Problem Set 5
Differential Topology

Bennett Rennier
bennett@brennier.com

October 5, 2018

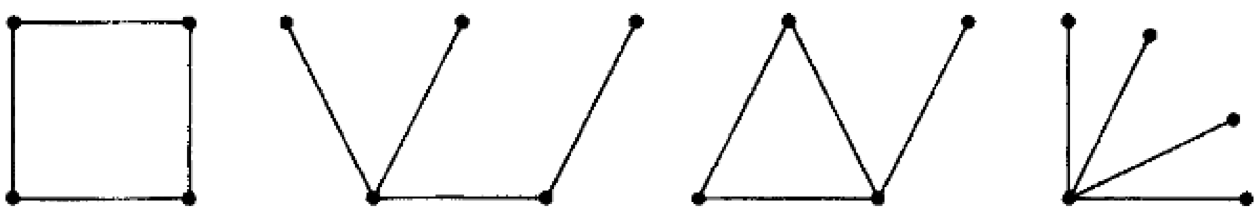
Gamelin-Green, Chapter 2, Section 8

Ex 3 Consider the two tangent open discs $\{(x,y) : x^2+y^2<1\}$ and $\{(x,y) : (x-2)^2+y^2<1\}$ in $\mathbb{R}^2$. Is the union the discs a connected subset of $\mathbb{R}^2$? Is the union of their closures a connected subset of $\mathbb{R}^2$? Is the union of one disc and the closure of the other a connected subset of $\mathbb{R}^2$?

Proof. We will refer to the disc centered at the origin as S and the other as S'. We note these sets are both open in $\mathbb{R}^2$. This means they're open in the subspace topology induced $S \cup S'$. Thus, S and S' are themselves disjoint non-empty open sets that partition the space $S \cup S'$.

We recall that path-connected implies connected (See Ex 1 of the next section if you aren't convinced). Since S, S', $\overline{S} = \{(x,y) : x^2+y^2 \leq 1\}$, and $\overline{S'} = \{(x,y) : (x-2)^2+y^2 \leq 1\}$ are all convex sets, they are individually path-connected. Now, if we look at the path $\gamma : [0,1] \to \mathbb{R}$ where $\gamma(t) = (2t, 0)$, we see that $\mathrm{Im}(\gamma) = (0,2) \times \{0\} \subseteq S \cup S' \cup \{(1,0)\}$. Since the point $(1,0)$ is a member of $\overline{S}$ and $\overline{S'}$, we see that γ is a valid path in the spaces $\overline{S} \cup S'$, $S \cup \overline{S'}$, and $\overline{S} \cup \overline{S'}$. In any of these cases, as γ is a path between two path-connected sets, we see that any two points can be connected by a path. Thus, all these spaces are connected. $\square$

Ex 6 Each of the topological spaces in the figure below is the union of four closed intervals. Are any of the spaces homeomorphic?



Proof. For a topological space X, we call a point $x \in X$ a n-th degree *cut-vertex* if X is connected (i.e. only has one component) but $X \setminus \{x\}$ has n components. Let $\phi : X \to Y$ be a homeomorphism of topological spaces where X has only one component and $x \in X$ is a cut-vertex of degree n. Since the number of components of a space is preserved under homeomorphism, we know Y must also be connected. We note that the restriction $\phi|_{X \setminus \{x\}} : X \setminus \{x\} \to Y \setminus \{\phi(x)\}$ is also a homeomorphism. Again, since the number of components is perserved under homeomorphism, we have that $Y \setminus \{y\}$ has n components. Thus, $y \in Y$ is also a cut-vertex of degree n. This proves having a cut-vertex of degree n is perserved under homeomorphism.

We can see that the first figure has no cut-vertices of any degree, the second has a cut-vertex of degree 3 but none of degree 4, the third has cut-vertices of degree 2 but none of degree 3, and the fourth has a cut-vertex of degree 4. This proves that none of them can be homeomorphic to each other. ☐

Ex 10 Show by counterexample that a connected component of a topological space is not necessarily open.

Proof. Let $X = \{\frac{1}{2^n} : n \in \mathbb{N}_0\} \cup \{0\}$ be a topological subspace of $\mathbb{R}$. I claim that the set $\{0\}$ is a connected component of X that is not open. Since it's a singleton, it is trivally connected. Let S be a connected set of X that contains 0. Let's first look at some of the important open sets of X. We see that for any point $\frac{1}{2^k}$ in X, the open set $U_k = B_{\frac{1}{2^{k+1}}}(\frac{1}{2^k}) \cap X$ is simply $\{\frac{1}{2^k}\}$. And secondly, we see that the open set $V_k = B_{\frac{1}{2^k}}(0) \cap X$ is simply the set $\{\frac{1}{2^n} : n \geq k\} \cup \{0\}$. Thus, if S contains any point of the form $\frac{1}{2^k}$, then S can be separated by the open sets U_k and $(\cup_{i \in \mathbb{N} \setminus k} U_i) \cup B_{k-1}$. This proves that S must be the set $\{0\}$, proving that $\{0\}$ is a maximally connected set.

Since $\frac{1}{2^n}$ is a sequence converging to zero, the set $\{\frac{1}{2^n} : n \in \mathbb{N}_0\}$ does not contain all of its limit points, proving that the set isn't closed. This means the complement $\{0\}$ cannot be open. This proves the claim. ☐

Gamelin-Green, Chapter 2, Section 9

Ex 1 Prove that any subinterval of $\mathbb{R}$ is path-connected.

Proof. Let (a, b) be an open interval of $\mathbb{R}$ and let $u, v \in (a, b)$. We see then that the path $\gamma : [0, 1] \to \mathbb{R}$ where $\gamma(t) = (1 - t)u + tv$ is a path from u to v such that $\text{Im}(\gamma) = [u, v] \subseteq (a, b)$. Since $u, v \in (a, b)$ where arbitrary, we have that (a, b) is path-connected. The proof that $(a, b]$ and that $[a, b]$ are path-connected follows by the same argument. ☐

Ex 3 Prove that if X is path-connected and $f : X \to Y$ is a map, then $f(X)$ is path-connected.

Proof. Let $u, v \in f(X)$. Let $x \in f^{-1}(u)$ and $y \in f^{-1}(v)$ (there may be more than one such x or y; just choose any one of them). Since $x, y \in X$, there exists a path $\gamma : [0, 1] \to X$ from x to y. Thus, the map $f \circ \gamma : [0, 1] \to f(X)$ is a path from $f(\gamma(0)) = f(x) = u$ to $f(\gamma(1)) = f(y) = v$. Since $u, v \in f(X)$ were arbitrary, $f(X)$ is path-connected. ☐

Ex 6 An open subset of $\mathbb{R}^n$ is connected if and only if it is path-connected.

Proof. $\implies$) Let $U \subseteq \mathbb{R}^n$ be an connected open set, let $x \in U$, and let $P \subseteq U$ be the set of all points y such that there is a path between x and y in U. We see that if $y \in P \subseteq U$, then there's an $r > 0$ such that $y \in B_r(y) \subseteq U$. We note that $B_r(x)$ is convex, so there is straight-line path between y and any point in $B_r(y)$. Since there's a path from x to y by the definition of P, this means there's a path from x to any point in $B_r(y)$. This proves that $B_r(y) \subseteq P$, meaning P is open.

Now consider a point $y \in P^c$. This means there is no path from x to y. Since $y \in U$, there's an $r > 0$ such that $y \in B_r(y) \subseteq U$. Now, if there were a path between x and some point in $B_r(y)$, then since $B_r(y)$ is convex, there'd be a path from x to y. This can't happen by definition of y, so it must be that there is no path between x and any point of $B_r(y)$. This proves that $B_r(y) \subseteq P^c$, meaning P^c is open. Thus P is closed.

Since $x \in P$ using the trivial path and we have proven that P is both open and closed as a subset of a connected set U. This proves by connectedness that $P = U$. Thus, for any points $u, v \in U$, there is a path from x to u and from x to v. Concatenation of these paths gives us that there's a path from any $u \in U$ to any $v \in U$. This proves that U is path-connected.

$\Longleftarrow$) Let $X \subseteq \mathbb{R}^n$ be a path-connected set (doesn't have to be open). Suppose X were not connected. Then there exists non-empty disjoint open sets U, V such that $U \cup V = X$. Since X is path-connected, if we let $x \in U$ and $y \in V$, then there exists a path $\gamma : [0,1] \to \mathbb{R}^n$ from x to y. This means that $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are disjoint open sets. They're non-empty as $0 \in \gamma^{-1}(U)$ and $1 \in \gamma^{-1}(V)$. Also, $\gamma^{-1}(U) \cup \gamma^{-1}(V) = \gamma^{-1}(U \cup V) = \gamma^{-1}(X) = [0,1]$. Thus, $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are non-empty, disjoint open sets of $[0,1]$ whose union is all of $[0,1]$. This would imply that $[0,1]$ is not connected, which we know is not the case. This leads to a contradiction, which proves that X must be connected. $\square$

Guillemin-Pollack, Chapter 1, Section 4

Ex 1 If $f : X \to Y$ is a submersion and U is an open set of X, show that $f(U)$ is open in Y.

Proof. Let U be an open set of X and let $x \in X$. By the Local Submersion Theorem, f is locally equivalent to the canonical submersion near x. That is, there exist local coordinates around x and $f(x)$ such that $f(x_1, \ldots, x_k) = f(x_1, \ldots, x_\ell)$. Since this is a local property, we can always restrict these local coordinates to an open set U_x contained in U. Since the projection map is an open map, this means f takes U_x to an open set of $f(x)$ inside $f(U)$. Thus, we have that

$$f(U) = f\left(\bigcup_{x \in U} U_x\right) = \bigcup_{x \in U} f(U_x).$$

Since each U_x is open, this proves that $f(U)$ is open as well.

Note) I'll add in a proof that for $k \geq \ell$ the projection map $\pi : \mathbb{R}^k \to \mathbb{R}^\ell$ onto the first ℓ coordinates is an open map. Let U' be an open set of $\mathbb{R}^k$. Since $\mathbb{R}^k \simeq \mathbb{R}^\ell \times \mathbb{R}^{k-\ell}$, by the definition of the product topology, we have that for any $x \in U'$, there are open sets $U_x \subseteq \mathbb{R}^\ell$ and $V_x \subseteq \mathbb{R}^{k-\ell}$ such that $x \in U_x \times V_x \subseteq U$. Since $\pi(U_x \times V_x) = U_x$ is open, we have that

$$\pi(U') = \pi\left(\bigcup_{x \in U'} U_x \times V_x\right) = \bigcup_{x \in U'} \pi(U_x \times V_x) = \bigcup_{x \in U'} U_x.$$

This proves that $\pi(U')$ is open, meaning π is an open map. $\square$

Ex 2

a) If X is compact and Y connected, show every submersion $f : X \to Y$ is surjective.

b) Show that there exist no submersions of compact manifolds into Euclidean spaces.

Proof.

a) Since f is a submersion, we know by Ex 1 that f is an open map. In particular, $f(X)$ is an open set. Since X is compact we also know that $f(X)$ is compact and thus closed (as we are in an Hausdorff space). Assuming that $X \neq \varnothing$ (pretty safe assumption in my opinion), we also know that $f(X) \neq \varnothing$. Since Y is connected and $f(X)$ is open, closed, and non-empty, it must be that $f(X) = Y$, proving that f is surjective.

b) Suppose $f : X \to \mathbb{R}^k$ is a submersion for some compact manifold X. Since $\mathbb{R}^k$ is connected, we have by part (a) that f is surjective. But this would imply that $f(X) = \mathbb{R}^k$ is compact, which is a contradiction. Thus, there exist no submersions of compact manifolds into an Euclidean space. $\square$

Ex 6 Let p be any homogeneous polynomial in k-variables. Homogeneity means

$$p(tx_1, \ldots, tx_k) = t^m p(x_1, \ldots, x_k).$$

Prove that the set of points x, where $p(x) = a$, is a $k-1$ dimensional submanifold of $\mathbb{R}^k$, provided that $a \neq 0$. Show that manifolds obtained with $a > 0$ are all diffeomorphic, as are those with $a < 0$.

Proof. We need only to show that the values $\mathbb{R} \setminus \{0\}$ are regular values of the function, i.e. df_x is surjective for $f(x) \neq 0$. Since df_x is a linear map from $\mathbb{R}^k$, a k-dimensional space, to $\mathbb{R}$, a 1-dimensional space, we need only to show that it's not the zero map. Let $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and let $\gamma : (1-\varepsilon, 1+\varepsilon) \to \mathbb{R}^k$ be a path such that $\gamma(t) = (tx_1, \ldots, tx_k)$. We see that $\gamma(1) = x$ and that $\gamma'(1) = x$. This means

$$df_x(x) = \frac{d}{dt} f(\gamma(t))|_{t=1} = \frac{d}{dt} f(tx_1, \ldots, tx_k)|_{t=1} = \frac{d}{dt} t^m f(x_1, \ldots, x_k)|_{t=1} = mt^{m-1} f(x)|_{t=1} = mf(x).$$

This proves that if $f(x) \neq 0$, then df_x is not the zero map and is therefore surjective. This means that for $a \neq 0$, a is a regular value, meaning $f^{-1}(a)$ is a $k-1$ dimensional manifold as we wanted.

Let M_a be the manifold corresponding to $\phi^{-1}(a)$ for $a \neq 0$. Let $\ell > 0$ and let $\phi : \mathbb{R}^k \to \mathbb{R}^k$ be the map where $\phi(x_1, \ldots, x_k) = (\sqrt[m]{1/\ell} \cdot x_1, \ldots, \sqrt[m]{1/\ell} \cdot x_k)$. We see that ϕ is smooth (as it's linear) and that for $x \in M_\ell$

$$f(\phi(x)) = f(\sqrt[m]{1/\ell} \cdot x_k) = (\sqrt[m]{1/\ell})^m f(x) = 1/\ell \cdot \ell = 1.$$

Thus, ϕ maps M_ℓ to M_1. Simiarly, the inverse map $\phi^{-1}(x) = \sqrt[m]{\ell} \cdot x$ smoothly maps M_1 to M_ℓ. This proves that all M_a where $a > 0$ are diffeomorphic to M_1. Similarly, we can use the same map and its inverse to show that $M_{-\ell}$ is diffeomorphic to M_{-1}, proving that all M_a where $a < 0$ are diffeomorphic to M_{-1}. $\square$

Ex 10 Verify that the tangent space to $O(n)$ at the identity matrix I is the vector space of skew-symmetric $n \times n$ matrices–that is, matrices A satisfying $A^t = -A$.

Proof. From class, we found that $O(n)$ can be realized as $f^{-1}(\mathrm{Id}_n)$ where $f : M(n) \to M(n)$ is the smooth map $f(A) = A^T A$. We recall that the tangent space to $O(n)$ at the identity matrix can be found as the kernel of the map df_{Id_n}. To find this, let $A \in M(n)$ and let $\gamma : (-\varepsilon, \varepsilon) \to M(n)$ be the path $\gamma(t) = \mathrm{Id}_n + tA$. We note that $\gamma(0) = \mathrm{Id}_n$ and that $\gamma'(0) = A$. This means that

$$\begin{aligned} df_{\mathrm{Id}_n}(A) &= \frac{d}{dt} f(\gamma(t))|_{t=0} = \frac{d}{dt}(\mathrm{Id}_n + tA)^t(\mathrm{Id}_n + tA)|_{t=0} = \frac{d}{dt}(\mathrm{Id}_n^t + (tA)^t)(\mathrm{Id}_n + tA)|_{t=0} \\ &= \frac{d}{dt}(\mathrm{Id}_n + tA^t)(\mathrm{Id}_n + tA)|_{t=0} = \frac{d}{dt} \mathrm{Id}_n^2 + tA^t \mathrm{Id}_n + tA \mathrm{Id}_n + t^2 AA^t|_{t=0} \\ &= \frac{d}{dt} \mathrm{Id}_n + tA^t + tA + t^2 AA^t|_{t=0} = A^t + A + 2tAA^t|_{t=0} = A^t + A. \end{aligned}$$

Thus, the kernel of df_{Id_n} is the set of all matrices such that $A^t = -A$, i.e. the skew-symmetric matrices. $\square$

Ex 11

a) The $n \times n$ matrices with determinant $+1$ form a group denoted $SL(n)$. Prove that $SL(n)$ is a submanifold of $M(n)$ and thus is a Lie group.

b) Check that the tangent space to $SL(n)$ at the identity matrix consists of all matrcies with trace equal to zero.

Proof.

a) We note that $M(n)$ is isomorphic to $\mathbb{R}^{n^2}$. Under this isomorphism, the determinant map can be viewed as a polynomial of n^2 variables corresponding to the entries of a matrix in $M(n)$. We further note that for any $A \in M(n) \simeq \mathbb{R}^{n^2}$ and $t \in \mathbb{R}$, we have that $\det(tA) = t^n \det(A)$. This means the determinant map is a homogeneous polynomial. By Ex 6, this means that every $a \neq 0$ is a regular value of det. In particular, $a = 1$ is a regular value, proving that $\det^{-1}(1) = SL(n)$ is a submanifold of $M(n)$.

b) We recall that the tangent space to $SL(n)$ at the identity matrix can be found as the kernel of the map $d\det_{\mathrm{Id}_n}$. To find this, let $A \in M(n)$ and let $\gamma : (-\varepsilon, \varepsilon) \to M(n)$ be the path $\gamma(t) = e^{At}$. We note that $\gamma(0) = \mathrm{Id}_n$ and that $\gamma'(0) = A$. This means that

$$\begin{aligned} d\det_{\mathrm{Id}_n}(A) &= \frac{d}{dt}\det(\gamma(t))|_{t=0} = \frac{d}{dt}\det(e^{tA})|_{t=0} = \frac{d}{dt}e^{\mathrm{Tr}(tA)}|_{t=0} = \frac{d}{dt}e^{t\,\mathrm{Tr}(A)}|_{t=0} \\ &= \mathrm{Tr}(A)e^{t\,\mathrm{Tr}(A)}|_{t=0} = \mathrm{Tr}(A). \end{aligned}$$

Thus, the kernel of $d\det_{\mathrm{Id}_n}$ is the set of all matrices with trace equal to zero as we wanted.

Note) The identity $\det(e^A) = e^{\mathrm{Tr}(A)}$ is taken as an assumption in this proof. To give a general outline of how to prove this, consider a matrix A as a matrix over the complex numbers, change basis so that A is an upper triangular matrix (which is always possible over $\mathbb{C}$), and then the numbers on the diagonal are the eigenvalues of A, say λ_i for $i \leq n$. Once we have that, we prove that the eigenvalues of e^A are simply e^{λ_i} for $i \leq n$. Then we get that

$$\det(e^A) = \prod_{i \leq n} e^{\lambda_i} = e^{\left(\sum_{i \leq n} \lambda_i\right)} = e^{\mathrm{Tr}(A)}$$

as we want. ∎