Real Analysis
Semester 1

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Preface

These problems were assigned to me Dr. Paul Goodie as I took his class Real Analysis I during the Fall Semester of 2016 at the University of Oklahoma. All the problems found in this document come from the book “Real Analysis for Graduate Students” by Richard F. Bass. The only exception to this are the Preliminary Problems, which were designed by Dr. Goodie as a sort of pre-test. I hope that you find these solutions helpful.

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Chapter 1

Preliminary Problems

Ex 1 a) For sequences $(x_n), (y_n)$ of real numbers, prove that

$$\limsup\{x_n\} + \liminf\{y_n\} \leq \limsup\{x_n + y_n\} \leq \limsup\{x_n\} + \limsup\{y_n\}$$

b) Give a specific example where both inequalities are strict.

Proof. a) If $\alpha = \limsup\{x_n\}$, then for an $\epsilon \geq 0$, there are an infinite number of j's where $x_j \leq \alpha - \frac{\epsilon}{2}$. If there were only a finite number of such j's, then let n be the last one. This would mean that $\sup\limits_{k\geq n}\{x_k\} \leq \alpha - \frac{\epsilon}{2}$, which would mean that $\limsup\{x_n\} \leq \alpha - \frac{\epsilon}{2} < \alpha$. This is a contraction. If $\beta = \liminf\{y_n\}$, then for any $\epsilon > 0$, there is an N, such that for all $n \geq N$

$$y_n \geq \inf_{k\geq n}\{y_k\} \geq \beta - \frac{\epsilon}{2}$$

This means there are an infinite number of n where $x_n > \alpha - \frac{\epsilon}{2}$ and $y_n \geq \beta - \frac{\epsilon}{2}$. Thus, for $n \geq N$, it means that $x_n + y_n \geq \alpha + \beta - \epsilon$ for inifinitely many n. This means that $\sup\{x_n + y_n\} \geq \alpha + \beta - \epsilon$. Since ϵ was arbitrary, this means that $\limsup\{x_n + y_n\} \geq \limsup\{x_n\} + \liminf\{y_n\}$. This proves the first inequality.

Let $x_j \in \{x_k\}_{k\geq n}$ and $y_j \in \{y_k\}_{k\geq n}$. Then, this means that $x_j \leq \sup\limits_{k\geq n}\{x_k\}$ and that $y_j \leq \sup\limits_{k\geq n}\{x_k\}$. Thus, $x_j + y_j \leq \sup\limits_{k\geq n}\{x_k\} + \sup\limits_{k\geq n}\{y_k\}$. If you take the sup off all possible j's of both sides, you get $\sup\limits_{j\geq n}\{x_j + y_j\} \leq \sup\limits_{j\geq n}\{\sup\limits_{k\geq n}\{x_k\} + \sup\limits_{k\geq n}\{y_k\}\}$. The RHS was already "supped" and therefore was just a constant, so the additional sup doesn't do anything. Thus $\sup\limits_{j\geq n}\{x_j + y_j\} \leq \sup\limits_{k\geq n}\{x_k\} + \sup\limits_{k\geq n}\{y_k\}$. Taking the limit as $n \to \infty$, one gets the second inequality.

b) Let $y = \begin{cases} n & \text{if } n \text{ is odd} \\ -n & \text{if } n \text{ is even} \end{cases}$ and $x_n = \begin{cases} -n & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$.

Computing the the limsups and liminfs, we get

$\limsup\{x_n\} + \liminf\{y_n\} = 1 - \infty = \infty$

$\limsup\{x_n + y_n\} = 0$ as $x_n + y_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 - n & \text{if } n \text{ is odd} \end{cases}$

$\limsup\{x_n\} + \limsup\{y_n\} = 1 + \infty = \infty$

As can be seen, the inequalities are strict for these x_n and y_n.

□

Ex 2 a) Let $p > 1$ be an integer and x a real number with $0 < x < 1$. Show that there is a sequence of integers (a_n) with $0 \leq a_n < p$ such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

b) Show that the above sequence a_n is unique except when x is of the form $\frac{q}{p^n}$ for some integer q and that, in this case, there are exactly two such sequences.

c) Show, conversely, that if $a_n$is any sequence of integers with $0 \leq a_n < p$, the series

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

converges to a number x with $0 \leq x \leq 1$.

Proof. Incomplete at the moment □

Ex 3 Let f be a mapping from a set X to the set of subsets of X. By considering the set $E = \{x \in X : x \notin f(x)\}$, show that there is a subset of X which is not in the range of f.

Proof. Let $f : X \to P(X)$ and let $E = \{x \in X \mid x \notin f(x)\}$. Assume that f is surjective. Since $E \in P(X)$, then there exists an x' such that $f(x') = E$. Now, x' must be either in E or not in E. Say $x' \in E$, then $x' \in E = f(x')$. However, since $x' \in E$, this means by definition of E that $x' \notin f(x')$. Thus, we have a contradiction. Now say $x' \notin E$, then $x' \notin E = f(x')$. But, by definition of E, this means that $x' \in E$. This is also a contradiction. Thus, there is no $x \in X$ such that $f(x') = E$, which means f is not surjective. This shows there's a subset of X which is not in the range of f. □

Ex 4 If $S \subseteq X$ is uncountable and $A \subseteq X$ is countable, show that $S \cap A^c$ is uncountable.

Proof. Assume that $S \cap A^c$ is countable. Then $(S \cap A^c) \cup A$ is the union of two countable sets, and is thus countable. By distributivity, $(S \cap A^c) \cup A = (S \cup A) \cap (A \cup A^c) = (S \cup A) \cap X = S \cup A$. $S \cup A$ is clearly uncountable, as $S \subseteq S \cup A$. This is a contradiction. Thus, $S \cap A^C$ must be uncountable. □

Ex 5 a) Is the set of rationals open or closed in the set of real numbers?

b) Which sets of real numbers are both open and closed?

Proof. a) Every open interval on the real line contains both rationals and irrationals. Therefore, every open neighborhood of $q \in \mathbb{Q}$ contains irrationals. This shows that $\mathbb{Q}$ is not open. Similarly, if $\mathbb{Q}$ were closed, then $\mathbb{R} \setminus \mathbb{Q}$ would be open. But, every open neighborhood of $i \in \mathbb{R} \setminus \mathbb{Q}$ contains a rational number. Therefore, $\mathbb{R} \setminus \mathbb{Q}$ is not open, and $\mathbb{Q}$ is not closed. This means that $\mathbb{Q}$ is neither open nor closed.

b) $\varnothing$ is trivally open and closed. Similarly, it's complement, $\mathbb{R}$ is then open and closed. Assume that there exists an additional open and closed set $\varnothing \neq A \neq \mathbb{R}$. Then the complement A^c would also be open and closed, and not equal to $\varnothing$ or $\mathbb{R}$. Now fix $a \in A$ and $b \in A^c$. Without loss of generality (since $(A^c)^c = A$), let $a < b$. Now let $C = \{x \in \mathbb{R} \mid [a, x] \subseteq A\}$. C is nonempty, as $a \in C$. Also, C is bound above by b, as $b \notin A$. Thus, C has a least upper bound. Let's call it α.

Say $\alpha \in A$, then, since A is open, there'd be a ball of radius $r > 0$, such that $(\alpha - r, \alpha + r) \subseteq A$. Then $[\alpha, \alpha + \frac{r}{2}] \subseteq (\alpha - r, \alpha + r) \subseteq A$. This means $[a, \alpha] \cup \left[\alpha, \alpha + \frac{r}{2}\right] = \left[a, \alpha + \frac{r}{2}\right] \subseteq A$. This means that $\alpha + \frac{r}{2} \in C$, a contradiction, as α was the least upper bound of C. This means that $\alpha \notin A$.

Say $\alpha \in A^c$, then, since A^c is open, there'd be a ball of radius $r > 0$, such that $(\alpha - r, \alpha + r) \subseteq A^c$. Then $[\alpha - \frac{r}{2}, \alpha] \subseteq (\alpha - r, \alpha + r) \subseteq A^c$. This means that $\alpha - \frac{r}{2} \notin A$, which means $\alpha - \frac{r}{2} \notin C$. However $\alpha - \frac{r}{2} < \alpha$ and α is supposed to be the least upper bound. This is a contradiction, thus $\alpha \notin A^c$. This means that $\alpha \in A$. From the last paragraph, we proved that $\alpha \notin A$. This contradiction shows that there is no such set A. This means that $\varnothing$ and $\mathbb{R}$ are the only two open and closed sets in $\mathbb{R}$.

□

Ex 6 Prove that a set X is infinite if and only if there is a proper subset of X of the same cardinality as X.

Proof. Suppose X is infinite. Then create an injective function $\phi : \mathbb{N} \to X$. It's easy to see such a function exists, as you can choose an element of X for each $n \in \mathbb{N}$, and you'll never run out of elements, as that would mean that there was a bijection between $\{1 \ldots n\}$ and X. From this injective function let $\psi : X \to X$ where

$$\psi = \begin{cases} \phi\,(n+1) & x = \phi\,(n) \text{ for some } n \in \mathbb{N} \\ x & x \notin \operatorname{Im}(\phi) \end{cases}$$

One can see that $\phi\,(0) \notin \operatorname{Im}(\psi)$, so $\operatorname{Im}(\psi) \subsetneq X$, and also that ψ is injective, as ϕ was injective. Combined with the inclusion map from $\operatorname{Im}(\psi)$ to X, which is necessarily injective, we see that there's a bijection between X and $\operatorname{Im}(\psi) \subsetneq X$. This assumes the Axiom of Choice.

Let X be a set with a proper subset A. This means there exists an $x \in X$, such that $x \notin A$. Let's say they have the same cardinality. This means that there's a bijection $\phi : X \to A$. Construct the function $\psi : \mathbb{N} \to X$ as follows: $\psi\,(0) = a$ and $\psi\,(i+1) = \phi(\psi(i))$ for all $0 \neq n \in \mathbb{N}$. (Tip: It might be more intuitive to view it as a sequence.) Since ϕ is injective and $a \notin \operatorname{Im}(\phi)$, we see that ψ is injective. We also see that through the inclusion map, $\{1 \ldots n\}$ is injective to $\mathbb{N}$. If the cardinality of X were finite, then there'd be a bijection

$f : X \to \{1 \dots n\}$ for some $n \in \mathbb{N}$. Thus, the composition, $f \circ \psi : \mathbb{N} \to \{1 \dots n\}$ would be injective, and thus there's a bijection between $\mathbb{N}$ and $\{1 \dots n\}$. This is a contradiction, which means X must be infinite. $\square$

Chapter 2

Families of sets

Ex 2.1 Find an example of a set X and a monotone class $\mathcal{M}$ consisting of subsets of X such that $\varnothing \in \mathcal{M}, X \in \mathcal{M}$, but $\mathcal{M}$ is not a σ-algebra.

Proof. Let $X = \mathbb{R}$. Consider the monotone class $\mathcal{M}$ which is the set of unbounded intervals of $\mathbb{R}$ along with the empty set. We see that this trivally includes $\varnothing$ and $\mathbb{R}$. This is definitely a monotone class. If we have a sequence of $A_i \in \mathcal{M}$, where $A_{i+1} \subseteq A_i$, look at $A_i \downarrow A$. Say that A is nonempty and bounded below by α and above by β. This means for some $A_j \in \mathcal{M}$, A_j is bounded above by β and for some A_k, A_k is bounded below by α, thus $A_{\max(j,k)}$ is bounded above by β and below by α and is nonempty. This means that it's not in $\mathcal{M}$ which is a contradiction. Thus, $A \in \mathcal{M}$.

Similarly, let's say we have a sequence of $A_i \in \mathcal{M}$, where $A_i \subseteq A_{i+1}$, and look at $A_i \uparrow A$. The union of unbounded intervals is definitely unbounded, thus $A \in \mathcal{M}$.

However, say that $a \leq b$, then this means that $(a, \infty) \cap (-\infty, b) = (a, b)$. Thus, $\mathcal{M}$ is not closed under finite intersection, which means that $\mathcal{M}$ is not a σ-algebra. $\square$

Ex 2.2 Find an example of a set X and two σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, each consisting of subsets of X, such that $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-algebra.

Proof. Let $X = \{1, 2, 3\}$. Then we see that $\mathcal{A}_1 = \{\varnothing, X, \{1\}, \{2, 3\}\}$ and that $\mathcal{A}_2 = \{\varnothing, X, \{2\}, \{1, 3\}\}$. We can clearly see that these are closed under complementation and contained X and $\varnothing$. It's also easy to check that they are closed under countable intersection. However, $\mathcal{A}_1 \cup \mathcal{A}_2 = \{\varnothing, X, \{1\}, \{2\}, \{2, 3\}, \{1, 3\}\}$, which is not a σ-algebra, as $\{1\} \cup \{2\} = \{1, 2\} \notin \mathcal{A}_1 \cup \mathcal{A}_2$. $\square$

Ex 2.3 Suppose $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \ldots$ are σ-algebras consisting of subsets of a set X. Is $\cup_{i=1}^{\infty} \mathcal{A}_i$ necessarily a σ-algebra? If not, give a counterexample.

Proof. $\cup_{i=1}^{\infty} \mathcal{A}_i$ is not necessarily a σ-algebra. For a counterexample, let $\mathcal{A}_n = \mathcal{P}(\{1, \ldots, n\})$, where $\mathcal{P}$ is the powerset function. These are trivally σ-algebras. We also see easily that $\mathcal{A}_i \subseteq \mathcal{A}_{i+1}$. Look at $\mathcal{A} = \cup_{i=1}^{\infty} A_i$. We know that $\{n\} \in \mathcal{A}_n$, as $\{n\}$ is a subset of $\{1, \ldots, n\}$. However, $\cup_{n=1}^{\infty} \{n\} = \mathbb{N}$. Since all the members of each $\mathcal{A}_i$ are finite, this means that $\mathbb{N}$ is not a member of any $\mathcal{A}_i$. Thus, $\mathbb{N} \notin \cup_{i=1}^{\infty} \mathcal{A}_i$. This shows that $\mathcal{A}$ is not closed under countable union. $\square$

Ex 2.5 Let $(Y, \mathcal{A})$ be a measurable space and let f map X into Y, but do not assume that f is one-to-one. Define $\mathcal{B} = \{f^{-1}(A) \mid A \in \mathcal{A}\}$. Prove that $\mathcal{B}$ is a σ-algebra of subsets of X.

Proof. Firstly, we see that since $Y \in \mathcal{A}$, then $f^{-1}(Y) = X \in \mathcal{B}$. Also, since $\varnothing \in \mathcal{A}$, then $f^{-1}(\varnothing) = \varnothing \in \mathcal{B}$.

Now, let $B \in \mathcal{B}$. This means $B = f^{-1}(A_1)$ for some $A \in \mathcal{A}$. Since $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$. Thus, $f^{-1}(A^c) \in \mathcal{B}$. We shall prove that $f^{-1}(A^c) = f^{-1}(A)^c$. Let $x \in f^{-1}(A^c)$. This means that $f(x) \in A^c$, which means that $f(x) \notin A$. This means that $x \notin f^{-1}(A)$, and finally we see that $x \in f^{-1}(A)^c$. This shows that $f^{-1}(A^c) \subseteq f^{-1}(A)^c$. Using the same reasoning in the opposite order, we conclude that $f^{-1}(A^c) = f^{-1}(A)^c$. Thus, this $f^{-1}(A^c) \in \mathcal{B}$, is actually equal to $f^{-1}(A)^c = B^c \in \mathcal{B}$. This shows that $\mathcal{B}$ is closed under complements.

Now, let $B_i \in \mathcal{B}$, where $i \in \mathbb{N}$. This means that $B_i = f^{-1}(A_i)$ for some $A_i \in \mathcal{A}$. This means that $\cup_i A_i \in \mathcal{A}$, which means $f^{-1}(\cup_i A_i) \in \mathcal{B}$. Let $x \in f^{-1}(\cup_i A_i)$. This means that $f(x) \in \cup_i A_i$. This means that $f(x) \in A_n$ for some $n \in \mathbb{N}$. Thus, $x \in f^{-1}(A_n)$, which means that $x \in \cup_i f^{-1}(A_i)$. Let $x \in \cup_i f^{-1}(A_i)$. Then this means that $x \in f^{-1}(A_n)$ for some $n \in \mathbb{N}$. Thus, $f(x) \in A_n$, which means that $f(x) \in \cup_i A_i$. Thus, $x \in f^{-1}(\cup_i A_i)$. This proves that $f^{-1}(\cup_i A_i)$ which is in $\mathcal{B}$, is equal to $\cup_i f^{-1}(A_i) = \cup_i B_i \in \mathcal{B}$. Thus, $\mathcal{B}$ is closed under countable union. This shows that $\mathcal{B}$ is indeed a σ-algebra. □

Ex 2.6 Suppose $\mathcal{A}$ is a σ-algebra with the property that whenever $A \in \mathcal{A}$ is non-empty, there exist $B, C \in \mathcal{A}$ with $B \cap C = \varnothing$. $B \cup C = A$, and neither B nor C is empty. Prove that $\mathcal{A}$ is uncountable.

Proof. First off, a simple counterexample is if $\mathcal{A} = \{\varnothing\}$ is the trival σ-algebra over $X = \varnothing$. Excluding this, though, we continue on:

Assume that $|\mathcal{A}| = n$. And let $A \in \mathcal{A}$ be any nonempty element. This means there exists sets $B, C \in \mathcal{A}$, such that they are nonempty and disjoint and their union is equal to A. Do this same thing to the set $B \in \mathcal{A}$. These two sets that union to B and are nonempty can neither be C nor A. As if one were either of these two sets, the union of such two sets would contain elements not in B, which is a contradiction as their union is exactly B. Do this n times. This gives us more than n elements which must be in $\mathcal{A}$, which is a contradiction, as $|\mathcal{A}| = n$. Thus, $\mathcal{A}$ is not finite. Since $\mathcal{A}$ is not finite, by Exe 2.8, $\mathcal{A}$ is not countable. Thus, $\mathcal{A}$ must be uncountable. □

Ex 2.7 Suppose $\mathcal{F}$ is a collection of real-valued functions on X such that the constant functions are in $\mathcal{F}$ and $f + g$, fg, and cf are in $\mathcal{F}$ whenever $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$. Suppose $f \in \mathcal{F}$ whenever $f_n \to f$ and each $f_n \in \mathcal{F}$. Define the function

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Prove that $\mathcal{A} = \{A \subseteq X \mid \chi_A \in \mathcal{F}\}$ is a σ-algebra.

Proof. We see that $\varnothing \in \mathcal{A}$ as $\chi_\varnothing = 0$, a constant function, which we assumed to be in $\mathcal{F}$. Also, $X \in \mathcal{A}$, as $\chi_X = 1$, also a constant function which we assumed to be in $\mathcal{F}$.

Let $A \in \mathcal{A}$. This means that $\chi_A \in \mathcal{F}$. Look at the function $-(\chi_A - 1)$. We see that this function is in $\mathcal{F}$, as $\mathcal{F}$ is closed under function addition and scalar multiplication. Also, one sees that this function is equivalent to χ_{A^c}. Thus, since this means $\chi_{A^c} \in \mathcal{F}$, we know that $A^c \in \mathcal{A}$. This shows closure under addition.

Let $A, B \in \mathcal{A}$. Thus $\chi_A, \chi_B \in \mathcal{F}$. We see that $\chi_A \chi_B$ is one iff $x \in A$ and $x \in B$ or, in other words, $x \in A \cap B$. Thus, $\chi_A \chi_B = \chi_{A \cap B}$. We also see that $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$. Since $\mathcal{F}$ is closed under addition, multiplication, and scalar multiplication, we see that $\chi_{A \cup B} \in \mathcal{F}$, which means that $A \cup B \in \mathcal{A}$.

Let $A_i \in \mathcal{A}$. Then let $B_n = \cup_{i=1}^n A_i$. We see that $B_2 = A_1 \cup A_2$ is in $\mathcal{A}$, as proven by the last paragraph. By induction, $B_n = A_n \cup B_{n-1}$ is in $\mathcal{A}$, as $A_n \in \mathcal{A}$ and $B_{n-1} \in \mathcal{A}$ by the inductive hypothesis. Thus, $B_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. We see that $\cup_{i=1}^\infty A_i = \lim_{i\to\infty} B_i$. Since $B_i \in \mathcal{A}$, this means that $\chi_{B_i} \in \mathcal{F}$. Thus, since $\mathcal{F}$ is closed under limits, this means that $\lim_{i\to\infty} \chi_{B_i} \in \mathcal{F}$. Thus, $\lim_{i\to\infty} B_i \in \mathcal{F}$, which shows that $\cup_{i=1}^\infty A_i \in \mathcal{A}$. Thus, $\mathcal{A}$ is closed under countable union. This shows that $\mathcal{A}$ is a σ-algebra. $\square$

Ex 2.8 Does there exist a σ-algebra which has countably many elements, but not finitely many?

Proof. Let's assume that $\mathcal{A}$ is an infinitely countable σ-algebra on a set X. If X were finite, then $\mathcal{P}(X)$ would be finite, and since a σ-algebra is a collection of subsets of X, this is means $\mathcal{A}$ is finite, a contradiction. Thus, X must be infinite. Define $B_x = \cap_{x \in A \in \mathcal{A}} A$. That is, the intersection of all members of $\mathcal{A}$ that contain x. We see that this is well-defined, as there are only countably infinitely many members of $\mathcal{A}$, which means the intersection is a countable intersection.

We see that $\{B_x\}_{x \in X}$ is a collection of subsets of X. Claim: This collection defines a partition of X. We see that obviously $\{B_x\}_{x \in X}$ covers X, as $x \in B_x$ for all $x \in X$. Now let $x, y \in X$, and look at the intersection of B_x and B_y. If $x \notin B_y$, then this means that $B_x \setminus B_y \in \mathcal{A}$ is a smaller set in $\mathcal{A}$ that contains x. This is a contradiction to the definition of B_x. This means that $x \in B_y$. By similar argument, we see that $y \in B_x$. Since these are the smallest sets that contain x and y respectively, we see that $x \in B_x \subseteq B_y$ and that $y \in B_y \subseteq B_x$. This proves that $B_x = B_y$, which means that $\{B_x\}_{x \in X}$ are a set of disjoint sets that union to all of X. This also shows that $\{B_x\}_{x \in X}$ is a partition.

Let $A \in \mathcal{A}$. We see obviously that $A \subseteq \cup_{x \in A} B_x$, as each B_x contains x. Now, say there is a y in $\cup_{x \in X} B_x$ that isn't in A, that is that containment is strict. This means that $y \in B_x$ for some x. Since $x \in A$ and $y \notin A$, this means that $x \in A \cap B_x$ is strictly smaller set than B_x that contains x, as it doesn't contain y. This is a contradiction to the definition of B_x, thus the containment is not strict, that is that $\cup_{x \in A} B_x = A$. Thus, every A can be written in this form.

Since every $A \in \mathcal{A}$ can be written as a union of sets of the form B_x. This means that if $\{B_x\}_{x \in X}$ were finite, then $\mathcal{A}$ would be finite as well, which is a contradiction. Thus, this partition is not finite and must be countably infinite, as it's a subset of $\mathcal{A}$. However, we know that $\mathcal{A}$ contains all possible unions of the sets in $\{B_x\}_{x \in X}$. Since all of these sets are disjoint, and there's countably infinitely many of them, the number of possible unions is equal to the number of elements in the powerset of $\{B_x\}_{x \in X}$ (each subset corresponding to which combination of elements to union together). Since the powerset must be strictly greater in cardinality than the already countably infinite set $\{B_x\}_{x\in X}$, we see that it must result in an uncountably infinite set. Since these are all in $\mathcal{A}$, as $\mathcal{A}$ is closed under countable unions, we see that $\mathcal{A}$ is uncountably infinite. $\square$

Chapter 3

Measures

Ex 3.1 Suppose $(X, \mathcal{A})$ is a measurable space and μ is a non-negative set function that is finitely additive and such that $\mu(\varnothing) = 0$ and $\mu(B)$ is finite for some non-empty $B \in \mathcal{A}$. Supppose that whenever A_i is an increasing sequence of sets in $\mathcal{A}$, then $\mu(\cup_i A_i) = \lim_{i\to\infty} \mu(A_i)$. Show that μ is a measure.

Proof. Let $A_i \in \mathcal{A}$ be mutually disjoint. Now, let $B_n = \cup_{i=1}^n A_i$. This means that $B_{n-1} = \cup_{i=1}^{n-1} A_i \subseteq \cup_{i=1}^n A_i = B_n$. This shows that B_n is increasing. We also see that $\cup_{i=1}^\infty B_i = \cup_{i=1}^\infty A_i$. Now we see that

$$\mu(\cup_{i=1}^\infty A_i) = \mu(\cup_{i=1}^\infty B_i) = \lim_{i\to\infty} \mu(B_i) = \lim_{i\to\infty} \mu\left(\cup_{n=1}^i A_n\right) = \lim_{i\to\infty} \sum_{n=1}^i \mu(A_n) = \sum_{n=1}^\infty \mu(A_n)$$

Thus, we have proven that μ is a measure. □

Ex 3.2 Suppose $(X, \mathcal{A})$ is a measurable space and μ is a non-negative set function that is finitely additive and such that $\mu(\varnothing) = 0$ and $\mu(X) < \infty$. Suppose that whenever A_i is a sequence of sets in $\mathcal{A}$ that decrease to $\varnothing$, then $\lim_{i\to\infty} \mu(A_i) = 0$. Show that μ is a measure.

Proof. Let $A_i \in \mathcal{A}$ be mutually disjoint. Now, let $B_n = \cup_{i=n+1}^\infty A_i$. We see that $B_{n+1} = \cup_{i=n+2}^\infty A_i \subseteq \cup_{i=n+1}^\infty A_i = B_n$. Thus, this is a decreasing sequence. Let $x \in \cap_{n=1}^\infty B_n$. This means that $x \in B_n$ for all n. Thus, $x \in \cup_{i=n+1}^\infty A_i$ for all n. Take $n = 1$, this means that $x \in \cup_{i=2}^\infty A_i$. Thus, $x \in A_j$ for some $j \geq 2$. Also, x is not in any other A_i as they're mutually disjoint. But we see that if we take $n = j$, we get that $x \in \cup_{i=j+1}^\infty A_i$. This must mean that x is in another A_i where $i \geq j$. This is a contradiction. Thus, there is no such x. This proves that $\cap_{n=1}^\infty B_n = \varnothing$. Thus, we know that $\lim_{i\to\infty} \mu(B_i) = 0$.

Now we see that

$$\mu(\cup_{i=1}^\infty A_i) = \mu\left((\cup_{i=1}^n A_i) \cup \left(\cup_{i=n+1}^\infty A_i\right)\right) = \mu(\cup_{i=1}^n A_i) + \mu\left(\cup_{i=n+1}^\infty A_i\right) = \sum_{i=1}^n \mu(A_i) + \mu(B_n)$$

Now if we take the limit as $n \to \infty$, we see that

$$\lim_{n\to\infty} \left(\sum_{i=1}^n \mu(A_i) + \mu(B_n)\right) = \lim_{n\to\infty} \left(\sum_{i=1}^n \mu(A_i)\right) + \lim_{n\to\infty} \mu(B_n) = \sum_{i=1}^\infty \mu(A_i) + 0 = \sum_{i=1}^\infty \mu(A_i)$$

Splitting up the limit is true because the left half is always finite as $A_i \subseteq X$, which means $\mu(A_i) \le \mu(X) < \infty$. This proves that μ is a measure. $\square$

Ex 3.3 Let X be an uncountable set and let $\mathcal{A}$ be a collection of subsets A of X such that either A or A^c is countable. Define $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is uncountable. Prove that μ is a measure.

Proof. We see that $\mu(\varnothing) = 0$, as $\varnothing$ is countable. Suppose $A_i \in \mathcal{A}$ is mutually disjoint. If A_i is countable for each i, then $\cup_{i=1}^{\infty} A_i$ is countable. Thus, $\sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} 0 = 0 = \mu(\cup_{i=1}^{\infty} A_i)$. This means that axiom 2 holds if all the A_i's are countable.

Suppose there's an A_k which is uncountable. This means that $\cup_{i=1}^{\infty} A_i$ must be uncountable. Thus, $\mu(\cup_{i=1}^{\infty} A) = 1$. We see that the sum $\sum_{i=1}^{\infty} \mu(A_i) = 1$, as the only term that is nonzero is A_k.Thus, axiom 2 still holds if there is only one A_i which is uncountable.

Suppose there were at least two A_i's that were uncountable. This means that there's an A_j and an A_k that are both uncountable. Then, since A_i's are mutually disjoint, this means that A_j and A_k must be mutually disjoint. Thus, $A_j \subseteq A_k^c$. However, since A_k is uncountable, then A_k^c must be countable, otherwise A_k wouldn't be in $\mathcal{A}$. Thus, A_j must also be countable. This is a contradiction, as A_j was assumed to be uncountable. This proves that in a set of disjoint A_i's, at most one can be uncountable. This covers all cases, and thus proves that μ is a measure. $\square$

Ex 3.4 Suppose $(X, \mathcal{A}, \mu)$ is a measure space and $A, B \in \mathcal{A}$. Prove that

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$$

Proof. We see that $A = (A \cap B) \cup (A \setminus B)$ and that $A \cap B$ and $A \setminus B$ are disjoint. Similarly $B = (A \cap B) \cup (B \setminus A)$, which are also disjoint. Thus

$$\mu(A) + \mu(B\ ) = \mu((A \cap B) \cup (A \setminus B)) + \mu((A \cap B) \cup (B \setminus A))$$

$$= \mu(A \cap B) + \mu(A \cap B) + \mu(A \setminus B) + \mu(B \setminus A)$$

We see that $A \cup B$ can be broken down into the disjoint union of $A \cap B$, $A \setminus B$, and $B \setminus A$. Thus, these last three terms can be combined into $A \cup B$. This proves that $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$. $\square$

Ex 3.6 Prove that if $(X, \mathcal{A}, \mu)$ is a measure space, $B \in \mathcal{A}$, and we define $\upsilon(A) = \mu(A \cap B)$ for $A \in \mathcal{A}$, then υ is a measure.

Proof. We see that $\upsilon(\varnothing) = \mu(\varnothing \cap B) = \mu(\varnothing) = 0$. Thus, υ satisfies the first axiom. We also see that if $A_i \in \mathcal{A}$ are disjoint sets, then $A_i \cap B$ are disjoint, and so

$$\upsilon(\cup_{i=1}^{\infty} A_i) = \mu((\cup_{i=1}^{\infty} A_i) \cap B) = \mu(\cup_{i=1}^{\infty}(A_i \cap B)) = \sum_{i=1}^{\infty} \mu(A_i \cap B) = \sum_{i=1}^{\infty} \upsilon(A_i)$$

The second equality holds because intersection distributes over unions. Thus, υ is a measure. $\square$

Ex 3.7 Suppose $\mu_1, \mu_2, \ldots$ are measures on a measurable space $(X, \mathcal{A})$ and $\mu_n(A) \uparrow$ for each $A \in \mathcal{A}$. Define $\mu(A) = \lim_{n\to\infty} \mu_n(A)$. Is μ neccessarily a measure? If not, give a counterexample. What if $\mu_n(A) \downarrow$ for each $A \in \mathcal{A}$ and $\mu_1(X) < \infty$?

Proof. We obviously see that $\mu(\varnothing) = \lim_{n\to\infty} \mu_n(\varnothing) = \lim_{n\to\infty} 0 = 0$. Now, let $A_i \in \mathcal{A}$ be disjoint sets. This means that

$$\mu\left(\cup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu_n\left(\cup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_n(A_i) = \lim_{n\to\infty} \lim_{j\to\infty} \sum_{i=1}^{j} \mu_n(A_i)$$

Since μ_n and the partial sums are both monotonically increasing, both of these limits are equivalent to the supremum. This means that

$$\lim_{n\to\infty} \lim_{j\to\infty} \sum_{i=1}^{j} \mu_n(A_i) = \sup_{n\in\mathbb{N}} \sup_{j\in\mathbb{N}} \sum_{i=1}^{j} \mu_n(A_i) = \sup_{j\in\mathbb{N}} \sup_{n\in\mathbb{N}} \sum_{i=1}^{j} \mu_n(A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

Thus, μ must be a measure.

Consider $\lambda_n(A) = \mu_1(A) - \mu_n(A)$. We see that $\lambda_n(A) \geq 0$ and that $\lambda_n(A) \neq \infty$, as $\mu_n(A) \leq \mu_1(A) < \infty$ for all $A \in \mathcal{A}$ and $n \in \mathbb{N}$. Also, we see that $\lambda_n = \mu_1(A) - \mu_n(A) \leq \mu_1(A) - \mu_{n+1}(A) = \lambda_{n+1}(A)$. Thus, since $\lambda_n(A) \uparrow$ for each $A \in \mathcal{A}$, by the first part, $\lambda(A) = \lim_{n\to\infty} \lambda_n(A) = \mu_1(A) - \lim_{n\to\infty} \mu_n(A)$, is a measure. Let $\mu(A) = \lim_{n\to\infty} \mu_n(A)$. Thus, since $\mu_n(A) \leq \mu_1(A) \leq \mu_1(X) < \infty$ for every $n \in \mathbb{N}$, we see that $\mu(A) < \infty$ for every $A \in \mathcal{A}$. We also see that $\lambda(A) = \mu_1(A) - \mu(A) < \infty$, as both μ_1 and μ are finite. This means we can normal addition and substraction and thus get $\mu(A) = \mu_1(A) - \lambda(A)$.

Since λ and μ_1 are measures, we see that $\mu(\varnothing) = \mu_1(\varnothing) - \lambda(\varnothing) = 0 - 0 = 0$. We also see that for disjoint $A_i \in \mathcal{A}$ that $\mu\left(\cup_{i=1}^{\infty} A_i\right) = \mu_1\left(\cup_{i=1}^{\infty} A_i\right) - \lambda\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_1(A_i) + \sum_{i=1}^{\infty} \lambda(A_i) = \sum_{i=1}^{\infty} (\mu_1(A_i) - \lambda(A_i)) = \sum_{i=1}^{\infty} \mu(A_i)$. This proves that $\mu(A) = \lim_{n\to\infty} \mu_n(A)$ is, in fact, a measure. $\square$

Ex 3.8 Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathcal{N}$ be the collection of null sets with respect to $\mathcal{A}$ and μ, and let $\mathcal{B} = \sigma(\mathcal{A} \cup \mathcal{N})$. Show that $B \in \mathcal{B}$ if and only if there exists $A \in \mathcal{A}$ and $N \in \mathcal{N}$ such that $B = A \cup N$. Define $\bar{\mu}(B) = \mu(A)$ if $B = A \cup N$ with $A \in \mathcal{A}$ and $N \in \mathcal{N}$. Prove that $\bar{\mu}(B)$ is uniquely defined for each $B \in \mathcal{B}$, that $\bar{\mu}$ is a measure on $\mathcal{B}$, that $(X, \mathcal{B}, \bar{\mu})$ is complete, and that $(X, \mathcal{B}, \bar{\mu})$ is the completion of $(X, \mathcal{A}, \mu)$.

Proof. We see that the first goal is to prove that $\sigma(\mathcal{A} \cup \mathcal{N}) = \mathcal{A} \cup \mathcal{N}$. We see that, by definition, $\mathcal{A} \cup \mathcal{N} \subseteq \sigma(\mathcal{A} \cup \mathcal{N})$. Now to prove that $\mathcal{A} \cup \mathcal{N}$ is a σ-algebra:

Includes empty and whole space: Since $\varnothing$ is a null set, we see that $\varnothing = \varnothing \cup \varnothing \in \mathcal{A} \cup \mathcal{N}$. Also, we see that since $X \in \mathcal{A}$, that $X = X \cup \varnothing \in \mathcal{A} \cup \mathcal{N}$. Thus, $\varnothing, X \in \mathcal{A} \cup \mathcal{N}$.

Closure under complements: Let $C \in \mathcal{A} \cup \mathcal{N}$. Thus, $C = A \cup N$ for some $A \in \mathcal{A}$ and $N \in \mathcal{N}$. This means that $C^c = (A \cup N)^c = A^c \cap N^c$. We see that since $N \in \mathcal{N}$, there's a set $N' \in \mathcal{A}$, such that $N \subseteq N'$ and $\mu(N') = 0$. Since $N \subseteq N'$, we see that $N^c = X \setminus N = (X \setminus N') \cup (N' \setminus N) = N'^c \cup (N' \setminus N)$. This means that $C^c = A^c \cap (N'^c \cup (N' \setminus N)) = (A^c \cap N'^c) \cup (A^c \cap (N' \setminus N))$. Since $A \in \mathcal{A}$ and $N' \in \mathcal{A}$, this means that $A^c \cap N'^c \in \mathcal{A}$. Also, since $N' \setminus N \subseteq N'$, we see that $A^c \cap (N' \setminus N) \subseteq N'$, and since $\mu(N') = 0$, that means this set is in $\mathcal{N}$. Thus, $C^c \in \mathcal{A} \cup \mathcal{N}$.

Closure under countable union: Let $C_i \in \mathcal{A} \cup \mathcal{N}$. Thus, $C_i = A_i \cup N_i$ for some $A_i \in \mathcal{A}$ and $N_i \in \mathcal{N}$. We see that $\cup_{i=1}^{\infty} C_i = \cup_{i=1}^{\infty} A_i \cup N_i = \cup_{i=1}^{\infty} A_i \bigcup \cup_{i=1}^{\infty} N_i$. We see that $\cup_{i=1}^{\infty} A_i = A$ for some $A \in \mathcal{A}$, as $\mathcal{A}$ is a sigma algebra. We also see that for each N_i, there's a superset N_i' such that $\mu(N_i') = 0$. Thus, since $\cup_{i=1}^{\infty} N_i \subseteq \cup_{i=1}^{\infty} N_i'$ and since

$$\mu\left(\cup_{i=1}^{\infty} N_i'\right) = \sum_{i=1}^{\infty} \mu(N_i') = \sum_{i=1}^{\infty} 0 = 0$$

this means $\cup_{i=1}^{\infty} N_i$ is a null set, and is thus in $\mathcal{N}$, call this set N. This proves that $\cup_{i=1}^{\infty} C_i = A \cup N \in \mathcal{A} \cup \mathcal{N}$. Thus, $\mathcal{A} \cup \mathcal{N}$ is closed under countable union.

This proves that $\mathcal{A} \cup \mathcal{N}$ is a σ-algebra, and since $\mathcal{B} = \sigma(\mathcal{A} \cup \mathcal{N})$ is the smallest σ-algebra containing $\mathcal{A} \cup \mathcal{N}$, we see that $\mathcal{B} = \mathcal{A} \cup \mathcal{N}$.

$\bar{\mu}$ is well-defined: Let $B_1, B_2 \in \mathcal{B}$. Thus, $B_1 = A_1 \cup N_1$ and $B_2 = A_2 \cup N_2$ for some $A_1, A_2 \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$. Since $N_1, N_2 \in \mathcal{N}$, we know that for some N_1' and N_2', that $N_1 \subseteq N_1'$, $N_2 \subseteq N_2'$, and that $\mu(N_1') = \mu(N_2') = 0$. Now suppose that $B_1 = B_2$. This means that $A_1 \subseteq A_1 \cup N_1 = A_2 \cup N_2 \subseteq A_2 \cup N_2'$. Thus, $\bar{\mu}(B_1) = \mu(A_1) \leq \mu(A_2 \cup N_2') = \mu(A_2) + \mu(N_2') = \mu(A_2) + 0 = \bar{\mu}(B_2)$. This proves that $\bar{\mu}$ is well-defined.

$\bar{\mu}$ is a measure: We see that $\bar{\mu}(\varnothing) = \mu(\varnothing) = 0$. We also see that if $A_i \cup N_i \in \mathcal{B}$ are disjoint sets, then

$$\bar{\mu}\left(\cup_{i=1}^{\infty}(A_i \cup N_i)\right) = \bar{\mu}\left(\left(\cup_{i=1}^{\infty} A_i\right) \cup \left(\cup_{i=1}^{\infty} N_i\right)\right) = \mu\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \bar{\mu}(A_i \cup N_i)$$

This proves that $\bar{\mu}$ is a measure.

$(X, \mathcal{B}, \bar{\mu})$ is complete: Since $\varnothing \in \mathcal{N}$, we see that for all $A \in \mathcal{A}$, $\bar{\mu}(A) = \bar{\mu}(A \cup \varnothing) = \mu(A)$. Thus, if $(X, \mathcal{B}, \bar{\mu})$ is complete, it must be the completion of $(X, \mathcal{A}, \mu)$. Let $C \subseteq X$ be a $\bar{\mu}$-null set. This means that there's a $B = A \cup N \in \mathcal{B}$, where $A \in \mathcal{A}$, $N \in \mathcal{N}$, $C \subseteq B$ and $\bar{\mu}(B) = 0$. This means that $0 = \bar{\mu}(B) = \bar{\mu}(A \cup N) = \mu(A)$. Also, since $N \in \mathcal{N}$, there's a $N' \in \mathcal{A}$, where $N \subseteq N'$ and $\mu(N') = 0$. This means that $C \subseteq B = A \cup N \subseteq A \cup N'$. We see that $\mu(A \cup N') = \mu(A) + \mu(N') = 0 + 0 = 0$. Thus, since $C \subseteq A \cup N'$ and $\mu(A \cup N') = 0$, this means that C is a μ-null set. Thus, $C \in \mathcal{N}$. Since $\varnothing \in \mathcal{A}$, we see that $C = \varnothing \cup C \in \mathcal{A} \cup \mathcal{N} = \mathcal{B}$. $\square$

Chapter 4

Construction of measures

Ex 4.1 Let μ be a measure on the Borel σ-algebra of $\mathbb{R}$ such that $\mu(K) < \infty$ whenever K is compact, define $\alpha(x) = \mu((0,x])$ if $x \geq 0$ and $\alpha(x) = -\mu((x,0])$ if $x < 0$. Show that μ is the Lebesgue-Stieltjes measure corresponding to α.

Proof. Let $A \subseteq B$, where $\mu(B) < \infty$. Then, $B = (B \cap A) \cup (B \setminus A) = A \cup (B \setminus A)$. We see that $\mu(B) = \mu(A) + \mu(B \setminus A)$. Since $\mu(B) < \infty$, this means that $\mu(A) \leq \mu(B) < \infty$, and thus $\mu(B \setminus A) = \mu(B) - \mu(A)$. This will be important in the next paragraph.

For the Lebesgue-Stieltjes measure, we know that $\ell((a,b]) = \alpha(b) - \alpha(a)$. Since $[a,b]$ is compact, and $(0,b]$ and $(a,0]$ are subsets of this set, then they must be finite as well. Thus, there are three cases:

$$0 \leq a \leq b \implies \alpha(b) - \alpha(a) = \mu((0,b]) - \mu((0,a]) = \mu((0,b] \setminus (0,a]) = \mu((a,b])$$

$$a < 0 \leq b \implies \alpha(b) - \alpha(a) = \mu((0,b]) + \mu((a,0]) = \mu((0,b] \cup (a,0)) = \mu((a,b])$$

$$a \leq b < 0 \implies \alpha(b) - \alpha(a) = -\mu((b,0]) + \mu((a,0]) = \mu((a,0] \setminus (b,0]) = \mu((a,b])$$

We see that $b < 0$ and $a \geq 0$ is impossible, as $a \leq b$. This proves that $\ell((a,b]) = \mu((a,b])$.

Now, let A be a m-measurable set. We see then that there exists $B = \cup_{i=1}^{\infty} B_i$, where $B_i = (c_i, d_i] \in \mathcal{C}$, $A \subseteq B$, and $m(B) \leq m(A) + \varepsilon$. (That is to say, there exists a set of half-closed intervals that is arbitrarily close to the infinmum.) Thus, we see that

$$m(B) = \sum_{i=1}^{\infty} m(B_i) = \sum_{i=1}^{\infty} (\alpha(d_i) - \alpha(c_i)) = \sum_{i=1}^{\infty} \mu(B_i) = \mu(\cup_{i=1}^{\infty} B_i) = \mu(B)$$

Thus, since $A \subseteq B$, we see that $\mu(A) \leq \mu(B) = m(B) \leq m(A) + \varepsilon$. Since ε was abitrary, we see that $\mu(A) \leq m(A)$.

Since A is measurable, then A^c is measurable. By a similar argument, $\mu(A^c) \leq m(A^c)$. This means that $\mu(A) + m(A) + \mu(A^c) \leq \mu(A) + m(A) + m(A^c)$, which means that $\mu(A \cup A^c) + m(A) \leq \mu(A) + m(A \cup A^c)$. Thus, $m(A) \leq \mu(A)$. This proves that $\mu(A) = \mu(A)$ for all m-measurable sets. $\square$

Ex 4.2 Let m be Lebesgue measure and A a Lebesgue measurable subset of $\mathbb{R}$ with $m(A) < \infty$. Let $\varepsilon > 0$. Show there exist G open and F closed such that $F \subseteq A \subseteq G$ and $m(G \setminus F) < \varepsilon$.

Proof. Let $B = \cup_{i=1}^{\infty} B_i$ where $B_i = (c_i, d_i]$, $A \subseteq B$, and $\sum_{i=1}^{\infty} \ell(B_i) \le m(A) + \frac{\varepsilon}{4}$. Note that $m(B) = m(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \ell(B_i)$. Thus, $m(B) \le m(A) + \frac{\varepsilon}{4}$. Let $G_i = B_i \cup (d_i, e_i)$, where $\ell((d_i, e_i)) = e_i - d_i < \frac{\varepsilon}{2^{i+2}}$. We see that $G_i = (c_i, e_i)$, and thus $\cup_{i=1}^{\infty} G_i = G$ is open, as it's the union of open intervals. We also see that $A \subseteq B = \cup_{i=1}^{\infty} B_i \subseteq \cup_{i=1}^{\infty} G_i = G$. Thus, $A \subseteq G$ and G is an open set. We see that the measure of G can be computed by the following:

$$
\begin{aligned}
m(G) = m(\cup_{i=1}^{\infty} G_i) = \sum_{i=1}^{\infty} m(G_i) = \sum_{i=1}^{\infty} (e_i - c_i) = \sum_{i=1}^{\infty} ((e_i - d_i) + (d_i - c_i)) \\
\le \sum_{i=1}^{\infty} \left(\frac{\varepsilon}{2^{i+2}} + \ell(B_i) \right) = \sum_{i=1}^{\infty} \ell(B_i) + \frac{\varepsilon}{4} = m(B) + \frac{\varepsilon}{4} \le m(A) + \frac{\varepsilon}{2}
\end{aligned}
$$

Since A was measurable, then A^c is measurable. By the same argument, there's a V that is open that contains A^c and $m(V) \le m(A^c) + \frac{\varepsilon}{2}$. Let $V = F^c$. Thus, F is a closed set that is contained in A. We see that

$$
\begin{aligned}
m(G \setminus F) = m(G \cup F^c) = m(G) + m(F^c) - m(\varnothing) = m(G) + m(F^c) - m(A \cup A^c) \\
= (m(G) - m(A)) + (m(V) - m(A^c)) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

This proves the above statement. (Lots of ε's!) □

Ex 4.3 If $(X, \mathcal{A}, \mu)$ is a measure space, define

$$\mu^*(A) = \inf\{\mu(B) \mid A \subseteq B, B \in \mathcal{A}\}$$

for all subsets A of X. Show that μ^* is an outer measure. Show that each set in $\mathcal{A}$ is μ^*-measurable and μ^* agrees with the measure μ on $\mathcal{A}$.

Proof. Suppose $A \in \mathcal{A}$. We see that $A \subseteq A \in \mathcal{A}$, which means that $\mu^*(A) \le \mu(A)$. Now let B be a set in $\mathcal{A}$ such that $A \subseteq B$. We see that $\mu(A) \le \mu(B)$. Thus, $\mu(A)$ is a lower bound for all such $\mu(B)$, where $A \subseteq B$. This proves that for $A \in \mathcal{A}$, $\mu^*(A) = \mu(A)$. We see for a special case that $\mu^*(\varnothing) = \mu(\varnothing) = 0$.

Assume that $A \subseteq B$. Let $\mathbf{A} = \{S \in \mathcal{A} \mid A \subseteq S\}$ and $\mathbf{B} = \{S \in \mathcal{A} \mid B \subseteq S\}$. We see that if $C \in \mathbf{B}$, then $C \in \mathcal{A}$ and $B \subseteq C$. Since $A \subseteq B$, this means that $A \subseteq B \subseteq C$. This proves that $C \in \mathbf{A}$. This means that $\mathbf{B} \subseteq \mathbf{A}$. Since $\mathbf{B} \subseteq \mathbf{A}$, this means that $\mu^*(A) = \inf\{\mu(S) \mid S \in \mathbf{A}\} \le \inf\{\mu(S) \mid S \in \mathbf{B}\} = \mu^*(B)$.

Let $A = \cup_{n=1}^{\infty} A_n$ and let $\varepsilon > 0$. Suppose $\mu^*(A_n) = \infty$ for some n. Then as $A_n \subseteq \cup_{n=1}^{\infty} A_n$, by the last paragraph, this means that $\mu^*(\cup_{n=1}^{\infty} A_n) \ge \mu^*(A_n) = \infty \ge \sum_{n=1}^{\infty} A_n$ trivally. Thus, we may assume that $\mu^*(A_n)$ is finite for all n. Choose $B_n \in \mathcal{A}$, where $A_n \subseteq B_n$ and $\mu(B_n) \le \mu^*(A_n) + \varepsilon 2^{-n}$. Since $A \subseteq B = \cup_{n=1}^{\infty} B_n$, we see that $\mu^*(A) \le \mu^*(B) = \mu(B) \le \sum_{n=1}^{\infty} B_n \le \sum_{n=1}^{\infty} A_n + \varepsilon$. Since ε was arbitrary, this means that $\mu^*(A) = \mu^*(\cup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu^*(A_n)$. This proves that μ^* is an outer measure, and since we've already proven that $\mu^*(A) = \mu(A)$ for all $A \in \mathcal{A}$, we're done. □

Ex 4.4 Let m be Lebesgue-Stieltjes measure corresponding to a right continuous increasing function α. Show that for each x,

$$m(\{x\}) = \alpha(x) - \alpha(x-)$$

Proof. We see that $\{x\}$ is a Borel set, and is thus measurable under m. This means that

$$m(\{x\}) = \lim_{n\to\infty} m\left(\left(x - \frac{1}{n}, x\right]\right) = \lim_{n\to\infty} \alpha(x) - \alpha\left(x - \frac{1}{n}\right) = \alpha(x) - \alpha\left(x^-\right)$$

This proves the statement. $\square$

Ex 4.5 Suppose m is Lebesgue measure. Define $x + A = \{x + y \mid y \in A\}$ and $cA = \{cy \mid y \in A\}$ for $x \in \mathbb{R}$ and c a real number. Show that if A is a Lebesgue measurable set, then $m(x + A) = m(A)$ and $m(cA) = |c| m(A)$.

Proof. Let $B_i = (c_i, d_i]$ be a covering of A. Notice that since $A \subseteq \cup_{i=1}^{\infty} B_i$, then $x + A \subseteq \cup_{i=1}^{\infty} x + B_i$. We also see that $\ell(x + B_i) = \ell((x + c_i, x + d_i]) = (x + d_i) - (x + c_i) = d_i - c_i = \ell((c_i, d_i]) = \ell(B_i)$. Thus, $\sum_{i=1}^{\infty} \ell(B_i) = \sum_{i=1}^{\infty} \ell(x + B_i)$. Since $x + A \subseteq x + \cup_{i=1}^{\infty} B_i$, this shows that $m^*(x + A) \leq m^*(x + \cup_{i=1}^{\infty} B_i) \leq \sum_{i=1}^{\infty} m^*(x + B_i) = \sum_{i=1}^{\infty} \ell(x + B_i) = \sum_{i=1}^{\infty} \ell(B_i)$. Since $m^*(x + A)$ is less than any half-open covering of A, this means that $m^*(x + A) \leq m^*(A)$. If we let $x + A = C$, we see that $A = -x + C$. Thus, we can do a similar argument on C and $-x + C$, and see that $m^*(-x + C) \leq m^*(C)$, which means that $m^*(A) \leq m^*(x + A)$. This proves that $m^*(A) = m^*(x + A)$.

If $c \geq 0$, then $\sum_{i=1}^{\infty} \ell(cB_i) = \sum_{i=1}^{\infty} (c \cdot c_i, c \cdot d_i] = \sum_{i=1}^{\infty} c(d_i - c_i) = c \sum_{i=1}^{\infty} d_i - c_i = c \sum_{i=1}^{\infty} \ell(B_i)$. If $c < 0$, then $\sum_{i=1}^{\infty} \ell(cB_i) = \sum_{i=1}^{\infty} \ell(c \cdot (c_i, d_i]) = \sum_{i=1}^{\infty} \ell((c \cdot c_i, c \cdot d_i]) = \sum_{i=1}^{\infty} \ell([(-c)(-d_i), (-c)(-c_i))) = \sum_{i=1}^{\infty} -c(-c_i + d_i) = -c \sum_{i=1}^{\infty} \ell(B_i)$. Thus, $\sum_{i=1}^{\infty} \ell(cB_i) = |c| \sum_{i=1}^{\infty} \ell(B_i)$. Thus, using a similar argument as above, we see that $m^*(cA) = |c| m^*(A)$.

Now to prove that $x + A$ is Lebesgue measurable. Since A is Lebesgue measurable, then for any E,

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c)$$

Let $E = -x + F$. This means that

$$m^*(-x + F) = m^*((-x + F) \cap A) + m^*((-x + F) \cap A^c)$$

Let $y \in (-x + F) \cap A$. This means that $y \in -x + F$ and $y \in A$. Thus, $y + x \in F$ and $y + x \in x + A$. This proves that $y + x \in F \cap (x + A)$ and thus $y \in -x + (F \cap (x + A))$. This is reversible, and so $m^*((-x + F) \cap A) = m^*(-x + (F \cap (x + A))) = m^*(F \cap (x + A))$. Similarly, $m^*((-x + F) \cap A^c) = m^*(F \cap (x + A^c))$. If $y \in x + A^c$, then $y - x \notin A$, which means that $y \notin x + A$, and thus $y \in (x + A)^c$. This is reversible, so $m^*((-x + F) \cap A) = m^*(F \cap (x + A^c)) = m^*(F \cap (x + A)^c)$. Thus,

$$m^*(-x + F) = m^*(F) = m^*(F \cap (x + A)) + m^*(F \cap (x + A)^c)$$

Since E was arbitrary, so is F, and thus, $x + A$ is measurable.

Let $E = \frac{1}{c} F$. Similarly, we get that

$$m^*\left(\frac{1}{c} F\right) = m^*\left(\frac{1}{c} F \cap A\right) + m^*\left(\frac{1}{c} F \cap A^c\right)$$

Let $y \in \frac{1}{c} F \cap A$, then $cy \in F$ and $cy \in cA$. This means that $cy \in F \cap cA$, and thus that $y \in \frac{1}{c}(F \cap cA)$. This is all reversible once again. We also see that if $y \in cA^c$, then

$\frac{1}{c}y \in A^c$ which means that $\frac{1}{c}y \notin A$, and then that $y \notin cA$, and thus $y \in (cA)^c$. This is again, reversible. This shows that

$$m^*\left(\frac{1}{c}F\right) = m^*\left(\frac{1}{c}(F \cap cA)\right) + m^*\left(\frac{1}{c}(F \cap (cA)^c)\right)$$

Pulling out the $\frac{1}{c}$ as a $|\frac{1}{c}|$ and dividing by $|\frac{1}{c}|$, this proves that cA is measurable. □

Ex 4.6 Let m be Lebesgue measure. Suppose for each n, A_n is a Lebesgue measurable subset of $[0,1]$. Let B consist of those points x that are in infinitely many of the A_n.

1) Show B is Lebesgue measurable

2) If $m(A_n) > \delta > 0$ for each n, show $m(B) \geq \delta$

3) If $\sum_{n=1}^{\infty} m(A_n) < \infty$ prove that $m(B) = 0$

4) Give an example where $\sum_{n=1}^{\infty} m(A_n) = \infty$, but $m(B) = 0$

Proof. 1) We see that

$$B = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

To prove this, let $x \in B$. Then x is in infinitely many A_n. This means that $x \in \cup_{k=n}^{\infty} A_k$ for all n, which means that $x \in \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$. Now, if $x \notin B$, that is that x is only in finitely many A_n (perhaps none of them), say $A_{n_1}, \ldots, A_{n_j}$, then $x \notin \cup_{k=n_j+1}^{\infty} A_k$, which means that $x \notin \cap_{n=1}^{\infty} \cup_{k=1}^{\infty} A_k$. This proves that the two sets are equal. Since the Lebesgue measurable sets form a σ-algebra, they are closed under countable unions and countable intersections. Thus, B is Lebesgue measurable.

2) We see that $m(B) = \lim_{j\to\infty} m\left(\cap_{n=1}^{j} \cup_{k=n}^{\infty} A_k\right)$. Let $B_n = \cup_{k=n}^{\infty} A_k$. We see that $B_{n+1} \subseteq B_n$. Thus, $\cap_{n=1}^{j} B_n = B_j$. Since $A_j \subseteq B_j$, we see that $m\left(\cap_{n=1}^{j} \cup_{k=n}^{\infty} A_k\right) = m\left(\cap_{n=1}^{j} B_j\right) = m(B_j) \geq m(A_j) > \delta$ for any j. Thus, $m(B) = \lim_{j\to\infty} m\left(\cap_{n=1}^{j} \cup_{k=n}^{\infty} A_k\right) \geq \delta$.

3) Let $\varepsilon > 0$. If $\sum_{n=1}^{\infty} m(A_n) < \infty$, then there's a k such that $\sum_{n=1}^{\infty} m(A_n) - \sum_{n=1}^{k} m(A_n) < \varepsilon$. Notice that

$$\sum_{n=1}^{\infty} m(A_n) = \sum_{n=k}^{\infty} m(A_n) + \sum_{n=1}^{k} m(A_n)$$

Thus, since these are all finite, we see that $\sum_{n=k}^{\infty} m(A_k) = \sum_{n=1}^{\infty} m(A_n) - \sum_{n=1}^{k} m(A_n) < \varepsilon$. Since $B \subseteq \cup_{n=k}^{\infty} A_k$ for any k, we see that $m(B) \leq m(\cup_{n=k}^{\infty} A_n) \leq \sum_{n=k}^{\infty} A_n < \varepsilon$. Thus, $m(B) < \varepsilon$ for any $\varepsilon > 0$. This proves that $m(B) = 0$.

4) Let A_n be the Fat Cantor set on $[0,1]$, where $m(A_n) = \frac{1}{n}$. Thus, $\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$. I think this set works, but I do not know how to prove that $m(B) = 0$.

□

Ex 4.7 Suppose $\varepsilon \in (0,1)$ and m is Lebesgue measure. Find a measurable set $E \subseteq [0,1]$ such that the closure of E is $[0,1]$ and $m(E) = \varepsilon$.

Proof. Let $Q = \mathbb{Q} \cap [0,1]$, and let $E = (0,\varepsilon) \cup Q$. Since $(0,\varepsilon) \subseteq [0,1]$ and $Q \subseteq [0,1]$, this means that $E \subseteq [0,1]$. Since $\bar{Q} = [0,1]$ and $Q \subseteq E$, we see that $[0,1] \subseteq \bar{E}$. Since $E \subseteq [0,1]$ and $[0,1]$ is closed, that means $\bar{E} \subseteq [0,1]$. This proves that $\bar{E} = [0,1]$.

We see that $m(E) \leq m(Q) + m((0,\varepsilon)) = 0 + \varepsilon = \varepsilon$. Also, since $(0,\varepsilon) \subseteq E$, then $m((0,\varepsilon)) = \varepsilon \leq m(E)$. Thus, $m(E) = \varepsilon$. This proves the statement. □

Ex 4.10 Let $\varepsilon \in (0,1)$, let m be Lebesgue measure, and suppose A is a Borel measurable subset of $\mathbb{R}$. Prove that if

$$m(A \cap I) \leq (1-\varepsilon)\, m(I)$$

for every interval I, then $m(A) = 0$.

Proof. Let $A_n = A \cap [-n,n]$. This means that $m(A_n) \leq 2n$. Let $\{J_i\}$ be a collection of half-open/half-closed intervals covering A_n. Let $\{O_i\}$ be the same interval of $\{J_i\}$ except we remove the point on the closed side. Notice that $m(O_i) = m(J_i)$. If any of the points we removed from J_i was a point in A_n, then remove that point too from A_n and call the result A'_n. Since we're only removing at most countably many points, notice that $m(A'_n) = m(A_n) \leq \infty$, and that $A'_n \subseteq \cup_{i=1}^{\infty} O_i$. Now we see that,

$$m(A_n) = m(A'_n) = m(A'_n \cap \cup_{i=1}^{\infty} O_i) \leq m(A \cap \cup_{i=1}^{\infty} O_i) = m(\cup_{i=1}^{\infty} (A \cap O_i)) \leq \sum_{i=1}^{\infty} m(A \cap O_i)$$

Since O_i is an open interval, we see that

$$m(A_n) \leq \sum_{i=1}^{\infty} m(A \cap O_i) = \sum_{i=1}^{\infty} (1-\varepsilon)\, m(O_i) = (1-\varepsilon) \sum_{i=1}^{\infty} m(J_i) = (1-\varepsilon) \sum_{i=1}^{\infty} \ell(J_i)$$

If we take the infinmum over all such J_i's, we see that $m(A_n) \leq (1-\varepsilon)\, m(A_n)$. This is only true if $m(A_n) = 0$. We see that

$$m(A) = m(\cup_{i=1}^{\infty} A_n) \leq \sum_{i=1}^{\infty} m(A_n) = \sum_{i=1}^{\infty} 0 = 0$$

Thus, $m(A) = 0$. □

Ex 4.11 Suppose m is Lebesgue measure and A is a Borel measurable subset of $\mathbb{R}$ with $m(A) > 0$. Prove that if

$$B = \{x - y \mid x, y \in A\}$$

then B contains a non-empty open interval centered at the origin. This is known as the Steinhaus theorem.

Proof. Suppose there was no nonempty open interval centered at the origin contained in B. Then there is some element in $\left(\frac{1}{-n}, \frac{1}{n}\right)$ that's not in B. Let this element be x_n. Suppose $(x_n + A) \cap A \neq \varnothing$, then let w be in this set. This means that $w \in A$ and $w \in x_n + A$, and thus $w - x_n \in A$. However, this would mean that $w - (w - x_n) = x_n \in A - A$, a contradiction. This shows that $(x_n + A) \cap A = \varnothing$, which implies that $m((x_n + A) \cap A) = m(x_n + A) + m(A) = 2m(A)$. However, $m(A) = \lim_{n\to 0} m((x_n + A) \cap A) = \lim_{n \to 0} 2m(A) = 2m(A)$. This means that $2m(A) = m(A)$, which is a contradiction, as $m(A) > 0$. Thus, there must be some open interval centered at the origin contained in B. □

Ex 4.12 Let m be Lebesgue measure. Construct a Borel subset A of $\mathbb{R}$ such that $0 < m(A \cap I) < m(I)$ for every open interval I.

Proof. Enumerate all closed intervals with rational endpoints as I_k. For I_k, construct the fat cantor set in the first half and the interval and call it A_k. Do the same for the second half of the interval and call it B_k. We see that A_k and B_k are disjoint. Let $A = \cup_{n=1}^{\infty} A_n$. If I is an open interval, then it contains at least two rationals, and thus contains an I_k for some k. Thus, $A_k \in I$ and $B_k \in I$. This means that

$$0 < m(A_k) \leq m(A \cap I) < m(A \cap I) + m(B_k) \leq m(I)$$

We see that $m(A \cap I) < m(A \cap I) + m(B_n)$ as A and B_n are disjoint. Thus $0 < m(A \cap I) < m(I)$. $\square$

Ex 4.13 Let N be the non-measurable set defined in Section 4.4. Prove that if $A \subseteq N$ and A is Lebesgue measurable, then $m(A) = 0$.

Proof. Let $Q = \mathbb{Q} \cap [0, 1]$. Recall that $\cup_{q \in Q}(q + N) \subseteq [-1, 2]$, and that $q + N$ is disjoint for all q. Thus, $A \subseteq [-1, 2]$, and $q + A$ is disjoint for all q. Since A is measurable, this means that $\sum_{q \in Q} m(q + A) = m\left(\cup_{q \in Q}(q + A)\right) \leq m([-1, 2]) = 3$. Since $m(q + A) = m(A)$, this means that $\sum_{q \in Q} m(A) < 3$. This can only happen if $m(A) = 0$. This proves the statement. $\square$

Ex 4.14 Let m be Lebesgue measure. Prove that if A is a Lebesgue measurable subset of $\mathbb{R}$ and $m(A) > 0$, then there is a subset of A that is non-measurable.

Proof. Note that $\cup_{q \in \mathbb{Q}}(q + N) = \mathbb{R}$, where N is the set from Section 4.4. Thus we see that $\left(\cup_{q \in \mathbb{Q}}(q + N)\right) \cap A = A$. This means that

$$m(A) = m\left(\left(\bigcup_{q \in \mathbb{Q}} q + N\right) \bigcap A\right) = m\left(\bigcup_{q \in \mathbb{Q}}((q + N) \cap A)\right) \leq \sum_{q \in \mathbb{Q}} m((q + N) \cap A)$$

However, $(q + N) \cap A \subseteq q + N$. Using a slight variation of Exercise 13, this means that $(q + N) \cap A$ is either nonmeasurable or has measure zero. Suppose $(q + N) \cap A$ has measure zero for all $q \in \mathbb{Q}$, then $m(A) \leq \sum_{q \in \mathbb{Q}} m((q + N) \cap A) = \sum_{q \in \mathbb{Q}} 0 = 0$ This is a contradiction, as $m(A) > 0$. Thus, $(q + N) \cap A \subseteq A$ must be nonmeasurable for some $q \in \mathbb{Q}$. $\square$

Ex 4.18 Suppose $A \subseteq \mathbb{R}$ has Lebesgue measure 0. Prove that there exists $c \in \mathbb{R}$ such that $A \cap (c + \mathbb{Q}) = \varnothing$, where $c + \mathbb{Q} = \{c + x \mid x \in \mathbb{Q}\}$ and $\mathbb{Q}$ is the rational numbers.

Proof. Suppose $A \subseteq \mathbb{R}$ has Lebesgue measure 0. Also, assume that for every $c \in \mathbb{R}$, $A \cap (c + \mathbb{Q}) \neq \varnothing$. For each $c \in \mathbb{R}$, let w_c be an element of $A \cap (c + \mathbb{Q})$, and let C be the collection of these w_c's. We see that $m^*(C) = m^*(C + q)$. Let $x \in [0, 1]$. Since $x \in \mathbb{R}$, then there's a $w_x = x + q$ for some rational q. Thus, $x = w_x - q$ for some rational q, which means that $x \in K - q$. This means that $[0, 1] \subseteq \cup_{q \in \mathbb{Q}} K + q$. This proves that $1 \leq m^*\left(\cup_{q \in \mathbb{Q}} K + q\right) \leq \sum_{q \in Q} m^*(K + q) = \sum_{q \in \mathbb{Q}} m^*(K)$. This implies that $m^*(K) > 0$. Since $K \subseteq A$, this means that $0 < m^*(K) \leq m^*(A) = m(A)$. However, this is a contradiction as $m(A) = 0$. This means that there must be some $c \in \mathbb{R}$ such that $A \cap (c + \mathbb{Q}) = \varnothing$. $\square$

Chapter 5

Measurable functions

Ex 5.1 Suppose $(X, \mathcal{A})$ is a measurable space, f is a real-valued function, and $\{x \mid f(x) > r\} \in \mathcal{A}$ for each rational number r. Prove that f is measurable.

Proof. For any $a \in \mathbb{R}$, we see that from the denseness of $\mathbb{Q}$ in $\mathbb{R}$,

$$\{x \in X \mid f(x) > a\} = \bigcap_{q \in (a,\infty) \cap \mathbb{Q}} \{x \in X \mid f(x) > q\}$$

Since $\mathbb{Q}$ is countable, the intersection is a countable intersection of elements of the σ-algebra $\mathcal{A}$. Thus, the intersection is in $\mathcal{A}$, proving that f is measurable. $\square$

Ex 5.2 Let $f : (0, 1) \to \mathbb{R}$ be such that for every $x \in (0, 1)$ there exists $r > 0$ and a Borel measurable function g, both depending on x, such that f and g agree on $(x-r, x+r) \cap (0, 1)$. Prove that f is Borel measurable.

Proof. We see that for any $n \geq 2$

$$\left[\frac{1}{n}, 1 - \frac{1}{n}\right] \subseteq \bigcup_{x \in (0,1)} (x - r_x, x + r_x)$$

where r_x is the r in the question that depends in x. By compactness, there must be a finite subcovering. Let's denote it by $\{(x_i - r_i, x_i + r_i) \mid 0 < i \leq m\}$ for some $m \in \mathbb{N}$. Let g_i be the Borel measurable set that agrees with f on the interval $(x_i - r_i, x_i + r_i)$. We see then that for any $a \in \mathbb{R}$

$$B_i = \{x \in (x_i - r_i, x_i + r_i) \mid f(x) > a\} = \{x \in (x_i - r_i, x_i + r_i) \mid g_i(x) > a\}$$

is a Borel set. Thus,

$$C_n = \{x \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \mid f(x) > a\} = \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \cap \bigcup_{i=1}^{m} B_i$$

is a Borel set for $n = 1, 2, \ldots$. Finally, we see that

$$\{x \in (0, 1) \mid f(x) > a\} = \bigcup_{n=1}^{\infty} C_n$$

is a Borel set for each $a \in \mathbb{R}$. This proves that f is Borel measurable. $\square$

Ex 5.3 Suppose f is measurable and $f(x) > 0$ for all x. Let $g(x) = 1/f(x)$. Prove that g is a measurable function.

Proof. Since f is positive, then it's clear that g is also positive. Thus, if $a \leq 0$, then $\{x \in X \mid g(x) > a\} = X \in \mathcal{A}$. If $a > 0$, we see that

$$\{x \in X \mid g(x) > a\} = \{x \in X \mid f(x) < \frac{1}{a}\}$$

which is in $\mathcal{A}$, by Proposition 5.5. This proves that g is measurable. $\square$

Ex 5.4 Suppose f_n are measurable functions. Prove that a

$$A = \{x \mid \lim_{n\to\infty} f_n(x) \text{ exists }\}$$

is a measurable set.

Proof. By Proposition 5.8, we know that $\limsup f_n$ and $\liminf f_n$ are both measurable functions, if they are finite. Also, if they are both finite, then by Proposition 5.7, $\limsup f_n - \liminf f_n$ is measurable as well. It follows that

$$A_1 = \{x \mid \lim_{n\to\infty} f_n(x) \text{ exists and is finite }\} = \{x \mid \limsup f_n(x) - \liminf f_n(x) = 0\}$$

$$A_2 = \{x \mid \lim_{n\to\infty} f_n(x) = \infty\} = \bigcap_{i=1}^{\infty} \{x \mid \liminf f_n(x) > i\}$$

$$A_2 = \{x \mid \lim_{n\to\infty} f_n(x) = -\infty\} = \bigcap_{i=1}^{\infty} \{x \mid \liminf f_n(x) < -i\}$$

are measurable sets. Thus, $A = A_1 \cup A_2 \cup A_3$ is also measurable. $\square$

Ex 5.8 Give an example of a collection of measurable non-negative functions $\{f_\alpha\}_{\alpha\in A}$ such that if g is defined by $g(x) = \sup_{\alpha\in A} f_\alpha(x)$, then g is finite for all x but g is non-measurable. (A can be uncountable.)

Proof. Consider $(\mathbb{R}, \mathcal{A})$, where $\mathcal{A}$ is the Lebesgue σ-algebra. Let E be the Vitali set constructed in section 4.4. For each $e \in E$, let $f_e = \chi_{\{e\}}$. Then, we see that each f_e is measurable as sets comprising one point are null sets and hence measurable. It's clear to see that, for any $x \in \mathbb{R}$,

$$g(x) = \sup_{e\in E} f_e(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

and so $g \in \chi_E$, which is non-measurable. $\square$

Ex 5.9 Suppose $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $g : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that $g \circ f$ is Lebesgue measurable. Is this true if g is Borel measurable instead of continuous? Is this true if g is Lebesgue measurable instead of continuous?

Proof. If g is continuous, then it is Borel measurable by Proposition 5.6. If g is Borel measurable and f is Lebesgue measurable and if $a \in \mathbb{R}$, we see that $(g \circ f)^{-1}((a, \infty)) = f^{-1}(g^{-1}((a, \infty)))$. By Proposition 5.11, $g^{-1}((a, \infty))$ is a Borel set, and by the same proposition, we see that $f^{-1}(g^{-1}((a, \infty)))$ is Lebesgue measurable. This answers the first two parts. Now we will give a counterexample to the last question.

Let $\varphi : [0, 1] \to [0, 1]$ be the Cantor-Lebesgue function and let $\psi(x) = x + \varphi(x)$. It is clear that $\psi : [0, 1] \to [0, 2]$. Since φ is continuous and x is continuous, this means that φ is continuous as well. Since φ is monotonically increasing, ψ is strictly increasing, which implies injectivity. Since ψ is continuous and $\psi(0) = 0$ and $\psi(1) = 2$, then ψ is surjective as well. Finally, the continuity of ψ^{-1} follows from it is the inverse of a continuous bijection between compact sets.

Now, let C be the Cantor set in $[0, 1]$. Recall that φ is constant on open intervals contained in the complement of the Cantor set. Thus, if I is such an interval, then $m(\psi(I)) = m(I + c_I)$, where c_I is the constant given by $\varphi(x) = c_I$ for all $x \in I$. Thus, $m(\psi(I)) = m(I)$. The monotonicity and continuity of ψ shows that disjoint open intervals in $[0, 1]$ are mapped into disjoint open intervals of $[0, 2]$. Thus, a $m(\psi([0, 1] \setminus C)) = m([0, 1] \setminus C) = 1$ which means that $m(\psi(C)) = 2 - m(\psi([0, 1] \setminus C)) = 1$. Since $\psi(C)$ is closed and has positive measure, by Question 4.14, we see that there's a non-measurable set $D \subseteq \psi(C) \subseteq [0, 2]$.

Let $E \subseteq [0, 1]$ where $E = \psi^{-1}(D)$ and let $g = \chi_E$. Since $D \subseteq \psi(C)$, we see that $E \subseteq C$, and thus E is a null set and therefore measurable. This proves that g is a measurable function. Let $f = \psi^{-1}$ and remember that $f : [0, 2] \to [0, 1]$ is continuous. We see that since $g : [0, 1] \to \{0, 1\}$, we have that $g \circ f : [0, 2] \to \{0, 1\}$ is the composition of a Lebesgue measurable function and a continuous function. However, a

$$(g \circ f)(x) = \chi_E(f(x)) = \chi_{f^{-1}(E)}(x) = \chi_{\psi(E)}(x) = \chi_D(x)$$

which is clearly non-measurable. This shows that even if f is continuous and g is Lebesgue measurable, then $g \circ f$ is not necessarily Lebesgue measurable. $\square$

Chapter 6

The Lebesgue integral

Ex 6.2 Let X be a set and $\mathcal{A}$ the collection of all subsets of X. Pick $y \in X$ and let δ_y be the point mass at y, defined in Example 3.4. Prove that if $f : X \to \mathbb{R}$, then

$$\int f \, d\delta_y = f(y)$$

Proof. Since $\mathcal{A}$ is the set of all subsets, f is trivally measurable. If $f(y) < 0$, let f be f^-, otherwise, take f to be f^+ in the following. Let s be a simple function where $0 \le s \le f$. Represent s in the canonical form of

$$s = \sum_{i=1}^{n} a_i \chi_{E_i}$$

where the E_i's are disjoint. Since the E_i's are disjoint, y is in at most one of them. If y is in none of them then the integral is 0. If y is in one of them, suppose E_j, then

$$\int s \, d\delta_y = \sum_{i=1}^{n} a_i \delta_y(E_i) = a_j = s(y) \le f(y)$$

Since $\int f \, d\delta_y$ is by definition the supremum of such simple functions, this proves that $\int f \, d\delta_y \le f(y)$. Consider the simple function $s = f(y)\chi_{\{y\}}$. We see that $0 \le s \le f$, and so, $\int f \, d\delta_y \ge \int s \, d\delta_y = f(y)$. Thus, $\int f \, d\delta_y = f(y)$. $\square$

Ex 6.3 Let X be the positive integers and $\mathcal{A}$ the collection of all subsets of X. If $f : X \to \mathbb{R}$ is non-negative and μ is counting measure defined in Example 3.2, prove that

$$\int f \, d\mu = \sum_{k=1}^{\infty} f(k)$$

This exercise is very useful because it allows one to derive many conclusions about series from analogous results about general measure spaces.

Proof. Again, since the $\mathcal{A}$ is the set of all subsets, f is trivally measurable. Let $s_n = \sum_{k=1}^n f(k)\chi_{\{k\}}$. We see that s_n is simple and that $s_n \le f$. Thus, $\int s\, d\mu \le \int f\, d\mu$. Also, we see that $\int s_n\, d\mu = \sum_{k=1}^n f(k)\mu(\{k\}) = \sum_{k=1}^n f(k)$. This means that $\sum_{k=1}^n f(k) \le \int f\, d\mu$ for all n, and thus $\sum_k^\infty f(k) \le \int f\, d\mu$.

Let $s = \sum_{k=1}^n a_k\chi_{E_k}$ be a simple function represented in its canonical form where $0 \le s \le f$. If $x \in E_j$, then, since s is canonical, it doesn't appear in any other E_k. This means that $s(x) = a_j$, and since $s \le f$, this shows that $a_j \le f(x)$ where $x \in E_j$. With this, we see that

$$\int s\, d\mu = \sum_{k=1}^n a_k|E_k| = \sum_{k=1}^n a_k \sum_{x\in E_k} 1 = \sum_{k=1}^n \sum_{x\in E_k} a_k \le$$

$$\sum_{k=1}^n \sum_{x\in E_k} f(x) \le \sum_{x\in \cup_k E_k} f(x) \le \sum_{x\in X} f(x) \le \sum_{k=1}^\infty f(k)$$

Since $\sum_{k=1}^\infty f(k)$ is greater than any simple function less than or equal to f, it's greater than the supremum of all such simple functions, which is, by definition $\int f\, d\mu$. Thus, $\int f\, d\mu \le \sum_{k=1}^\infty f(k)$. This proves that $\int f\, d\mu = \sum_{k=1}^\infty f(k)$. □

Ex 6.5 Let f be a non-negative measurable function. Prove that

$$\lim_{n\to\infty} \int (f \wedge n) = \int f$$

Proof. We see that $f \wedge n \le f$, and thus $\int (f\wedge n)\, d\mu \le \int f\, d\mu$ for any n. This proves that $\lim_{n\to\infty} \int (f\wedge n)\, d\mu \le \int f\, d\mu$.

Let $s = \sum_{k=1}^n a_k\chi_{E_k}$ be a simple function in its canonical form where $0 \le s \le f$. Let $x \in X$. Then x lies in at most one of these E_k's. If it's in none, then $s(x) = 0$, if it's in one, then $s(x) = a_j$ for some j. Thus, for any $x \in X$, $s(x) \le \max\{a_k\}$. Let n be an integer greater than this maximum. Since $s \le f$ and $s \le n$, this means that $s \le f \wedge n$. Thus, $\int s\, d\mu \le \int (f\wedge n)\, d\mu \le \lim_{n\to\infty} \int (f\wedge n)\, d\mu$. If one takes the supremum of all such s, we see that $\int f\, d\mu \le \lim_{n\to\infty} \int (f\wedge n)\, d\mu$. This proves the statement. □

Ex 6.6 Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose μ is σ-finite. Suppose f is integrable. Prove that given ε there exists δ such that

$$\int_A |f(x)|\, \mu(dx) < \varepsilon$$

whenever $\mu(A) < \delta$.

Proof. Let $\varepsilon > 0$. We see that since $|f|\chi_A \le |f|$ that $\int |f|\chi_A\, d\mu \le \int |f|\, d\mu < \infty$. This proves that $|f|\chi_A$ is integrable. Since it's finite, this means that there's a simple function s such that $0 \le s \le |f|\chi_A$ and where $\int |f|\chi_A\, d\mu - \int s\, d\mu < \frac{\varepsilon}{2}$.

Since $0 \le s \le |f|\chi_A$, we can see that this means that $s(x) = 0$ for all $x \in A$. Thus, $s = s\chi_A$. Let $\sum_{k=1}^n a_k\chi_{E_k}$ be the conanical form of s. This means that $s = s\chi_A = \chi_A \sum_{k=1}^n a_k\chi_{E_k} = \sum_{k=1}^n a_k\chi_{E_k\cap A}$. Thus

$$\int s\, d\mu = \int s\chi_A\, d\mu = \sum_{k=1}^n a_k\mu(A \cap E_k)$$

If we let $\mu(A) < \delta = \frac{\varepsilon}{2\sum_{k=1}^n a_k}$, we see that

$$\int s\,d\mu \le \sum_{k=1}^n a_k\mu(A) = \mu(A)\sum_{k=1}^n a_k < \frac{\varepsilon}{2\sum_{k=1}^n a_k}\sum_{k=1}^n a_k = \frac{\varepsilon}{2}$$

This shows that

$$\int_A |f|\,d\mu = \int |f|\chi_A\,d\mu < \frac{\varepsilon}{2} + \int s\,d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This proves the statement. $\square$

Ex 6.8 If f_n is a sequence of non-negative integrable functions such that $f_n(x)$ decreases to $f(x)$ for every x, prove that

$$\int f_n\,d\mu \to \int f\,d\mu$$

Proof. Since f is the limit of decreasing non-negative functions, we see that $f \ge 0$. Thus, $0 \le f \le f_n$. This means that $\int |f|\,d\mu = \int f\,d\mu \le \int f_n\,d\mu = \int |f_n|\,d\mu$. Since f_n is integrable, this means that f is integrable.

Let $g_n = f_1 - f_n$. Since f_n was decreasing, then g_n is increasing. We see that $g_n = f_1 - f_n \ge 0$, and also that that $g_n \uparrow (f_1 - f_n)$. Thus, using the Monotone Convergence Theorem and the fact that the Lebesgue integral is linear on integrable functions, we see that:

$$\int f_1\,d\mu - \lim_{n\to\infty}\int f_n\,d\mu = \lim_{n\to\infty}\left(\int f_1\,d\mu - \int f_n\,d\mu\right) = \lim_{n\to\infty}\int (f_1 - f_n)\,d\mu =$$

$$\int \lim_{n\to\infty}(f_1 - f_n)\,d\mu = \int (f_1 - f)\,d\mu = \int f_1\,d\mu - \int f\,d\mu$$

Since f_1 is integrable, its integral is finite. Thus, we can subtract it from both sides and multiply by -1, which gives that

$$\lim_{n\to\infty}\int f_n\,d\mu = \int f\,d\mu$$

$\square$

Chapter 7

Limit theorems

Ex 7.3 Suppose f is integrable. Prove that if either $A_n \uparrow A$ or $A_n \downarrow A$, then $\int_{A_n} f\,d\mu \to \int_A f\,d\mu$.

Proof. Let $A_n \uparrow A$ or $A_n \downarrow A$. Either way, we see that $f\chi_{A_n} \to f\chi_A$ as $n \to \infty$ and that $|f\chi_{A_n}| \leq |f|$. Since $|f|$ is integrable, it dominates the $f\chi_{A_n}$'s. By the Dominated Convergence Theorem, this proves that

$$\lim_{n\to\infty} \int f\chi_{A_n}\,d\mu = \int \lim_{n\to\infty} f\chi_{A_n}\,d\mu = \int f\chi_A\,d\mu$$

which proves the statement for both cases. $\square$

Ex 7.4 Show that if $\sum_{n=1}^{\infty} \int |f_n|\,d\mu < \infty$, then $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely almost everywhere, is integrable, and its integral is equal to $\sum_{n=1}^{\infty} \int f_n\,d\mu$. (NOTE: There were some typos in the original that I've corrected.)

Proof. Let $\varphi(x) = \sum_{n=1}^{\infty} |f_n(x)|$. Since each $|f_n|$ is positive and $|\varphi| = \varphi$, this means that by Proposition 7.6:

$$\int |\varphi|\,d\mu = \int \varphi\,d\mu = \int \sum_{n=1}^{\infty} |f_n|\,d\mu = \sum_{n=1}^{\infty} \int |f_n|\,d\mu < \infty$$

Since $\sum_{n=1}^{\infty} f_n(x) \leq \sum_{n=1}^{\infty} |f_n(x)| = \varphi$, this proves that $\sum_{n=1}^{\infty} f_n(x)$ is integrable. Since $\sum_{n=1}^{\infty} f_n(x)$ is integrable, it's finite almost everywhere, and thus it converges absolutely almost everywhere. Now let $g_k = \sum_{n=1}^{k} f_n$. We see that

$$|g_k| = \left|\sum_{n=1}^{k} f_n\right| \leq \sum_{n=1}^{k} |f_n| \leq \sum_{n=1}^{\infty} |f_n| = \varphi$$

Since $g_k \to \sum_{n=1}^{\infty} f_n(x)$ and g_k is dominated by φ, which is an integrable function, then by the Dominated Convergence Theorem, we see that

$$\int \sum_{n=1}^{\infty} f_n\,d\mu = \lim_{k\to\infty} \int g_k\,d\mu = \lim_{k\to\infty} \int \sum_{n=1}^{k} f_n\,d\mu = \lim_{k\to\infty} \sum_{n=1}^{k} \int f_n\,d\mu = \sum_{n=1}^{\infty} \int f_n\,d\mu$$

$\square$

Ex 7.7 Suppose $(X, \mathcal{A}, \mu)$ is a measure space, f and each f_n is integrable and non-negative, $f_n \to f$ almost everywhere, and $\int f_n \, d\mu \to \int f \, d\mu$. Prove that for each $A \in \mathcal{A}$

$$\int_A f_n \, d\mu \to \int_A f \, d\mu$$

Proof. Since $f_n - f_n\chi_A$ is clearly positive and integrable, then by Fatou's Lemma, we see that

$$\int f - \limsup \int_A f_n = \liminf \int (f_n - f_n\chi_A) \geq \int \liminf (f_n - f_n\chi_A) = \int f - \int_A f$$

Since f is integrable, $\int f$ is finite. This means we can cancel them and get

$$\limsup \int_A f_n \, d\mu \leq \int_A f \, d\mu$$

Similarly, using $f_n + f_n\chi_A$, which is also clearly positive and integrable, we get that

$$\int_A f \, d\mu \leq \liminf \int_A f_n \, d\mu$$

Thus

$$\limsup \int_A f_n \, d\mu \leq \int_A f \, d\mu \leq \liminf \int_A f_n \, d\mu$$

Since $\limsup x_n \geq \liminf x_n$ for any sequence, this means that these inequalities are really equalities. Since liminf and limsup agree, that means

$$\int_A f_n \, d\mu \to \int_A f \, d\mu$$

$\square$

Ex 7.17 Prove that for $p > 0$

$$\sum_{k=1}^{\infty} \frac{1}{(p+k)^2} = -\int_0^1 \frac{x^p}{1-x} \log x \, dx$$

For this problem, you may use the Fundamental Theorem of Calculus.

Proof. For $0 < x < 1$, we see that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

which means that

$$-\int_0^1 \frac{x^p}{1-x} \log x \, dx = -\int_0^1 \sum_{n=0}^{\infty} x^{n+p} \log x \, dx = \int_0^1 \sum_{n=0}^{\infty} x^{n+p} \left(-\log x\right) dx$$

Since this is non-negative over $0 < x < 1$, by Proposition 7.6, we see that

$$\int_0^1 \sum_{n=0}^{\infty} x^{n+p} (-\log x)\, dx = -\sum_{n=0}^{\infty} \int_0^1 x^{n+p} \log x\, dx$$

Using integration by parts, letting $u = \log x$ and $dv = x^{n+p} dx$, we get that

$$\int_0^1 x^{n+p} \log x\, dx = [uv]_0^1 - \int_0^1 v\, du = \left[\log x \frac{x^{n+p+1}}{n+p+1}\right]_0^1 - \int_0^1 \frac{x^{n+p+1}}{n+p+1} \cdot \frac{1}{x}\, dx$$

This means that

$$\int_0^1 x^{n+p} \log x\, dx = \log 1 \frac{1^{n+p+1}}{n+p+1} - \lim_{x \to 0} \log x \frac{x^{n+p+1}}{n+p+1} - \frac{1}{n+p+1} \int_0^1 x^{n+p+1}\, dx$$

Since $p > 0$, then $n + p + 1 > 0$, which means that $x^{n+p+1} \log x \to 0$ as $x \to 0$. This shows that

$$\int_0^1 x^{n+p} \log x\, dx = 0 - 0 - \frac{1}{(n+p+1)^2}$$

Thus, we finally get that

$$-\int_0^1 \frac{x^p}{1-x} \log x\, dx = -\sum_{n=0}^{\infty} \int_0^1 x^{n+p} \log x\, dx = -\sum_{n=0}^{\infty} -\frac{1}{(n+p+1)^2} = \sum_{k=1}^{\infty} \frac{1}{(k+p)^2}$$

$\square$