Problem Set 8
Topology II

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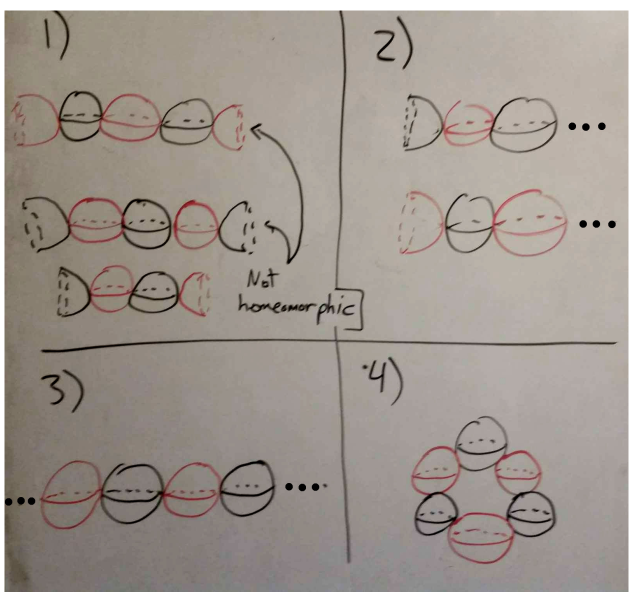
Ex 1. Find all connected covering spaces of $\mathbb{R}\mathrm{P}^2 \vee \mathbb{R}\mathrm{P}^2$.

Proof. We note that $\pi_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}_2$. Since this group has exactly two subgroups, the space $\mathbb{R}\mathrm{P}^2$ has exactly 2 covering spaces, namely $\mathbb{R}\mathrm{P}^2$ itself and the sphere S^2. Since any covering space of $\mathbb{R}\mathrm{P}^2 \vee \mathbb{R}\mathrm{P}^2$ restricts to a covering space of both summands, it must be the union of copies of $\mathbb{R}\mathrm{P}^2$ and S^2. Let a and b be the generators of each summand. Since for some neighborhood of the common basepoint of $\mathbb{R}\mathrm{P}^2 \vee \mathbb{R}\mathrm{P}^2$ the pre-image must look like disjoint copies of it, the copies of $\mathbb{R}\mathrm{P}^2$ and S^2 that make up the covering space must intersect at the pre-images of the basepoint. $\mathbb{R}\mathrm{P}^2$ has only one such point while S^2 has two. This makes classifying the combinations easy to describe: 1) it can be a sequence of spheres chained together with $\mathbb{R}\mathrm{P}^2$ at the ends (note that with an odd number of such spheres there are two different non-homeomorphic arrangements), 2) an infinite sequence terminated with a $\mathbb{R}\mathrm{P}^2$ on one end, 3) a bi-inifinite sequence of spheres (this is the universal cover), or 4) a loop of an even number of spheres. The following picture shows these possible arrangements:



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Ex 2. Let $p : \tilde{X} \to X$ be a simply connected covering space of X. Let $A \subseteq X$ be connected, locally path-connected subspace. Note that $p^{-1}(A)$ might be disconnected (e.g., if A is a point). For any component $\tilde{A} \subseteq p^{-1}(A)$, prove that the restriction of p gives a covering $\tilde{A} \to A$ and the subgroup of $\pi_1(A)$ it corresponds to is the kernel of the natural map $\pi_1(A) \to \pi_1(X)$.

Proof. Let $q : \tilde{A} \to A$ be the restriction of the map $p : \tilde{X} \to X$. The restriction of a continuous function is continuous, so we need only to prove that every element has an evenly-covered neighborhood. Let $x \in A$ and U be neighborhood of x such that U is evenly-covered by p; call these pre-images U_α. We see that

$$p^{-1}(U \cap A) = p^{-1}(U) \cap p^{-1}(A) = \cup_\alpha U_\alpha \cap p^{-1}(A).$$

where each $U_\alpha \cap p^{-1}(A)$ is disjoint from the others in $\tilde{A}$ and is mapped homeomorphically onto $U \cap A$ by q. Thus, $U \cap A$ is a neighborhood of x that is evenly-covered by q; this proves that q is a covering.

We see that this gives us the following commuting square:

$$\begin{tikzcd} \tilde{A} \arrow[r, hook, "\tilde{\imath}"] \arrow[d, "q"] & \tilde{X} \arrow[d, "p"] \\ A \arrow[r, hook, "i"] & X \end{tikzcd}$$

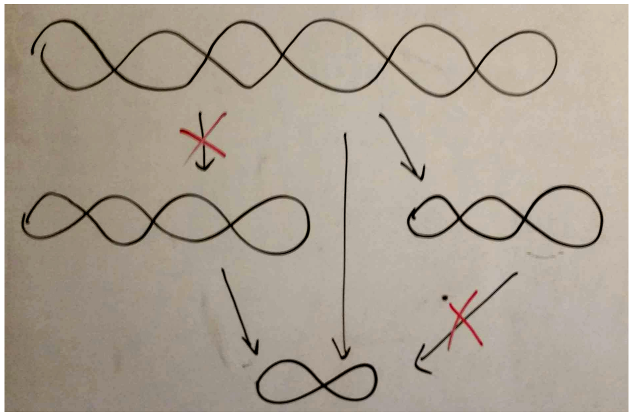
From this square we see that $p\tilde{\imath} = iq$ which means that $p_*\tilde{\imath}_* = i_*q_*$. As $\tilde{X}$ is simply-connected, $\pi_1(\tilde{X})$ is trivial, which means that $\tilde{\imath}_*$ is actually the zero map. This means that i_*q_* is also the zero map, proving that $q_*(\pi_1(A)) \subseteq \ker(i_*)$.

For the converse, let $f : I \to A$ be some path such that $if : I \to X$ is nullhomotopic. By the lifting criterion, the path if lifts to a path $\tilde{if}$. If we choose the basepoint of the lift as the one in $\tilde{A}$, then we have that $\tilde{if}$ is a path in $\tilde{A}$ (as $\mathrm{im}(if) \subseteq A$). This means that $\tilde{if}$ is actually a lift of f through q as well. Thus, we have that $[f] = q_*([\tilde{if}]) \in q_*(\pi_1(A))$ as desired. □

Ex 3. Given a composition of maps $X \xrightarrow{p} Y \xrightarrow{q} Z$ such that any two of the three maps are coverings, prove that the third is as well. When can you prove that the third covering is normal?

Proof. We note that if U is an evenly-covered neighborhood, then any sub-neighborhood is also evenly-covered. Suppose that p and q are coverings and let $z \in Z$. We also denote the elements of $q^{-1}(z)$ as z_α. Since q is a covering, there is an evenly-covered neighborhood U of z. Since p is also a covering, there is an evenly-covered neighborhood of each z_α; call these sets V_α. The set $W = \cap_a q(V_\alpha) \cap U$ is then a neighborhood evenly-covered by $q \circ p$, since it lies inside the evenly-covered neighborhood U and each component of $q^{-1}(W)$ lies inside the evenly-covered neighborhood V_α.

~~The other conjectures, that $q, q \circ p$ being covers implies p is a covering and that $p, q \circ q$ being covers implies q is a covering are not true by the following counter-examples:~~ I have just realized that these are not actually counter-examples. I still could not prove why the pre-images of the non-covering map have to line up in a ?-to-1 manner.



If both p and q are normal coverings, then that means that $\pi_1(Z) \trianglelefteq \pi_1(Y) \trianglelefteq \pi_1(X)$. Unfortunately, this is not enough to conclude that $\pi_1(Z) \trianglelefteq \pi_1(X)$. If $\pi_1(Z)$ were a characteristic subgroup, then this would hold. $\square$

Ex 4. Let X be a connected, locally path-connected space with a covering space action by a group G.

a) Give a subgroup $H \subseteq G$, prove that $X \to X/H$ and $X/H \to X/G$ are normal covering spaces.

b) Prove that for any path-connected space Y and fits into covers $X \to Y$ and $Y \to X/G$, there is a subgroup $H \subseteq G$ realizing this cover.

c) Prove such covers X/H_1 and X/H_2 are equivalent if and only if H_1 and H_2 are conjuage subgroups of G.

d) The cover $X/H \to X/G$ is normal if and only if H is a normal subgroup of G, and in this case the group of deck transformations is the group G/H.

Proof. Proof not completed. $\square$

Ex 5. Let $\alpha : G \times X \to X$ be a group action on a Hausdorff space X. Prove that this α is a covering space action if and only if it is free and properly discontinuous.

Proof. Suppose that α is free and properly discontinuous. Let $x \in X$ and U be a neighborhood of x where $S = \{g \in G : U \cap g(U) \neq \varnothing\}$ is finite. For each $g \in S$, we let V_g be an open set containing $g.x$ where $\{V_g : g \in S\}$ are disjoint; we can do this because X is Hausdorff. If we then let

$$V = \bigcap_{g \in S} g^{-1}(V_g).$$

Then V is a neighborhood of x such that $g(V) \cap g'(V) = \varnothing$ for $g \neq g'$. This proves that α is a covering space action.

Now suppose that α is a covering space action. We see that this trivially implies that α is properly discontinuous, so we only need to prove that α is free. To do this, let $g \in G$ have some fixed point, i.e. $g(x) = x$ for some $x \in X$. Since α is a covering space action, there's some neighborhood U of x such that $h(U) \cap h'(U) = \varnothing$ for $h \neq h'$ in G. Since $g(x) = x = e(x)$, it must be that g is actually the identity element. This proves that α is free as desired. $\square$