Problem Set 8
Real Analysis I

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Ex 6.2 Let X be a set and $\mathcal{A}$ the collection of all subsets of X. Pick $y \in X$ and let δ_y be the point mass at y, defined in Example 3.4. Prove that if $f : X \to \mathbb{R}$, then

$$\int f \, d\delta_y = f(y)$$

Proof. Since $\mathcal{A}$ is the set of all subsets, f is trivally measurable. If $f(y) < 0$, let f be f^-, otherwise, take f to be f^+ in the following. Let s be a simple function where $0 \leq s \leq f$. Represent s in the canonical form of

$$s = \sum_{i=1}^{n} a_i \chi_{E_i}$$

where the E_i's are disjoint. Since the E_i's are disjoint, y is in at most one of them. If y is in none of them then the integral is 0. If y is in one of them, suppose E_j, then

$$\int s \, d\delta_y = \sum_{i=1}^{n} a_i \delta_y(E_i) = a_j = s(y) \leq f(y)$$

Since $\int f \, d\delta_y$ is by definition the supremum of such simple functions, this proves that $\int f \, d\delta_y \leq f(y)$. Consider the simple function $s = f(y)\chi_{\{y\}}$. We see that $0 \leq s \leq f$, and so, $\int f \, d\delta_y \geq \int s \, d\delta_y = f(y)$. Thus, $\int f \, d\delta_y = f(y)$. □

Ex 6.3 Let X be the positive integers and $\mathcal{A}$ the collection of all subsets of X. If $f : X \to \mathbb{R}$ is non-negative and μ is counting measure defined in Example 3.2, prove that

$$\int f \, d\mu = \sum_{k=1}^{\infty} f(k)$$

This exercise is very useful because it allows one to derive many conclusions about series from analogous results about general measure spaces.

Proof. Again, since the $\mathcal{A}$ is the set of all subsets, f is trivally measurable. Let $s_n = \sum_{k=1}^n f(k)\chi_{\{k\}}$. We see that s_n is simple and that $s_n \leq f$. Thus, $\int s\,d\mu \leq \int f\,d\mu$. Also, we see that $\int s_n\,d\mu = \sum_{k=1}^n f(k)\mu(\{k\}) = \sum_{k=1}^n f(k)$. This means that $\sum_{k=1}^n f(k) \leq \int f\,d\mu$ for all n, and thus $\sum_k^\infty f(k) \leq \int f\,d\mu$.

Let $s = \sum_{k=1}^n a_k\chi_{E_k}$ be a simple function represented in its canonical form where $0 \leq s \leq f$. If $x \in E_j$, then, since s is canonical, it doesn't appear in any other E_k. This means that $s(x) = a_j$, and since $s \leq f$, this shows that $a_j \leq f(x)$ where $x \in E_j$. With this, we see that

$$\int s\,d\mu = \sum_{k=1}^n a_k|E_k| = \sum_{k=1}^n a_k \sum_{x\in E_k} 1 = \sum_{k=1}^n \sum_{x\in E_k} a_k \leq$$

$$\sum_{k=1}^n \sum_{x\in E_k} f(x) \leq \sum_{x\in\cup_k E_k} f(x) \leq \sum_{x\in X} f(x) \leq \sum_{k=1}^\infty f(k)$$

Since $\sum_{k=1}^\infty f(k)$ is greater than any simple function less than or equal to f, it's greater than the supremum of all such simple functions, which is, by definition $\int f\,d\mu$. Thus, $\int f\,d\mu \leq \sum_{k=1}^\infty f(k)$. This proves that $\int f\,d\mu = \sum_{k=1}^\infty f(k)$. $\square$

Ex 6.5 Let f be a non-negative measurable function. Prove that

$$\lim_{n\to\infty} \int (f \wedge n) = \int f$$

Proof. We see that $f \wedge n \leq f$, and thus $\int (f\wedge n)\,d\mu \leq \int f\,d\mu$ for any n. This proves that $\lim_{n\to\infty} \int (f\wedge n)\,d\mu \leq \int f\,d\mu$.

Let $s = \sum_{k=1}^n a_k\chi_{E_k}$ be a simple function in its canonical form where $0 \leq s \leq f$. Let $x \in X$. Then x lies in at most one of these E_k's. If it's in none, then $s(x) = 0$, if it's in one, then $s(x) = a_j$ for some j. Thus, for any $x \in X$, $s(x) \leq \max\{a_k\}$. Let n be an integer greater than this maximum. Since $s \leq f$ and $s \leq n$, this means that $s \leq f \wedge n$. Thus, $\int s\,d\mu \leq \int (f\wedge n)\,d\mu \leq \lim_{n\to\infty} \int (f\wedge n)\,d\mu$. If one takes the supremum of all such s, we see that $\int f\,d\mu \leq \lim_{n\to\infty} \int (f\wedge n)\,d\mu$. This proves the statement. $\square$

Ex 6.6 Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose μ is σ-finite. Suppose f is integrable. Prove that given ε there exists δ such that

$$\int_A |f(x)|\,\mu(dx) < \varepsilon$$

whenever $\mu(A) < \delta$.

Proof. Let $\varepsilon > 0$. We see that since $|f|\chi_A \leq |f|$ that $\int |f|\chi_A\,d\mu \leq \int |f|\,d\mu < \infty$. This proves that $|f|\chi_A$ is integrable. Since it's finite, this means that there's a simple function s such that $0 \leq s \leq |f|\chi_A$ and where $\int |f|\chi_A\,d\mu - \int s\,d\mu < \frac{\varepsilon}{2}$.

Since $0 \leq s \leq |f|\chi_A$, we can see that this means that $s(x) = 0$ for all $x \in A$. Thus, $s = s\chi_A$. Let $\sum_{k=1}^n a_k\chi_{E_k}$ be the conanical form of s. This means that $s = s\chi_A = \chi_A \sum_{k=1}^n a_k\chi_{E_k} = \sum_{k=1}^n a_k\chi_{E_k\cap A}$. Thus

$$\int s\,d\mu = \int s\chi_A\,d\mu = \sum_{k=1}^n a_k\mu(A \cap E_k)$$

If we let $\mu(A) < \delta = \frac{\varepsilon}{2\sum_{k=1}^{n} a_k}$, we see that

$$\int s\,d\mu \leq \sum_{k=1}^{n} a_k \mu(A) = \mu(A) \sum_{k=1}^{n} a_k < \frac{\varepsilon}{2\sum_{k=1}^{n} a_k} \sum_{k=1}^{n} a_k = \frac{\varepsilon}{2}$$

This shows that

$$\int_A |f|\,d\mu = \int |f|\chi_A\,d\mu < \frac{\varepsilon}{2} + \int s\,d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This proves the statement. □

Ex 6.8 If f_n is a sequence of non-negative integrable functions such that $f_n(x)$ decreases to $f(x)$ for every x, prove that

$$\int f_n\,d\mu \to \int f\,d\mu$$

Proof. Since f is the limit of decreasing non-negative functions, we see that $f \geq 0$. Thus, $0 \leq f \leq f_n$. This means that $\int |f|\,d\mu = \int f\,d\mu \leq \int f_n\,d\mu = \int |f_n|\,d\mu$. Since f_n is integrable, this means that f is integrable.

Let $g_n = f_1 - f_n$. Since f_n was decreasing, then g_n is increasing. We see that $g_n = f_1 - f_n \geq 0$, and also that that $g_n \uparrow (f_1 - f_n)$. Thus, using the Monotone Convergence Theorem and the fact that the Lebesgue integral is linear on integrable functions, we see that:

$$\int f_1\,d\mu - \lim_{n\to\infty} \int f_n\,d\mu = \lim_{n\to\infty} \left(\int f_1\,d\mu - \int f_n\,d\mu \right) = \lim_{n\to\infty} \int (f_1 - f_n)\,d\mu =$$

$$\int \lim_{n\to\infty} (f_1 - f_n)\,d\mu = \int (f_1 - f)\,d\mu = \int f_1\,d\mu - \int f\,d\mu$$

Since f_1 is integrable, its integral is finite. Thus, we can subtract it from both sides and multiply by -1, which gives that

$$\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu$$

□