Problem Set 4
Real Analysis II

Bennett Rennier
barennier@gmail.com

January 15, 2018

Ex 15.1 Show that L^∞ is complete.

Proof. Let $\{f_i\}$ be a Cauchy sequence in L^∞. This means that for every k, there exists an $N \in \mathbb{N}$, such that for all $n, m \geq N$, we get that

$$|f_n - f_m| < \frac{1}{k}$$

for all $x \in A^c_{n,m,k}$, where $A_{n,m,k}$ is a set of measure zero. If we let

$$A = \bigcup_{n,m,k \in \mathbb{N}} A_{n,m,k}$$

then we see that $\mu(A)$ is still zero. We define $f : X \to \mathbb{R}$ by

$$f(x) = \lim_{i \to \infty} f_i(x)$$

By letting $m \to \infty$ in the very first equation, we get that

$$|f_n - f| \leq \frac{1}{k}$$

for all $n \geq N$ and $x \in A^c$. This proves that $f \in L^\infty$ and that $f_n \to f$ almost everywhere. Thus, L^∞ is complete. □

Ex 15.2 Prove that the collection of simple functions is dense in L^p.

Proof. Let $1 \leq p < \infty$ and let $f \in L^p$. We assume that f is nonnegative, as if not, we can just decompose f into nonnegative and negative parts. Since f is nonnegative, this means there exists simple functions $s_n(x)$, such that $0 \leq s_n(x) \leq f(x)$ and $s_n \to f$ pointwise. Since $s(x) \leq f(x)$, that means that $\|s\|_p \leq \|f\|_p < \infty$. This proves that $s \in L^p$. We also see that since $f \in L^p$ that

$$|f - s_n|^p \leq |f|^p \in L^1(x)$$

which proves that $|f - s_n|^p$ is integrable. Using the dominated convergence theorem, we get that

$$\lim_{n\to\infty} \int |f - s_n|^p \, dm = \int \lim_{n\to\infty} |f - s_n|^p \, dm = 0$$

which proves the statement.

Now let $f \in L^\infty$. Suppose that $\|f\|_\infty = m$. We may assume that f is bounded, as the measure of $A = \{x \mid f(x) > m\}$ is zero, which means that f is equivalent to the function

$$f' = \begin{cases} f(x) & x \notin A \\ 0 & x \in A \end{cases}$$

Using this assumption, we see that f is bounded and that $\|f\|_\infty = \sup_{x\in X} |f(x)| < \infty$. Let $s_n(x)$ be defined similarly as to before. We see that since all simple functions are bounded that $\|s_n\|_\infty = \sup_{x\in X} |s(x)| < \infty$, which means that $s \in L^p$ and that $|f - s_n|$ is bounded. This means that $\lim_{n\to\infty} \|f - s_n\| = \lim_{n\to\infty} \sup_{x\in X} |f(x) - s_n(x)| = 0$, since $s_n \to f$ pointwise. This proves that the simple functions are dense in L^p. □

Ex 15.3 Prove the equality

$$\int |f(x)|^p \, dx = \int_0^\infty pt^{p-1} m(\{x \mid |f(x)| \geq t\}) \, dt$$

for $p \geq 1$.

Proof. Let $A_t = \{x \mid |f(x)| \geq t\}$. Then we see that

$$\int_0^\infty pt^{p-1} m(A_t) \, dt = \int_0^\infty pt^{p-1} \int \chi_{A_t} \, dx \, dt = \int_0^\infty \int pt^{p-1} \chi_{A_t} \, dx \, dt$$

Since we are dealing with σ-finite measures and $pt^{p-1}\chi_{A_t}$ is nonnegative for $t \geq 0$, we can interchange the integrals by Fubini's Theorem. Thus, we have

$$\int_0^\infty \int pt^{p-1} \chi_{A_t} \, dx \, dt = \int \int_0^\infty pt^{p-1} \chi_{A_t} \, dt \, dx$$

If we fix x, we should get that

$$\int_0^\infty pt^{p-1} \chi_{A_t} \, dt = |f(x)|^p$$

however, I'm sure how to prove this. If we take this equality on faith, we'd get that

$$\int_0^\infty pt^{p-1} m(A_t) \, dt = \int |f(x)|^p \, dx$$

for $p \geq 1$. □

Ex 15.5 When does equality hold in Hölder's inequality? When does equality hold in the Minkowski inequality?

[Incomplete]

Ex 15.8 Prove that if p and q are conjugate exponents, $f_n \to f$ in L^p, and $g \in L^q$, then

$$\int f_n g \to \int fg$$

[Incomplete]