, \mathcal{A})$, where $\mathcal{A}$ is the Lebesgue σ-algebra. Let E be the Vitali set constructed in a past chapter. For each $e \in E$, let $f_e = \chi_{\{e\}}$. Then, we see that each f_e is measurable as sets comprising one point are null sets and hence measurable. It's clear to see that, for any $x \in \mathbb{R}$,

$$g(x) = \sup_{e \in E} f_e(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

and so $g \in \chi_E$, which is non-measurable. □

Ex 5.9 Suppose $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $g : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that $g \circ f$ is Lebesgue measurable. Is this true if g is Borel measurable instead of continuous? Is this true if g is Lebesgue measurable instead of continuous?

Proof. If g is continuous, then it is Borel measurable by Proposition 5.6. If g is Borel measurable and f is Lebesgue measurable and if $a \in \mathbb{R}$, we see that $(g \circ f)^{-1}((a, \infty)) = f^{-1}(g^{-1}((a, \infty)))$. By Proposition 5.11, $g^{-1}((a, \infty))$ is a Borel set, and by the same proposition, we see that $f^{-1}(g^{-1}((a, \infty)))$ is Lebesgue measurable. This answers the first two parts. Now we will give a counterexample to the last question.

Let $\varphi : [0, 1] \to [0, 1]$ be the Cantor-Lebesgue function and let $\psi(x) = x + \varphi(x)$. It is clear that $\psi : [0, 1] \to [0, 2]$. Since φ is continuous and x is continuous, this means that φ is continuous as well. Since φ is monotonically increasing, ψ is strictly increasing, which implies injectivity. Since ψ is continuous and $\psi(0) = 0$ and $\psi(1) = 2$, then ψ is surjective as well. Finally, the continuity of ψ^{-1} follows from it is the inverse of a continuous bijection between compact sets.

Now, let C be the Cantor set in $[0, 1]$. Recall that φ is constant on open intervals contained in the complement of the Cantor set. Thus, if I is such an interval, then $m(\psi(I)) = m(I + c_I)$, where c_I is the constant given by $\varphi(x) = c_I$ for all $x \in I$. Thus, $m(\psi(I)) = m(I)$. The monotonicity and continuity of ψ shows that disjoint open intervals in $[0, 1]$ are mapped into disjoint open intervals of $[0, 2]$. Thus, a $m(\psi([0, 1] \setminus C)) = m([0, 1] \setminus C) = 1$ which means that $m(\psi(C)) = 2 - m(\psi([0, 1] \setminus C)) = 1$. Since $\psi(C)$ is closed and has positive measure, by Question 4.14, we see that there's a non-measurable set $D \subseteq \psi(C) \subseteq [0, 2]$.

Let $E \subseteq [0, 1]$ where $E = \psi^{-1}(D)$ and let $g = \chi_E$. Since $D \subseteq \psi(C)$, we see that $E \subseteq C$, and thus E is a null set and therefore measurable. This proves that g is a measurable function. Let $f = \psi^{-1}$ and remember that $f : [0, 2] \to [0, 1]$ is continuous. We see that since $g : [0, 1] \to \{0, 1\}$, we have that $g \circ f : [0, 2] \to \{0, 1\}$ is the composition of a Lebesgue measurable function and a continuous function. However, a

$$(g \circ f)(x) = \chi_E(f(x)) = \chi_{f^{-1}(E)}(x) = \chi_{\psi(E)}(x) = \chi_D(x)$$

which is clearly non-measurable. This shows that even if f is continuous and g is Lebesgue measurable, then $g \circ f$ is not necessarily Lebesgue measurable. $\square$