Problem Set 4
Algebra III

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Ex 1. Let K be an infinite-degree field extension of k. Consider the following subset R of $M_2(K)$,

$$R = \{(a_{ij}) \in M_2(K) : a_{11}, a_{12} \in K, a_{22} \in k, a_{21} = 0\} = \begin{bmatrix} K & K \\ 0 & k \end{bmatrix}.$$

Verify that R is a subring of $M_2(K)$ and prove that it is left artinian and left noetherian but neither right artinian nor right noetherian. [This is problem 49 on pg 26 in the book.]

Proof. We recall that left (resp. right) artinian implies left (resp. right) noetherian, so we need only to prove that R is left artinian but not right noetherian. We see that R is a subring as it's clearly closed under addition for $c, f \in k$ and $a, b, d, e \in K$ we have that

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} ad & ae + bf \\ 0 & cf \end{bmatrix} \in \begin{bmatrix} K & K \\ 0 & k \end{bmatrix}.$$

Let F be a field inbetween k and K. Consider the set $I_F = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$. We see that this is clearly closed under addition. We also see that

$$\begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K & K \\ 0 & k \end{bmatrix} = \begin{bmatrix} 0 & Fk \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}.$$

This proves that I_F is a right ideal of R. Since K is infinite dimensional over k, we can find a sequence of intermediate fields F_i each containing the last for all $i \in \mathbb{N}$. This means that I_{F_i} is a strictly ascending chain of ideals. This proves that R is not right Noetherian.

Now to prove left Artinian. Let I be the ideal

$$\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}.$$

and let

$$I_1 \supseteq I_2 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

be a descending chain of ideals. Looking at the surjective homomorphism $\varphi : R \to K \oplus k$ where

$$\varphi\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = (a_{11}, a_{22}),$$

we see that $\ker(\varphi) = I$. Thus, by the First Isomorphism Theorem, we have that $R/I \simeq K \oplus k$ is an Artinian ring. Now if every I_n in our chain contains an element which is non-zero in either the a_{12}'s or in the a_{21}'s place, then they all contain the ideal I. This puts them in a one-to-one correspondence with ideals in R/I forming a new descending chain. Since R/I is Artinian, the new chain of ideals eventually terminates, so the original chain must terminate as well. If, on the other hand, there is some k such that $I \subsetneq I_k$, this must mean that I_k contains no elements in either the a_{12}'s or the a_{21}'s place. That is I_k is a subset of the following ideal

$$\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}.$$

Since of element of the above ideal generates the whole ideal, it doesn't contain any subideals. Thus, I_{k+1} must the the zero ideal. Either way, our descending chain must eventually terminate, proving that R is left Artinian. $\square$

Ex 2. Generalized 2.5.4 as follows: Let R and S be rings, M and R-module and N an S-module, $f : R \to S$ a ring isomorphism and $h : M \to N$ a bijective additive map satisfying $h(rm) = f(r)h(m)$ for all $r \in R$ and $m \in M$. Construct a ring isomorphism between $\mathrm{End}_R(M)$ and $\mathrm{End}_S(N)$ and show that this is a k-algebra isomorphism if R and S are k-algebras and f is a k-algebra isomorphism.

Proof. We see that

$$h(h^{-1}(a) + h^{-1}(b)) = h(h^{-1}(a)) + h(h^{-1}(b)) = a + b$$

which means $h^{-1}(a + b) = h^{-1}(a) + h^{-1}(b)$, proving that h^{-1} is additive. We also see that

$$h(f^{-1}(s)h^{-1}(n)) = f(f^{-1}(s))h(h^{-1}(n)) = sn$$

so $h^{-1}(sn) = f^{-1}(s)h^{-1}(n)$ as well.

Now consider the map $\Phi : \mathrm{End}_R(M) \to \mathrm{End}_S(N)$ where $\Phi(\varphi) = h \circ \varphi \circ h^{-1}$. This is well-defined as $h \circ \varphi \circ h^{-1}$ is the composition of additive maps and we see that

$$\begin{aligned}(h \circ \varphi \circ h^{-1})(sn) &= (h \circ \varphi)(h^{-1}(sn)) = (h \circ \varphi)(f^{-1}(s)h^{-1}(n))) = h(\varphi(f^{-1}(s)h^{-1}(n))) \\ &= h(f^{-1}(s)\varphi(h^{-1}(n))) = f(f^{-1}(s))h(\varphi(h^{-1}(n))) = s(h \circ \varphi \circ h^{-1})(n),\end{aligned}$$

which proves that $h \circ \varphi \circ h^{-1} \in \mathrm{End}_S(N)$. We see that

$$\Phi(\varphi \circ \psi) = h \circ (\varphi \circ \psi) \circ h^{-1} = (h \circ \varphi \circ h^{-1}) \circ (h \circ \psi \circ h^{-1}) = \Phi(\varphi) \circ \Phi(\psi)$$

which proves that Φ is multiplicative and that

$$\begin{aligned}\Phi(\varphi + \psi)(n) &= (h \circ (\varphi + \psi) \circ h^{-1})(n) = h((\varphi + \psi)((h^{-1}(n))) = h(\varphi(h^{-1}(n)) + \psi(h^{-1}(n))) \\ &= \Phi(\varphi)(n) + \Phi(\psi)(n) = (\Phi(\varphi) + \Phi(\psi))(n)\end{aligned}$$

which proves that Φ is additive as well; this proves that Φ is a ring homomorphism. We can easily see that the inverse of Φ is simply $\Psi : \mathrm{End}_S(N) \to \mathrm{End}_R(M)$ where $\Psi(\psi) = h^{-1}\psi \circ h$. This proves that Φ is a ring isomorphism.

If R and S are k-algebras, then $\mathrm{End}_R(M)$ and $\mathrm{End}_S(N)$ are k-algebras where k is identified with $k\,\mathrm{Id}_M$ and $k\,\mathrm{Id}_N$ respectively. We see that if f is also a k-algebra isomorphism, then

$$\begin{aligned}\Phi(k\,\mathrm{Id}_M)(n) &= (h \circ (k\,\mathrm{Id}_M) \circ h^{-1})(n) = h(k\,\mathrm{Id}_M(h^{-1}(n))) = f(k)h(\mathrm{Id}_M(h^{-1}(n))) \\ &= kh(h^{-1}(n)) = kn.\end{aligned}$$

This proves that $\Phi(k\,\mathrm{Id}_M) = k\,\mathrm{Id}_n$, meaning Φ is also a k-algebra isomorphism. $\square$

Ex 3. Keep the notations introduced in Prop 2.6.1. Set $S_i = \mathrm{End}_R(M_i^{n_i})$ and $S = S_1 \times \cdots \times S_n$. Show that the map $\varphi : \mathrm{End}_R(M) \to S$, where $f \mapsto (f_1, \ldots, f_n)$ introduced in 2.6.1 is a ring isomorphism. You may use Prop (1) - (3) in that proof.

Proof. We see that

$$\begin{aligned}\varphi(f+g) &= ((f+g)_1, \ldots, (f+g)_n) = (p_1(f+g)h_1, \ldots, p_n(f+g)h_n) \\ &= (p_1 f h_1 + p_1 g h_1, \ldots, p_n f h_n + p_n g h_n) = (f_1 + g_1, \ldots, f_n + g_n) \\ &= (f_1, \ldots, f_n) + (g_1, \ldots, g_n) = \varphi(f) + \varphi(g)\end{aligned}$$

so φ is additive. We also see that

$$\begin{aligned}\varphi(fg) &= ((fg)_1, \ldots, (fg)_n) = (p_1(fg)h_1, \ldots, p_n(fg)h_n) \\ &= (p_1 f \,\mathrm{Id}_{M_1}\, g h_1, \ldots, p_n f \,\mathrm{Id}_{M_n}\, g h_n) \\ &= ((p_1 f h_1)(p_1 g h_1), \ldots, (p_n f h_n)(p_n g h_n)) \\ &= (f_1 g_1, \ldots, f_n g_n) = (f_1, \ldots, f_n)(g_1, \ldots, g_n) = \varphi(f)\varphi(g)\end{aligned}$$

which proves that φ is multiplicative.

Let $f_i \in \mathrm{End}_R(M_i^{n_i})$ for each $i \leq n$. We then let $f = \sum_{i \leq n} h_i f_i p_i$. We see that

$$\begin{aligned}\varphi(f) = \varphi\left(\sum_{i \leq n} h_i f_i p_i\right) &= \left(p_1\left(\sum_{i \leq n} h_i f_i p_i\right) h_1, \ldots, p_1\left(\sum_{i \leq n} h_i f_i p_i\right) h_n\right) \\ &= \left(\sum_{i \leq n} p_1 h_i f_i p_i h_1, \ldots, \sum_{i \leq n} p_n h_i f_i p_i h_n\right) \\ &= \left(\sum_{i \leq n} \delta_{1i} \,\mathrm{Id}_{M_i^{n_i}}\, f_i \delta_{i1} \,\mathrm{Id}_{M_i^{n_i}}, \ldots, \sum_{i \leq n} \delta_{ni} \,\mathrm{Id}_{M_i^{n_i}}\, f_i \delta_{in} \,\mathrm{Id}_{M_i^{n_i}}\right) \\ &= \left(\mathrm{Id}_{M_i^{n_i}}\, f_1 \,\mathrm{Id}_{M_i^{n_i}}, \ldots, \mathrm{Id}_{M_n^{n_n}}\, f_n \,\mathrm{Id}_{M_n^{n_n}}\right) = (f_1, \ldots, f_n)\end{aligned}$$

which proves that φ is surjective.

Now suppose $f \in \mathrm{End}_R(M)$ such that $\varphi(f) = (f_1, \ldots, f_n) = (0, \ldots, 0)$, where 0 is the zero map of $\mathrm{End}_R(M_i^{n_i})$. This means that $p_i f h_i = 0$ for each $i \leq n$, implying that $h_i p_i f h_i p_i = 0$ as well. Similar to the proof that $\sum_{i \leq n} h_i p_i = \mathrm{Id}_m$, we obtain that

$$0 = \sum_{i \leq n} h_i p_i f h_i p_i = \mathrm{Id}_m f \,\mathrm{Id}_m = f.$$

This proves that the kernel of φ is trivial, and thus φ is injective. This concludes the proof that φ is a ring isomorphism. □

Ex 4. Let R be a nonzero commutative semisimple ring.

a) Show that R is isomorphic to a finite direct product of fields.

b) Determine the length $\ell(R)$ in terms of this direct product.

c) List all ideals of R. How many are there? Which of them are maximal?

Proof.

a) By the Artin-Wedderburn Theorem, we know that $R \simeq \times_{i\leq n} M_{n_i}(D_i)$ where $n, n_i \in \mathbb{N}$ and each D_i is a skew-field. This means that they have isomorphic centers, so we get that

$$R = Z(R) \simeq Z\left(\bigtimes_{i\leq n} M_{n_i}(D_i)\right) = \bigtimes_{i\leq n} Z(M_{n_i}(D_i)) \simeq \bigtimes_{i\leq n} Z(D_i).$$

Since the center of a skewfield is a field, we have that R is isomorphic to the direct product of fields. Note that by the uniqueness of Artin-Wedderburn, we have that $n_i = 1$ and that D_i were fields to begin with.

b) We showed in the previous part that $R \simeq \times_{i\leq n} k_i$, where k_i are fields. Thus, the length of R is equal to the length of $\times_{i\leq n} k_i$. In class, we showed the length of a semisimple ring can be found as the number of minimal ideals. Since the ideals of a product are the product of ideals and the only ideals of a field are $\{0\}$ and itself, we have that the only ideals of $\times_{i\leq n} k_i$ are $\times_{i\leq n} I_i$ where each I_i is either $\{0\}$ or k_i. We can clearly see from this that the only minimal ideals are $k_j \times \{0\}^{n-1}$ for each $j \leq n$. Since there are n minimal ideals, we have that $\ell(R) = \ell(\times_{i\leq n} k_i) = n$.

c) Let $\varphi : \times_{i\leq n} k_i \to R$ be a ring isomorphism. From this, the ideals of R are simply $\varphi(I)$, where I is an ideal of $\times_{i\leq n} k_i$. In part (b), we identified that the ideals of $\times_{i\leq n} k_i$ are $\times_{i\leq n} I_i$ where each I_i is either $\{0\}$ or k_i. This gives all the ideals of R. By simple combinatorics, we see that there are 2^n ideals of R. The maximal ones are the images of the maximal ideals of $\times_{i\leq n} k_i$, which we can easily see are the ideals $\times_{i\leq n, i\neq j} k_i \times \{0\}$ for each $j \leq n$.

□

Ex 5. Assume that k is a field and R a semisimple k-algebra.

a) Prove that R is commutative or a k-division algebra if $\dim_k(R) = 3$. Is the conclusion true if we drop the assumption "semisimple"?

b) If $\dim_k(R) = 4$ and R contains a nonzero nilpotent element, show that R is isomorphic (as a k-algebra) to $M_2(k)$.

c) Show that R is isomorphic to $M_p(k)$ if p is prime, $\dim_k(R) = p^2$, and R is simple and contains a nonzero nilpotent element.

Proof.

a) By the Artian-Wedderburn Theorem, we know that $R \simeq \times_{i\leq n} M_{n_i}(D_i)$ where $n, n_i \in \mathbb{N}$ and each D_i is a k-division algebra. We see that if $\dim(R) = 3$ and if $n_j \geq 2$ for some $j \leq n$, then

$$3 = \dim(R) = \dim\left(\bigtimes_{i\leq n} M_{n_i}(D_i)\right) = \sum_{i\leq n} \dim(M_{n_i}(D_i)) = \sum_{i\leq n} n_i^2 \dim(D_i) \geq \sum_{i\leq n} n_i^2 \geq n_j^2 \geq 4,$$

which is a contradiction. Thus, we have that $R \simeq \times_{i\leq k} M_1(D_i) \simeq \times_{i\leq k} D_i$. This proves that R is the direct product of k-division rings. This gives us three possibilities: 1) R is isomorphic to a 3-dimensional k-division algebra, 2) R is isomorphic to the product of a 2-dimensional k-algebra and k, or 3) $R = k^3$. Now if D is a 2-dimensional k-algebra, then we can find a basis D of $\{e_1, e_2\}$ where ke_1 is identified with k. Since $k \subseteq Z(D)$, we have that $e_1 e_2 = e_2 e_1$.

Since e_2 commutes with the basis elements, we have that D is actually commutative. This proves R is either a 3-dimensional k-division algebra or R is commutative.

This is not true if we drop semi-simplicity, though. Take for example the set of upper triangular 2×2 matrices over some field k, that is matrices of the form

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

where $a, b, c \in k$. We can clearly see that this is three-dimensional; it can be generated by $\{e_{11}, e_{12}, e_{22}\}$ where e_{ij} is an elementary matrix. We also see that

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which proves that this k-algebra is non-commutative. This k-algebra is also not a k-division algebra as it contains matrices of determinant 0, which don't have inverses.

b) By the Artian-Wedderburn Theorem, we know that $R \simeq \times_{i \leq n} M_{n_i}(D_i)$ where $n, n_i \in \mathbb{N}$ and each D_i is a k-division algebra. By dimension counting, we know that

$$4 = \dim(R) = \sum_{i \leq n} n_i^2 \dim_k(D_i).$$

Since $\dim_k(D_i) \geq 1$, we see that either $n_i = 1$ for all i or that $n_j = 2$ for some j, $n_i = 0$ for $i \neq j$, and $\dim_k(D_j) = 1$. This means that either R is isomorphic to the direct product of division rings or that $R \simeq M_2(k)$. Since division rings don't contain nilpotent elements, we see that the former case cannot happen, thus it must be that $R \simeq M_2(k)$.

c) By the Artian-Wedderburn Theorem, we know that $R \simeq \times_{i \leq n} M_{n_i}(D_i)$ where $n, n_i \in \mathbb{N}$ and each D_i is a k-division algebra. Since R is simple, it must be that $k = 1$, meaning $R \simeq M_n(D)$ for some k-divison algebra D. If $n = 1$, then R would be isomorphic to a division ring, but division rings don't contain any nonzero nilpotent elements, which is a contradiction. Thus, it must be that $n > 1$. We see that $p^2 = \dim_k(R) = \dim_k(M_n(D)) = n^2 \dim_k(D)$. From this, we know that n divides p. Since $n \neq 1$, it must be that $n = p$. This means that $\dim_k(D) = 1$, proving that $D \simeq k$. Thus, we have that $R \simeq M_p(k)$ as desired.

□

Ex 6.

a) Write down, up to isomorphism, all semisimple 16-dimensional $\mathbb{C}$-algebras.

b) We shall prove later that every finite-dimensional non-commutative $\mathbb{R}$-division algebra is isomorphic to the standard quaternion division algebra $\mathbb{H}$. Using this, write down, up to isomorphism, all semisimple 10-dimensional $\mathbb{R}$-algebras.

Proof.

a) Let D be a finite-dimensional $\mathbb{C}$-division algebra. If we let $x \in D$, we see that the subring generated by x and $\mathbb{C} \subseteq D$ is commutative, meaning its a field over $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, this subring must actually be $\mathbb{C}$ itself, so $x \in \mathbb{C}$. Since this is true for all x, we have that $D = \mathbb{C}$.

By the Artian-Wedderburn Theorem, we know that if R is a semisimple $\mathbb{C}$-algebra, then $R \simeq \times_{i\leq n} M_{n_i}(D_i)$ where $n, n_i \in \mathbb{N}$ and each D_i is a $\mathbb{C}$-division algebra. But we just proved that the only $\mathbb{C}$-division algebra is $\mathbb{C}$ itself. Thus, we have that $R \simeq \times_{i\leq n} M_{n_i}(\mathbb{C})$. If $\dim(R) = 16$, then we have that $\sum_{i\leq n} n_i^2 = 16$. This does not leave very many possibilites. Using this we can see that the only possible semisimple 16-dimensional $\mathbb{C}$-algebras are $M_4(\mathbb{C})$, $M_3(\mathbb{C}) \times \mathbb{C}^7$, $M_3(\mathbb{C}) \times M_2(\mathbb{C}) \times \mathbb{C}^3$, $M_2(\mathbb{C}) \times \mathbb{C}^{12}$, $M_2(\mathbb{C})^2 \times \mathbb{C}^8$, $M_2(\mathbb{C})^3 \times \mathbb{C}^4$, $M_2(\mathbb{C})^4$, and $\mathbb{C}^{16}$.

b) If R is a finite-dimensional commutative $\mathbb{R}$-division algebra, then R is actually a field and so is a field extension of $\mathbb{R}$. Since $\mathbb{C}$ is the algebraic closure, we have that $R \subseteq \mathbb{C}$. Since $\mathbb{C}$ is a field extension of degree 2, there are no intermediate fields, so it must be that $R = \mathbb{R}$ or $R = \mathbb{C}$. Thus, the only finite-dimensional $\mathbb{R}$-division algebras are $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$.

By the Artian-Wedderburn Theorem, we know that if R is a semisimple $\mathbb{R}$-algebra, then $R \simeq \times_{i\leq n} M_{n_i}(D_i)$ where $n, n_i \in \mathbb{N}$ and each D_i is a $\mathbb{R}$-division algebra. But we just proved that the only $\mathbb{R}$-division algebras are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. This proves that $D_i = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ for each $i \leq n$. If $\dim(R) = 10$, then we have that $\sum_{i\leq n} n_i^2 \dim_k(D_i) = 16$. This still leaves quite a few possibilites, but via brute force, we see that the only combinations are $M_3(\mathbb{R}) \times \mathbb{R}$, $M_2(\mathbb{C}) \times \mathbb{R}^2$, $M_2(\mathbb{C}) \times \mathbb{C}$, $M_2(\mathbb{R}) \times \mathbb{R}^6$, $M_2(\mathbb{R}) \times \mathbb{C} \times \mathbb{R}^4$, $M_2(\mathbb{R}) \times \mathbb{C}^2 \times \mathbb{R}^2$, $M_2(\mathbb{R}) \times \mathbb{C}^3$, $M_2(\mathbb{R}) \times \mathbb{H} \times \mathbb{R}^2$, $M_2(\mathbb{R}) \times \mathbb{H} \times \mathbb{C}$, $M_2(\mathbb{R}) \times M_2(\mathbb{R}) \times \mathbb{R}^2$, $M_2(\mathbb{R}) \times M_2(\mathbb{R}) \times \mathbb{C}$, and all direct products of $\mathbb{H}$, $\mathbb{C}$, $\mathbb{R}$ whose dimension add up to 16.

$\square$