Graph Theory
Problem Set 3

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Ex 1.2.7 Prove that a bipartite graph has a unique bipartition (except for interchanging the two partite sets) if and only if it is connected.

Proof. Let G be a disconnected bipartite graph and let X and Y be distinct connected components of G. Then, since G is bipartite, X and Y must be bipartite. If we let X_1 and X_2 be a bipartition of X and let Y_1 and Y_2 be a bipartition of Y, then we have that $X_1 \cup Y_1, X_2 \cup Y_2$ and $X_1 \cup Y_2, X_2 \cup Y_1$ are two different bipartitions of G. Thus, G does not have a unique bipartition.

Now let G be a bipartite graph with two distinct bipartitions U, V and X, Y. Since these bipartitions are distinct, we have that $U \cap X$, $U \cap Y$, $V \cap X$, and $V \cap Y$ are non-empty. We use these sets to construct the sets $H_1 = (X \cap U) \cup (Y \cap V)$ and $H_2 = (X \cap V) \cup (Y \cap U)$, which we know are non-empty. Let $x \in H_1$ and let $y \in H_2$. This means that x is either in both X and U or in both Y and V. Similarly, y is either in both X and V or Y and U. We see that in all cases, x and y are both in one of the four sets. But that means that they are independent from each other and thus have no edge connecting them. Since x and y were arbitrary, there is no edge that connects the sets H_1 and H_2, which proves that G is disconnected. $\square$

Ex 1.2.8 Determine the values of m and n such that $K_{m,n}$ is Eulerian.

Proof. By Theorem 1.2.26, a graph is Eulerian if and only if it has at most one nontrivial component and its vertices all have even degree. Since $K_{m,n}$ is connected, it always has exactly one nontrivial component. Thus, we only need to find when the graph $K_{m,n}$ is even.

Now, we know that $K_{m,n}$ decomposes into two disjoint independent sets U and V where $|U| = n$ and $|V| = m$. If we let $u \in U$ be an arbitrary vertex, by the definition of $K_{m,n}$, the only edges incident to u are the edges between u and every vertex of V. Since the degree of u needs to be even, it follows that $|V| = m$ must be even. Similarly, $|U| = n$ must be even as well. Conversely, if n, m are even, it follows easily that each vertex is even. Thus, $K_{m,n}$ is Eulerian if and only if m and n are even. $\square$

Ex 1.2.10 Prove or disprove:

a) Every Eulerian bipartite graph has an even number of edges.

b) Every Eulerian simple graph with an even number of vertices has an even number of edges.

Proof.

a) Let G be a Eulerian bipartite graph. This means that for some disjoint independent vertex sets U and V, we have that $V(G) = U \cup V$. Now, let $(v_i)_{i \in I}$ be a Eulerian trail (i.e. is closed and contains all edges once), where $I = \{0, 1, \ldots, n\}$ for some n. Without loss of generality, suppose $v_0 \in U$. Since U is an independent set, it must be that $v_1 \in V$. We see then that if i is odd, then $v_i \in V$ and if i is even, then $v_i \in U$. Since $v_n = v_0 \in U$, then this means that n must be even. This means that our Eulerian trail has an even number of edges. Since a Eulerian trail contains all edges exactly once, this means that $|E(G)|$ is even as well.

b) This is false. A simple counterexample is the graph $K_3 \times K_1$ (a triangle with a edgeless vertex), which is Eulerian, has an even number of vertices, but only has 3 edges.

□

Ex 1.2.11 Prove or disprove: If G is a Eulerian graph with edges e, f that share a vertex, then G has a Eulerian circuit in which e, f appear consecutively.

Proof. This is false. Take for example the following "bowtie" graph:

We clearly see that the closed trail 3253143 is Eulerian, which proves that the graph is Eulerian. However, if one starts at the vertex 1, 3, or 4, it is clearly impossible to travel through e and f consecutively. Additionally, if one starts at the vertex 2 (or 5), the only way to travel through e and f consecultively would be to go around the cycle 235, which means that one will end back up at vertex 2 (or 5) and not be able to complete the circuit. Thus, the proposition is false for this graph. $\square$

Ex 1.2.15 Let W be a closed walk of length at least 1 that does not contain a cycle. Prove that some edge of W repeats immediately (once in each direction).

Proof. We note that it's impossible for a simple graph to contain a closed walk of length 1. We shall now procede by induction on the length of W. For a closed walk of length 2, it must be that W travels to an adajecent vertex and then immediately returns, which means that it immediately repeats an edge. Now, assume that W is of length n and that all smaller walks with no cycle have such an immediate edge repetition. Now, it must be that W repeats a vertex, otherwise W itself would be a cycle (as not repeating a vertex also implies not repeating an edge). We call this repeated vertex v. We can then deconstruct W into two closed walks which both start and end at v. Since these closed walks are of smaller size than W and also do not contain a cycle (as otherwise W would've contained a cycle), they must contain an immediate edge repetition. Thus, W must contain an immediate edge repetition. $\square$