Problem Set 12
Graph Theory I

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Ex 4.2.3 Let G be the digraph with vertex set $\{1, 2, \ldots, 12\}$ in which $i \to j$ if and only if i divides j. Determine $\kappa(1, 12)$ and $\kappa'(1, 12)$.

Proof. Firstly, we note that $\kappa(1, 12)$ is undefined as 1 and 12 are adjacent in G. Next, we see that there are 5 edge-disjoint ways to get from 1 to 12, either through 2, 3, 4, 6 or directly to 12. This means that we need to delete at least five edges in order to separate 1 and 12. Deleting the five edges that go into 12 (from 1, 2, 3, 4, and 6) will certainly separate 1 and 12, and thus $\kappa'(1, 12) = 5$. □

Ex 4.2.4 Prove or disprove: If P is a u, v-path in a 2-connected graph G, then there is a u, v-path Q that is internally disjoint from P.

Proof. False. Take the graph K_4 with vertices $\{a, b, c, d\}$ and remove the edge ab. We will call this graph G. We see that G is 2-connected as it is connected and has no cut-vertex. Additionally if one considers the path $P = acdb$, then it is impossible for another path it be internally disjoint from P, as P goes through all vertices and since $d(a, b) = 2$, every path from a to b must have an internal vertex. □

Ex 4.2.5 Let G be a simple graph, and let $H(G)$ be the graph with vertex set $V(G)$ such that $uv \in E(H)$ if and only if u, v appear on a common cycle in G. Characterize the graphs G such that H is a clique.

Proof. H is a clique $\iff$ for all $u, v \in V(G)$, there is a cycle containing u and v $\iff$ G is 2-connected. □

Ex 4.2.8 Prove that a simple graph G is 2-connected if and only if for every triple (x, y, z) of distinct vertices, G has an x, z-path through y.

Proof. Assume that G is 2-connected. By Theorem 4.2.23 (Fan Lemma), for any $v \in V(G)$ and set of vertices $U = \{x, z\}$, there is a y, U-fan. If we combine the two paths of the fan, we create an x, z-path through y as desired.

Conversely, we see if we have such a condition, then G must be connected. Suppose that G had a cut vertex at x. Let $y, z \in V(G)$ be arbitrary vertices distinct from each other and x. Then, by the condition, there is a x, z-path through y. This means that even in $G - x$, there is still a path from y to z, which proves that x is not a cut-vertex. This is a contradiction, proving that G has no cut-vertex and is thus 2-connected. $\square$

Ex 4.2.9 Prove that a graph G with at least four vertices is 2-connected if and only if for every pair X, Y of disjoint vertex subsets with $|X|, |Y| \geq 2$, there exist two completely disjoint paths P_1, P_2 in G such that each has an endpoint in X and an endpoint in Y and no internal vertex in X or Y.

Proof. ($\implies$) Let G be 2-connected graph with at least four vertices. Let X, Y be disjoint vertex subsets with $|X|, |Y| \geq 2$. If there are no edges within X, then we add an edge. We do the same for Y and call the resulting graph G'. We note that adding edges doesn't not change 2-connectedness, which means that G' is still 2-connected. By Theorem 4.2.4, this is equivalent to $\delta(G') \geq 1$ (unimportant to the proof) and that every pair of edges in G' lie on a common cycle. By taking an edge in X and an edge in Y, this means they lie on some common cycle. By taking this cycle and removing the edges in X and Y, we create two disjoint paths from X to Y in G.

($\impliedby$) Suppose that the condition holds. If G is disconnected, then we can take two vertices x_1 and y_1 in different components of G. If possible, we choose another vertex y_2 in the same component as y_1 and choose x_2 arbitrarily. Thus, there can be no path from x_2 to either y_1 or y_2, which is a contradiction to the condition. If it's not possible to choose y_2 in the same component as y_1, then that means that y_1 is an isolated vertex. Thus, we can choose x_2 and y_2 arbitrarily and see that it's impossible for there to be a path from y_1 to either x_1 or x_2, again a contradiction to the condition. This proves that G is connected and thus that $\delta(G) \geq 1$.

Let $e_1, e_2 \in E(G)$ be arbitrary. Then, if we take X to be the the set of vertices incident to e_1 and Y to be the set of vertices incident to e_2, then there are two completely disjoint paths from X to Y. By adding the edges e_1 and e_2 to these paths, we form a cycle containing e_1 and e_2. Since these edges were arbitrary, we have that every pair of edges lies on a common cycle. By Theorem 4.2.4, this combined with the fact that $\delta(G) \geq 1$ is equivalent to G being 2-connected. $\square$