Problem Set 1
Complex Analysis I

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January 15, 2018

Ex 8 A field F is said to be *ordered* if there is a distinguished subset $P \subseteq F$ with the following properties:

a) if $a, b \in P$, then $a + b \in P$ and $a \cdot b \in P$;

b) if $a \in F$, then precisely one of the following holds:

$$a \in P \quad \text{or} \quad -a \in P \quad \text{or} \quad a = 0.$$

Verify that $\mathbb{R}$ is ordered when $P \subseteq \mathbb{R}$ is taken to be $\{x \in \mathbb{R} : x > 0\}$. Prove that there is no choice of $P \subseteq \mathbb{C}$ which makes $\mathbb{C}$ ordered.

Proof. We see that if $x > 0$ and $y > 0$, then $x + y > 0$. Similarly, if $x > 0$ and $y > 0$, then $xy > 0$. Finally, for all $x \in \mathbb{R}$, either $x = 0$, $x > 0$ ($x \in P$), or $x < 0$ ($x \in -P$).

To prove that $\mathbb{C}$ is not ordered, suppose that there does exist such a set P. By (ii), either $1 \in P$ or $-1 \in P$. However, if $-1 \in P$, then $(-1)(-1) = 1 \in P$, by property (i). This is a contradiction, so it must be that $1 \in P$. Similarly by (ii), either $i \in P$ or $-i \in P$. However, either way we'll have that $i \cdot i = (-i)(-i) = -1 \in P$, a contradiction. Thus, there can exist no such set P. $\square$

Ex 34 If f is a C^1 function on the open set $U \subseteq \mathbb{C}$, then prove that

$$\overline{\frac{\partial}{\partial z} f} = \frac{\partial}{\partial \overline{z}} \overline{f}.$$

Proof. We see that

$$\begin{aligned}\overline{\frac{\partial}{\partial z} f} &= \overline{\frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y)} = \frac{1}{2}(u_x + v_y) - \frac{i}{2}(v_x - u_y) \\ &= \frac{1}{2}(u_x - (-v)_y) + \frac{i}{2}((-v)_x + u_y) = \frac{\partial}{\partial z}(u - iv) = \frac{\partial}{\partial \overline{z}} \overline{f}.\end{aligned}$$ $\square$

Ex 50 Let F be holomorphic on a connected open set $U \subseteq \mathbb{C}$. Suppose that G_1, G_2 are holomorphic on U and that

$$\frac{\partial G_1}{\partial z} = F = \frac{\partial G_2}{\partial z}.$$

Prove that $G_1 - G_2 = c$ for some $c \in \mathbb{C}$.

Proof. First, we prove that if f is holomorphic and $U \subseteq \mathbb{C}$ is an open, connected set such that $\frac{\partial f}{\partial z} = 0$ for all $z \in U$, then $f = c$ for some $c \in \mathbb{C}$. To prove this, let $x_0 \in U$ and let $S = \{x \in U : f(x) = f(x_0)\}$. Then:

a) S is non-empty. This is obvious as $x_0 \in S$.

b) S is open. To prove this, let $y \in S$ and let $B_r(y)$ be an open ball in U around y of radius r. Then, since one can get to any point in $B_r(y)$ from y through a horizontal and then vertical line segment, then it follows from class that $B_r(y) \subseteq S$.

c) S is closed. To prove this let $x_i \to x$ be a sequence in S which converges to x. Then, since f is continouous, we have that $f(x_i) \to f(x)$, which means that $f(x_0) \to f(x)$ and thus that $f(x) = f(x_0)$. This proves that $x \in S$.

Since S is non-empty, open, and closed, and since U is connected, it must be that $S = U$, which means that $f(x) = f(x_0)$ for all $x \in U$. Now, we see that

$$\frac{\partial}{\partial z}(G_1 - G_2) = \frac{\partial G_1}{\partial z} - \frac{\partial G_2}{\partial z} = F - F = 0$$

on U. By our previous proof, this means that $G_1 - G_2 = c$ for some $c \in \mathbb{C}$. $\square$

Ex 52 The function $f(z) = \frac{1}{z}$ is holomorphic on $U = \{z \in \mathbb{C} : 1 < |z| < 2\}$. Prove that f does not have a holomorphic antiderivative on U.

Proof. Let $F(z) = \log|z| + i\arg(z)$. We note that $\arg(z)$ has the same derivative as $\arctan(y/x)$ for $z = x+iy$. We also see that $\arg(z)$ is continuous on $\Omega = U \setminus \{(x,0) : x \geq 0\}$. We let $p(x,y) = \log|z|$ and let $q(x,y) = \arg(x+iy)$. From this, we have that

$$\frac{\partial p}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{\frac{1}{2}}{(x^2+y^2)^{3/2}} \cdot 2x = \frac{x}{x^2+y^2}$$
$$\frac{\partial p}{\partial y} = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{\frac{1}{2}}{(x^2+y^2)^{3/2}} \cdot 2y = \frac{y}{x^2+y^2}$$

and that

$$\frac{\partial q}{\partial x} = \frac{\partial}{\partial x}\arctan(y/x) = \frac{1}{1+(y/x)^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$
$$\frac{\partial q}{\partial y} = \frac{\partial}{\partial y}\arctan(y/x) = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} = \frac{1}{x+y/x} = \frac{x}{x^2+y^2}.$$

From this, we see that $\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}$ and that $\frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x}$, which proves that F is holomorphic over Ω. Finally, we have that

$$\begin{aligned}\frac{\partial F}{\partial z} &= \frac{\partial p}{\partial x} + i\frac{\partial q}{\partial x} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} \cdot \frac{x+iy}{x+iy} \\ &= \frac{x^2+y^2}{(x^2+y^2)(x+iy)} = \frac{1}{x+iy} = \frac{1}{z}.\end{aligned}$$

which proves that F is a holomorphic antiderivative on Ω.

Now, suppose there were a function $G(z)$ that was the holomorphic antiderivative of $\frac{1}{z}$ on U. Then G restricted to Ω would a holomorphic antiderivative on Ω. By the uniqueness of antiderivatives up to a constant, this implies that $G = F + c$ for some $c \in \mathbb{C}$. However,

$$\lim_{y\to 0^+} \arg(x+iy) = 0 \neq 2\pi = \lim_{y\to 0^-} \arg(x+iy)$$

for all $x > 0$. This means there is no way to make $\arg(z)$ continouous over U, and thus that G cannot be continuous over U. This proves that $\frac{1}{z}$ has no holomorphic antiderivative. $\square$

Ex 54 Let f be a holomorphic function on an open set $U \subseteq \mathbb{C}$ and assume that f has a holomorphic antiderivative F. Does it follow that F has a holomorphic antiderivative?

Proof. Let $F(z) = -\frac{1}{z}$, which is holomorphic on U according to Ex 52. We see that

$$\begin{aligned}\frac{d}{dz}F(z) &= \frac{\partial}{\partial x}F(x,y) = \frac{\partial}{\partial x}\left(\frac{-1}{x+iy}\right) = \frac{\partial}{\partial x}\left(\frac{-(x-iy)}{x^2+y^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) + i\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) \\ &= -\left(\frac{1}{x^2+y^2} - \frac{x}{(x^2+y^2)^2} \cdot 2x\right) + i\left(\frac{-y}{(x^2+y^2)^2} \cdot 2x\right) \\ &= \frac{-(x^2+y^2)}{(x^2+y^2)^2} + \frac{2x^2}{(x^2+y^2)^2} - \frac{2xyi}{(x^2+y^2)^2} \\ &= \frac{2x^2 - 2xyi - x^2 - y^2}{(x^2+y^2)^2} = \frac{x^2 - 2xyi - y^2}{(x^2+y^2)^2} = \frac{(x-iy)^2}{(x^2+y^2)^2} \\ &= \frac{(x-iy)^2}{(x-iy)^2(x+iy)^2} = \frac{1}{(x+iy)^2} = \frac{1}{z^2}\end{aligned}$$

on U. This means that $\frac{1}{z^2}$ has a holomorphic antiderivative (i.e. $-\frac{1}{z}$); however, we also know from Ex 52 that $\frac{1}{z}$ has no holomorphic antiderivative, which proves that $-\frac{1}{z}$ has no holomorphic antiderivative by the linearity of the antiderivative operator. Thus, $\frac{1}{z^2}$ is a counterexample to the claim. $\square$