Problem Set 3
Real Analysis II

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Section 14

Ex 5 Prove that f is Lipschitz continuous with constant M if and only if f is absolutely continuous and $|f'| \leq M$ almost everywhere.

Proof. Suppose that f is absolutely continuous and $|f'| \leq M$ almost everywhere. Since f is absolutely continuous, then we know that

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Using this fact, we can see that

$$|f(x) - f(y)| = \left|\int_x^y f'(t)\,dt\right| \leq \int_y^x |f'(t)|\,dt \leq \int_y^x M\,dt = M(x-y)$$

which proves that f is Lipschitz continuous with respect to the constant M.

Now suppose that f is Lipschitz continuous with constant M. Let $\varepsilon > 0$ and let $\delta = \varepsilon/M$. We see that for any collection of disjoint intervals $\{[a_i, b_i]\}_{i\leq n}$ where $\sum_{i=1}^n |b_i - a_i| < \delta$, we have that

$$\sum_{i=1}^n |f(b_i) - f(a_i)| < \sum_{i=1}^n M|b_i - a_i| < M\delta = \varepsilon$$

This proves that f is absolutely continuous. To show that $|f'| \leq M$, we see that

$$|f'(x)| = \left|\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right| = \lim_{h\to 0} \frac{|f(x+h) - f(x)|}{|h|} \leq \lim_{h\to 0} \frac{M|h|}{|h|} = \lim_{h\to 0} M = M$$

which concludes the proof. $\square$

Ex 6 Suppose F_n is a sequence of increasing nonnegative right continuous functions on $[0,1]$ such that $\sup_n F_n(1) < \infty$. Let $F = \sum_{n=1}^{\infty} F_n$ and suppose $F(1) < \infty$. Prove that

$$F'(x) = \sum_{n=1}^{\infty} F_n'(x)$$

for almost every x.

Proof. If each f_n then surely $f = \sum_{n=1}^{\infty} f_n$ is increasing. First, we see that

$$\int_y^1 \left(f'(x) - \sum_{n=1}^{\infty} f_n'(x) \right) dx = \int_y^1 f'(x)\,dx + \int_1^y \sum_{n=1}^{\infty} f_n'(x)\,dx$$

Since $f_n(x)$ is increasing for each n, this means that $f_n'(x) \geq 0$ for all n and x. Thus, we can exchange the integral and the summation to get that

$$\int_y^1 \left(f'(x) - \sum_{n=1}^{\infty} f_n'(x) \right) dx = \int_y^1 f'(x)\,dx + \sum_{n=1}^{\infty} \int_1^y f_n'(x)\,dx$$

Again, since f_n and f are increasing we have that

$$\int_y^1 \left(f'(x) - \sum_{n=1}^{\infty} f_n'(x) \right) dx \leq f(1) - f(y) + \sum_{n=1}^{\infty} (f_n(y) - f_n(1))$$

Since $f_n(1)$ is bounded above, we get that

$$\int_y^1 \left(f'(x) - \sum_{n=1}^{\infty} f_n'(x) \right) dx \leq f(1) - f(y) + \sum_{n=1}^{\infty} f_n(y) - \sum_{n=1}^{\infty} f_n(1) = f(1) - f(y) + f(y) - f(1) = 0$$

Now to prove that $f'(x) \geq \sum_{n=1}^{\infty} f_n'(x)$. We see that

$$f'(x) = \lim_{h\to\infty} \frac{f(x+h) - f(x)}{h} = \lim_{h\to\infty} \frac{\sum_{n=1}^{\infty} f(x+h) - \sum_{n=1}^{\infty} f(x)}{h}$$

Even if $\sum_{n=1}^{\infty} f(x) = \infty$, then we still have that

$$\lim_{h\to\infty} \frac{\sum_{n=1}^{\infty} f(x+h) - \sum_{n=1}^{\infty} f(x)}{h} \leq \lim_{h\to\infty} \frac{\sum_{n=1}^{\infty} (f_n(x+h) - f_n(x))}{h} = \sum_{n=1}^{\infty} f_n'(x)$$

This proves that the function $f'(x) - \sum_{n=1}^{\infty} f_n(x)$ is nonnegative. This means that

$$\lim_{y\to\infty} \left[\chi_{[y,1]} (f'(x) - \sum_{n=1}^{\infty} f_n(x)) \right]$$

is the limit of increasing (in that each term is bigger than the last) nonnegative functions. Thus, by the monotone convergence theorem.

$$0 = \lim_{y\to 0} \int_y^1 \left(f'(x) - \sum_{n=1}^{\infty} f_n'(x)\,dx \right) = \int_0^1 \left(f'(x) - \sum_{n=1}^{\infty} f_n'(x) \right) dx$$

which means that $f'(x) - \sum_{n=1}^{\infty} f_n'(x) = 0$ almost everywhere as the function is nonnegative. This proves the statement. $\square$

Ex 8 If f is real-valued and differentiable at each point of $[0,1]$, is f necessarily absolutely continuous on $[0,1]$? If not, find a counterexample.

Proof. No, this is not true. As a counterexample, consider $f(x) = x^2 \sin \frac{\pi}{x^4}$ where $f(0) = \lim_{x\to 0^+} f(x) = 0$. We see that this function is differentiable everywhere except possibly at zero. However, looking at

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{h^2 \sin \frac{\pi}{h^4}}{h} = \lim_{h\to 0} h \sin \frac{\pi}{h^4} = 0$$

we see that f is actually differentiable at 0. Using the normal calculus, we find that

$$f'(x) = 2x \sin \frac{\pi}{x^4} + x^2 \cos \frac{\pi}{x^4} \frac{-4}{x^5} = 2x \sin \frac{\pi}{x^4} - \frac{4 \cos \frac{\pi}{x^4}}{x^3}$$

which is not integrable because of the $\frac{1}{x^3}$ term. Since all absolutely continuous functions have an integrable deriviative, then this proves that f cannot be absolutely continuous. □

Additional Problems

Ex 1 Prove that if f is of bounded variation on $[a,b]$ and the function $V(x) = V(f;a,x)$ is absolutely continuous on $[a,b]$, then f is absolutely continuous on $[a,b]$.

Proof. Let $\varepsilon > 0$ and let δ be a sufficiently small number so as to invoke the absolutely continuous property of $V(x)$. Now let $\{[a_i,b_i]\}_{i\le n}$ be a disjoint collection of disjoint intervals such that $\sum_{i=1}^n |b_i - a_i| < \delta$.

We can view $\{a_i, b_i\}$ as a partition on $[a_i,b_i]$, which means that $|f(b_i)-f(a_i)| \le V(f;a_i,b_i)$ since $V(f;a_i,b_i)$ is the supremum over any partition of $[a_i,b_i]$. We also proved in class that $V(f;a,a_i) + V(f;a_i,b_i) = V(f;a,b_i)$, which means that $V(f;a_i,b_i) = V(f;a,b_i) - V(f;a,a_i) = V(b_i) - V(a_i)$, which is positive as $V(x)$ is increasing. This proves that

$$\sum_{i=1}^{\infty} |f(b_i) - f(a_i)| \le \sum_{i=1}^{\infty} V(f;a_i,b_i) = \sum_{i=1}^{\infty} |V(b_i) - V(a_i)| < \varepsilon$$

as $V(x)$ is absolutely continuous. This proves that using the same δ, f is also absolutely continuous. □

Ex 2 Suppose f is of bounded variation on $[a,b]$.

(i) Prove that

$$\int_a^b |f'|\,dx \le V(f;a,b)$$

(ii) Prove that if

$$\int_a^b |f'|\,dx = V(f;a,b)$$

then f is absolutely continuous on $[a,b]$.

Proof. a) Since f is of bounded variation on $[a,b]$, then we know from class that $V'(x) = |f'(x)|$ almost everywhere. If $x_1 < x_2$, then $V(x_1) = V(f;a,x_1) = V(f;a,x_2) - V(f;x_1,x_2) \leq V(f;a,x_2) = V(x_2)$. This means that

$$\int_a^b |f'|\,dx = \int_a^b V'(x)\,dx \leq V(b) - V(a) = V(b) = V(f;a,b)$$

as $V(a) = V(f;a,a)$ is trivally zero. This proves the statement.

b) We see that

$$\begin{aligned}
V(f;a,b) - \int_a^b |f'|\,dx = 0& \\
&\implies (V(f;a,x) + V(f;x,b)) - \left(\int_a^x |f'|\,dx + \int_x^b |f'|\,dx\right) = 0 \\
&\implies \left(V(f;a,x) - \int_a^x |f'|\,dx\right) + \left(V(f;x,b) - \int_x^b |f'|\,dx\right) = 0 \\
&\implies \left(V(f;a,x) - \int_a^x V'(x)\,dx\right) + \left(V(f;x,b) - \int_x^b V'(x)\,dx\right) = 0 \quad (1)
\end{aligned}$$

However,

$$V(f;a,x) - \int_a^x V'(x)\,dx \geq V(f;a,x) - (V(x) - V(b)) = V(b) \geq 0$$

and similarly $V(f;x,b) - \int_x^b V'(x)\,dx \geq 0$. Given the previous equation, this proves that

$$V(f;a,x) = \int_a^x V'(x)\,dx \quad (2)$$

$$V(f;x,b) = \int_x^b V'(x)\,dx \quad (3)$$

which means that $\int_c^d |f'|\,dx = V(f;c,d)$ for any $[c,d] \subseteq [0,1]$.

Now let $\varepsilon > 0$ and let $\{[a_i,b_i]\}_{i\leq n}$ where $\sum_{i=1}^n |b_i - a_i| \leq \delta$ (to be determined later). We see then that

$$\sum_{i=1}^n |f(b_i) - f(a_i)| \leq \sum_{i=1}^n V(f;a_i,b_i) = \sum_{i=1}^n \int_{a_i}^{b_i} |f'(x)|\,dx = \int_{\cup_i [a_i,b_i]} |f'|\,dx$$

Since f' is integrable (as it's integral is the total variation which is bounded), then we know that we can find a δ small enough such that for any measurable set E where $m(E) < \delta$, we get that $\int_E |f'|\,dx < \varepsilon$. Let our original δ be this δ. This proves that

$$\sum_{i=1}^n |b_i - a_i| < \delta \implies \sum_{i=1}^n |f(b_i) - f(a_i)| \leq \int_{\cup [a_i,b_i]} |f'|\,dx < \varepsilon$$

which proves that f is absolutely continuous.

□