Problem Set 6
Real Analysis II

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Ex 19.1 For $f, g \in L^2([0,1])$, let $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$. Let $H = \mathcal{C}[0,1]$ be the functions that are continuous on $[0,1]$. Is H a Hilbert space with respect to the norm defined in terms of the inner product $\langle\cdot,\cdot\rangle$? Justify your answer.

Proof. Let $f_n(x)$ be the function that is zero on $[0, 1/2 - 1/n]$, one on $[1/2 + 1/n]$ and linear on $[1/2 - 1/n, 1/2 + 1/n]$ where $n \geq 2$. I have drawn a picture to give a better intuition for the function below:

We see that $f_n(x)$ is clearly continuous and its square is bounded by $\chi_{[0,1]}$, which proves that it's in L^2. Now let $f = \chi_{[1/2,1]}$. We see that $|f_n - f|$ is given by the triangle drawn below.

If we call this function $T_n(x)$, we see then that $T_n(x)^2$ is bounded above by $T_n(x)$, since $T_n(x) < 1$ for all $x \in [0,1]$. We clearly see that the area of this function goes to zero, as the base shrinks. This proves that

$$||f_n - f||^2 = \langle f_n - f, f_n - f\rangle = \int_0^1 |f_n - f|^2\,dx = \int_0^1 T_n(x)^2\,dx \leq \int_0^1 T_n(x)\,dx$$

Since $T_n(x)$ is bounded by an integrable function (namely, $\chi_{[0,1]}$), by the DCT, we get that

$$\lim_{n\to\infty} ||f_n - f||^2 \le \lim_{n\to\infty} \int_0^1 T_n(x)\,dx = \int_0^1 \lim_{n\to\infty} T_n(x)\,dx = 0$$

which proves that $f_n \to f$ in L^2. However, $f = \chi_{[1/2,1]}$ is not continuous and is not equal to any continuous function almost everywhere. This proves that H is not complete, and thus not a Hilbert space. □

Ex 19.2 Suppose H is a Hilbert space with a countable basis. Suppose $||x_n|| \to ||x||$ as $n \to \infty$ and $\langle x_n, y\rangle \to \langle x, y\rangle$ as $n \to \infty$ for every $y \in H$. Prove that $||x_n - x|| \to 0$ as $n \to \infty$.

Proof. We see that

$$||x_n - x||^2 = \langle x_n - x, x_n - x\rangle = \langle x_n, x_n\rangle - \langle x_n, x\rangle - \overline{\langle x_n, x\rangle} + \langle x, x\rangle$$

which shows that

$$\begin{aligned}\lim_{n\to\infty} ||x_n - x||^2 &= \lim_{n\to\infty} ||x_n||^2 - \langle x_n, x\rangle - \overline{\langle x_n, x\rangle} + \langle x, x\rangle \\ &= \lim_{n\to\infty} ||x_n||^2 - \langle x_n, x\rangle - \overline{\langle x_n, x\rangle} + \langle x, x\rangle = ||x||^2 - \langle x, x\rangle - \overline{\langle x, x\rangle} + \langle x, x\rangle \\ &= \langle x, x\rangle - \langle x, x\rangle - \langle x, x\rangle + \langle x, x\rangle = 0\end{aligned}$$

Thus, $||x_n - x|| \to 0$ as $n \to \infty$. □

Ex 19.4 Give an example of a subspace M of a Hilbert space H such that $M \neq H$ but $M^\perp = \{0\}$.

Proof. Let $M \subseteq \ell^2$ be the collection of sequences for which all but finitely many elements are zero. This is clearly a subspace of ℓ^2. We see that $M \neq H$, as $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots) \in \ell^2 \setminus M$. Let $\{e_n\}$ be the standard basis vectors. We see then that $e_n \in M$ for each $n \in \mathbb{N}$. Let $x = (x_1, x_2, \dots) \in M^\perp$. Since $e_n \in M$, that means that $\langle x, e_n\rangle = \sum_{i=1}^\infty x_i \delta_n(i) = x_n = 0$ for all $n \in \mathbb{N}$. Thus, $x = 0$ all along. This proves that $M^\perp = \{0\}$. □

Ex 19.11 Suppose $\{e_n\}$ is an orthonormal basis for a separable Hilbert space and $\{f_n\}$ is an orthonormal set such that $\sum_n ||e_n - f_n|| < 1$. Prove that $\{f_n\}$ is a basis.

Proof. First, we note that

$$\sum_{n\in\mathbb{N}} ||e_n - f_n|| < 1 \implies \sum_{n\in\mathbb{N}} ||e_n - f_n||^2 < 1$$

Now let $0 \neq x \in H$ be such that $\langle x, f_n\rangle = 0$ for all $n \in \mathbb{N}$. Using the fact that e_n is a basis, we see that

$$\begin{aligned}||x||^2 = \sum_{n=1}^\infty |\langle x, e_n\rangle|^2 = \sum_{n=1}^\infty |\langle x, e_n\rangle - \langle x, f_n\rangle|^2 &= \sum_{n=1}^\infty |\langle x, e_n - f_n\rangle|^2 \\ &\le \sum_{n=1}^\infty ||x||^2 ||f_n - e_n||^2 = ||x||^2 \sum_{n=1}^\infty ||f_n - e_n||^2\end{aligned}$$

Dividing by $||x||^2$ (which we can do because $x \neq 0$) we get that

$$||f_n - e_n||^2 \geq 1$$

which is a contradiction. Thus, x must be zero. This proves that $\{f_n\}$ is basis. $\square$

Ex 19.12 Use Parseval's identity with the function $f(x) = x$ on $[0, 2\pi)$ to derive the formula

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

Proof. Section 19.4 shows that the set $\{\frac{e^{inx}}{\sqrt{2\pi}}\}_{n\in\mathbb{Z}}$ forms an orthonomal basis of $L^2([0, 2\pi])$. This means that Parseval's identity holds, that is

$$||f(x)||^2 = \sum_{n\in\mathbb{Z}} \left| \langle f(x), \frac{e^{inx}}{\sqrt{2\pi}} \rangle \right|^2$$

for all $f(x) \in L^2([0, 2\pi])$. Using $f(x) = x$, we see that

$$||x||^2 = \int_0^{2\pi} x^2\, dx = \left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{(2\pi)^3}{3} = \frac{8\pi^3}{3}$$

and that

$$\langle x, \frac{e^{inx}}{\sqrt{2\pi}} \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} x e^{-inx}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{-xe^{-inx}}{in} \Big|_0^{2\pi} + \frac{1}{in} \int_0^{2\pi} e^{-inx}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{(in)^2}[e^{in2\pi} - e^{in0}] - \frac{2\pi}{in} \right] = -\frac{\sqrt{2\pi}}{in} = \frac{\sqrt{2\pi}}{n} i$$

However, this doesn't work when $n = 0$, so we see that

$$\langle x, \frac{1}{\sqrt{2\pi}} \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} x\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{x^2}{2}\right]_0^{2\pi} = \frac{(2\pi)^2}{2\sqrt{2\pi}} = \pi\sqrt{2\pi}$$

Finally, this proves that

$$\frac{8\pi^3}{3} = |\pi\sqrt{2\pi}|^2 + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left| \frac{\sqrt{2\pi}}{n} i \right|^2 = 2\pi^3 + 2\sum_{n\in\mathbb{N}} \frac{2\pi}{n^2}$$

After dividing by 4π, we get that

$$\frac{2\pi^2}{3} = \frac{\pi^2}{2} + \sum_{n\in\mathbb{N}} \frac{1}{n^2}$$

which means that

$$\sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{4\pi^2 - 3\pi^2}{6} = \frac{\pi^2}{6}$$

as desired. $\square$