Problem Set 4
Real Analysis 1

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Ex 4.1 Let μ be a measure on the Borel σ-algebra of $\mathbb{R}$ such that $\mu(K) < \infty$ whenever K is compact, define $\alpha(x) = \mu((0, x])$ if $x \geq 0$ and $\alpha(x) = -\mu((x, 0])$ if $x < 0$. Show that μ is the Lebesgue-Stieltjes measure corresponding to α.

Proof. Let $A \subseteq B$, where $\mu(B) < \infty$. Then, $B = (B \cap A) \cup (B \setminus A) = A \cup (B \setminus A)$. We see that $\mu(B) = \mu(A) + \mu(B \setminus A)$. Since $\mu(B) < \infty$, this means that $\mu(A) \leq \mu(B) < \infty$, and thus $\mu(B \setminus A) = \mu(B) - \mu(A)$. This will be important in the next paragraph.

For the Lebesgue-Stieltjes measure, we know that $\ell((a, b]) = \alpha(b) - \alpha(a)$. Since $[a, b]$ is compact, and $(0, b]$ and $(a, 0]$ are subsets of this set, then they must be finite as well. Thus, there are three cases:

$$0 \leq a \leq b \implies \alpha(b) - \alpha(a) = \mu((0, b]) - \mu((0, a]) = \mu((0, b] \setminus (0, a]) = \mu((a, b])$$

$$a < 0 \leq b \implies \alpha(b) - \alpha(a) = \mu((0, b]) + \mu((a, 0]) = \mu((0, b] \cup (a, 0)) = \mu((a, b])$$

$$a \leq b < 0 \implies \alpha(b) - \alpha(a) = -\mu((b, 0]) + \mu((a, 0]) = \mu((a, 0] \setminus (b, 0]) = \mu((a, b])$$

We see that $b < 0$ and $a \geq 0$ is impossible, as $a \leq b$. This proves that $\ell((a, b]) = \mu((a, b])$.

Now, let A be a m-measurable set. We see then that there exists $B = \cup_{i=1}^{\infty} B_i$, where $B_i = (c_i, d_i] \in \mathcal{C}$, $A \subseteq B$, and $m(B) \leq m(A) + \varepsilon$. (That is to say, there exists a set of half-closed intervals that is arbitrarily close to the infinmum.) Thus, we see that

$$m(B) = \sum_{i=1}^{\infty} m(B_i) = \sum_{i=1}^{\infty} (\alpha(d_i) - \alpha(c_i)) = \sum_{i=1}^{\infty} \mu(B_i) = \mu(\cup_{i=1}^{\infty} B_i) = \mu(B)$$

Thus, since $A \subseteq B$, we see that $\mu(A) \leq \mu(B) = m(B) \leq m(A) + \varepsilon$. Since ε was abitrary, we see that $\mu(A) \leq m(A)$.

Since A is measurable, then A^c is measurable. By a similar argument, $\mu(A^c) \leq m(A^c)$. This means that $\mu(A) + m(A) + \mu(A^c) \leq \mu(A) + m(A) + m(A^c)$, which means that $\mu(A \cup A^c) + m(A) \leq \mu(A) + m(A \cup A^c)$. Thus, $m(A) \leq \mu(A)$. This proves that $\mu(A) = \mu(A)$ for all m-measurable sets. $\square$

Ex 4.2 Let m be Lebesgue measure and A a Lebesgue measurable subset of $\mathbb{R}$ with $m(A) < \infty$. Let $\varepsilon > 0$. Show there exist G open and F closed such that $F \subseteq A \subseteq G$ and $m(G \setminus F) < \varepsilon$.

Proof. Let $B = \cup_{i=1}^{\infty} B_i$ where $B_i = (c_i, d_i]$, $A \subseteq B$, and $\sum_{i=1}^{\infty} \ell(B_i) \leq m(A) + \frac{\varepsilon}{4}$. Note that $m(B) = m(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \ell(B_i)$. Thus, $m(B) \leq m(A) + \frac{\varepsilon}{4}$. Let $G_i = B_i \cup (d_i, e_i)$, where $\ell((d_i, e_i)) = e_i - d_i < \frac{\varepsilon}{2^{i+2}}$. We see that $G_i = (c_i, e_i)$, and thus $\cup_{i=1}^{\infty} G_i = G$ is open, as it's the union of open intervals. We also see that $A \subseteq B = \cup_{i=1}^{\infty} B_i \subseteq \cup_{i=1}^{\infty} G_i = G$. Thus, $A \subseteq G$ and G is an open set. We see that the measure of G can be computed by the following:

$$\begin{aligned} m(G) = m(\cup_{i=1}^{\infty} G_i) = \sum_{i=1}^{\infty} m(G_i) = \sum_{i=1}^{\infty} (e_i - c_i) = \sum_{i=1}^{\infty} ((e_i - d_i) + (d_i - c_i)) \\ \leq \sum_{i=1}^{\infty} \left(\frac{\varepsilon}{2^{i+2}} + \ell(B_i) \right) = \sum_{i=1}^{\infty} \ell(B_i) + \frac{\varepsilon}{4} = m(B) + \frac{\varepsilon}{4} \leq m(A) + \frac{\varepsilon}{2} \end{aligned}$$

Since A was measurable, then A^c is measurable. By the same argument, there's a V that is open that contains A^c and $m(V) \leq m(A^c) + \frac{\varepsilon}{2}$. Let $V = F^c$. Thus, F is a closed set that is contained in A. We see that

$$\begin{aligned} m(G \setminus F) = m(G \cup F^c) = m(G) + m(F^c) - m(\varnothing) = m(G) + m(F^c) - m(A \cup A^c) \\ = (m(G) - m(A)) + (m(V) - m(A^c)) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This proves the above statement. (Lots of ε's!) □

Ex 4.3 If $(X, \mathcal{A}, \mu)$ is a measure space, define

$$\mu^*(A) = \inf\{\mu(B) \mid A \subseteq B, B \in \mathcal{A}\}$$

for all subsets A of X. Show that μ^* is an outer measure. Show that each set in $\mathcal{A}$ is μ^*-measurable and μ^* agrees with the measure μ on $\mathcal{A}$.

Proof. Suppose $A \in \mathcal{A}$. We see that $A \subseteq A \in \mathcal{A}$, which means that $\mu^*(A) \leq \mu(A)$. Now let B be a set in $\mathcal{A}$ such that $A \subseteq B$. We see that $\mu(A) \leq \mu(B)$. Thus, $\mu(A)$ is a lower bound for all such $\mu(B)$, where $A \subseteq B$. This proves that for $A \in \mathcal{A}$, $\mu^*(A) = \mu(A)$. We see for a special case that $\mu^*(\varnothing) = \mu(\varnothing) = 0$.

Assume that $A \subseteq B$. Let $\mathbf{A} = \{S \in \mathcal{A} \mid A \subseteq S\}$ and $\mathbf{B} = \{S \in \mathcal{A} \mid B \subseteq S\}$. We see that if $C \in \mathbf{B}$, then $C \in \mathcal{A}$ and $B \subseteq C$. Since $A \subseteq B$, this means that $A \subseteq B \subseteq C$. This proves that $C \in \mathbf{A}$. This means that $\mathbf{B} \subseteq \mathbf{A}$. Since $\mathbf{B} \subseteq \mathbf{A}$, this means that $\mu^*(A) = \inf\{\mu(S) \mid S \in \mathbf{A}\} \leq \inf\{\mu(S) \mid S \in \mathbf{B}\} = \mu^*(B)$.

Let $A = \cup_{n=1}^{\infty} A_n$ and let $\varepsilon > 0$. Suppose $\mu^*(A_n) = \infty$ for some n. Then as $A_n \subseteq \cup_{n=1}^{\infty} A_n$, by the last paragraph, this means that $\mu^*(\cup_{n=1}^{\infty} A_n) \geq \mu^*(A_n) = \infty \geq \sum_{n=1}^{\infty} A_n$ trivally. Thus, we may assume that $\mu^*(A_n)$ is finite for all n. Choose $B_n \in \mathcal{A}$, where $A_n \subseteq B_n$ and $\mu(B_n) \leq \mu^*(A_n) + \varepsilon 2^{-n}$. Since $A \subseteq B = \cup_{n=1}^{\infty} B_n$, we see that $\mu^*(A) \leq \mu^*(B) = \mu(B) \leq \sum_{n=1}^{\infty} B_n \leq \sum_{n=1}^{\infty} A_n + \varepsilon$. Since ε was arbitrary, this means that $\mu^*(A) = \mu^*(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$. This proves that μ^* is an outer measure, and since we've already proven that $\mu^*(A) = \mu(A)$ for all $A \in \mathcal{A}$, we're done. □

Ex 4.4 Let m be Lebesgue-Stieltjes measure corresponding to a right continuous increasing function α. Show that for each x,

$$m(\{x\}) = \alpha(x) - \alpha(x-)$$

Proof. We see that $\{x\}$ is a Borel set, and is thus measurable under m. This means that

$$m\left(\{x\}\right)=\lim_{n\to\infty}m\left(\left(x-\frac{1}{n},x\right]\right)=\lim_{n\to\infty}\alpha\left(x\right)-\alpha\left(x-\frac{1}{n}\right)=\alpha\left(x\right)-\alpha\left(x^{-}\right)$$

This proves the statement. $\square$