Problem Set 1
Real Analysis I

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Exercise 1.

a) For sequences $(x_n), (y_n)$ of real numbers, prove that

$$\limsup\{x_n\} + \liminf\{y_n\} \leq \limsup\{x_n + y_n\} \leq \limsup\{x_n\} + \limsup\{y_n\}$$

b) Give a specific example where both inequalities are strict.

Proof.

a) If $\alpha = \limsup\{x_n\}$, then for an $\epsilon \geq 0$, there are an infinite number of j's where $x_j \leq \alpha - \frac{\epsilon}{2}$. If there were only a finite number of such j's, then let n be the last one. This would mean that $\sup_{k\geq n}\{x_k\} \leq \alpha - \frac{\epsilon}{2}$, which would mean that $\limsup\{x_n\} \leq \alpha - \frac{\epsilon}{2} < \alpha$. This is a contraction. If $\beta = \liminf\{y_n\}$, then for any $\epsilon > 0$, there is an N, such that for all $n \geq N$

$$y_n \geq \inf_{k\geq n}\{y_k\} \geq \beta - \frac{\epsilon}{2}$$

This means there are an infinite number of n where $x_n > \alpha - \frac{\epsilon}{2}$ and $y_n \geq \beta - \frac{\epsilon}{2}$. Thus, for $n \geq N$, it means that $x_n + y_n \geq \alpha + \beta - \epsilon$ for inifinitely many n. This means that $\sup\{x_n + y_n\} \geq \alpha + \beta - \epsilon$. Since ϵ was arbitrary, this means that $\limsup\{x_n + y_n\} \geq \limsup\{x_n\} + \liminf\{y_n\}$. This proves the first inequality.

Let $x_j \in \{x_k\}_{k\geq n}$ and $y_j \in \{y_k\}_{k\geq n}$. Then, this means that $x_j \leq \sup_{k\geq n}\{x_k\}$ and that $y_j \leq \sup_{k\geq n}\{x_k\}$. Thus, $x_j + y_j \leq \sup_{k\geq n}\{x_k\} + \sup_{k\geq n}\{y_k\}$. If you take the sup off all possible j's of both sides, you get $\sup_{j\geq n}\{x_j + y_j\} \leq \sup_{j\geq n}\{\sup_{k\geq n}\{x_k\} + \sup_{k\geq n}\{y_k\}\}$. The RHS was already "supped" and therefore was just a constant, so the additional sup doesn't do anything. Thus $\sup_{j\geq n}\{x_j + y_j\} \leq \sup_{k\geq n}\{x_k\} + \sup_{k\geq n}\{y_k\}$. Taking the limit as $n \to \infty$, one gets the second inequality.

b) Let $y = \begin{cases} n & \text{if } n \text{ is odd} \\ -n & \text{if } n \text{ is even} \end{cases}$ and $x_n = \begin{cases} -n & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$.

Computing the the limsups and liminfs, we get

$\limsup\{x_n\} + \liminf\{y_n\} = 1 - \infty = \infty$

$\limsup\{x_n + y_n\} = 0$ as $x_n + y_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1-n & \text{if } n \text{ is odd} \end{cases}$

$\limsup\{x_n\} + \limsup\{y_n\} = 1 + \infty = \infty$

As can be seen, the inequalities are strict for these x_n and y_n.

□

Exercise 2. a) Let $p > 1$ be an integer and x a real number with $0 < x < 1$. Show that there is a sequence of integers (a_n) with $0 \le a_n < p$ such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

b) Show that the above sequence a_n is unique except when x is of the form $\frac{q}{p^n}$ for some integer q and that, in this case, there are exactly two such sequences.

c) Show, conversely, that if $a_n$is any sequence of integers with $0 \le a_n < p$, the series

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

converges to a number x with $0 \le x \le 1$.

Proof. a) (This proof needs to be cleaned up.) Look at the partial sum $S_k = \sum_{n=1}^{k} \frac{a_n}{p^n}$. For a_1, let it be the max number such that $\frac{a_1}{p} \le x$ and that $a_1 < p$. This definitely exists as if $a_1 = 0$, then $\frac{a_1}{p} = 0 \le x$. Now let a_n be the max number such that $S_{n-1} + \frac{a_n}{p^n} \le x$ and that $a_n < p$. This definitely exists as if $a_n = 0$, then $S_{n-1} + \frac{0}{p^n} = S_{n-1} \le x$ and each S_k for $k < n$ has already been constructed to be less than x. By part (c) of this problem, every such sequence must converge, and by construction, this sequence must converge to something less than or equal to x. Suppose it converged to $x - \epsilon$ for $\epsilon > 0$. Look at $\frac{1}{\epsilon} > 0$. There must exist a p^k for $k \in \mathbb{Z}^+$, such that $\frac{1}{\epsilon} < p^k$ (as $p > 1$). Let k be the least such integer. This means that $\epsilon > \frac{1}{p^k}$. We see that by $S_k + \frac{1}{p_k}$, we've altered the sequence a_n by adding 1 to a_k. If $a_k = p - 1$, then add 1 to a_{k+1}. If $a_j = p - 1$ for all $j \ge k$, then by part (b), this means that (a_n) converges to the same thing as a sequence which is 0 for all $j \ge k$. Take this new sequence to be the new a_n and add 1 to its kth position. Let l be whatever position we eventually added the 1, and (b_n) be the sequence that we've obtained by adding 1 to a_l. Note that $l \ge k$. Then (b_n) converges to $x - \epsilon + \frac{1}{p^l} > x - \epsilon$. Also, we know that $x - \epsilon + \frac{1}{p^l} \le x - \epsilon + \frac{1}{p^k} < x$, as $\epsilon > \frac{1}{p^k}$. This is a contradiction, as (a_n) was constructed to be the greatest sequence such that it converged to something less than or equal to x. This must mean that a_n is converged to x all along.

b) (This proof is also partially incomplete.) Essentially, let (a_n) and (b_n) be sequences that are not equal, but converge to the same x. Then, let k be the first position that they differ. If they differ by more than 1 in the kth position, then they differ by $\frac{2}{p^k}$ so far, which is more than the rest of the tail combined can be. Thus, they converge to different things, a contradiction. If they only differ by 1, then look at the $k+1$th place. Say that (b_n) was the sequence that was higher in the kth position, without loss of generality. If $b_{k+1} \neq 0$ and $a_{k+1} \neq p-1$, then their difference at this position is at most $\frac{p-2}{p^{k+1}}$. Their difference so far would be at least $\frac{1}{p^k} - \frac{p-2}{p^{k+1}} = \frac{p-p+2}{p^{k+1}} = \frac{2}{p^{k+1}}$, and as mentioned before, this is too great of a distance for the rest of the tail after the $k+1$th position to overcome. Repeat this process for the proceding positions. (TODO: Next, show that if this does occur, then they converge to the same thing.) Since this only happens to x when one of its representations is all 0's after a certain position, this means that $x = S_n$ for some $n \in \mathbb{N}$, as the rest of the sequence contributes nothing to the partial sum. We see that $S_k = \frac{a_k}{p^k} + \frac{a_{k-1}}{p^{k-1}} \cdots + \frac{a_1}{p} = \frac{a_k + pa_{k-1} \cdots + p^{k-1}a_1}{p^k}$. Thus, $x = \frac{q}{p^k}$ for some integers q and k

c) Firstly, since $\frac{a_n}{p^n} \geq 0$ for all n, we see that the partial sums $S_k = \sum_{n=1}^{k} \frac{a_n}{p^n}$ are monotonically increasing. Thus, we need only to prove that it's bounded by 0 and 1. We see that $0 \leq \frac{a_n}{p^n} \leq \frac{p-1}{p^n}$. Thus, by the comparison test, $\sum_{n=1}^{\infty} 0 = 0 \leq \sum_{n=1}^{\infty} \frac{a_n}{p^n} \leq \sum_{n=1}^{\infty} \frac{p-1}{p^n}$. We see that$\sum_{n=1}^{\infty} \frac{p-1}{p^n} = \sum_{n=1}^{\infty} \left(\frac{1}{p^{n-1}} - \frac{1}{p^n}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{p^n} - \frac{1}{p^n}\right) + \frac{1}{p^0} = 1$. Thus $\sum_{n=1}^{\infty} \frac{a_n}{p^n} \leq 1$. Since this means that S_k is increasing and bounded, this means that S_k converges to a number between 0 and 1.

□

Exercise 3. Let f be a mapping from a set X to the set of subsets of X. By considering the set $E = \{x \in X : x \notin f(x)\}$, show that there is a subset of X which is not in the range of f.

Proof. Let $f : X \to P(X)$ and let $E = \left\{x \in X \;\middle|\; x \notin f(x)\right\}$. Assume that f is surjective. Since $E \in P(X)$, then there exists an x' such that $f(x') = E$. Now, x' must be either in E or not in E. Say $x' \in E$, then $x' \in E = f(x')$. However, since $x' \in E$, this means by definition of E that $x' \notin f(x')$. Thus, we have a contradiction. Now say $x' \notin E$, then $x' \notin E = f(x')$. But, by definition of E, this means that $x' \in E$. This is also a contradiction. Thus, there is no $x \in X$ such that $f(x') = E$, which means f is not surjective. This shows there's a subset of X which is not in the range of f. □

Exercise 4. If $S \subseteq X$ is uncountable and $A \subseteq X$ is countable, show that $S \cap A^c$ is uncountable.

Proof. Assume that $S \cap A^c$ is countable. Then $(S \cap A^c) \cup A$ is the union of two countable sets, and is thus countable. By distributivity, $(S \cap A^c) \cup A = (S \cup A) \cap (A \cup A^c) = (S \cup A) \cap X = S \cup A$. $S \cup A$ is clearly uncountable, as $S \subseteq S \cup A$. This is a contradiction. Thus, $S \cap A^C$ must be uncountable. □

Exercise 5. a) Is the set of rationals open or closed in the set of real numbers?

b) Which sets of real numbers are both open and closed?

Proof. a) Every open interval on the real line contains both rationals and irrationals. Therefore, every open neighborhood of $q \in \mathbb{Q}$ contains irrationals. This shows that $\mathbb{Q}$ is not open. Similarly, if $\mathbb{Q}$ were closed, then $\mathbb{R} \setminus \mathbb{Q}$ would be open. But, every open neighborhood of $i \in \mathbb{R} \setminus \mathbb{Q}$ contains a rational number. Therefore, $\mathbb{R} \setminus \mathbb{Q}$ is not open, and $\mathbb{Q}$ is not closed. This means that $\mathbb{Q}$ is neither open nor closed.

b) $\varnothing$ is trivally open and closed. Similarly, it's complement, $\mathbb{R}$ is then open and closed. Assume that there exists an additional open and closed set $\varnothing \neq A \neq \mathbb{R}$. Then the complement A^c would also be open and closed, and not equal to $\varnothing$ or $\mathbb{R}$. Now fix $a \in A$ and $b \in A^c$. Without loss of generality (since $(A^c)^c = A$), let $a < b$. Now let $C = \{x \in \mathbb{R} \bigm| [a, x] \subseteq A\}$. C is nonempty, as $a \in C$. Also, C is bound above by b, as $b \notin A$. Thus, C has a least upper bound. Let's call it α.

Say $\alpha \in A$, then, since A is open, there'd be a ball of radius $r > 0$, such that $(\alpha - r, \alpha + r) \subseteq A$. Then $[\alpha, \alpha + \frac{r}{2}] \subseteq (\alpha - r, \alpha + r) \subseteq A$. This means $[a, \alpha] \cup \left[\alpha, \alpha + \frac{r}{2}\right] = \left[a, \alpha + \frac{r}{2}\right] \subseteq A$. This means that $\alpha + \frac{r}{2} \in C$, a contradiction, as α was the least upper bound of C. This means that $\alpha \notin A$.

Say $\alpha \in A^c$, then, since A^c is open, there'd be a ball of radius $r > 0$, such that $(\alpha - r, \alpha + r) \subseteq A^c$. Then $[\alpha - \frac{r}{2}, \alpha] \subseteq (\alpha - r, \alpha + r) \subseteq A^c$. This means that $\alpha - \frac{r}{2} \notin A$, which means $\alpha - \frac{r}{2} \notin C$. However $\alpha - \frac{r}{2} < \alpha$ and α is supposed to be the least upper bound. This is a contradiction, thus $\alpha \notin A^c$. This means that $\alpha \in A$. From the last paragraph, we proved that $\alpha \notin A$. This contradiction shows that there is no such set A. This means that $\varnothing$ and $\mathbb{R}$ are the only two open and closed sets in $\mathbb{R}$.

□

Exercise 6. Prove that a set X is infinite if and only if there is a proper subset of X of the same cardinality as X.

Proof. Suppose X is infinite. Then create an injective function $\phi : \mathbb{N} \to X$. It's easy to see such a function exists, as you can choose an element of X for each $n \in \mathbb{N}$, and you'll never run out of elements, as that would mean that there was a bijection between $\{1 \ldots n\}$ and X. From this injective function let $\psi : X \to X$ where

$$\psi = \begin{cases} \phi\,(n+1) & x = \phi\,(n) \text{ for some } n \in \mathbb{N} \\ x & x \notin \operatorname{Im}(\phi) \end{cases}$$

One can see that $\phi\,(0) \notin \operatorname{Im}(\psi)$, so $\operatorname{Im}(\psi) \subsetneq X$, and also that ψ is injective, as ϕ was injective. Combined with the inclusion map from $\operatorname{Im}(\psi)$ to X, which is necessarily injective, we see that there's a bijection between X and $\operatorname{Im}(\psi) \subsetneq X$. This assumes the Axiom of Choice.

Let X be a set with a proper subset A. This means there exists an $x \in X$, such that $x \notin A$. Let's say they have the same cardinality. This means that there's a bijection $\phi : X \to A$. Construct the function $\psi : \mathbb{N} \to X$ as follows: $\psi\,(0) = a$ and $\psi\,(i+1) = \phi(\psi(i))$ for all $0 \neq n \in \mathbb{N}$. (Tip: It might be more intuitive to view it as a sequence.) Since ϕ is injective and $a \notin \mathrm{Im}(\phi)$, we see that ψ is injective. We also see that through the inclusion map, $\{1 \ldots n\}$ is injective to $\mathbb{N}$. If the cardinality of X were finite, then there'd be a bijection $f : X \to \{1 \ldots n\}$ for some $n \in \mathbb{N}$. Thus, the composition, $f \circ \psi : \mathbb{N} \to \{1 \ldots n\}$ would be injective, and thus there's a bijection between $\mathbb{N}$ and $\{1 \ldots n\}$. This is a contradiction, which means X must be infinite. $\square$