Problem Set 3
Differential Topology

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Ex 1 If $k < l$, we can consider $\mathbb{R}^k$ to be the subset $\{(a_1, \ldots, a_k, 0, \ldots, 0)\}$ in $\mathbb{R}^l$. Show that smooth functions on $\mathbb{R}^k$, considered as a subset of $\mathbb{R}^l$ are same as usual.

Proof. Let $\iota : \mathbb{R}^k \to \mathbb{R}^l$ be the inclusion map, that is $\iota(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0)$ and let $p : \mathbb{R}^l \to \mathbb{R}^k$ be the projection map, that is $p(x_1, \ldots, x_l) = (x_1, \ldots, x_k)$. These maps can easily be seen to be smooth. Now let $f : \mathbb{R}^k \to Y$ be a smooth function. By Exercise 3, the composition $f \circ p : \mathbb{R}^l \to Y$, i.e. the function $(f \circ p)(x_1, \ldots, x_l) = (f(x_1), \ldots, f(x_k))$ is smooth. Similarly, if $g : \mathbb{R}^l \to Y$ is a smooth function on the subset $\mathbb{R}^k$ of $\mathbb{R}^l$, we have by Exercise 3 again that $g \circ \iota : \mathbb{R}^k \to Y$ is a smooth function, where $(g \circ \iota)(x_1, \ldots, x_k) = (g(x_1), \ldots, g(x_k), 0, \ldots, 0)$. Since the compositions $(g \circ \iota \circ p)_{\mathbb{R}^k} = g$ and $f \circ p \circ i = f$, we have that the smooth maps on $\mathbb{R}^k$ are the same as those on $\mathbb{R}^k$ considered as a subset of $\mathbb{R}^l$. $\square$

Ex 2 Suppose that X is a subset of $\mathbb{R}^N$ and Z is a subset of X. Show that the restriction of Z of any smooth map on X is a smooth map on Z.

Proof. Let $z \in Z$ and let $f : X \to Y$ be smooth. Since $z \in X$ there is an open set $U \subseteq \mathbb{R}^N$ of z and a smooth function $F : U \to Y$ such that $F|_X = f$. Since $Z \subseteq X$, this means that $F|_Z = f|_Z$ as well. Thus, for each point $z \in Z$, we have an open neighborhood U of z and a smooth function $F : U \to Y$ such that $F|_Z = f|_Z$. This proves that $f|_Z$ is smooth. $\square$

Ex 3 Let $X \subseteq \mathbb{R}^N$, $Y \subseteq \mathbb{R}^M$, $Z \subseteq \mathbb{R}^L$ be arbitrary subsets and let $f : X \to Y$, $g : Y \to Z$ be smooth maps. then the composite $g \circ f : X \to Z$ is smooth. If f and g are diffeomorphisms, so is $g \circ f$.

Proof. Let $x \in X$. Since f is smooth there is an open set $U \subseteq \mathbb{R}^N$ containing x and a smooth function $F : U \to Y$ where $F|_X = f$. Similarly, since g is smooth at $f(x)$, there is an open set $V \subseteq \mathbb{R}^M$ of $f(x)$ and $G : V \to Z$ where $G|_Y = g$. Now let W be an open set of $(g \circ f)(x)$ in Z. Since $G^{-1}(W)$ is open, we see that $V' = G^{-1}(W) \cap V$ and $U' = G^{-1}(V') \cap U$ are open sets. By Exercise 2, $F' = F|_{U'}$ and $G' = G|_{V'}$ are smooth functions. This ensures that $F'(U') \subseteq V'$ and that $G'(V') \subseteq W$.

Then, we have that $G' \circ F'$ is a smooth function such that for any $x \in X$,

$$(G' \circ F')(x) = G'(F'(x)) = G'(f(x)) = g(f(x)) = (g \circ f)(x).$$

(We can convert the G' to g because $f(x) \in Y$.) Thus, we have a neighborhood U of x where $G' \circ F' : U' \to Z$ is a smooth function and $(G' \circ F')|_X = g \circ f$. Since x was arbitrary, this proves that $g \circ f$ is smooth.

If f and g are diffeomorphisms then $g \circ f$ is smooth and has inverse $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, which is also smooth since diffeomorphisms have smooth inverses. This proves that $g \circ f$ is smooth. $\square$

Ex 4

a) Let B_a be the open ball $\{x : |x|^2 < a\}$ in $\mathbb{R}^k$. Show that the map

$$x \mapsto \frac{ax}{\sqrt{a^2 - |x|^2}}$$

is a diffeomorphism of B_a onto $\mathbb{R}^k$.

b) Suppose that X is a k-dimensional manifold. Show that every point in X has a neighborhood diffeomorphic to all of $\mathbb{R}^k$. Thus local parametrizations may always be chosen with all of $\mathbb{R}^k$ for their domains.

Proof.

a) The map

$$x \mapsto \frac{ax}{\sqrt{a^2 + |x|^2}}$$

is the inverse of the given function (I verified this on the board but typing it out seems like a waste.) We see that for $x \in B_a$, the function $a^2 \pm |x|^2 = a^2 \pm \sum_{n=1}^{N} x_i^2$ is smooth and always positive. Since both $\sqrt{x}$ and $1/x$ are smooth on the positive real numbers , we get that the composition $1/\sqrt{a^2 \pm |x|^2}$ is smooth. Finally, since ax and $1/\sqrt{a^2 \pm |x|^2}$ are smooth, their product is smooth. This proves that both the considered map and its inverse are smooth function, and thus that the map is a diffeomorphism.

b) Let $x \in X$. Since X is a manifold, there is a neighborhood U of x such that there is a diffeomorphism $\phi : U \to V$, where V is an open subset of $\mathbb{R}^k$. Without loss of generality, we may assume that $\phi(x) = 0$ (as translation is also a diffeomorphism). Since V is open, there is some $r > 0$ such that $B_r(0) \subseteq V$. By part (a), there is a diffeomorphism ψ from $\mathbb{R}^k$ to $B_r(0)$. Thus, $\phi_x \circ \psi$ is a diffeomorphism from $\mathbb{R}^k$ to $\phi^{-1}(B_r(0))$, which is a open neighborhood of x. As $x \in X$ was arbitrary, this proves the statement. $\square$

Ex 5 Show that every k-dimensional vector subspace V of $\mathbb{R}^N$ is a manifold diffeomorphic to $\mathbb{R}^k$ and that all linear maps on V are smooth.

Proof. Let V be a k-dimensional vector space and $\{e_i\}_{i \leq k}$ be a basis of V. We see that the map $f : V \to \mathbb{R}^k$ where $f(e_i) = (0, \dots 1, \dots, 0)$ (that is, zero in every component except for a 1 in the ith component). This uniquely defines a linear map which is smooth because for $v = \sum_i \alpha_i e_i$ we have

$$f(v) = f(\alpha_1 e_1 + \cdots + \alpha_k e_k) = \alpha_1 f(e_1) + \cdots + \alpha_k f(e_k) = (\alpha_1, \dots, \alpha_k).$$

This function is obviously a bijection and by what we did in class, the jacobian of f is simply the matrix representation of f. Thus, f is smooth. Since f is a bijection, its inverse is also a linear map. Since the matrix of f^{-1} is just the inverse matrix of f, we see that the jacobian of f^{-1} is just the inverse matrix of f; meaning f^{-1} is smooth. Thus, f is a diffeomorphism.

Let $f : V \to V$ be a linear map on V where $\{e_i\}_{i\leq k}$ is a basis of V. Since $\mathbb{R}^N$ is an N dimensional vector space, we can extend this basis with the elements $\{d_{k+1}, d_{k+2}, \dots, d_N\}$ to be a basis of $\mathbb{R}^N$. Thus, the linear map $g : \mathbb{R}^N \to V$ where $g(e_i) = f(e_i)$ and $g(d_j) = d_j$ is a linear map that agrees with f on V. Since linear maps from $\mathbb{R}^N$ are $\mathbb{R}^N$ are smooth (similar argument to the previous paragraph), g is smooth. As $g|_V = f$, this proves that f is smooth as well. □

Ex 6 A smooth bijective map of manifolds need not be a diffeomorphism. In fact, show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is an example.

Proof. We see that $g(x) = x^{1/3}$ is the inverse of f. This function is continuous, but its derivative

$$g'(x) = \frac{d}{dx}x^{1/3} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

is undefined at $x = 0$. Since g does not have a continuous derivative, g is not smooth. Thus, f is not a diffeomorphism. □

Ex 9 Explicitly exhibit enough parametrizations to cover $S^1 \times S^1 \in \mathbb{R}^4$.

Proof. Let $\phi_\theta : (-\pi/2, \pi/2) \to S^1$ be the map $\phi_\theta(x) = (\cos(x + \theta),\ \sin(x + \theta))$. We see that this is a smooth map between $(-\pi/2, \pi/2)$ and the arc of the circle between the angles $(\theta - \pi/2, \theta + \pi/2)$. This function has the smooth inverse

$$\phi_\theta^{-1}(x_1, x_2) = \tan^{-1}\left(\frac{\sin(-\theta)x_1 + \cos(-\theta)x_2}{\cos(-\theta)x_1 - \sin(-\theta)x_2}\right)$$

which looks complicated, but simply rotates the circle so that the arc formed by the angles $(\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$ (i.e. the image of ϕ_θ) gets mapped to the arc formed by $(-\frac{\pi}{2}, \frac{\pi}{2})$, and then we take the inverse tangent to map back to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. This proves that ϕ_θ is a diffeomorphism between the interval $(-\pi/2, \pi/2)$ and the arc of the circle with angles $(\theta - \pi/2, \theta + \pi/2)$. Thus, the maps $\phi_0, \phi_{\pi/2}, \phi_\pi, \phi_{3\pi/2}$ are a parametrizaion of S^1.

From this, we can see that the maps $\{\phi_{\theta_1} \times \phi_{\theta_2} : \theta_1, \theta_2 \in \{0, \pi/2, \pi, 3\pi/2\}\}$ are a parametrization of $S^1 \times S^1$ using the squares $(-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$. □

Ex 10 The torus on the set of points in $\mathbb{R}^3$ at distance b from the circle of radius a in the xy plane, where $0 < b < a$. Prove that these tori are all diffeomorphic to $S^1 \times S^1$. Also draw the cases $b = a$ and $b > a$; why are these not manifolds?

Proof. We note that S^1 is diffeomorphic to the circle of radius α (call this S_α) via the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x, y) = (\alpha x, \alpha y)$. Thus, the torus $S^1 \times S^1$ is diffeomorphic to $S_a \times S_b$ as desired. If this isn't explicit enough, we note that we can parametrize the circle around the point $(0, a, 0)$ with radius b in the yz plane by

$$f(\theta) = (0, a + b\cos(\theta), b\sin(\theta)).$$

Then, we can create the surface of revolution around the z-axis to get

$$f(\theta, \phi) = (\ (a + b\cos(\theta))\cos(\phi),\ (a + b\cos(\theta))\sin(\phi),\ b\sin(\theta)\ )$$

as an explicit diffeomorphism between the angles in $S^1 \times S^1$ and the described tori. Note that if $a = b$, then

$$f(\pi, \phi) = (\ (a + b(-1))\cos(\phi),\ (a + b(-1))\sin(\phi),\ b \cdot 0\ ) = (0, 0, 0)$$

which would mean that f isn't injective. This holds true for when $a < b$ as well. The reason for this is if you actually graph such a "surface", you'd see that it self-intersects. It cannot be a manifold because manifolds are locally diffeomorphic to $\mathbb{R}^n$ for some n and you cannot smoothly map at self-intersections; there's not a unique derivative at that point. $\square$

Ex 18

a) An extremely useful function $f : \mathbb{R} \to \mathbb{R}$ is

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

Prove that f is smooth.

b) Show that $g(x) = f(x-a)f(b-x)$ is a smooth function, positive on (a, b) and zero elsewhere. Then

$$h(x) = \frac{\int_{-\infty}^{x} g\,dx}{\int_{-\infty}^{\infty} g\,dx}$$

is a smooth function satisfying $h(x) = 0$ for $x < a$, $h(x) = 1$ for $x > b$, and $0 < h(x) < 1$ for $x \in (a, b)$.

c) Now construct a smooth function on $\mathbb{R}^k$ that equals 1 on the ball of radius a, zero outside the ball of radius b, and is strictly between 0 and 1 at intermediate points.

Proof.

a) Seeing that this function is smooth at $x \neq 0$ is quite easy and also seeing that it's differentiable at $x = 0$ is easy as well. Proving that it's infinitely differentiable at $x = 0$ turns out to be a considerable amount of calculation, though. I'll skip this part.

b) We see that g is the product of two translated smooth functions, so g is smooth as well. We also see that for $x > b$, $b - x < 0$ so $g(x) = f(x-a)f(b-x) = f(x-a) \cdot 0 = 0$. Similarly, for $x < a$, $g(x) = 0$. We see that $g(x)$ is positive for $x \in (a, b)$ as both $x - a > 0$ and $b - x > 0$, so as $f(x-a) > 0$ and $f(b-x) > 0$.

We see h is smooth as it's the integral of a smooth function divided by a constant. For $x < a$, we know that $g(x) = 0$, so $\int_{-\infty}^{x} g\,dx = 0$ as well, showing that $h(x) = 0$ for such x. For $x > b$, we get that

$$h(x) = \frac{\int_{-\infty}^{x} g\,dx}{\int_{-\infty}^{\infty} g\,dx} = \frac{\int_{a}^{b} g\,dx}{\int_{a}^{b} g\,dx} = 1$$

as $g(x) = 0$ for $x > b$. Finally $h(x)$ is monotonically increasing on (a, b) as $g(x)$ is positive on this region.

c) Take the function $f : \mathbb{R}^k \to \mathbb{R}^k$ where $f(x_1, \ldots, x_k) = (h(x_1), \ldots, h(x_k))$. This simply extends h radially about the the origin, giving us the desired function. $\square$

Ex 2 Let $\gamma : U \subseteq \mathbb{R} \to \mathbb{R}^k$ be a smooth map. For $t \in U$, let $T_tU = \mathbb{R}$ be the canonical basis vector. Prove that $d\gamma_t(1)$ is equal to the usual derivative $\gamma'(t)$.

Proof. We have that

$$d\gamma_t(1) = \begin{bmatrix} \frac{d\gamma_1}{dt}(t) \\ \frac{d\gamma_2}{dt}(t) \\ \vdots \\ \frac{d\gamma_k}{dt}(t) \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} \frac{d\gamma_1}{dt}(t) & \frac{d\gamma_2}{dt}(t) & \dots & \frac{d\gamma_k}{dt}(t) \end{bmatrix} = \begin{bmatrix} \gamma_1'(t) & \gamma_2'(t) & \dots & \gamma_k'(t) \end{bmatrix} = \gamma'(t)$$

as desired. □

Ex 3 Let $f : U \to V$ be a smooth map between open subsets $U \subseteq \mathbb{R}^p$ and $V \subseteq \mathbb{R}^q$ and let $x \in U$. Prove that for a vector $v \in T_xU$, the differential $df_x(v)$ is equal to

$$df_x(v) = \frac{d}{dt} f(\gamma(t))|_{t=0},$$

where $\gamma : (-\varepsilon, \varepsilon) \to U$ is any differential function such that $\gamma(0) = x$ and $\gamma'(0) = v$.

Proof. We see that

$$\frac{d}{dt}\, f(\gamma(t))|_{t=0} = \left(\frac{d}{dt} f_1(\gamma(t)), \dots, \frac{d}{dt} f_q(\gamma(t))\right)\Bigg|_{t=0} = \big(\Delta f_1(\gamma(t)) \cdot \gamma'(t), \dots, \Delta f_q(\gamma(t)) \cdot \gamma'(t)\big)\Big|_{t=0}$$

$$= \big(\Delta f_1(\gamma(0)) \cdot \gamma'(0), \dots, \Delta f_q(\gamma(0)) \cdot \gamma'(0)\big) = (\Delta f_1(x) \cdot v, \dots, \Delta f_q(x) \cdot v)$$

$$= \left(\sum_{i=1}^{p} \frac{\partial f_1}{\partial x_i}(x) v_i, \dots, \sum_{i=1}^{p} \frac{\partial f_q}{\partial x_i}(x) v_i\right) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_q}{\partial x_1} & \cdots & \frac{\partial f_q}{\partial x_p} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix} = df_x \cdot v = df_x(v)$$

as desired. □