Problem Set 5
Real Analysis II

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Ex 16.1 Find the Fourier Transform of $\chi_{[a,b]}$ and in particular, find the Fourier transform of $\chi_{[-n,n]}$.

Proof. We see that

$$\mathcal{F}(\chi_{[a,b]}) = \int_{\mathbb{R}} e^{iux}\chi_{[a,b]}\,dx = \int_a^b e^{iu\cdot x}\,dx = \left[\frac{e^{iux}}{iu}\right]_{x=a}^{b} = \frac{e^{iub}-e^{iua}}{iu}$$

in the general case. If we focus in particular on $\chi_{[-n,n]}$, then we get that

$$\begin{aligned}\mathcal{F}(\chi_{[-n,n]}) = \frac{e^{iun}-e^{-iun}}{iu} &= \frac{\cos(un)+i\sin(un)-(\cos(-un)+i\sin(-un))}{iu}\\ = \frac{\cos(un)+i\sin(un)-\cos(un)+i\sin(un)}{iu} &= \frac{2i\sin(un)}{iu} = \frac{2\sin(un)}{u}\end{aligned}$$

□

Ex 16.2 Find a real-valued function $f \in L^1$ such that $\hat{f} \notin L^1$.

Proof. We easily see that $\chi_{[-1,1]}$ is an integrable function over the real numbers. As per the previous exercise, we also know that $\mathcal{F}(\chi_{[-1,1]}) = 2\frac{\sin x}{x}$. Now, I claim that $2\frac{\sin x}{x}$ is not in L^1. To show this, we first see that for all $n \in \mathbb{N}$, we have that

$$\int_{(n-1)}^{n\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{1}{n\pi}\int_{(n-1)\pi}^{n\pi} |\sin x|\,dx = \frac{2}{n\pi}$$

This means that

$$\int_0^{m\pi} \left|\frac{\sin x}{x}\right| dx \geq \sum_{n=1}^{m\pi} \frac{2}{n\pi}$$

If we then let $m \to \infty$, we get that

$$\int_0^{\infty} \left|\frac{\sin x}{x}\right| dx \geq \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n} = \infty$$

as the sum is the harmonic series. This proves that $\frac{\sin x}{x} \notin L^1$. □

Ex 16.4 If f is integrable, real-valued, and all the partial derivatives $f_j = \partial f/\partial x_j$ are integrable, prove that the Fourier transform of f_j is given by $\hat{f}_j(u) = -iu_j\hat{f}(u)$.

Proof. Firstly, we can rearrange the integrals by Fubini's Theorem (as the function is integrable) like so:

$$\hat{f}_j(u) = \int_{\mathbb{R}^n} e^{iu\cdot x} f_j(x)\,dx = \int_{\mathbb{R}_1}\int_{\mathbb{R}_2}\cdots\int_{\mathbb{R}_n} e^{iu\cdot x} f_j(x)\,dx_n \ldots dx_2\,dx_1$$
$$= \int_{\mathbb{R}_1}\int_{\mathbb{R}_2}\cdots\int_{\mathbb{R}_n}\int_{\mathbb{R}_j} e^{iu\cdot x} f_j(x)\,dx_j\,dx_n \ldots dx_2\,dx_1$$

Applying integration by parts on the innermost integral, we get that

$$\int_{\mathbb{R}_j} e^{iu\cdot x} f_j(x)\,dx_j = \lim_{a_j\to\infty}\left[f(a)e^{iu\cdot a} - f(-a)e^{iu\cdot -a}\right] - \int_{\mathbb{R}_j} iu_j e^{iu\cdot x} f(x)\,dx_j$$

Since f_j is integrable, then we can use the same argument in Proposition 16.3 to see that $f(a) \to 0$ as $a_j \to \infty$. This proves that

$$\int_{\mathbb{R}_j} e^{iu\cdot x} f_j(x)\,dx_j = -\int_{\mathbb{R}_j} iu_j e^{iu\cdot x} f(x)\,dx_j$$

Which means that

$$\hat{f}_j(u) = \int_{\mathbb{R}_1}\int_{\mathbb{R}_2}\cdots\int_{\mathbb{R}_n}\int_{\mathbb{R}_j} e^{iu\cdot x} f_j(x)\,dx_j\,dx_n \ldots dx_2\,dx_1$$
$$= \int_{\mathbb{R}_1}\int_{\mathbb{R}_2}\cdots\int_{\mathbb{R}_n}\int_{\mathbb{R}_j} -iu_j e^{iu\cdot x} f(x)\,dx_j\,dx_n \ldots dx_2\,dx_1$$
$$= -iu_j\int_{\mathbb{R}_1}\int_{\mathbb{R}_2}\cdots\int_{\mathbb{R}_n} e^{iu\cdot x} f(x)\,dx_n \ldots dx_2\,dx_1 = -iu_j\hat{f}(u)$$

□

Ex 16.7 If f is real-valued and continuously differentiable on $\mathbb{R}$, prove that

$$\left(\int |f|^2\,dx\right)^2 \leq 4\left(\int |xf(x)|^2\,dx\right)\left(\int |f'|^2\,dx\right)$$

Proof. We know that

$$\langle f, g \rangle = \int |f(x)g(x)|\, dx$$

is a valid inner product on integrable functions. Using the Cauchy-Schwarz on this inner product, we see that

$$\langle xf(x), f'(x) \rangle^2 \leq \langle xf(x), xf(x) \rangle \langle f'(x), f'(x) \rangle$$

which means that

$$\int |xf(x)f'(x)|\, dx \leq \int |xf(x)|^2\, dx \int |f'(x)|^2\, dx$$

Now in order to prove the inequality, we must show that

$$\int |f(x)|^2\, dx \leq 4 \int |xf(x)f'(x)|\, dx$$

However, this isn't true in general, so I'm not sure how to prove this. I am fairly sure that it has to do with the Cauchy-Schwarz inequality under a different inner product that I'm not seeing. □