Problem Set 5
Algebra III

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Ex 1. Let R be a quaternion algebra over a field k of characteristic different from 2. Prove that R is either a k-division algebra or isomorphic (as k-algebras) to $M_2(k)$.

Proof. From the first homework, we proved that R is a simple k-algebra of dimension 4. If we let I be a left ideal of R, then not only is I an R-module, but I is also a k-module via the inclusion map of k into R. This means that I is a k-vector space and thus a k-vector subspace of R. As R is a finite dimensional k-vector space, we have that any descending chain of ideals (which are vector subspaces) is eventually constant. This proves that R is semi-simple. Thus as R is simple and semi-simple, so by Artin-Wedderburn, we have that $R \simeq M_n(D)$ for some $n \in \mathbb{N}$ and division algebra D. If $n = 2$, then $\dim(D) = 1$, so $R \simeq M_2(D) \simeq M_2(k)$. Otherwise, $n = 1$ and $\dim(D) = 4$. This means $R \simeq D$, proving that R is a k-division algebra. □

Ex 2. Let D be a skew-field, n a natural number, $R = M_n(D)$ and M a finitely-generated R-module. Note that this naturally provides M with the structure of a D-vector space.

a) What are the possible values for the dimension $\dim_D(M)$?

b) Prove that necessary and sufficient condition for M to be a simple R-module.

Proof.

a) We note that $R = M_n(D)$ is semisimple, so all R-modules are semisimple. This means that $M = \oplus_{i \leq k} M_i$ where M_i are simple R-modules. But we know that any simple R-module is isomorphic to the simple R-modules that appear in the decomposition of $M_n(D)$. Since $M_n(D) = \oplus_{i \leq n} D^n$, we have that any simple R-module is isomorphic to D^n so $M_i \simeq D^n$ for each $i \leq k$. This proves that

$$\dim_D(M) = \sum_{i \leq k} \dim_D(M_i) = \sum_{i \leq k} \dim_D(D^n) = \sum_{i \leq k} n = kn.$$

This proves that the possible values for $\dim_D(M)$ are kn for some $k \geq 0$.

b) We note that if $\dim_D(M) = n$, then $M \simeq \oplus_{1 \leq i \leq 1} D^n = D^n$, which is a simple R-module. If $\dim_D(M) \neq n$, then either $\dim_D(M) = 0$, so $M = \{0\}$ and is not simple or $\dim_D(M) = kn$ for some $k > 1$, in which case $M \simeq \oplus_{i \leq k} D^n$, which is not simple. Thus, M is a simple R-module if and only if $\dim_D(M) = n$. □

Ex 3. For a ring R, show that any nil-ideal I is contained in $J(R)$.

Proof. Let $x \in I$. This means that x is nilpotent, so $x^n = 1$ for some n. We recall that $x \in J(R)$ if and only if $1 - rx$ has a left inverse for any $r \in R$. Now let $r \in R$. Since $rx \in I$, we see that $(rx)^n = 0$ for some $n \in \mathbb{N}$. This means that

$$(1 + rx + \cdots + (rx)^{n-1})(1 - rx) = 1 - (rx)^n = 1.$$

As $1 - rx$ has a left inverse, we have that $x \in J(R)$. Since $x \in I$ was arbitary, $I \subseteq J(R)$. $\square$

Ex 4. Determine the Jacobson radical of $J(\mathbb{Z}_n)$.

Proof. Let $\pi : \mathbb{Z} \to \mathbb{Z}_n$ be the canonical projection map. By the Correspondence Theorem, π is an inclusion-preserving bijection between the ideals of $\mathbb{Z}$ that contain $n\mathbb{Z}$ and the ideals of $\mathbb{Z}_n$. In particular, this means that π is a bijection between the maximal ideals $\mathbb{Z}$ that contain $n\mathbb{Z}$ and the maximal ideals of $\mathbb{Z}_n$. Since the maximum ideals of $\mathbb{Z}$ are $p\mathbb{Z}$ where p is prime, if we let $n = p_1^{e_1} \cdots p_n^{e_k}$ then the maximal ideals containing $n\mathbb{Z}$ are $p\mathbb{Z}$ where p divides n. Thus, the maximum ideals of $\mathbb{Z}_n$ are $\pi(p\mathbb{Z}) = p\mathbb{Z}/n\mathbb{Z}$ where p divides n. From this we see that

$$J(\mathbb{Z}_n) = \bigcap_{\substack{I \subseteq \mathbb{Z}_n \\ I \text{ max}}} I = \bigcap_{p|n} p\mathbb{Z}/n\mathbb{Z} = \bigcap_{p|n} \pi(p\mathbb{Z}) = \pi(\bigcap_{p|n} p\mathbb{Z}) = \pi(p_1 \cdots p_k\mathbb{Z}) = (p_1 \cdots p_k)\mathbb{Z}/n\mathbb{Z}.$$

This proves that $J(\mathbb{Z}_n)$ is the ideal $(p_1 \cdots p_k)/n\mathbb{Z}$. $\square$

Ex 5. Let k be a field of characteristic $p > 0$ and G a finite cyclic group of order n.

a) Show that $k[G]$ is isomorphic (as k-algebras) to $k[x]/(x^n - 1)$.

b) Determine the Jacobson radical of $k[G]$.

Proof.

a) We note that $k[x]/(x^n - 1)$ is isomorphic to the field $k[r_1, \ldots, r_n]$ where the set $\{r_i\}_{i \leq n}$ are the nth roots of unity in the algebraic closure of k. Since the nth roots of unity form a cyclic group of order n, we have that

$$k[G] \simeq k[r_1, \ldots, r_n] \simeq \frac{k[x]}{(x^n - 1)}$$

as desired.

b) We note that if n is relatively prime to p, then we know that $k[G]$ is semi-simple. Thus, $J(k[G]) = \{0\}$. Now if $n = p^m$ for some m, then

$$k[G] \simeq \frac{k[x]}{(x^{p^m} - 1)} = \frac{k[x]}{((x-1)^{p^m})}.$$

By the Correspondence Theorem, the ideals of $k[x]/((x-1)^{p^m})$ are in correspondence with the ideals of $k[x]$ containing $((x-1)^{p^m})$. Since maximal ideals look like $(x - c)$ for some $c \in k$, we have that the only maximal ideal of $k[x]$ containing $((x-1)^{p^m})$ is $(x - 1)$. As this correspondence of ideals respects the ordering of inclusion, the only maximal ideal of $k[x]/((x-1)^{p^m})$ is $(x-1)/((x-1)^{p^m}) = (x-1) + ((x-1)^{p^m})$. Using the isomorphism from part (a), this means the only maximal ideal of $k[G]$ is $(g - 1)$. Thus, $J(k[G]) = (g - 1)$.

[This could be wrong.] Now, let n be an arbitrary natural number. Then we know that $n = p^m \cdot \ell$ where p does not divide ℓ. By the Chinese Remainder Theorem, $G = \mathbb{Z}_{p^m} \times \mathbb{Z}_\ell$

generated by g^ℓ and g^{p^m} respectively. This means that $k[G] = k[\mathbb{Z}_{p^m}] \otimes k[\mathbb{Z}_\ell]$, so if we treat these purely as rings we get

$$J(k[G]) = J(k[\mathbb{Z}_{p^m}] \otimes k[\mathbb{Z}_\ell]) = J(k[\mathbb{Z}_{p^m}] \times k[\mathbb{Z}_\ell]) = J(k[\mathbb{Z}_{p^m}]) \times J(k[\mathbb{Z}_\ell]) = (g^\ell - 1) \times \{0\} = (g^\ell - 1).$$

□

Ex 6. For $n \in \mathbb{N}$ and a field k, consider the following k-subalgebra of $M_n(k)$

$$R = \{(a_{ij}) \in M_n(k) : a_{ij} = 0 \text{ for all } i > j\}.$$

a) Determine the Jacobson radical $J(R)$.

b) Determine the structure of the ring $R/J(R)$.

Proof.

a) Let $A = (a_{ij}) \in R$ where $a_{ii} = 0$ for $i \leq n$. Then A is strictly upper triangular and thus nilpotent. In Exercise 3, we proved that any nilpotent element is in $J(R)$, so we have that $A \in J(R)$.

   Now let $A = (a_{ij}) \in J(R)$ where $a_{kk} \neq 0$ for some $k \leq n$. This means that for any $B = (b_{ij}) \in R$ where $b_{kk} = a_{kk}^{-1}$, then

$$\det(1 - BA) = \prod_{i \leq n} (1 - b_{ii} a_{ii}) = (1 - a_{kk}^{-1} a_{kk}) \prod_{k \neq i \leq n} (1 - b_{ii} a_{ii}) = 0 \cdot \prod_{k \neq i \leq n} (1 - b_{ii} a_{ii}) = 0.$$

   This proves that $1 - BA$ has no inverse, meaning $A \notin J(R)$. Thus, $J(R)$ is exactly the ideal of strictly upper triangular matrices.

b) Let $\phi : R \to k^n$ be a ring homomorphism where $\phi(a_{ij}) = (a_{11}, \dots, a_{nn})$. We see then that ϕ is surjective and that $\ker(\phi) = \{(a_{ij}) \in M_n(k) : a_{ii} = 0\} = J(R)$. This proves that $R/J(R) \simeq k^n$ as rings.

□

Ex 7. Let $f : R \to S$ be a homomorphism between non-zero rings.

a) Prove that $f(J(R))$ is contained in $J(S)$ if f is surjective.

b) Give an example where f is surjective and $f(J(R))$ is different from $J(S)$.

c) Given an example where $f(J(R))$ is not contained in $J(S)$.

Proof.

a) Let I be a maximal ideal of R. Suppose that $f(I)$ were not a maximal ideal of S. Then there'd be an ideal J such that $f(I) \subsetneq J \subsetneq S$. Applying f^{-1}, since we know that $f^{-1}(S) = R$, we get that $I \subsetneq f^{-1}(J) \subsetneq R$, which contradicts the maximality of I. Thus, $f(I)$ is a maximal ideal of R. We see then that

$$f(J(R)) = f\Big(\bigcap_{\substack{I \subseteq R \\ I \text{ max}}} I\Big) = \bigcap_{\substack{I \subseteq R \\ I \text{ max}}} f(I) \subseteq \bigcap_{\substack{J \subseteq S \\ J \text{ max}}} J = J(S)$$

   which proves the statement.

b) Consider the surjective map $\pi : \mathbb{Z} \to \mathbb{Z}_{p^2}$ where p is prime. By Exercise 4, we determined that $J(\mathbb{Z}_{p^2}) = p\mathbb{Z}/p^2\mathbb{Z}$. Since $J(\mathbb{Z}) = \cap_{p \text{ prime}} \mathbb{Z}_p = \{0\}$, we see that $f(J(\mathbb{Z})) = f(\{0\}) = \{0\} \neq p\mathbb{Z}/p^2\mathbb{Z}$.

c) Let ϕ be the inclusion map between R, the ring of upper triangular $n \times n$ matrices over k, into the ring $M_n(k)$. By Excerise 6, we know that $J(R)$ is the set of strictly upper triangular matrices. However, $J(M_n(k)) = \{0\}$ as $M_n(k)$ is semi-simple. We see that $\phi(J(R)) = J(R)$ is not contained in $J(M_n(k)) = \{0\}$.

$\square$