Problem Set 6
Topology II

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Ex 1. For a cover $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$, let $H \subseteq \pi_1(X, x_0)$ be the image of the map incduced p_* on fundamental groups.

a) Construct a well-defined map $N(H) \to \mathrm{Deck}(p)$ where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.

b) Prove this map is a homomorphism.

c) Prove the map is surjective.

d) Prove the kernel of this map is H.

e) Conclude that $\mathrm{Deck}(p) \simeq N(H)/H$, where $H = \mathrm{im}(p_*)$.

Proof.

a) Let $[\gamma] \in N(H) \subseteq \pi_1(X, x_0)$. Since γ is a loop at x_0, if we lift γ to $\tilde{\gamma}$ in such a way that $\tilde{\gamma}(0) = \tilde{x}_0$, then it must be that $\tilde{\gamma}(1)$ is lifted to some other (possibly the same) element of the fiber of x_0, call this element $\tilde{x}_1$. Now let $\tilde{\alpha}$ be any loop with basepoint $\tilde{x}_0$. This means its projection α is in H, as H is $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

   Since $[\gamma] \in N(H)$, we have that $[\overline{\gamma}\alpha\gamma]$ is in H as well. Thus, the lift $\widetilde{\overline{\gamma}\alpha\gamma}$ starting at $\tilde{x}_1$ is a loop that starts at $\tilde{x}_1$, goes to $\tilde{x}_0$, follows $\tilde{\alpha}$, and then returns back to $\tilde{x}_1$. Since $\tilde{\alpha}$ was an arbitrary loop at $\tilde{x}_0$, this proves that $\pi_1(\tilde{X}, \tilde{x}_0) \subseteq \pi_1(\tilde{X}, \tilde{x}_1)$, meaning $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_1))$. Using the lifting criterion, this means that there's a deck transformation that sends $\tilde{x}_0$ to $\tilde{x}_1$. Since knowing where one element goes uniquely determines a deck transformtion, we'll let $\varphi : N(H) \to \mathrm{Deck}(p)$ send $[\gamma]$ to this deck transformation.

b) Let $[\gamma], [\gamma'] \in N(H)$ where $\varphi([\gamma]) = \tau$ is the deck transformation that takes $\tilde{x}_0$ to $\tilde{x}_1$ and $\varphi([\gamma']) = \tau'$ is the deck transformation that takes $\tilde{x}_0$ to $\tilde{x}_2$. We see that $\gamma\gamma'$ lifts to $\tilde{\gamma}\tau(\tilde{\gamma}')$ which starts at $\tilde{x}_0$ and ends at $\tau(\tilde{x}_2) = \tau\tau'(\tilde{x}_0)$. This proves that $\tau\tau'$ is the deck transformation corresponding to $[\gamma\gamma']$. Thus,

$$\varphi([\gamma][\gamma']) = \varphi([\gamma\gamma']) = \tau\tau' = \varphi([\gamma])\varphi([\gamma']).$$

c) Let τ be a deck transformation from $\tilde{x}_0$ to $\tilde{x}_1$. Since $p \circ \tau = p$, we have that

$$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_*(\tau_*(\pi_1(\tilde{X}, \tilde{x}_0))) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_1)).$$

Similarly, since $\mathrm{Deck}(p)$ is a group, there's a deck transformation τ^{-1} that takes $\tilde{x}_1$ to $\tilde{x}_0$. Using the same technique, we can get the reverse inclusion to see that

$$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_*(\pi_1(\tilde{X}, \tilde{x}_1)).$$

Now, we let $\tilde{\gamma}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$ (which means that γ is a loop at x_0) and $[\alpha] \in H$. Since $[\alpha] \in H$, we can lift α to a loop $\tilde{\alpha}$ with basepoint $\tilde{x}_0$. We can then concatenate to see that $\bar{\tilde{\gamma}}\tilde{\alpha}\tilde{\gamma}$ is a loop at $\tilde{x}_1$, which means that $[\bar{\tilde{\gamma}}\tilde{\alpha}\tilde{\gamma}] \in \pi_1(\tilde{X}, \tilde{x}_1)$. This means that

$$p_*[\bar{\tilde{\gamma}}\tilde{\alpha}\tilde{\gamma}] = [\gamma^{-1}\alpha\gamma] = [\gamma^{-1}][\alpha][\gamma] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$$

which proves that $\gamma \in N(H)$. We see that this process is very similar to the reverse of (a) and that this γ has the property that $\varphi([\gamma]) = \tau$. This proves that φ is surjective.

d) Let γ be in the kernel of φ. We saw in (a) that when we lift γ so that $\tilde{\gamma}(0) = \tilde{x}_0$, then $\varphi(\gamma)$ is a deck transformation that takes $\tilde{x}_0$ to $\tilde{\gamma}(1)$. However, since γ is in the kernel of φ, we know that $\varphi(\gamma)$ is the identity deck transformation. Thus, $\tilde{\gamma}(1)$ must be also $\tilde{x}(0)$. This means that γ is a loop at x_0 that can be lifted to a loop at $\tilde{x}_0$, which is equivalent to saying that $\gamma \in H$.

e) This follows immediately from applying the First Isomorphism Theorem to φ. □

Ex 2. For a path-connected, locally path-connected, and semilocally simply-connected pointed space (X, x_0), we define $\tilde{X}$ to be the path-homotopy classes of curves $\gamma : I \to X$ in X that start at the basepoint x_0. We also defined a map $p : \tilde{X} \to X$ by sending $[\gamma] \mapsto \gamma(1)$. Finally, we defined a basis for the topology on $\tilde{X}$ as the collection of

$$(U, [\gamma]) = \{[\gamma \cdot \eta] : \eta : I \to U, \eta(0) = \gamma(1)\},$$

where U runs over path-connected open sets such that the map induced by inclusion $U \subseteq X$ on π_1 is trivial, and where $[\gamma]$ runs over elements in $\tilde{X}$ such that $\gamma(1) \in U$. Check that this forms a basis and then prove that p is a covering.

Proof. Firstly, we check if these sets form a basis. We see that these sets cover the space $\tilde{X}$ by noting that if $[\gamma]$ is a path in $\tilde{X}$, then $[\gamma] \in (U, [\gamma])$ where U is some path-connected open set containing $\gamma(1)$. Such a U is guaranteed since X is locally path-connected.

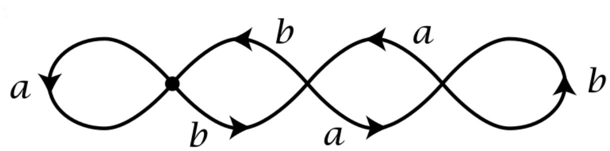
Now, to prove the second citerion these sets need to satisfy in order to be a basis, let $(U, [\alpha])$ and $(V, [\beta])$ be examples of such sets where γ is an element in the intersection. First, since $\gamma \in (U, [\alpha])$, we know that $[\gamma] = [\alpha \cdot \eta_1]$ for some $\eta_1 : I \to U$. Similarly, $[\gamma] = [\beta \cdot \eta_2]$ for some $\eta_2 : I \to V$. We see in particular that this means that $\gamma(1) \in U \cap V$. Let W be the path-connected component of $U \cap V$ that contains $\gamma(1)$. I claim that $(W, [\gamma])$ is then a open set that contains γ and is contained in the intersection of $(U, [\alpha])$ and $(V, [\beta])$.

Since $W \subseteq U$, we see that the chain of maps on π_1 induced by inclusion $\pi(W) \to \pi(U) \to \pi(\tilde{X})$ is trivial since the last such map in the chain is trivial. This proves that $(W, [\gamma])$ is in our set of proposed basis elements. To prove that $(W, [\gamma])$ is really contained in both $(U, [\alpha])$ and $(V, [\beta])$, let δ be an arbitrary path in $(W, [\gamma])$. This means that $[\delta] = [\gamma \cdot \eta]$ for some $\eta : I \to W$. Since we already know that $[\gamma] = [\alpha \cdot \eta_1]$, we see then that $[\delta] = [\alpha \cdot (\eta_1 \cdot \eta)]$. As $W \subseteq U$, the path $\eta_1 \cdot \eta$ is contained in U, meaning that $\delta \in (U, [\alpha])$ by definition. Similarly, $\delta \in (V, [\beta])$. Thus, since δ was arbitary, we have proved that $(W, [\gamma]) \subseteq (U, [\alpha]) \cap (V, [\beta])$ as desired.

Now we must prove that $p : \tilde{X} \to X$ is a covering. We first note that this map is surjective as any path from x_0 to x gets mapped to the point x and such a path always exists since X is path-connected. We also note that if $(U,[\gamma])$ is one of the basis elements of $\tilde{X}$, then the restriction $p:(U,[\gamma])\to U$ is surjective (using the same argument as the last sentence) and injective as different η's joining $\gamma(1)$ to a point $x\in U$ are all homotopic as $\pi_1(U)\to\pi_1(X)$ is trivial. This map is also a homeomorphism since it restricts to a bijection between open subsets $(V,[\gamma'])\subseteq(U,[\gamma])$ and $V\subseteq U$.

Using the fact that X is locally path-connected, for any $x\in X$ there's a path-connected open set U that contains x where U can be made as small as we'd like. We see that $p^{-1}(U)$ is simply the sets $(U,[\gamma_i])$, where the γ_i's are not homotopic to each other. Since these are our basis elements of $\tilde{X}$, this proves that p is continuous. Furthermore, by the previous paragraph, p restricted to each of these sets is homeomorphism onto U. This proves that p is indeed a covering. $\square$

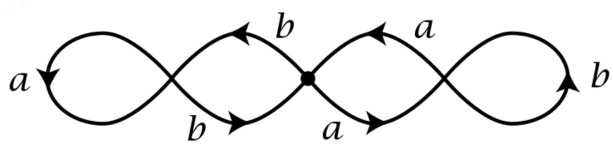
Ex 3. Let $X=S^1\vee S^1$ and let $p:\tilde{X}\to X$ be the following 3-sheeted cover with trivial deck group:



$\langle a, b^2, ba^2b^{-1}, baba^{-1}b^{-1}\rangle$

We note that the deck group is not transitive in this case, so we know that $H=p_*(\pi_1(\tilde{X}))$ is not normal. Prove this by hand.

Proof. To prove that H is not normal, we need to find a conjugate group that is not equal to H. If we change the basepoint of the covering space to be the middle intersection then we get the following space:



$\langle a^2, b^2, aba^{-1}, bab^{-1}\rangle$

We call the group of this space H' and see that it's conjugate to H, specifically $H'=b^{-1}Hb$. To prove that $H\neq H'$, we need only to prove that $a\notin H'$. This is easy to see as any element of H' is a loop in the covering space based at the new basepoint, and there is no possible loop at the new basepoint that travels only a. In fact, there are no single edge loops at the new basepoint at all. This proves that $a\notin H'$ and thus that $H\neq H'$ as desired. $\square$

Ex 4. Let H denote the skew field of quaternions $a+bi+cj+dk$ where $a,b,c,d\in\mathbb{R}$ and let $S^3\subseteq H$ denote the subset for which $|q|=\sqrt{q\bar{q}}=1$. Let $\mathrm{im}(H)$ denote the skew subfield of imaginary quaternions, i.e., those of the form $ai+bj+ck$.

a) Prove that S^3 acts on $\mathrm{im}(H)$ by conjugation.

b) Prove that these actions are orthogonal. That is, prove that the linear maps $v\mapsto qv\bar{q}$ (where $q\in S^3$) are orthogonal with respect to the standard inner product on H.

c) Prove that the resulting homomorphism $S^3 \to \mathrm{SO}(3)$ has kernel $\{\pm 1\}$.

d) Prove that this homomorphism is surjective.

e) Conclude that there is a free (and therefore properly discontinuous) action of $\mathbb{Z}_2$ on S^3 and that the resulting projection map gives a map $S^3 \to \mathrm{SO}(3)$ that is simultaneously a 2-to-1 covering map and a surjective group homomorphism.

Proof.

a) We first note that if $x \in H$, then $\bar{\bar{x}} = x$ and that $\overline{xy} = \bar{y}\bar{x}$. We also see that $x \in \mathrm{im}(H)$ if and only if $\bar{x} = -x$. Using these facts, we can easily see that if $q \in S^3$ and $x \in \mathrm{im}(H)$, then

$$\overline{qx\bar{q}} = \bar{\bar{q}}\bar{x}\bar{q} = q(-x)\bar{q} = -qx\bar{q}$$

which proves that $qx\bar{q} \in \mathrm{im}(H)$. Since conjugation by the identity $1 \in S^3$ is the identity and we have that

$$p.(q.x) = p.(qx\bar{q}) = p(qx\bar{q})\bar{p} = (pq)x(\overline{pq}) = (pq).x$$

we see that S^3 indeed acts on $\mathrm{im}(H)$ via conjugation.

b) Let $q \in S^3$ and let $T : \mathrm{im}(H) \to \mathrm{im}(H)$ where $T(v) = qv\bar{q}$. It's clear that T is a linear map. We first note if $v \in \mathrm{im}(H)$, then $|v|^2 = v\bar{v} = -v^2$. Using this, we can see that T is norm-preserving as

$$|qv\bar{q}|^2 = (qv\bar{q})(\overline{qv\bar{q}}) = (qv\bar{q})(-qv\bar{q}) = q(-v^2)\bar{q} = q|v|^2\bar{q} = q\bar{q}|v|^2 = |v|^2.$$

Since H is an inner-product space, an operator is norm-preserving if and only if it preserves inner-products (this follows from the parallelogram law). Thus, T preserves inner-products as well, making it an orthogonal linear map.

c) Suppose $q \in S^3$ and $(x \mapsto qx\bar{q}) \in \mathrm{SO}(3)$ is the identity map. This means that $x = qx\bar{q}$ for all $x \in \mathrm{im}(H)$. Multipying on the right by q gives us that $xq = qx$. Thus, q is in the center of $\mathrm{im}\, H$. As everything commutes with elements of $\mathrm{Re}(H)$, this means that q is in the center of all of H.

Now since $ij = -ji$ and $jk = -kj$, we know that i, j, k are not in the center of H, meaning that no element of $\mathrm{im}(H)$ is in the center either. Since the real subset of H commutes with everything, we see that the center of H is exactly $\mathrm{Re}(H)$. If we let q be in $S^3 \cap \mathrm{Re}(H)$, we see that $\sqrt{q\bar{q}} = \sqrt{q^2} = \pm 1$. This proves that the kernel is contained in $\{\pm 1\}$. A simple check proves that both 1 and -1 do indeed map to the identity map of $\mathrm{SO}(3)$, which means the kernel is exactly $\{\pm 1\}$ as desired.

d) I'm not sure how to prove this; my knowledge of Lie groups is a little rusty.

e) The previous parts imply that $S^3/\mathbb{Z}_2$ is isomorphic to $\mathrm{SO}(3)$ using the First Isomorphism Theorem. We see that $\mathbb{Z}_2$ acts on S^3 where $0.q = q$ and $1.q = -q$. We note that this action is free since $1.q$ does not fix any points. Additionally, the action is trivially properly discontinuous since $\mathbb{Z}_2$ is finite. This means that S^3 is a 2-to-1 covering space of $S^3/\mathbb{Z}^2 \simeq \mathrm{SO}(3)$. Since this projection simply sends antipodal elements to the same element, it must be the same map constructed in the previous parts. Thus, this projection map is both a 2-to-1 covering map and a surjective group homomorphism as desired. $\square$

Ex 5.

a) Prove that the symmetry group of the tetrahedron has 24 elements, compute it.

b) Find the subgroup of rotational symmetries, and prove that it has order 12. Call it T_{12}.

c) Summarize the proof that the binary tetrahedral group T^*_{24}, defined as the preimage of T_{12} under the homomorphism $S^3 \to \mathrm{SO}(3)$, acts freely on S^3 and hence produces a space X with $\pi_1(X) \simeq T^*_{24}$.

d) Using a presentation of T^*_{24}, construct a CW complex Y with $\pi_1(Y) \simeq T^*_{24}$. Now, which of these spaces do you like better?

Proof. Proof not completed. $\square$