# Problem Set 7
# Abstract Algebra II

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## Section 12.1

**Ex 1** Let $M$ be a module over the integral domain $R$.

a) Suppose $x$ is a nonzero torsion element in $M$. Show that $x$ and 0 are "linearly dependent." Conclude that the rank of $\operatorname{Tor}(M)$ is 0, so that in particular any torsion $R$-module has rank 0.

b) Show that the rank of $M$ is the same as the rank of the (torsion free) quotient $M/\operatorname{Tor} M$.

*Proof.* a) Since $x$ is a nonzero torsion element of $M$, this means that $r.x = 0$ for some nonzero $r$. Thus, we have that $r.x + r.0 = 0 + 0 = 0$, which proves that $x$ and 0 are linearly dependent (as we can represent zero using nonzero coefficients).

Now suppose that $A \subseteq \operatorname{Tor}(M)$ is a nonempty set of linearly independent elements. Then, for each nonzero element $x \in A$, there's a $r$ such that $r.x = 0$. If we use these $r$'s as the coefficients of these elements, then we've found a way to represent 0 using not-all-zero coefficients. [Note: If $0 \in A$, then we can give it any coefficient we want, so that doesn't matter.] This a contradiction, which proves that no such $A$ exists. It follows then that the rank of $\operatorname{Tor}(M)$ is zero.

b) Let the rank of $M$ be $m$ and let $x_1, \dots, x_m \in M$ be a maximal linearly independent set. We see that $r.x_i \neq 0$ for all $r \in R$, as otherwise, $\{x_i\}$ wouldn't be linearly independent. This proves that $x_i \notin \operatorname{Tor} M$. We then see that

$$\lambda_1(x_1 + \operatorname{Tor} M) + \cdots + \lambda_m(x_m + \operatorname{Tor} M) = \lambda_1 x_1 + \cdots + \lambda_m x_m + \operatorname{Tor} M = 0 + \operatorname{Tor} M$$

which means that

$$\lambda_1 x_1 + \cdots + \lambda_m x_m \in \operatorname{Tor} M$$

which proves there is some $0 \neq r \in R$ such that

$$r(\lambda_1 x_1 + \cdots + \lambda_m x_m) = 0$$

Since $x_1, \ldots, x_m$ are linearly independent, this proves that $r\lambda_i = 0$. Since $R$ is an integral domain and $r \neq 0$, this proves that $\lambda_i = 0$. Thus, $x_1 + \operatorname{Tor} M, \ldots, x_m + \operatorname{Tor} M$ is linearly independent in $M/\operatorname{Tor} M$. Thus, $m \leq \operatorname{rank}(M/\operatorname{Tor} M)$.

Now, conversely, let $y_1 + \operatorname{Tor} M, \ldots, y_n + \operatorname{Tor} M$ be a maximal linearly independent set in $M/\operatorname{Tor} M$. We then see that

$$\begin{aligned}
&\lambda_1 y_1 + \cdots + \lambda_n y_n = 0 \\
&\implies \lambda_1 y_1 + \cdots + \lambda_n y_n + \operatorname{Tor} M = 0 + \operatorname{Tor} M \\
&\implies \lambda_1 (y_1 + \operatorname{Tor} M) + \cdots + \lambda_n (y_n + \operatorname{Tor} M) = \operatorname{Tor} M \\
&\implies \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0
\end{aligned}$$

which proves $y_1, \ldots, y_n$ is then a linearly independent set of $M$. Thus, $\operatorname{rank}(M/\operatorname{Tor} M) \leq m$. This proves that $\operatorname{rank}(M/\operatorname{Tor} M) = \operatorname{rank}(M)$.

□

**Ex 2** Let $M$ be a module over the integral domain $R$.

a) Suppose that $M$ has rank $n$ and that $x_1, x_2, \ldots, x_n$ is any maximal set of linearly independent elements of $M$. Let $N = Rx_1 + \cdots + Rx_n$ be the submodule generated by $x_1, x_2, \ldots, x_n$. Prove that $N$ is isomorphic to $R^n$ and that the quotient $M/N$ is a torsion $R$-module (equivalently, the elements $x_1, \ldots, x_n$ are linearly independent and for any $y \in M$ there is a nonzero element $r \in R$ such that $ry$ can be written as a linear combination $r_1 x_1 + \cdots + r_n x_n$ of the $x_i$).

b) Prove conversely that if $M$ contains a submodule $N$ that is free of rank $n$ (i.e., $N \simeq R^n$) such that the quotient $M/N$ is a trosion $R$-module, then $M$ has rank $n$. [Let $y_1, y_2, \ldots, y_{n+1}$ be any $n+1$ elements of $M$. Use the fact that $M/N$ is torsion to write $r_i y_i$ as a linear combination of a basis for $N$ for some nonzero elements $r_1, \ldots, r_{n+1}$ of $R$. Use an argument as in the proof of Proposition 3 to see that the $r_i y_i$, and hence also the $y_i$, are linearly dependent.]

*Proof.* a) First, we prove that $N \simeq R^n$. Let $\varphi : R^n \to N$ where $\varphi(r_1, \ldots, r_n) = r_1 x_1 + \cdots + r_n x_n$. This map is clearly surjective. I claim that it's also surjective. To prove this, suppose $(r_1, \ldots, r_n) \in \ker \varphi$. Then this means that $\varphi(r_1, \ldots, r_n) = r_1 x_1 + \cdots + r_n x_n = 0$. Since $x_1, \ldots, x_n$ are linearly independent, this must mean that $r_1 = r_2 = \cdots = r_n = 0$. Thus, $\ker \varphi = \{(0, \ldots, 0)\}$, which proves injective. We see that this map is a $R$-module homomorphism as

$$\begin{aligned}
\varphi((r_1, \ldots, r_n) - \lambda(s_1, \ldots, s_n)) &= \varphi(r_1 - \lambda s_1, \ldots, r_n - \lambda s_n) \\
= (r_1 - \lambda s_1)x_1 + \cdots + (r_n - \lambda s_n)x_n &= (r_1 x_1 + \cdots + r_n x_n) - \lambda(s_1 x_1 + \cdots + s_n x_n) \\
&= \varphi(r_1, \ldots, r_n) - \lambda\varphi(s_1, \ldots, s_n)
\end{aligned}$$

which finally proves that $N \simeq R^n$. Now to prove that $M/N$ is torsion. Let $x + N \in M/N$. If $x \in N$, then $1.(x + N) = N$, which would mean that $x$ is torsion. Now suppose that

$x \notin N$. This means that $x \neq x_i$ (as otherwise that would mean that $x \in N$). This means that $x_1, \ldots, x_n, x$ can't be linearly independent in $M$, as the rank of $M$ is only $n$. Thus, $r_0 x = r_1 x_1 + \cdots + r_n x_n$ for some $r_i \in R$, which means that $r_0 x \in N$. This proves that $r_0.(x+N) = r_0 x + N = N$, which shows that $M/N$ is a torsion $R$-module.

□

**Ex 3** Let $R$ be an integral domain and let $A$ and $B$ be $R$-modules of ranks $m$ and $n$ respectively. Prove that the rank of $A \oplus B$ is $m+n$. [Use the previous exercise.]

*Proof.* By the previous exercise, we know that there exists free submodules $A' \subseteq A$ and $B' \subseteq B$ with free ranks $m$ and $n$ respectively, such that $A/A'$ and $B/B'$ are torsion. We see then that $A' \oplus B' \subseteq A \oplus B$ is then free. We also see that $(A \oplus B)/(A' \oplus B') \simeq_R (A/A') \oplus (B/B')$. Since $R$ is an integral domain, then the direct sum of torsion modules is torsion. This proves that $(A \oplus B)/(A' \oplus B')$ is torsion. Since $A' \oplus B'$ is free and has rank $n+m$, then $A \oplus B$ has rank $n+m$. □

**Ex 5** Let $R = \mathbb{Z}[x]$ and let $M = (2, x)$ be the ideal generated by 2 and $x$, considered as a submodule of $R$. Show that $\{2, x\}$ is not a basis of $M$. [Find a nontrivial $R$-linear dependence between these two elements.] Show that the rank of $M$ is 1 but that $M$ is not free of rank 1.

*Proof.* We see that $2.x + (-x).2 = 2x - 2x = 0$, which proves that 2 and $x$ are not $R$-linearly independent. Thus, $\{2, x\}$ is not a basis of $M$.

Suppose that $a, b \in M$ are nonzero. Since $M \subseteq R$, we have that $b.a + (-a).b = ab + ab = 0$. This proves that the rank of $M$ is at most 1. Since $\mathbb{Z}[x]$ is an integral domain, any singleton is linearly independent. Thus, $M$ has exactly rank 1.

Suppose that $M$ was free of rank 1. This means that $M = Ra$ for some $a$. This would mean that $2 = ra$ for some $r \in R$, which proves that $a = \{\pm 1, \pm 2\}$ as $R = \mathbb{Z}[x]$. If $a = \pm 1$, this would mean that $Ra = a$. This is a contradiction, as we know that $(2, x) \neq \mathbb{Z}[x]$. Suppose then that $a = 2$. That would mean that $M = 2\mathbb{Z}[x]$. However, this is a contradiction as $x \notin 2\mathbb{Z}[x]$. Similarly, we have a contradiction if $a = -2$. This proves that $M$ is not free of rank 1 as a $\mathbb{Z}[x]$-module. □

# Additional Exercises

**Ex A** As in class, a *graded ring* is a ring $R$ where $R = \oplus_{d \in D} R_d$ as abelian groups under the $+$ operation and $R_{d_1} R_{d_2} \subseteq R_{d_1 + d_2}$ for all $d_1, d_2 \in D$. An element $r \in R$ is called *homogeneous of degree $d$* if $r \in R_d$ for some $d$. Show that the following are equivalent for an ideal $I$ of a graded ring $R$.

1) If we define $I_d = I \cap R_d$, then $I = \oplus_{d \in D} I_d$.

2) The ideal $I$ has a set of generators which are homogeneous.

3) If we define $(R/I)_d = R_d / I_d$, then $R/I = \oplus_{d \in D} (R/I)_d$ is a graded ring.

An ideal which satisfies these equivalent conditions is called a *graded ideal* or *homogeneous ideal.*

*Proof.* $1 \implies 2)$ Suppose that $I = \oplus_{d\in D} I_d$. Let $G = \cup_{d\in D} I_d$. Since $I_d \subseteq R_d$, that means that every element of $I_d$ is homogeneous, which means that $G$ is a set of homogeneous elements. If $x \in I = \oplus_{d\in D} I_d$, then that means that $x = \sum_{d\in D} h_d$, where $h_d \in I_d$ and $h_d = 0$ for all but finitely many $d$. This proves that $x \in \langle G \rangle$, and thus that $G$ a set of generators which are homogeneous.
$2 \implies 1)$ Suppose that $I$ has a set of generators which are homogeneous and let $H = \{h_j\}_{j\in J}$ be this set. Let $x \in I$. Then we see that $x = \sum_{j\in J} \lambda_j h_j$. Let $h'_d$ be the collection of the terms that have degree $d$. This means that $x = \sum_{d\in D} h'_d$. Since $h'_d \in I_d$, we have that $x \in \oplus_{d\in D} I_d$. We also trivally have that $\oplus_{d\in D} I_d \subseteq I$, which proves that $I = \oplus_{d\in D} I_d$.

□

**Ex B** A *homomorphism of graded rings* $\varphi : R \to S$ is a ring homomorphism which satisfies $\varphi(R_d) \subseteq S_d$ for all $d \in D$. Show that the kernel of such a homomorphism is a graded ideal.

*Proof.* Let $x_{k_1} + \cdots + x_{k_n} \in \ker\varphi$, where $x_{k_i} \in R_{k_i}$ and the $k_i$'s are distinct. Since $\varphi$ is a ring homomorphism, this means that $\varphi(x_{k_1} + \cdots + x_{k_n}) = \varphi(x_{k_1}) + \cdots + \varphi(x_{k_n}) = \varphi(0) = 0$. Since $\varphi$ preserves degree, this means that $\varphi(x_{k_i}) \in S_{k_i}$. Since $S = \oplus_d S_d$, there's only one way to represent zero using elements of differing degrees, that is, they must each be zero themselves. This proves that $\varphi(x_{k_i}) = 0$, which proves that $x_{k_i} \in \ker\varphi$. Thus, $\ker\varphi$ is a graded ideal. □

**Ex C** If $R$ is a graded ring and $M$ is an $R$-module, then we say $M$ is a *graded module* if $M = \oplus_{d\in D} M_d$ as abelian groups under the $+$ operation and $R_{d_1}.M_{d_2} \subseteq M_{d_1+d_2}$ for all $d_1, d_2 \in D$.

C1) Show that the annihilator of $M$ is a graded ideal of $R$.

C2) Decide on the correct definition of a homomorphism of graded $R$-modules and show that the kernel and image of such a map is again a graded module.

C3) Show that the collection of graded $R$-modules and graded $R$-module homomorphisms defines a category.

*Proof.* C1) Let $x_{k_1} + \cdots + x_{k_n} \in \mathrm{Ann}(M)$, where $x_{k_i} \in R_{k_i}$ and the $k_i$'s are distinct. Fix $j$ and let $m_j \in M_j \subseteq M$. This means that

$$(x_{k_1} + \cdots + x_{k_n}).m_j = x_{k_1}.m_j + \cdots + x_{k_n}.m_j = 0$$

Since $m_j \in J$ and $x_{k_i} \in R_{k_i}$, that means that $x_{k_i}.m_j \in M_{j+k_i}$. Since $j$ was fixed, that means that the $j + k_i$'s are distinct. Since we have homogeneous elements of distinct degrees adding to zero, then they must all be zero themselves. Thus, $x_{k_i}.m_j = 0$. If we vary $j$, we see that $x_{k_i}$ annihilates any homogeneous element of $M$. Since $M$ is the direct sum of sets of homogeneous elements, that means that $x_{k_i}$ annihilates all of $M$. Thus, $x_{k_i} \in \mathrm{Ann}(M)$, which proves that $\mathrm{Ann}(M)$ is a graded ideal.

□