Problem Set 4
Complex Analysis I

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Ex 42 Let f be holomorphic on a neighborhood of $\overline{D}(P,r)$. Suppose that f is not identically zero. Prove that f has at most finitely many zeros in $D(P,r)$.

Proof. Let Z be the set of zeros in $D(P,r)$ and suppose that Z is infinite. We see that $\overline{Z} \subseteq \overline{D}(P,r)$, which means that $\overline{Z}$ is compact. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of distinct such zeros. Since this sequence is inside the compact set $\overline{Z}$, it must have a convergent subsequence in $\overline{Z}$. Additionally, since we chose our sequence to be of distinct zeros, this convergent subsequence is not a constant sequence. As we have found a non-constant sequence of zeros that also converges to a zero of f, it must be that $f = 0$. This is a contradiction to our assumption, which means that Z must be a finite set. $\square$

Ex 44 If $f : D(0,1) \to \mathbb{C}$ is a function, f^2 is holomorphic, and f^3 is holomorphic, then prove that f is holomorphic.

Proof. We first note that the zeros of f are the same as the zeros of f^2 and f^3. We denote this set of zeros as Z. If $Z = \mathbb{C}$, then $f = 0$ is trivially holomorphic. Thus, we may assume that $f \neq 0$. Since f^2 and f^3 are holomorphic, we see that $f = f^3/f^2$ is holomorphic on $D(0,1) \setminus Z$. Now let $z_0 \in Z$. Since $f^2 \neq 0$ and f^2 is holomorphic, we know that Z must be an isolated set, which means that there exists an $r > 0$ such that f^2 and f^3 are nonzero on $D(z_0,r) \setminus \{z_0\}$. We also see that

$$|f(z)| = \left|\frac{f^3(z)}{f^2(z)}\right| = \frac{|f^3(z)|}{|f^2(z)|} = \frac{|f^2(z)|^{\frac{3}{2}}}{|f^2(z)|} = \left|f^2(z)\right|^{\frac{1}{2}}.$$

Since f^2, $\sqrt{z}$, and z^2 are continuous, we have that

$$\lim_{z\to z_0} |f(z)| = \lim_{z\to z_0} \left|f^2(z)\right|^{\frac{1}{2}} = \left|f^2(z_0)\right|^{\frac{1}{2}} = 0.$$

Thus, $\lim_{z\to z_0} f(z) = 0$. This means that f is continous at z_0. Since f is holomorphic on $D(z_0,r) \setminus \{z_0\}$ and is continuous at z_0, we have that f is actually holomorphic on all of $D(z_0,r)$. As $z_0 \in Z$ was arbitrary, f is holomorphic on all of $D(0,1)$. $\square$

Ex 45 Suppose that f is holomorphic on all of $\mathbb{C}$ and that

$$\lim_{n\to\infty}\left(\frac{d}{dz}\right)^n f(z)$$

exists, uniformly on compact sets, and that this limit is not identically zero. Then the limit function F must be a very particular kind of entire function. Can you say what kind?

Proof. Since the sequence $F = \lim_{n\to\infty}(\frac{d}{dz})^n f(z)$ converges uniformly on compact sets, then F is holomorphic and we can also interchange limits and derivative. Thus, we have that

$$F'(z) = \frac{d}{dz}\lim_{n\to\infty}\left(\frac{d}{dz}\right)^n f(z) = \lim_{n\to\infty}\left(\frac{d}{dz}\right)^{n+1} f(z) = F(z).$$

We know that the functions ce^z satisfy this relation, where $c \in \mathbb{C}$. Suppose that there were another function $g(z)$ not of this form that satisfies $g'(z) = g(z)$. Then we see that

$$\frac{d}{dz}g(z)e^{-z} = g'(z)e^{-z} - g(z)e^{-z} = g(z)e^{-z} - g(z)e^{-z} = 0,$$

which proves that $g(z)e^{-z} = c$ for some constant $c \in \mathbb{C}$. This means that $g(z) = ce^z$, which contradicts the fact that $g(z)$ is not of the form ce^z. Thus, it must be that $F(z) = ce^z$ for some constant $k \in \mathbb{C}$. $\square$