Problem Set 4
Topology II

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The following exercises are from Hatcher Section 1.1.

Ex 11 If X_0 is a path-component of a space X containing the basepoint x_0, show that the inclusion $X_0 \hookrightarrow X$ induces an isomorphism $\pi_1(X_0, x_0) \to \pi_1(X, x_0)$.

Proof. Let $i : X_0 \to X$ be the inclusion map. We know that $i_* : \pi_1(X_0, x_0) \to \pi_1(X, x_0)$ is a homomorphism, so we need only to prove that it's injective and surjective.

Injectivity) Let $f, g : (S^1, 1) \to (X_0, x_0)$ and suppose that $i_*[f] = [ig] = [ih] = i_*[h]$. This means there's a homotopy of paths $h : S^1 \times I \to X$ between ig and ih. Since $S^1 \times I$ is a cylinder, it's path-connected, which means that the image of h must be path-connected and thus be contained in X_0 (as x_0 is in the image). This means we can restrict the image of h to X_0 and obtain a homotopy of paths between g and h in X_0. This proves that $[g] = [h]$, meaning i_* is injective.

Surjectivity) Let $f : (S^1, 1) \to (X, x_0)$. Since S^1 is path-connected, the image of f must be path-connected and thus be contained in X_0 (as x_0 is in the image). This means we can restrict the image of f to X_0; let's call this function f'. Using this, we have that $i_*[f'] = [if'] = [f]$. Since f was arbitrary, we have proved that i_* is surjective. □

Ex 12 Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism φ_* of a map $\varphi : S^1 \to S^1$.

Proof. We note that $\pi_1(S^1) \simeq \mathbb{Z}$ and that any element of $\pi_1(S^1)$ can be represented in the form $[f_k]$ where $f_k : [0, 1] \to S^1$ is defined by $f_k(t) = e^{2\pi ikt}$. Since $\pi_1(S^1)$ is generated by the element $[f_1]$, any homomorphism is uniquely determined by where it sends $[f_1]$. Let h be a homomorphism and suppose h sends $[f_1]$ to $[f_k]$. Now, let $\varphi_k : S^1 \to S^1$ be the map $\varphi(z) = z^k$. We see then that

$$\varphi_{k*}[f_1] = [\varphi_k f_1] = [\varphi_k(e^{2\pi it})] = [e^{2\pi ikt}] = [f_k].$$

Since h and φ_{k*} both send the generator to the same element, it must be that $h = \varphi_{k*}$. Thus, any homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ is induced by a map $\varphi_k : S^1 \to S^1$. □

Ex 16 Show that there are no retractions $r : X \to A$ in the following cases:

a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.

b) $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.

c) $X = S^1 \times D^2$ and A the circle shown in the figure [Figure not shown].

d) $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.

e) X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$.

f) X the Möbius band and A its boundary circle.

Proof.

a) Suppose there were a retraction $r : \mathbb{R}^3 \to A$. This means that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, hook, "i_A"'] \arrow[rr, bend left, "\mathbb{1}_A"] & \mathbb{R}^3 \arrow[r, two heads, "r"'] & A \end{tikzcd}$$

Since A is homeomorphic to S^1, we know that $\pi_1(A) \simeq \pi_1(S^1) \simeq \mathbb{Z}$. Thus, if we then apply the π_1 functor to the previous diagram we get the following diagram (up to isomorphism):

$$\begin{tikzcd} \mathbb{Z} \arrow[r, hook, "(i_A)_*"'] \arrow[rr, bend left, "\mathbb{1}_\mathbb{Z}"] & 0 \arrow[r, two heads, "r_*"'] & \mathbb{Z} \end{tikzcd}$$

We know that r_* is surjective as $r_* \circ (i_A)_*$ is the identity on $\mathbb{Z}$. But this is a contradiction as there can be no surjective homomorphism from 0 to $\mathbb{Z}$. This proves that there is no such retraction.

b) The space $S^1 \times D^2$ is homotopy-equivalent to the space S^1 via the retraction onto the circle going through the center of the torus. This means that $\pi_1(S^1 \times D^2) \simeq \pi_1(S^1) \simeq \mathbb{Z}$. If there were a retraction from $S^1 \times D^2$ to $S^1 \times S^1$, then this would induce a surjection from $\pi_1(S^1 \times D^2) \simeq \mathbb{Z}$ onto $\pi_1(S^1 \times S^1) \simeq \mathbb{Z} \times \mathbb{Z}$. This is impossible, though, as the image of any homomorphism from $\mathbb{Z}$ is cyclic and $\mathbb{Z} \times \mathbb{Z}$ is not cyclic. This proves that there is no such retraction.

c) Suppose there were a retraction $r : S^1 \times D^2 \to A$. This would mean that $r \circ i = \mathbb{1}_A$, where $i : A \to S^1 \times D^2$ is the inclusion map. If we apply the π_1 functor, we get that $r_* \circ i_* = \mathbb{1}_Z$, which would mean that i_* is injective. If we let $f : S^1 \to A$ be the generator of $\pi_1(A)$ (i.e. the loop goes around A once), we see that f is actually contractible in $S^1 \times D^2$. This means that $i_*[f] = [\text{const}_{x_0}]$ in $\pi_1(S^1 \times D^2)$. Since i_* has a non-trivial kernel, it can't be injective. This is a contradiction, proving that there is no such retraction.

d) Since $D^2 \vee D^2$ is path-connected, our choice of basepoint doesn't matter. As such, let x_0 be the identified point of the two disks. We see that any loop based at x_0 is homotopic to const_{x_0} via a straight-line homotopy. This means that $\pi_1(D^2 \vee D^2)$ is trivial. Let be map $r : S^1 \vee S^1 \to S^1$ that fixes the first circle and maps the second circle to the identified common point. We see that this map is continuous (the inverse images of open sets are either themselves or themselves union an open ε neighborhood of the second circle). Since $r \circ i = \mathbb{1}_{S^1}$ as well, we can apply the π_1 functor to see that $r_* \circ i_* = \mathbb{1}_{\mathbb{Z}}$. This proves that i_* must be injective. Since i_* is a injection from $\pi_1(S^1) \simeq \mathbb{Z}$ onto $\pi_1(S^1 \vee S^1)$, this means that $\pi_1(S^1 \vee S^1)$ is not trivial.

Similar to the previous parts, if there were a retraction from $D^2 \vee D^2$, there'd be a composition of maps $\pi_1(S^1 \vee S^1) \to \pi_1(D^2 \vee D^2) \to \pi_1(S^1 \vee S^1)$ that act as the identity map. However, since $\pi_1(S^1 \vee S^1)$ is not the trivial group and $\pi_1(D^2 \vee D^2)$ is, there are no such maps. Thus, there is no retraction from $D^2 \vee D^2$ to $S^1 \vee S^1$.

e) Let x_0 be the identified boundary points, which is also the identified point in $S^1 \vee S^1$. Let $f, g : S^1 \to S^1 \vee S^1$ be the loops with basepoint x_0, where the first loop goes around the first circle and the second loop goes around the second circle. The map $r = \mathbb{1}_{S^1} \vee \text{const}_{x_0} : S^1 \vee S^1 \to S^1$ is a retraction onto the first circle. Since $r_*[f] = [f] \neq [\text{const}_{x_0}]$ and $r_*[g] = [\text{const}_{x_0}]$, it must be that $[f] \neq [g]$ in $\pi_1(A)$. If there were a retraction from X to A, then similar to part (c), the map i_* induced by the inclusion map would be injective. However, we see that f and g are homotopic in X via the straight-line homotopy (I'm imagining the points being identified on the boundary of X as antipodal). This means that $i_*[f] = [if] = [ig] = i_*[g]$. This is a contradiction to i_* being injective. Thus, there is no such retraction.

f) Let S be the circle that goes through the center of the Möbius band. We see that X and S are homotopy-equivalent, as there is a straight-line deformation retraction from the Möbius band onto its core circle. This means that $\pi_1(X) \simeq \pi_1(S) \simeq \mathbb{Z}$. If we let f be the generator of $\pi_1(A)$ (i.e. the loop that goes around the boundary of the Möbius band once) and g be the generator of $\pi_1(X) \simeq \pi_1(S)$ (i.e. the loop that goes through the center of the Möbius band), we see that $i_*[f] = [if] = [g]^2$, since going around the boundary circle once is equivalent to going around the center circle twice. Suppose there were a retraction $r : X \to A$. This would mean that $r \circ i = \mathbb{1}_A$ and as such that $r_* \circ i_* = \mathbb{1}_{\pi_1(A)}$. However, then we would have that

$$[f] = \mathbb{1}_{\pi_1(A)}[f] = r_* i_*[f] = r_*[if] = r_*[g]^2.$$

This is a contradiction as all maps $\mathbb{Z} \to \mathbb{Z}$ are of the form $n \mapsto kn$ for some non-negative k, so there is no such map $r_* : \pi_1(X) \simeq \mathbb{Z} \to \mathbb{Z} \simeq \pi_1(A)$ that maps twice the generator onto the generator. □

Ex 17 Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \to S^1$.

Proof. We will index the circles for clarity, that is, consider the circles S^1_t and S^1_s described by $e^{i\pi t}$ and $e^{i\pi s}$ respectively, where $s, t \in [0, 1)$. We let $S^1_t \vee S^1_s$ be the space where $e^{i\pi 0_t}$ and $e^{i\pi 0_s}$ are identified. Now let $\varphi_k : S^1_t \vee S^1_s \to S^1_t$ be the map where $e^{i\pi t} \mapsto e^{i\pi t}$ and $e^{i\pi s} \mapsto e^{i\pi k t}$ for $k \in \mathbb{Z}$. This map is well-defined as $\varphi(e^{i\pi 0_s}) = e^{i\pi k 0_t} = e^{i\pi 0_t} = \varphi(e^{i\pi 0_t})$. The map is continuous as it is a continuous function on both circles individually. Since it is the identity on its image, this proves that φ_k is a retraction. I claim for $n, m \in \mathbb{Z}$ such that $n \neq m$, the retractions φ_n and φ_m are not homotopic.

To prove this, suppose that there is a homotopy between φ_n and φ_m. Now consider the loop $e^{i\pi s}$ in $S^1_t \vee S^1_s$ with basepoint $e^{i\pi 0_s}$. Since there's a homotopy between φ_n and φ_m, there is a homotopy between the two compositions $\varphi_n e^{i\pi s}$ and $\varphi_m e^{i\pi s}$. This would mean that

$$[e^{i\pi nt}] = [\varphi_n e^{i\pi s}] = [\varphi_m e^{i\pi s}] = [e^{i\pi mt}]$$

which is a contradiction as $e^{i\pi nt}$ and $e^{i\pi mt}$ are not homotopic in S^1_t. $\square$

Ex 18 Using Lemma 1.15, show that if a space X is obtained from a path-connected subspace A by attaching a cell e^n with $n \geq 2$, then the inclusion $A \hookrightarrow X$ induces a surjection on π_1. Apply this to show:

a) The wedge sum $S^1 \vee S^2$ has fundamental group $\mathbb{Z}$.

b) For a path-connected CW complex X, the inclusion map $X^1 \hookrightarrow X$ of its 1-skeleton induces a surjection $\pi_1(X^1) \to \pi_1(X)$.

[Addendum: Assumed that the CW complex has finitely many cells.]

Proof. From the first homework, attaching a cell to a path-connected space results in a path-connected space. This means, by the previous homework, and our choice of basepoint does not matter. Let $x_0 \in A$ be our basepoint then and let $f : I \to X$ be a path such that $f(0) = f(1) = x_0$. Again, from the first homework, there is a homotopy from this path to a path $g : I \to X$ that avoids the interior of e_n. Thus, we may view g as a path into A and $[g]$ as a member of $\pi_1(A, x_0)$. Since f and g are homotopic in X, we have that $i_*[g] = [ig] = [f]$. As f was an arbitrary loop in X, we have proved that i_* is a surjection from $\pi_1(A, x_0) \to \pi_1(X, x_0)$. We use this to prove the following.

a) If we view $S^1 \vee S^2$ as $S^1 \cup_\varphi D^2$ where φ sends the boundary of D^2 to a single point of S^1, then by the proved statement, we can see that the inclusion map $i : S^1 \to S^1 \vee S^2$ is a surjection from $\pi_1(S^1)$ onto $\pi_1(S^1 \vee S^2)$.

   Now let $c : S^1 \vee S^2 \to S^1$ be the map which sends every point of S^2 to the point where S^1 and S^2 are joined and which restricts to the identity on S^1. This map is continuous (the inverse images of open sets of S^1 are either themselves or themselves union with some open epsilon neighborhood of S^2). Additionally, we have that $r \circ i = \mathbb{1}_{S^1}$. This means that $r_* \circ i_* = \mathbb{1}_{\pi_1(S^1)}$. Since the identity is a bijection, this means that i_* is injective. This means that i_* is an isomorphism of fundamental groups, giving us that

$$\pi_1(S^1 \vee S^2) \simeq \pi_1(S^1) \simeq \mathbb{Z}.$$

b) Since X is a CW complex we can represent X recursively, where $X_1 = X^1$ (the 1-skeleton of X), $X_n = X_{n-1} \cup_{\varphi_{n-1}} D^{a_{n-1}}$, and $a_{n-1} \geq 1$. We will then prove the claim via induction. By the first homework, a CW complex is path-connected if and only if its 1-skeleton is path-connected. This means that the inclusion map $X^1 = X_1 \hookrightarrow X_1 \cup_{\varphi_1} D^{a_1} = X_2$ induces a surjection on π_1 since $a_1 \geq 2$. This takes care of the base case. Now assume that $X_1 \hookrightarrow X_k$ induces a surjection on π_1. The inclusion map $X_k \hookrightarrow X_k \cup_{\varphi_k} D^{a_k} = X_{k+1}$ induces a surjection from $\pi_1(X_k)$ onto $\pi_1(X_{k+1})$ as $a_{k+1} \geq 2$. The composition of inclusions (which is the same map as the direct inclusion) $X_1 \hookrightarrow X_k \hookrightarrow X_{k+1}$ thus induces a surjection from $\pi_1(X_1)$ onto $\pi_1(X_{k+1})$. This completes the induction and proves the claim.

$\square$