Problem Set 7
Abstract Algebra 1

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Section 3.4

Ex 5 Prove that subgroups and quotient groups of a solvable group are solvable.

Ex 6 Prove part (1) of the Jordan-Holder Theorem by induction on $|G|$.

Ex 9 Prove the following special case of part (2) of the Jordan-Holder Theorem: assume the finite group G has two composition series

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \dots N_r = G \qquad 1 = M_0 \trianglelefteq M_1 \trianglelefteq M_2 = G$$

Show that $r = 2$ and that the list of composition factors is the same.

5.1

Ex 1 Show that the center of a direct product is the direct product of the centers.

$$Z(G_1 \times G_2 \times \cdots \times G_n) = Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n)$$

Deduce that a direct product of groups is abelian if and only if each of the factors is abelian.

Ex 5 Exhibit a nonnormal subgroup of $Q_8 \times Z_4$ (note that every subgroup of each factor is normal).

Ex 7 Let $G_1, G_2, \dots, G_n$ be groups and let π be a fixed element of S_n. Prove that the map

$$\varphi : G_1 \times \cdots \times G_n \to G_{\pi^{-1}(1)} \times \cdots \times G_{\pi^{-1}(n)}$$

defined by

$$\varphi_\pi(g_1, g_2, \dots, g_n) = \left(g_{\pi^{-1}(1)}, g_{\pi^{-1}(2)}, \dots, g_{\pi^{-1}(n)}\right)$$

is an isomorphism (so that changing the order of factors in a direct product does not change the isomorphism type).

Ex 8 Let $G_1 = G_2 = \cdots = G_n$ and let $G = G_1 \times \cdots \times G_n$. Under the notation of the preceding exercise show that $\varphi_\pi \in \mathrm{Aut}(G)$. Show also that the map $\pi \mapsto \varphi_\pi$ is an injective homomorphism of S_n into $\mathrm{Aut}(G)$.

Ex 9 Let G_i be a field F for all i and use the preceding exercise to show that the set of $n \times n$ matrices with one 1 in each row and each column is a subgroup of $\mathrm{GL}_n(F)$ isomorphic to S_n.

5.4

Ex 9 Prove that if p is an odd prime and P is a group of order p^3 then $P' = Z(P)$.

Ex 15 If A and B are normal subgroups of G and K is cyclic, prove that $G' \leq \mathrm{C}_G(K)$. (Recall that the automorphism group of a cyclic group is abelian.)