Problem Set 4
Homological Algebra

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Ex 1 Prove that an arbitrary direct product of injective modules is injective.

Proof. Let $\{A_i\}_{i\in I}$ be an arbitrary collection of injective modules and let $A = \prod_i A_i$ with $\pi_i : A \to A_i$ as the projection maps. Suppose we have the following diagram of R-modules:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "g"] & Y \\ & A & \end{tikzcd}$$

We see then that this means we have the following diagram for each A_i:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\pi_i g"] & Y \\ & A_i & \end{tikzcd}$$

Since each A_i is injective, this means there's a unique map $h_i : Y \to A_i$ such that

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\pi_i g"] & Y \arrow[dl, "h_i"] \\ & A_i & \end{tikzcd}$$

By the universal proprety of direct products, this means that there exists a unique $h : Y \to A$ such that $\pi_i h = h_i$. Since we have that $\pi_i g = h_i f = \pi_i h f$, we see that g and hf agree on every projection, proving that $g = hf$. Thus, we have that

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[dl, "h"] \\ & A & \end{tikzcd}$$

This proves that A is injective. $\square$

Ex 2 Let S be the arbitrary sum of injective R-modules, $j : A \to B$ an injective and $f : A \to S$ an arbitrary R-module homomorphism. If A is finitely-generated, show that there exists an extension $F : B \to S$ with $Fj = f$.

Proof. Let $S = \oplus_i S_i$. We call from Ex 7 on the previous homework that there is a canonical embedding from $\oplus_i \operatorname{Hom}_R(A, S_i)$ into $\operatorname{Hom}(A, S)$. Since A is finitely-generated, the image of maps from A to S must be contained in the direct sum of finitely many S_i. Since Hom commutes with finite direct sums, we see that the embedding is actually an isomorphism.

Thus, we can write $f \in \operatorname{Hom}(A, \oplus_i S_i)$ as $\oplus_i f_i \in \oplus_i \operatorname{Hom}(A, S_i)$. This gives us the following diagram for each S_i:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "j"] \arrow[d, "f_i"] & B \\ & S_i & \end{tikzcd}$$

Since S_i is an injective module, there means there exists a unique $h_i : B \to S_i$ such that $h_i g = f_i$. Again, as $\oplus_i \operatorname{Hom}_R(B, S_i)$ embeds into $\operatorname{Hom}_R(B, S)$, this means that we can form the function $\oplus_i h_i : B \to S$. Since $(\oplus h_i)j = \oplus(h_i j) = \oplus f_i = f$, we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "j"] \arrow[d, "f"] & B \arrow[dl, "\oplus_i h_i"] \\ & S & \end{tikzcd}$$

This proves that $F = \oplus_i h_i$ is exactly the extension we want. $\square$

Ex 3 Formulate and prove the dual of Lemma 2.3.5 from class using injective instead of projective modules.

Proof. The dual of Lemma 2.3.5 is that for any short exact sequence of R-modules $A' \to A \to A''$, there exist injective modules I', I, and I'' such that the following diagram commutes:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A' \arrow[r, "\phi"] \arrow[d, "\varepsilon'"] & A \arrow[r, "\psi"] \arrow[d, "\varepsilon"] & A'' \arrow[r] \arrow[d, "\varepsilon''"] & 0 \\ 0 \arrow[r] & I' \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & I'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{coker}(\varepsilon') \arrow[r] \arrow[d] & \operatorname{coker}(\varepsilon) \arrow[r] \arrow[d] & \operatorname{coker}(\varepsilon'') \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

Now, we first note that since injective resolutions exist, we can find injective modules I' and I'' such that $A' \xhookrightarrow{\varepsilon'} I'$ and $A'' \xhookrightarrow{\varepsilon''} I''$. We then form an exact sequence $I' \to I' \oplus I'' \to I''$ as well. Note that $I' \times I''$ is injective as it's the direct product of two injective modules. Now since I' is an injective module and f is an injective map, there exists a map $\gamma : A \to I'$ such that $\varepsilon' = \gamma\phi$. This gives us the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r, "\phi"] \arrow[d, "\varepsilon'"] & A \arrow[r, "\psi"] \arrow[dl, "\gamma"] & A'' \arrow[r] \arrow[d, "\varepsilon''"] & 0 \\
0 \arrow[r] & I' \arrow[r, "\iota"] & I' \times I'' \arrow[r, "\pi"] & I'' \arrow[r] & 0
\end{tikzcd}$$

Now we define the map $\varepsilon : A \to I' \times I''$ by $\varepsilon = \gamma \times \varepsilon''\psi$, that is $\varepsilon(a) = (\gamma(a), \varepsilon''\psi(a))$. We will now prove several things about ε

i) The map ε commutes with the rest of our diagram as

$$\varepsilon\phi = (\gamma \times \varepsilon''\psi)\phi = \gamma\phi \times \varepsilon''\psi\phi = \gamma\phi \times \varepsilon''0 = \gamma\phi \times 0 = \iota\gamma\phi = \iota\varepsilon'$$

and

$$\pi\varepsilon = \pi(\gamma \times \varepsilon''\psi) = \varepsilon''\psi.$$

ii) We see that

$$\ker(\varepsilon) = \ker(\gamma \times \varepsilon''\psi) = \ker(\gamma) \cap \ker(\varepsilon''\psi) = \ker(\gamma) \cap \ker(\psi) = \ker(\gamma) \cap \operatorname{Im}(\phi).$$

Now if $y \in \ker(\gamma) \cap \operatorname{Im}(\phi)$, then that means that $\phi(x) = y$ for some $x \in A'$, we obtain then that

$$0 = \gamma(y) = \gamma(\phi(x)) = \varepsilon'(x).$$

As ε' is injective, $x = 0$ and so $y = \phi(0) = 0$. This proves that $\ker(\varepsilon) = \ker(\gamma) \cap \operatorname{Im}(\phi) = 0$, i.e. ε is injective.

Using this facts, we now arrive at the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r, "\phi"] \arrow[d, "\varepsilon'"] & A \arrow[r, "\psi"] \arrow[d, "\varepsilon"] & A'' \arrow[r] \arrow[d, "\varepsilon''"] & 0 \\
0 \arrow[r] & I' \arrow[r, "\iota"] & I \times I'' \arrow[r, "\pi"] & I'' \arrow[r] & 0
\end{tikzcd}$$

We can add in the bottom row using the canonical projection from the injective modules onto their cokernels. We can then use the snake lemma on the top two rows to obtain maps inbetween the cokernels that commute with the rest of the diagram. Finally, since the columns and the top two rows are all exact, we can apply the Nine Lemma, to show that the bottom row must be exact as well, giving us our desired result. $\square$

Ex 4 Let $n > 1$ be a natural number, $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, and C the countably infinite direct sum of copies of $\mathbb{Z}_n$. Set $A = \mathbb{Z}$, considered as a $\mathbb{Z}$-module, and let B be the direct sum of A and C. Construct, including all the necessary arguments, a short exact sequence of $\mathbb{Z}$-modules $0 \to A \to B \to C \to 0$, which is **not** split.

Proof. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus (\mathbb{Z}_n)^{\mathbb{N}} \xrightarrow{g} (\mathbb{Z}_n)^{\mathbb{N}} \longrightarrow 0$$

where $f(x) = (nx, 0, 0, \dots)$ and $g(x, m_1, m_2, \dots) = (x + n\mathbb{Z}, m_1, m_2, \dots)$. Suppose that this short exact sequence splits. Then there would be a $\sigma : (\mathbb{Z}_n)^{\mathbb{N}} \to \mathbb{Z} \oplus (\mathbb{Z}_n)^{\mathbb{N}}$ such that $g\sigma = \mathrm{Id}_{(\mathbb{Z}_n)^{\mathbb{N}}}$. However, we see that $\pi_1\sigma : (\mathbb{Z}_n)^{\mathbb{N}} \to \mathbb{Z}$ must be the zero map, as every element of $(\mathbb{Z}_n)^{\mathbb{N}}$ has finite order but all the non-zero elements of $\mathbb{Z}$ have infinite order. This means that $(1, 0, 0, \dots)$ can't be in the image of $g\sigma$, which is a contradiction. Thus, the short exact sequence cannot be split. □

Ex 5

a) Prove that an arbitrary direct sum of projective resolutions is again projective and use this to show $\mathrm{Ext}^n_R(\oplus_{i\in I} A_i, B) \simeq \prod_{i\in I} \mathrm{Ext}^n_R(A, B)$ for any collection of R-modules $\{A_i\}_{i\in I}$.

b) Prove that an arbitrary direct product of injective resolutions is an injective resolution and use this to show $\mathrm{Ext}^n_R(A, \prod_{j\in J} B_j) \simeq \prod_{j\in J} \mathrm{Ext}^n_R(A, B_j)$ for any collection of R-modules $\{B_j\}_{j\in J}$.

Proof.

a) Consider a a collection of modules A_i with projective resolutions $\underline{P_i} \to A_i$. We can then take the direct sum of these chains to obtain a chain $\oplus_i \underline{P_i} \to \oplus_i A_i$. Since arbitrary direct sums of projective modules is projective, this chain is actually a projective resolution of $\oplus_i A_i$. We see then that

$$\begin{aligned}\mathrm{Ext}^n_R(\oplus_i A_i, B) &\simeq H_n(\mathrm{Hom}(\oplus_i \underline{P_i}, B)) \simeq H_n(\textstyle\prod_i \mathrm{Hom}(\underline{P_i}, B)) \\ &\simeq \textstyle\prod_i H_n(\mathrm{Hom}(\underline{P_i}, B)) = \prod_i \mathrm{Ext}^n_R(A_i, B)\end{aligned}$$

as desired.

b) Consider a a collection of modules A_j with injective resolutions $A_j \to \underline{Q_j}$. We can then take the direct product of these chains to obtain a chain $\prod_j A_j \to \prod_j \underline{Q_j}$. Since arbitrary direct products of injective modules is injective, this chain is actually a projective injective of $\prod_j A_j$. We see then that

$$\begin{aligned}\mathrm{Ext}^n_R(A, \textstyle\prod_j B_j) &\simeq H_n(\mathrm{Hom}(A, \textstyle\prod_j \underline{Q_j})) \simeq H_n(\textstyle\prod_j \mathrm{Hom}(A, \underline{Q_j})) \\ &\simeq \textstyle\prod_j H_n(\mathrm{Hom}(A, \underline{Q_j})) = \prod_j \mathrm{Ext}^n_R(A, B_j)\end{aligned}$$

as desired. □

Ex 6 Prove that $\mathrm{Tor}^R_n(A, \oplus_{j\in J} B_j) \simeq \oplus_{j\in J} \mathrm{Tor}^R_n(A, B_j)$ for any collection of R-modules $\{B_j\}_{j\in J}$.

Proof. Let $\underline{P}$ be a projection resolution of A. Recall that the homology functor is additive and we see that

$$\mathrm{Tor}_n(A, \oplus_{j\in J} B_j) = H_n(\underline{P} \otimes (\oplus_j B_j)) = H_n(\oplus_j (\underline{P} \otimes B_j)) = \oplus_j H_n(\underline{P} \otimes B_j) = \oplus_j \mathrm{Tor}_n(A, B_j)$$

as desired. $\square$

Ex 7 Let $R = k[x, y]$ where k is a field and let I be the ideal (x, y) in R.

a) Let $\alpha : R \to R^2$ be the map $\alpha(r) = (yr, -xr)$ and let $\beta : R^2 \to R$ be the map $\beta(r_1, r_2) = r_1x + r_2y$. Show that

$$0 \longrightarrow R \xrightarrow{\alpha} R^2 \xrightarrow{\beta} R \xrightarrow{\pi} k \longrightarrow 0$$

where the map $\pi : R \to R/I = k$ is the canonical projection, gives a free resolution of k as an R-module.

b) Use the resolution in (a) to show that $\mathrm{Tor}_2^R(k, k) \simeq k$.

c) Prove that $\mathrm{Tor}_1^R(k, I) \simeq k$. [Use the long exact sequence]

d) Conclude that the torsion free R-module I is not flat.

Proof.

a) It is easy to see that R and R^2 are free R-modules and that the given maps are R-module homomorphisms, so we need only to prove that the given sequence is exact at each point. First, if $\alpha(r) = (yr, -xr) = (0, 0)$, then it must be that both $yr = 0$, meaning $r = 0$. This proves that α is injective and that the sequence is exact at the first R.

   We see that $\beta(\alpha(r)) = \beta(yr, -xr) = yrx - xry = 0$, so $\mathrm{Im}(\alpha) \subseteq \ker(\beta)$. Now, if r_1, r_2 are such that $\beta(r_1, r_2) = r_1x + r_2y = 0$, then we see that $r_1x = -r_2y$. This means that $r_1/y = -r_2/x$. Now, if we let $r = r_1/y = -r_2/x$, then we see that $\alpha(r) = (yr, -xr) = (r_1, r_2)$, and so $\ker(\beta) \subseteq \mathrm{Im}(\alpha)$. This proves that the sequence is exact at R^2.

   We see that $\pi(\beta(r_1, r_2)) = \pi(r_1x + r_2y) = r_1x + r_2y + (x, y) = (x, y)$, so $\mathrm{Im}(\beta) \subseteq \ker(\pi)$. Now, if $r \in \ker(\pi)$, then $\pi(r) = r + (x, y) = (x, y)$, which means $r \in (x, y)$. Since R is commutative, elements in (x, y) have the form $r_1x + r_2y$ for some $r_1, r_2 \in R$. Thus, $\ker(\pi) \subseteq \mathrm{Im}(\beta)$, and so the sequence is exact at the second R.

   Finally, π is surjective because it is a projection map, and so we see that the given sequence is in fact an exact sequence. This proves that it is a free resolution of k.

b) [Incomplete]

$\square$

Ex 8 If $F : \mathcal{C} \to \mathcal{D}$ is an additive covariant functor between two module categories $\mathcal{C}$ and $\mathcal{D}$, prove that $F(0) = 0$, where 0 denotes the zero object in each category.

Proof. Let $0_M : M \to 0$ be the unique zero map for a given object M. If $M = 0$, this is the unique map from the zero object to itself. Since the identity map on the zero object satisfies this, it must be that $\mathrm{Id}_M = 0_M$. Additionally, if M is not the zero object, it is obvious that Id_M cannot be the zero map as its domain is not the zero object. Thus, we have that $M = 0$ if and only if $0_M = \mathrm{Id}_M$.

Now, since F is a functor, it must be that $F(\mathrm{Id}_M) = \mathrm{Id}_{FM}$. Furthermore, as F is additive, $F : \mathrm{Hom}(M, M) \to \mathrm{Hom}(FM, FM)$ is a homomorphism of abelian groups, meaning $F(0_M) = 0_{FM}$. This means that if $M = 0$, we have that

$$\mathrm{Id}_{FM} = F(\mathrm{Id}_M) = F(0_M) = 0_{FM},$$

which means that FM must be the zero object as well. $\square$