Problem Set 1
Real Analysis II

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Ex 8.5 Suppose f is a non-negative integrable function on a measurable space $(X, \mathcal{A}, \mu)$. Prove that

$$\lim_{t\to\infty} t\mu(\{x \mid f(x) \geq t\}) = 0$$

Proof. Let $A_t = \{x \mid f(x) \geq t\}$. We clearly see that $A_t \downarrow \varnothing$. We also see that

$$t\mu(\{x \mid f(x) \geq n\}) = t\mu(A_t) = t\int \chi_{A_t}\, d\mu = \int t\chi_{A_t}\, d\mu \leq \int_{A_t} f\, d\mu$$

Since f is integrable, by the dominated convergence theorem

$$\lim_{t\to\infty} t\int_{A_t} d\mu = \int \lim_{t\to\infty} t\chi_{A_t}\, d\mu = \int 0\, d\mu = 0$$

This proves that $\lim_{n\to\infty} n\mu(\{x \mid f(x) \geq n\}) = 0$. □

Ex 11.5 Prove the equality

$$\int_{-\infty}^{\infty} |f(x)|\, dx = \int_0^{\infty} m(\{x \mid |f(x)| \geq t\})\, dt$$

where m is Lebesgue measure.

Proof. Similar to the proof in Problem 8.5, we see that

$$\int_0^{\infty} m(\{x \mid |f(x)| \geq t\})\, dt = \int_0^{\infty}\int \chi_{A_t}\, dx\, dt = \int_0^{\infty}\int_{-\infty}^{\infty} \chi_{A_t}\, dx\, dt$$

where $A_t = \{x \mid |f(x)| \geq t\}$. Since $|f|$ is nonnegative and we are dealing with σ-finite measures, by Fubini's theorem, we can interchange the limits. We thus see that

$$\int_0^{\infty}\int_{-\infty}^{\infty} \chi_{A_t}\, dx\, dt = \int_{-\infty}^{\infty}\int_0^{\infty} \chi_{A_t}\, dt\, dx$$

If we fix x, then

$$\int_0^\infty \chi_{A_t}\,dt = 1 \cdot |f(x)|$$

which proves that

$$\int_{-\infty}^\infty \int_0^\infty \chi_{A_t}\,dt\,dx = \int_{-\infty}^\infty |f(x)|\,dx$$

$\square$

Ex 11.15 Let $X = \{1, 2, \dots\}$ and let μ be the counting measure on X. Define $f : X \times X \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1, & x = y \\ -1, & x = y + 1 \\ 0, & \text{otherwise} \end{cases}$$

Show that

$$\int_X \int_X f(x,y)\,\mu(dx)\,\mu(dy) \neq \int_X \int_X f(x,y)\,\mu(dy)\,\mu(dx)$$

Why is this not a contradiction to the Fubini Theorem?

Proof. We see that $\int_X f(x,y)\,\mu(dx) = g(y) = \begin{cases} 1, & y = 1 \\ 0, & \text{otherwise} \end{cases}$. If we integrate again we get that $\int_X g(y)\,\mu(dy) = 1$. We also see that $\int_X f(x,y)\,\mu(dy)) = 0$, which means that $\int_X \int_X f(x,y)\,\mu(dy)\,\mu(dx) = 0$. This proves the inequality. This is not a contradiction to the Fubini Theorem as the function is neither nonnegative nor is it integrable in the product measure. $\square$

Ex 11.16 Let $\{a_n\}$ and $\{r_n\}$ be two sequences of real numbers such that $\sum_{n=1}^\infty |a_n| < \infty$. Prove that

$$\sum_{n=1}^\infty \frac{a_n}{\sqrt{|x - r_n|}}$$

converges absolutely for almost every $x \in \mathbb{R}$.

Proof. Let

$$f(x) = \sum_{n=1}^\infty \frac{|a_n|}{\sqrt{|x - r_n|}}$$

Now let $k \in \mathbb{N}$ be a fixed number. We see that

$$\int_k^{k+1} f(x)\,dx = \sum_{n=1}^\infty \int_k^{k+1} \frac{|a_n|}{\sqrt{|x - r_n|}}\,dx = \sum_{n=1}^\infty |a_n| \int_k^{k+1} \frac{1}{\sqrt{|x - r_n|}}\,dx$$

which we can do as the counting measure and the Lebesgue measure are both σ-finite and the terms are all nonnegative. [Incomplete] $\square$

Ex 13.12 Suppose μ is a σ-finite measure and ν is a finite measure. Prove that if $\nu \ll \mu$ and $\nu \perp \mu$, then ν is the zero measure.

Proof. Since μ and ν are mutually singular, this means that there exists a set E such that $\mu(E) = \nu(E^c) = 0$. Let A be an arbitrary measurable set. We see that

$$\nu(A) = \nu(A \cap E) + \nu(A \cap E^c) \leq \nu(E) + \nu(E^c) = 0 + \nu(E) = \nu(E)$$

However, since $\nu \ll \mu$ and $\mu(E) = 0$, this means that $\nu(E) = 0$. Thus $\nu(A) = 0$ for all measurable sets $A \in \mathcal{A}$. This does not require that μ be σ-finite nor that ν be finite. (I believe that it doesn't even require that they be positive, but I'm not sure if mutually singular is even used with signed measures.) $\square$

Ex 13.13 Prove the decomposition in the Lebesgue decomposition theorem is unique.

Proof. Let μ be σ-finite and let ν be finite. Suppose Lebesgue decomposition wasn't unique, that is $\nu = \lambda + \rho = \alpha + \beta$, where $\lambda \perp \mu$, $\alpha \perp \mu$, $\rho \ll \mu$, and $\beta \ll \mu$. Since $\lambda \perp \mu$ and $\alpha \perp \mu$, then there exist measurable sets E and F such that $\lambda(E^c) = \mu(E) = \alpha(F^c) = \mu(F) = 0$. Let $G = E \cup F$. We now see that $\mu(G) = \mu(E \cup F) \leq \mu(E) + \mu(F) = 0$ and that $\lambda(G^c) = \lambda(E^c \cap F^c) \leq \lambda(E^c) = 0$. Similarly, $\alpha(G^c) = 0$.

Let $A \subseteq G$ be a measurable set. Since $\rho \ll \mu$, $\beta \ll \mu$, and $\mu(G) = 0$, this means that $\rho(A) \leq \rho(G) = 0$ and that $\beta(A) \leq \beta(G) = 0$. This means that $\nu(A) = \lambda(A) + 0 = \alpha(A) + 0$. Thus, $\lambda = \alpha$ on G. Since $\lambda(G^c) = \alpha(G^c) = 0$, we see that $\lambda = \alpha$ on all of X. Since $\lambda = \alpha$ on all of X and $\lambda + \rho = \alpha + \beta$, this proves that $\rho = \beta$. Thus, the decomposition in the Lebesgue decomposition theorem is unique. $\square$