Problem Set 3
Complex Analysis

Bennett Rennier
bennett@brennier.com

September 25, 2018

Ex 1 Define $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ by $f(z) = \left(\frac{1+z}{1-z}\right)^2$. Is f injective on $\mathbb{D}$? Prove your answer. Find $f(\mathbb{D})$.

Proof. We first note that $\phi(z) = \frac{1+z}{1-z}$ is a möbius transform. Since we see that

$$\begin{aligned}
\phi(-1) &= 0 \\
\phi(i) &= \frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{2i}{2} = i \\
\phi(-i) &= \frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{-2i}{2} = -i,
\end{aligned}$$

we know that ϕ takes the unit circle to the imaginary line. Additionally, since $\phi(0) = 1$, it maps $\mathbb{D}$ to the right half plane, that is $\{re^{i\theta} \in \mathbb{C} : r \geq 0, -\pi \leq \theta \leq \pi\}$. This proves that

$$\begin{aligned}
f(\mathbb{D}) = \phi(\mathbb{D})^2 &= \{re^{i\theta} : r \geq 0, -\pi \leq \theta \leq \pi\}^2 = \{r^2 e^{i2\theta} : r \geq 0, -\pi \leq \theta \leq \pi\} \\
&= \{re^{i\theta} : r \geq 0, -2\pi \leq \theta \leq 2\pi\} = \mathbb{C}.
\end{aligned}$$

Thus, $f(\mathbb{D}) = \mathbb{C}$; however, f is not injective as

$$\begin{aligned}
f(i) &= \phi(i)^2 = i^2 = -1 \\
f(-i) &= \phi(-i)^2 = (-i)^2 = -1.
\end{aligned}$$

□

Ex 2 Given open sets $U, V \in \mathbb{C}$, we call a function $f : U \to V$ a *conformal map* if it is a holomorphic bijection with continuous inverse (and thus f^{-1} is holomorphic). Let $\alpha \in (0, \pi]$ and set $\Omega_\alpha = \{z \in \mathbb{C} : z = |z|e^{i\theta} \text{ for some } -\alpha < \theta < \alpha\}$. Construct a conformal map from Ω_α to $\mathbb{D}$.

Proof. We note that $z^{\pi/\alpha}$ is a holomorphic function that injectively sends $\{re^{i\theta} : r > 0, -\alpha < \theta < \alpha\}$ to the set $\{re^{i\theta} : r > 0, -\pi < \theta < \pi\}$ with continuous inverse $z^{\alpha/\pi}$. That is, $z^{\pi/\alpha}$ conformally maps Ω_α to Ω_π. We see that $\phi(z) = \frac{z-1}{z+1}$ is a möbius transformation such that

$$\begin{aligned}
\phi(0) &= -1 \\
\phi(i) &= \frac{i-1}{i+1} = \frac{(i-1)(-i+1)}{(i+1)(-i+1)} = \frac{-(i-1)^2}{2} = \frac{2i}{2} = i \\
\phi(-i) &= \frac{-i-1}{-i+1} = \frac{-(i+1)^2}{(i+1)(-i+1)} = \frac{-2i}{2} = -i
\end{aligned}$$

which means ϕ maps the imaginary line to the unit circle. Additionally, since $\phi(1) = 0$, we know that it maps the right half plane (that is Ω_π) to the unit disk $\mathbb{D}$. Thus the map $z \mapsto \Omega_\pi(z^{\pi/\alpha})$ is a conformal map that sends Ω_α to $\mathbb{D}$. □

Ex 3 Construct a conformal map between $\{z \in \mathbb{C} : -\pi < \text{Im}(z) < \pi\}$ and $\mathbb{D}$.

Proof. We note that e^z is holomorphic, a bijection from $\{z \in \mathbb{C} : -\pi < \text{Im}(z) < \pi\}$ to $\{re^{i\theta} \in \mathbb{C} : r > 0, -\pi < \theta < \pi\}$, and has a continuous inverse $\log(z)$ using the branch cut of the nonnegative reals and $\log(1) = 0$. This proves that e^z is a conformal map from $\{z \in \mathbb{C} : -\pi < \text{Im}(z) < \pi\}$ to $\{re^{i\theta} \in \mathbb{C} : r > 0, -\pi < \theta < \pi\}$. Since this latter set is simply Ω_π as in problem (2) we can apply the same möbius transformation $\phi(z) = \frac{z-1}{z+1}$ t to conformally map to the unit disk $\mathbb{D}$. Thus, $z \mapsto \phi(e^z)$ is a conformal map that sends $\{z \in \mathbb{C} : -\pi < \text{Im}(z) < \pi\}$ to $\mathbb{D}$. □

Ex 4

a) For $a \in \mathbb{D}$, let $\phi_a(z) = \frac{z-a}{1-\overline{a}z}$. Prove that ϕ_a maps $\mathbb{D}$ to itself.

b) For $a, b \in \mathbb{D}$, prove that there is a conformal map $f : \mathbb{D} \to \mathbb{D}$ so that $f(a) = b$.

Proof.

a) We note that ϕ_a is a möbius transformation such that

$$|\phi(1)| = \frac{|1-a|}{|1-\overline{a}|} = \frac{|1-a|}{|\overline{1-a}|} = \frac{|1-a|}{|1-a|} = 1$$
$$|\phi(-1)| = \frac{|-1-a|}{|1+\overline{a}|} = \frac{|1+a|}{|\overline{1+a}|} = \frac{|1+a|}{|1+a|} = 1$$
$$|\phi(i)| = \frac{|i-a|}{|1-\overline{a}i|} = \frac{|i-a|}{|i||1-\overline{a}i|} = \frac{|i-a|}{|i+\overline{a}|} = \frac{|i-a|}{|\overline{-i+a}|} = \frac{|i-a|}{|i-a|} = 1.$$

Since möbius transformations take circles to circles and the points $-1, 1, i$ are three points whose unique circle that goes through all of them is the unit circle and they get mapped back to the unit circle, it must be that ϕ_a takes the unit circle to itelf. Additionally, since $|\phi(0)| = |-a| \leq 1$, we see that ϕ_a does indeed map $\mathbb{D}$ to $\mathbb{D}$.

b) By part (a), we see that ϕ_{-b} is a conformal map from $\mathbb{D}$ to $\mathbb{D}$ where $\phi_{-b}(0) = b$. Since the inverse of a möbius transformation and the composition of möbius transformations are again möbius transformations, we see that $\phi_{-b} \circ \phi_a^{-1}$ is a möbius transformation such that $(\phi_{-b} \circ \phi_a^{-1})(a) = \phi_{-b}(\phi_a^{-1}(a)) = \phi_{-b}(0) = b$. Thus, we have a conformal map satisfying the stated condition. □

Ex 5 Find a Möbius transformation that takes the first quadrant to the top half of the unit disk and satisfies $f(2) = i$.

Proof. Let $\phi_1(z) = z/2$ and $\phi_2(z) = \frac{i-z}{z+i}$. We note that both of these are möbius transformations. The first is clear to understand. We see for the latter that

$$\phi_2(0) = \frac{i}{i} = 1$$
$$\phi_2(1) = \frac{i-1}{1+i} = \frac{(1-i)(i-1)}{(1+i)(1-i)} = \frac{2i}{2} = i$$
$$\phi_2(-1) = \frac{i+1}{-1+i} = \frac{(-1-i)(i+1)}{(-1+i)(-1-i)} = \frac{-2i}{2} = -i$$

which means that ϕ_2 takes the real line to the unit circle. Additionally, as $\phi_2(i) = 0$, it maps the upper half of the plane to the unit disk $\mathbb{D}$. Since $\phi(-i) = \infty$ (on the Riemann sphere), we get that the imaginary axis is mapped to the real axis. Finally, with the knowledge that

$$\phi_2(i+1) = \frac{-1}{1+2i} = \frac{-(1-2i)}{(1+2i)(1-2i)} = \frac{2i-1}{1+4} = -\frac{1}{5} + \frac{2i}{5},$$

which is in the upper half of the unit disk $\mathbb{D}$, we have that ϕ_2 sends the first quadrant to the top half of the unit disk as desired. We see that ϕ_2 sends 1 to i instead of 2 to i, though. We can fix this with ϕ_1, as it's a möbius transformation that preserves the first quadrant and sends 2 to 1. Thus, $\phi_2 \circ \phi_1$ is our desired transformation that satisfies the condition. $\square$

Ex 6 Suppose that $U \subseteq \mathbb{C}$ is open and that $f : U \to \mathbb{C}$ and $\overline{f}$ are both holomorphic. Prove that f is constant.

Proof. Since f and $\overline{f}$ are both holomorphic, so must their product $f\overline{f} = |f|^2$. If we let $z_0 \in U$, then there must be an $r > 0$ such that $B_r(z_0) \subseteq U$. Let $\gamma_1 : (-r, r) \to U$ and $\gamma_2 : (-r, r) \to U$ such that $\gamma_1(t) = z_0 + t$ and $\gamma_2(t) = z_0 + it$. These two paths are clearly orthogonal at z_0, but their images under $|f|^2$ lie within the real line. Thus, the angle between their image must be either 0 or π. This proves that $|f|^2$ is not conformal at z_0. Since $|f|^2$ is holomorphic, though, it must be that $\frac{\partial |f|^2}{\partial z}(z_0) = 0$.

Since z_0 was arbitrary, we have that $\frac{\partial}{\partial z}|f|^2 = 0$ everywhere. This proves that $|f|^2$ is constant. Since $|f|$ must be nonnegative, this shows that $|f|$ is constant as well. By the previous homework, a holomorphic function with constant magnitude is constant. Thus, f is constant. $\square$

Ex 7 Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \sqrt{|\operatorname{Re}(z)\operatorname{Im}(z)|}$. Show that the Cauchy-Riemann equations are satisfied for f at $z = 0$, but that f is not differentiable at $z = 0$.

Proof. We see that for $h \in \mathbb{R}$ we get

$$\frac{\partial f}{\partial x}(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\sqrt{|h \cdot 0|}}{h} = 0$$
$$\frac{\partial f}{\partial y}(0) = \lim_{h\to 0} \frac{f(0+ih) - f(0)}{h} = \lim_{h\to 0} \frac{\sqrt{|0 \cdot h|}}{h} = 0.$$

This proves that f trivially satisfies the Cauchy-Riemann equations at $z = 0$. However, we see that

$$\lim_{h\to 0} \frac{f(0+(1+i)h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h+ih)}{h} = \lim_{h\to 0} \frac{\sqrt{|h \cdot h|}}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

which does not converge (as it depends on what direction you're approaching with h). Since the derivative along the direction of $i+1$ does not converge, the derivative in general cannot converge. Thus, f is not differentiable at $z = 0$. $\square$

Ex 8 Recall that if $U \subseteq \mathbb{C}$ is open and $u : U \to \mathbb{C}$ has continuous second parital derivatives, then we define the Laplacian of u by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial v^2}.$$

If u has continuous second partial derivatives and $\Delta u = 0$ throughout U, then we say that u is harmonic in U.

a) Let $U \subseteq \mathbb{C}$ be open and $u : U \to \mathbb{C}$ have continuous second partial derivatives. Recall that the polar coordinates on $\mathbb{C}$ are given by the map $P : [0, \infty) \times [-\pi, \pi] \to \mathbb{C}$ given $P(r, \theta) = r\cos\theta + ir\sin\theta$. Find a formula for Δu in polar coordinates.

b) Fix $a, b \in [0, +\infty]$ with $a < b$, and let $\mathbb{D}_{a,b} = \{z \in \mathbb{C} : a < |z| < b\}$. Suppose that $u : \mathbb{D}_{a,b} \to \mathbb{C}$ is harmonic and that $u(z) = f(|z|)$ for some twice differentiable function $f : (a, b) \to \mathbb{C}$. Prove that there are constants $\alpha, \beta \in \mathbb{C}$ so that $u(z) = \alpha \log|z| + \beta$.

c) Let $a, b \in [0, +\infty]$ with $a < b$. Suppose that $u : \mathbb{D}_{a,b} \to \mathbb{C}$ is harmonic. Prove that $v : \mathbb{D}_{a,b} \to \mathbb{C}$ given by $v(z) = \frac{1}{2\pi}\int_0^{2\pi} u(|z|e^{i\theta})\,d\theta$ is also harmonic.

d) Let $u : \mathbb{D} \to \mathbb{C}$ be harmonic. Prove that for all $r \in (0, 1)$ we have that $u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta$.

Proof.

a) This was a lot of computation involving partial derivatives. I'll just write the result here for the subsequent parts

$$\Delta u(r, \theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

b) We see that in polar coordinates this means that $u(re^{i\theta}) = f(|re^{i\theta}|) = f(r)$ (i.e. u does not depend on θ). Thus, we have that

$$0 = \Delta u(re^{i\theta}) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{df}{dr^2} + \frac{1}{r}\frac{df}{dr} + \frac{1}{r^2}\cdot 0 = f''(r) + \frac{f'(r)}{r}.$$

This means $0 = rf'' + f' = (rf)'$. Thus, $rf = \alpha$ for some constant α. Furthermore,

$$f(r) = \int \frac{\alpha}{r}\,dr = \alpha\log(r) + \beta.$$

This proves that

$$u(z) = u(re^{i\theta}) = f(r) = \alpha\log(r) + \beta = \alpha\log|z| + b$$

for some constants $\alpha, \beta \in \mathbb{C}$ as desired.

c) We see that

$$v(re^{i\theta}) = \frac{1}{2\pi}\int_0^{\pi} u(|re^{i\theta}|e^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{\pi} u(re^{i\theta})\,d\theta$$

which means that in polar coordinates, v does not depend on θ. Thus, $v = v(r)$. We see that by the Leibniz rule

$$v_r = \frac{1}{2\pi}\int_0^{2\pi} u_r(re^{i\theta})\,d\theta$$
$$v_{rr} = \frac{1}{2\pi}\int_0^{2\pi} u_{rr}(re^{i\theta})\,d\theta.$$

Thus, we have

$$\begin{aligned} v_{rr} + \frac{v_r}{r} + \frac{v_{\theta\theta}}{r^2} &= \frac{1}{2\pi}\int_0^{2\pi} u_{rr}\,d\theta + \frac{1}{2\pi r}\int_0^{2\pi} u_r\,d\theta + \frac{0_\theta}{r^2} = \frac{1}{2\pi}\int_0^{2\pi}\left(u_{rr} + \frac{u_r}{r}\right)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} -\frac{u_{\theta\theta}}{r^2}\,d\theta = \frac{-1}{2\pi r^2}(u_\theta(re^{2\pi i}) - u_\theta(re^{0i})) = \frac{-1}{2\pi r^2}(u_\theta(r) - u_\theta(r)) = 0. \end{aligned}$$

This proves that v is a harmonic function as well.

d) By part (c), we know that

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta$$

is a harmonic function on $\mathbb{D}_{0,1}$ that does not depend on r. By part (b), this means that

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta = \alpha\log(r)+\beta.$$

for some constants $\alpha,\beta\in\mathbb{C}$. If we differentiate, we get

$$\frac{1}{2\pi}\int_0^{2\pi} u_r(re^{i\theta})\,d\theta = \frac{\alpha}{r}$$

which means

$$\frac{r}{2\pi}\int_0^{2\pi} u_r(re^{i\theta})\,d\theta = \alpha.$$

Thus for $r=0$ we get that

$$\frac{0\cdot u_r(0)}{2\pi}\int_0^{2\pi} d\theta = 0\cdot u_r(0) = 0 = \alpha,$$

which proves that $\alpha=0$. This reduces our equation to

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta = \beta.$$

Again, by plugging in $r=0$, we get that

$$\frac{1}{2\pi}\int_0^{2\pi} u(0)\,d\theta = \frac{u(0)}{2\pi}\cdot 2\pi = u(0) = \beta.$$

Thus, we have that

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta = u(0)$$

as desired. □

Ex 9 Suppose that $U\subseteq\mathbb{C}$ is open, that $f:U\to\mathbb{C}$ is holomorphic, and that $|f|$ is harmonic. Prove that f is constant.

Proof. Trying to do this the "smart" way. Without loss of generality, we may assume $\mathbb{D}\subseteq U$ (we can just translate and scale if not). We know f and $|f|$ are harmonic, so by Ex 8, we have that

$$f(0) = \int_0^{2\pi} f(re^{i\theta})\,d\theta$$

and

$$|f(0)| = \int_0^{2\pi} |f(re^{i\theta})|\,d\theta$$

for $r\in(0,1)$. This means that

$$\left|\int_0^{2\pi} f(re^{i\theta})\,d\theta\right| = |f(0)| = \int_0^{2\pi} |f(re^{i\theta})|\,d\theta$$

for $r\in(0,1)$. Thus, for each $r\in(0,1)$, there exists a β_r such that $\beta_r f$ is non-negative valued. [Incomplete] □