Problem Set 3
Real Analysis I

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Exercise 3.1. Suppose $(X, \mathcal{A})$ is a measurable space and μ is a non-negative set function that is finitely additive and such that $\mu(\varnothing) = 0$ and $\mu(B)$ is finite for some non-empty $B \in \mathcal{A}$. Supppose that whenever A_i is an increasing sequence of sets in $\mathcal{A}$, then $\mu(\cup_i A_i) = \lim\limits_{i\to\infty} \mu(A_i)$. Show that μ is a measure.

Proof. Let $A_i \in \mathcal{A}$ be mutually disjoint. Now, let $B_n = \cup_{i=1}^n A_i$. This means that $B_{n-1} = \cup_{i=1}^{n-1} A_i \subseteq \cup_{i=1}^n A_i = B_n$. This shows that B_n is increasing. We also see that $\cup_{i=1}^\infty B_i = \cup_{i=1}^\infty A_i$. Now we see that

$$\mu(\cup_{i=1}^\infty A_i) = \mu(\cup_{i=1}^\infty B_i) = \lim_{i\to\infty} \mu(B_i) = \lim_{i\to\infty} \mu\left(\cup_{n=1}^i A_n\right) = \lim_{i\to\infty} \sum_{n=1}^{i} \mu(A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

Thus, we have proven that μ is a measure. □

Exercise 3.2. Suppose $(X, \mathcal{A})$ is a measurable space and μ is a non-negative set function that is finitely additive and such that $\mu(\varnothing) = 0$ and $\mu(X) < \infty$. Suppose that whenever A_i is a sequence of sets in $\mathcal{A}$ that decrease to $\varnothing$, then $\lim\limits_{i\to\infty} \mu(A_i) = 0$. Show that μ is a measure.

Proof. Let $A_i \in \mathcal{A}$ be mutually disjoint. Now, let $B_n = \cup_{i=n+1}^\infty A_i$. We see that $B_{n+1} = \cup_{i=n+2}^\infty A_i \subseteq \cup_{i=n+1}^\infty A_i = B_n$. Thus, this is a decreasing sequence. Let $x \in \cap_{n=1}^\infty B_n$. This means that $x \in B_n$ for all n. Thus, $x \in \cup_{i=n+1}^\infty A_i$ for all n. Take $n = 1$, this means that $x \in \cup_{i=2}^\infty A_i$. Thus, $x \in A_j$ for some $j \geq 2$. Also, x is not in any other A_i as they're mutually disjoint. But we see that if we take $n = j$, we get that $x \in \cup_{i=j+1}^\infty A_i$. This must mean that x is in another A_i where $i \geq j$. This is a contradiction. Thus, there is no such x. This proves that $\cap_{n=1}^\infty B_n = \varnothing$. Thus, we know that $\lim\limits_{i\to\infty} \mu(B_i) = 0$.

Now we see that

$$\mu(\cup_{i=1}^\infty A_i) = \mu\left((\cup_{i=1}^n A_i) \cup \left(\cup_{i=n+1}^\infty A_i\right)\right) = \mu(\cup_{i=1}^n A_i) + \mu\left(\cup_{i=n+1}^\infty A_i\right) = \sum_{i=1}^{n} \mu(A_i) + \mu(B_n)$$

Now if we take the limit as $n \to \infty$, we see that

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} \mu(A_i) + \mu(B_n)\right) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \mu(A_i)\right) + \lim_{n\to\infty} \mu(B_n) = \sum_{i=1}^{\infty} \mu(A_i) + 0 = \sum_{i=1}^{\infty} \mu(A_i)$$

Splitting up the limit is true because the left half is always finite as $A_i \subseteq X$, which means $\mu(A_i) \leq \mu(X) < \infty$. This proves that μ is a measure. $\square$

Exercise 3.3. Let X be an uncountable set and let $\mathcal{A}$ be a collection of subsets A of X such that either A or A^c is countable. Define $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is uncountable. Prove that μ is a measure.

Proof. We see that $\mu(\varnothing) = 0$, as $\varnothing$ is countable. Suppose $A_i \in \mathcal{A}$ is mutually disjoint. If A_i is countable for each i, then $\cup_{i=1}^{\infty} A_i$ is countable. Thus, $\sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} 0 = 0 = \mu(\cup_{i=1}^{\infty} A_i)$. This means that axiom 2 holds if all the A_i's are countable.

Suppose there's an A_k which is uncountable. This means that $\cup_{i=1}^{\infty} A_i$ must be uncountable. Thus, $\mu(\cup_{i=1}^{\infty} A) = 1$. We see that the sum $\sum_{i=1}^{\infty} \mu(A_i) = 1$, as the only term that is nonzero is A_k.Thus, axiom 2 still holds if there is only one A_i which is uncountable.

Suppose there were at least two A_i's that were uncountable. This means that there's an A_j and an A_k that are both uncountable. Then, since A_i's are mutually disjoint, this means that A_j and A_k must be mutually disjoint. Thus, $A_j \subseteq A_k^c$. However, since A_k is uncountable, then A_k^c must be countable, otherwise A_k wouldn't be in $\mathcal{A}$. Thus, A_j must also be countable. This is a contradiction, as A_j was assumed to be uncountable. This proves that in a set of disjoint A_i's, at most one can be uncountable. This covers all cases, and thus proves that μ is a measure. $\square$

Exercise 3.4. Suppose $(X, \mathcal{A}, \mu)$ is a measure space and $A, B \in \mathcal{A}$. Prove that

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$$

Proof. We see that $A = (A \cap B) \cup (A \setminus B)$ and that $A \cap B$ and $A \setminus B$ are disjoint. Similarly $B = (A \cap B) \cup (B \setminus A)$, which are also disjoint. Thus

$$\mu(A) + \mu(B\ ) = \mu((A \cap B) \cup (A \setminus B)) + \mu((A \cap B) \cup (B \setminus A))$$

$$= \mu(A \cap B) + \mu(A \cap B) + \mu(A \setminus B) + \mu(B \setminus A)$$

We see that $A \cup B$ can be broken down into the disjoint union of $A \cap B$, $A \setminus B$, and $B \setminus A$. Thus, these last three terms can be combined into $A \cup B$. This proves that $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$. $\square$

Exercise 3.6. Prove that if $(X, \mathcal{A}, \mu)$ is a measure space, $B \in \mathcal{A}$, and we define $\upsilon(A) = \mu(A \cap B)$ for $A \in \mathcal{A}$, then υ is a measure.

Proof. We see that $\upsilon(\varnothing) = \mu(\varnothing \cap B) = \mu(\varnothing) = 0$. Thus, υ satisfies the first axiom. We also see that if $A_i \in \mathcal{A}$ are disjoint sets, then $A_i \cap B$ are disjoint, and so

$$\upsilon(\cup_{i=1}^{\infty} A_i) = \mu((\cup_{i=1}^{\infty} A_i) \cap B) = \mu(\cup_{i=1}^{\infty}(A_i \cap B)) = \sum_{i=1}^{\infty} \mu(A_i \cap B) = \sum_{i=1}^{\infty} \upsilon(A_i)$$

The second equality holds because intersection distributes over unions. Thus, υ is a measure. $\square$

Exercise 3.7. Suppose $\mu_1, \mu_2, \ldots$ are measures on a measurable space $(X, \mathcal{A})$ and $\mu_n(A) \uparrow$ for each $A \in \mathcal{A}$. Define $\mu(A) = \lim_{n\to\infty} \mu_n(A)$. Is μ neccessarily a measure? If not, give a counterexample. What if $\mu_n(A) \downarrow$ for each $A \in \mathcal{A}$ and $\mu_1(X) < \infty$?

Proof. We obviously see that $\mu(\varnothing) = \lim_{n\to\infty} \mu_n(\varnothing) = \lim_{n\to\infty} 0 = 0$. Now, let $A_i \in \mathcal{A}$ be disjoint sets. This means that

$$\mu\left(\cup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu_n\left(\cup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_n(A_i) = \lim_{n\to\infty} \lim_{j\to\infty} \sum_{i=1}^{j} \mu_n(A_i)$$

Since μ_n and the partial sums are both monotonically increasing, both of these limits are equivalent to the supremum. This means that

$$\lim_{n\to\infty} \lim_{j\to\infty} \sum_{i=1}^{j} \mu_n(A_i) = \sup_{n\in\mathbb{N}} \sup_{j\in\mathbb{N}} \sum_{i=1}^{j} \mu_n(A_i) = \sup_{j\in\mathbb{N}} \sup_{n\in\mathbb{N}} \sum_{i=1}^{j} \mu_n(A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

Thus, μ must be a measure.

Consider $\lambda_n(A) = \mu_1(A) - \mu_n(A)$. We see that $\lambda_n(A) \geq 0$ and that $\lambda_n(A) \neq \infty$, as $\mu_n(A) \leq \mu_1(A) < \infty$ for all $A \in \mathcal{A}$ and $n \in \mathbb{N}$. Also, we see that $\lambda_n = \mu_1(A) - \mu_n(A) \leq \mu_1(A) - \mu_{n+1}(A) = \lambda_{n+1}(A)$. Thus, since $\lambda_n(A) \uparrow$ for each $A \in \mathcal{A}$, by the first part, $\lambda(A) = \lim_{n\to\infty} \lambda_n(A) = \mu_1(A) - \lim_{n\to\infty} \mu_n(A)$, is a measure. Let $\mu(A) = \lim_{n\to\infty} \mu_n(A)$. Thus, since $\mu_n(A) \leq \mu_1(A) \leq \mu_1(X) < \infty$ for every $n \in \mathbb{N}$, we see that $\mu(A) < \infty$ for every $A \in \mathcal{A}$. We also see that $\lambda(A) = \mu_1(A) - \mu(A) < \infty$, as both μ_1 and μ are finite. This means we can normal addition and substraction and thus get $\mu(A) = \mu_1(A) - \lambda(A)$.

Since λ and μ_1 are measures, we see that $\mu(\varnothing) = \mu_1(\varnothing) - \lambda(\varnothing) = 0 - 0 = 0$. We also see that for disjoint $A_i \in \mathcal{A}$ that $\mu\left(\cup_{i=1}^{\infty} A_i\right) = \mu_1\left(\cup_{i=1}^{\infty} A_i\right) - \lambda\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_1(A_i) + \sum_{i=1}^{\infty} \lambda(A_i) = \sum_{i=1}^{\infty} (\mu_1(A_i) - \lambda(A_i)) = \sum_{i=1}^{\infty} \mu(A_i)$. This proves that $\mu(A) = \lim_{n\to\infty} \mu_n(A)$ is, in fact, a measure. $\square$

Exercise 3.8. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathcal{N}$ be the collection of null sets with respect to $\mathcal{A}$ and μ, and let $\mathcal{B} = \sigma(\mathcal{A} \cup \mathcal{N})$. Show that $B \in \mathcal{B}$ if and only if there exists $A \in \mathcal{A}$ and $N \in \mathcal{N}$ such that $B = A \cup N$. Define $\bar{\mu}(B) = \mu(A)$ if $B = A \cup N$ with $A \in \mathcal{A}$ and $N \in \mathcal{N}$. Prove that $\bar{\mu}(B)$ is uniquely defined for each $B \in \mathcal{B}$, that $\bar{\mu}$ is a measure on $\mathcal{B}$, that $(X, \mathcal{B}, \bar{\mu})$ is complete, and that $(X, \mathcal{B}, \bar{\mu})$ is the completion of $(X, \mathcal{A}, \mu)$.

Proof. We see that the first goal is to prove that $\sigma(\mathcal{A} \cup \mathcal{N}) = \mathcal{A} \cup \mathcal{N}$. We see that, by definition, $\mathcal{A} \cup \mathcal{N} \subseteq \sigma(\mathcal{A} \cup \mathcal{N})$. Now to prove that $\mathcal{A} \cup \mathcal{N}$ is a σ-algebra:

Includes empty and whole space: Since $\varnothing$ is a null set, we see that $\varnothing = \varnothing \cup \varnothing \in \mathcal{A} \cup \mathcal{N}$. Also, we see that since $X \in \mathcal{A}$, that $X = X \cup \varnothing \in \mathcal{A} \cup \mathcal{N}$. Thus, $\varnothing, X \in \mathcal{A} \cup \mathcal{N}$.

Closure under complements: Let $C \in \mathcal{A} \cup \mathcal{N}$. Thus, $C = A \cup N$ for some $A \in \mathcal{A}$ and $N \in \mathcal{N}$. This means that $C^c = (A \cup N)^c = A^c \cap N^c$. We see that since $N \in \mathcal{N}$, there's a set $N' \in \mathcal{A}$, such that $N \subseteq N'$ and $\mu(N') = 0$. Since $N \subseteq N'$, we see that $N^c = X \setminus N = (X \setminus N') \cup (N' \setminus N) = N'^c \cup (N' \setminus N)$. This means that $C^c = A^c \cap (N'^c \cup (N' \setminus N)) = (A^c \cap N'^c) \cup (A^c \cap (N' \setminus N))$. Since $A \in \mathcal{A}$ and $N' \in \mathcal{A}$, this means that $A^c \cap N'^c \in \mathcal{A}$. Also, since $N' \setminus N \subseteq N'$, we see that $A^c \cap (N' \setminus N) \subseteq N'$, and since $\mu(N') = 0$, that means this set is in $\mathcal{N}$. Thus, $C^c \in \mathcal{A} \cup \mathcal{N}$.

Closure under countable union: Let $C_i \in \mathcal{A} \cup \mathcal{N}$. Thus, $C_i = A_i \cup N_i$ for some $A_i \in \mathcal{A}$ and $N_i \in \mathcal{N}$. We see that $\cup_{i=1}^{\infty} C_i = \cup_{i=1}^{\infty} A_i \cup N_i = \cup_{i=1}^{\infty} A_i \bigcup \cup_{i=1}^{\infty} N_i$. We see that $\cup_{i=1}^{\infty} A_i = A$ for some $A \in \mathcal{A}$, as $\mathcal{A}$ is a sigma algebra. We also see that for each N_i, there's a superset N_i' such that $\mu(N_i') = 0$. Thus, since $\cup_{i=1}^{\infty} N_i \subseteq \cup_{i=1}^{\infty} N_i'$ and since

$$\mu\left(\cup_{i=1}^{\infty} N_i'\right) = \sum_{i=1}^{\infty} \mu\left(N_i'\right) = \sum_{i=1}^{\infty} 0 = 0$$

this means $\cup_{i=1}^{\infty} N_i$ is a null set, and is thus in $\mathcal{N}$, call this set N. This proves that $\cup_{i=1}^{\infty} C_i = A \cup N \in \mathcal{A} \cup \mathcal{N}$. Thus, $\mathcal{A} \cup \mathcal{N}$ is closed under countable union.

This proves that $\mathcal{A} \cup \mathcal{N}$ is a σ-algebra, and since $\mathcal{B} = \sigma(\mathcal{A} \cup \mathcal{N})$ is the smallest σ-algebra containing $\mathcal{A} \cup \mathcal{N}$, we see that $\mathcal{B} = \mathcal{A} \cup \mathcal{N}$.

$\bar{\mu}$ is well-defined: Let $B_1, B_2 \in \mathcal{B}$. Thus, $B_1 = A_1 \cup N_1$ and $B_2 = A_2 \cup N_2$ for some $A_1, A_2 \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$. Since $N_1, N_2 \in \mathcal{N}$, we know that for some N_1' and N_2', that $N_1 \subseteq N_1'$, $N_2 \subseteq N_2'$, and that $\mu(N_1') = \mu(N_2') = 0$. Now suppose that $B_1 = B_2$. This means that $A_1 \subseteq A_1 \cup N_1 = A_2 \cup N_2 \subseteq A_2 \cup N_2'$. Thus, $\bar{\mu}(B_1) = \mu(A_1) \leq \mu(A_2 \cup N_2') = \mu(A_2) + \mu(N_2') = \mu(A_2) + 0 = \bar{\mu}(B_2)$. This proves that $\bar{\mu}$ is well-defined.

$\bar{\mu}$ is a measure: We see that $\bar{\mu}(\varnothing) = \mu(\varnothing) = 0$. We also see that if $A_i \cup N_i \in \mathcal{B}$ are disjoint sets, then

$$\bar{\mu}\left(\cup_{i=1}^{\infty}\left(A_i \cup N_i\right)\right) = \bar{\mu}\left(\left(\cup_{i=1}^{\infty} A_i\right) \cup \left(\cup_{i=1}^{\infty} N_i\right)\right) = \mu\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu\left(A_i\right) = \sum_{i=1}^{\infty} \bar{\mu}\left(A_i \cup N_i\right)$$

This proves that $\bar{\mu}$ is a measure.

$(X, \mathcal{B}, \bar{\mu})$ is complete: Since $\varnothing \in \mathcal{N}$, we see that for all $A \in \mathcal{A}$, $\bar{\mu}(A) = \bar{\mu}(A \cup \varnothing) = \mu(A)$. Thus, if $(X, \mathcal{B}, \bar{\mu})$ is complete, it must be the completion of $(X, \mathcal{A}, \mu)$. Let $C \subseteq X$ be a $\bar{\mu}$-null set. This means that there's a $B = A \cup N \in \mathcal{B}$, where $A \in \mathcal{A}$, $N \in \mathcal{N}$, $C \subseteq B$ and $\bar{\mu}(B) = 0$. This means that $0 = \bar{\mu}(B) = \bar{\mu}(A \cup N) = \mu(A)$. Also, since $N \in \mathcal{N}$, there's a $N' \in \mathcal{A}$, where $N \subseteq N'$ and $\mu(N') = 0$. This means that $C \subseteq B = A \cup N \subseteq A \cup N'$. We see that $\mu(A \cup N') = \mu(A) + \mu(N') = 0 + 0 = 0$. Thus, since $C \subseteq A \cup N'$ and $\mu(A \cup N') = 0$, this means that C is a μ-null set. Thus, $C \in \mathcal{N}$. Since $\varnothing \in \mathcal{A}$, we see that $C = \varnothing \cup C \in \mathcal{A} \cup \mathcal{N} = \mathcal{B}$.

$\square$