Problem Set 5
Complex Analysis I

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Ex 3 Let f be holomorphic on $U/\{P\}$, $P \in U$, U open. If f has an essential singularity at P, then what type of singularity does $1/f$ have at P? What about when f has a removable singularity or pole at P?

Ex 5 Let $P = 0$. Classify each of the following as having a removable singularity, a pole, or an essential singularity at P:

a) $\frac{1}{z}$,

b) $\sin \frac{1}{z}$,

c) $\frac{1}{z^3} - \cos z$,

d) $z \cdot e^{1/z} \cdot e^{-1/z^2}$,

e) $\frac{\sin z}{z}$,

f) $\frac{\cos z}{z}$.

Ex 8a Let $U = D(P, r) \backslash \{P\}$. Prove that if f is holomorphic on U and $\lim_{z \to P}(z-P) \cdot f(z) = 0$, then f continues holomorphically across P (to all of U).

Ex 9 Prove that if $f : D(P, r) \setminus \{P\} \to \mathbb{C}$ has an essential singularity at P, then for each positive integer N there is a sequence $\{z_n\} \subseteq D(P, r) \setminus \{P\}$ with $\lim_{n\to\infty} z_n = P$ and

$$|(z_n - P)^N \cdot f(z_n)| \geq N.$$

[Informally, we can say that f "blows up" faster than any positive power of $1/(z - P)$ along some sequence converging to P.]

Ex 18 Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant entire function. Define $g(z) = f(1/z)$. Prove that f is a polynomial if and only if g has a pole at 0. In other words, f is transcendental (nonpolynomial) if and only if g has an essential singularity at 0.

Ex 27c Calculate the first four terms of the Laurent expansion of

$$f(z) = z/[(z - 1)(z - 3)(z - 5)]$$

about $P = 1$ and specify the annulus of convergence of the expansion.

Ex 33abc Compute each of the following residues:

a) $\mathrm{Res}_f(2i)$, $\quad f(z) = \frac{z^2}{(z-2i)(z+3)}$,

b) $\mathrm{Res}_f(-3)$, $\quad f(z) = \frac{z^2+1}{(z+3)^2 z}$,

c) $\mathrm{Res}_f(i+1)$, $\quad f(z) = \frac{e^z}{(z-i-1)^3}$.