Problem Set 5
Complex Analysis

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October 29, 2018

Ex 1 Let $u : U \to \mathbb{R}$ be given. We define

$$\limsup_{z \to \partial U} u(z) = \inf_{K} \sup_{z \in U \setminus K} u(z)$$

where the infimum is over all compact subsets K of U. Show that

a) Set $L = \limsup_{z \to \partial U} u(z)$. Prove that for every $S \in \mathbb{R}$ with $S > L$, there is a compact $K \subseteq U$ such that $u(z) \leq S$ for all $z \in U \setminus K$. Also prove that there is a sequence $(z_n)_n$ in U tending to ∂U with $\limsup_{n \to \infty} u(z_n) \geq L$.

b) Prove that

$$\limsup_{z \to \partial U} u(z) = \sup_{(z_n)_n} \limsup_{n \to \infty} u(z_n)$$

where the limit supremum is over all sequences $(z_n)_n$ with $z_n \to \partial U$.

c) Suppose that Ω is a domain and that $u : \Omega \to \mathbb{C}$ is nonconstant and harmonic. Suppose $z_n \in \Omega$ is a sequence such that $|u(z_n)| \to \sup_{z \in \Omega} |u(z)|$. Prove that $z_n \to \partial\Omega$.

d) Suppose that $\Omega \subseteq \mathbb{C}$ is a domain and $u : \Omega \to \mathbb{C}$ is harmonic. Prove that

$$\sup_{z \in \Omega} |u(z)| = \limsup_{z \to \partial\Omega} |u(z)|.$$

Proof.

a) Suppose that there is no compact $K \subseteq U$ such that $u(z) \leq S$ for all $z \in U \setminus K$. This means that $\sup_{z \in U \setminus K} u(z) \geq S$ for any compact $K \subseteq U$. Thus, we'd have that

$$\limsup_{z \to \partial U} u(z) = \inf_{K} \sup_{z \in U \setminus K} u(z) \geq S > L = \lim_{z \to \partial U} u(z)$$

which is a contradiction. Thus, there is some compact $K \subseteq U$ such that $u(z) \leq S$ for all $z \in U \setminus K$.

Let $K_i = \overline{\{x \in U : d(x, \partial U) \geq \frac{1}{n}, |x| \leq i\}}$ where $d(x, \partial U) = \inf_{u \in U} |x - u|$. We see that K_i are closed and also bounded (as $K_i \subseteq \overline{B_i(0)}$), so they're compact. We also see that $K_i \subseteq K_{i+1}$. For each K_i, we have by the definition of L that $L \leq \sup_{z \in U \setminus K_i} u(z)$. This means there exists a $z_i \in U \setminus K_i$ such that $u(z_i) \geq L - \frac{1}{i}$. Take these z_i's as a sequence $(z_i)_{i \in \mathbb{N}}$. And I claim this sequence is what we seek.

We see that $\limsup_{n\to\infty} u(z_n) \geq \limsup_{n\to\infty} L - \frac{1}{n} = L$, so we need only to prove that $z_n \to \partial U$. Let K be a compact set and let $\delta = d(\partial U, K)$ and let $D = \operatorname{diam}(K)$, which is finite as K is compact. This means that $K \subseteq K_n$ for $n \geq D$ and $\frac{1}{n} \leq \delta$. Since $z_n \in U \setminus K_n$, we have that $z_n \notin K$ for all n greater than $\max(D, \frac{1}{\delta})$. This proves that $z_n \to \partial U$, concluding the proof of the statement.

b) In part (a) we found a sequence $(z_i)_{i\in\mathbb{N}}$ such that $z_n \to \partial U$ and $\limsup_{n\to\infty} u(z_n) \geq \limsup_{z\to\partial U} u(z)$. This proves that

$$\limsup_{z\to\partial U} u(z) \leq \sup_{(z_n)_n} \limsup_{n\to\infty} u(z_n).$$

For the reverse inequality, let $(z_n)_{n\in\mathbb{N}}$ be any sequence such that $z_n \to \partial U$. We see that for each K_i (as constructed in part (a)), there's an N_i such that $z_n \in U \setminus K_i$ for all $n \geq N_i$. This means that $\sup\{u(z_n) : n \geq N_i\} \leq \sup\{u(z) : z \in U \setminus K_i\}$. Since removing beginning terms of $(z_n)_{n\in\mathbb{N}}$ doesn't affect the limsup, we see that

$$\limsup_{n\to\infty} u(z_n) \leq \sup_{z\in U\setminus K_i} u(z)$$

for each K_i. From this and the fact that any compact set of U is contained in some K_i, we can conclude that

$$\limsup_{n\to\infty} u(z_n) \leq \inf_{K_i} \sup_{z\in U\setminus K_i} u(z) = \inf_{K_i} \sup_{z\in U\setminus K} u(z) = \limsup_{z\to\partial U} u(z),$$

which proves the statement.

c) Let $z_n \in \Omega$ be a sequence such that $|u(z_n)| \to \sup_{z\in\Omega} |u(z)|$ and let $K \subseteq \Omega$ be any compact set. We can construct the open neighborhood $U = \{z \in \Omega : z \in K \text{ or } d(z, K) < \delta\}$ where $\delta = \min(d(\partial\Omega, K), 1)$. Since u is nonconstant, by the Maximum Modulus Principle we have thate

$$\sup_{z\in K} |u(z)| < \sup_{z\in\partial U} |u(z)| \leq \sup_{z\in\Omega} |u(z)|.$$

Thus, in order for $|u(z_n)| \to \sup_{z\in\Omega} |u(z)|$, it must be that for some $N \geq 0$, $z_n \notin K$ for all $n \geq N$. This proves that $z_n \to \partial\Omega$.

d) We first see that

$$\limsup_{z\to\partial\Omega} |u(z)| = \inf_K \sup_{z\in U\setminus K} |u(z)| \leq \inf_K \sup_{z\in\Omega} |u(z)| = \sup_{z\in\Omega} |u(z)|.$$

Now let $(z_n)_{n\in\mathbb{N}}$ be a sequence such that $|z_n| \to \sup_{z\in\Omega} |u(z)|$. Then we know that $\limsup_{n\to\infty} |u(z_n)| = \sup_{z\in\Omega} |u(z)|$ and by part (c) we have that $z_n \to \partial\Omega$. This means

$$\sup_{z\in\Omega} |u(z)| = \limsup_{n\to\infty} |u(z_n)| \leq \sup_{(z_n)} \limsup_{n\to\infty} |u(z_n)| = \limsup_{z\to\partial U} |u(z)|$$

where the last equality is from part (b). This proves the statement. $\square$

Ex 2

a) Let $\phi : [a, b] \times [c, d] \to \mathbb{C}$ be a continuous function and define $g : [c, d] \to \mathbb{C}$ by

$$g(t) = \int_a^b \phi(s, t)\, ds.$$

Prove that g is continuous.

b) Let G be an open set and let γ be a C^1 curve in G. Suppose that $\phi : \{\gamma\} \times G \to \mathbb{C}$ is a continuous function and define $g : G \to \mathbb{C}$ by

$$g(z) = \int_\gamma \phi(w, z)\, dw.$$

Then we know that g is continuous. If $\frac{\partial \phi}{\partial z}$ exists for each (w, z) in $\{\gamma\} \times G$ and is continuous, then g is analytic and

$$g'(z) = \int_\gamma \frac{\partial \phi}{\partial z}(w, z)\, dw.$$

Proof.

a) Let $\varepsilon > 0$. Since ϕ is continuous, there exists a $\delta > 0$ such that

$$0 < ||x - y|| < \delta \implies |\phi(x) - \phi(y)| < \frac{\varepsilon}{b - a}$$

where $||x - y||$ is the norm in $\mathbb{R}^2$. We see then that for $t_0, t_1 \in [c, d]$ such that $0 \neq |t_0 - t_1| < \delta$ we have

$$\begin{aligned} |g(t_0) - g(t_1)| &= \left| \int_a^b \phi(s, t_0)\, ds - \int_a^b \phi(s, t_1)\, ds \right| = \left| \int_a^b (\phi(s, t_0) - \phi(s, t_1))\, ds \right| \\ &\leq \int_a^b |\phi(s, t_0) - \phi(s, t_1)|\, ds < \int_a^b \frac{\varepsilon}{b - a}\, ds = \varepsilon \end{aligned}$$

where the last inequality follows because

$$||(s, t_0) - (s, t_1)|| = ||(s - s, t_0 - t_1)|| = \sqrt{0 + (t_0 - t_1)^2} = |t_0 - t_1| < \delta.$$

This proves that $g(t)$ is a continuous function.

b) Let $z_0 \in G$ and let $D \subseteq G$ be a closed disk of radius $r > 0$ centered at z_0. Since ϕ_z is continuous, this means that its uniformly continuous on $\operatorname{Im}(\phi) \times D$ (as its compact). If we let $|h| \to 0$ be sufficiently closed to zero so that $z_0 + h \in D$, then

$$\begin{aligned} \left| \frac{\phi(w, z_0 + h) - \varphi(w, z_0)}{h} - \varphi_z(w, z_0) \right| &= \left| \frac{1}{h} \int_{z_0}^{z_0 + h} \varphi(w, z)\, dz - \varphi_z(w, z_0) \right| \\ &= \left| \frac{1}{h} h \int_0^1 \varphi(w, z_0 + sh)\, ds - \varphi_z(w, z_0) \right| \\ &= \left| \int_0^1 (\varphi(w, z_0 + sh) - \varphi_z(w, z_0))\, ds \right| \\ &\leq \int_0^1 |\varphi(w, z_0 + sh) - \varphi_z(w, z_0)|\, ds. \end{aligned}$$

By the uniformly continuity of φ_z, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\varphi_z(t, z_0 + sh) - \varphi_z(t, z_0)| < \varepsilon$. Thus, the difference above is strictly bounded by ε for any $w \in \operatorname{Im}(\varphi)$. This proves that $\frac{\varphi(w, z_0 + h) - \varphi(w, z_0)}{h} \to \varphi_z(w, z_0)$ converges uniformly with respect to w in

$\mathrm{Im}(\varphi) \times D$. Using this, we see that

$$\begin{aligned} g'(z_0) &= \lim_{|h|\to 0} \frac{g(z_0+h)-g(z_0)}{h} = \lim_{|h|\to 0} \frac{\int_\gamma \phi(w,z_0+h)\,dw - \int_\gamma \phi(w,z_0)\,dw}{h} \\ &= \lim_{|h|\to 0} \int_\gamma \frac{\phi(w,z_0+h)-\phi(w,z_0)}{h}\,dw = \int_\gamma \lim_{|h|\to 0} \frac{\phi(w,z_0+h)-\phi(w,z_0)}{h}\,dw \\ &= \int_\gamma \phi_z(w,z_0)\,dw, \end{aligned}$$

as we wanted. (We can interchange the limit and the integral because of the uniform convergence we just proved for $|h|$ sufficiently close to 0.) This proves the statement. □

Ex 3 Find all entire functions f such that $f(f(z)) = f(z)$ for all $z \in \mathbb{C}$.

Proof. We see that if f is constant, say $f(x) = c$, then $f(f(x)) = f(c) = c = f(x)$. This proves that constant maps satisfy the condition. Suppose that f is nonconstant then. Let $y \in \mathrm{Im}(f)$, meaning $y = f(x)$ for some $x \in \mathbb{C}$. From this, we get that $f(y) = f(f(x)) = f(x) = y$, which proves that $f|_{\mathrm{Im}(f)}$ is the identity map. Since f is nonconstant, by the Open Mapping Theorem, we have that $\mathrm{Im}(f) = f(\mathbb{C})$ is open. Let $y \in \mathrm{Im}(f)$ and let $r > 0$ be such that $B_r(y) \subseteq \mathrm{Im}(f)$. Thus, we have that $f(x) = x$ for all $x \in B_r(y)$. Since the set $B_r(y)$ has an accumlation point, we know the set $\{z \in \mathbb{C} : f(z) = z\}$ has an accumlation point in $\mathbb{C}$. This proves that $f(z) = z$ on all of $\mathbb{C}$. Hence, the only entire functions satisfying the condition that $f(f(z)) = f(z)$ for all $z \in \mathbb{C}$ are constants and the identity map. □

Ex 4 Suppose that $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $|a_0| > 1$ and $n \geq 1$. Show that there is a $w \in \mathbb{C}$ with $|w| > 1$ and $p(w) = 0$.

Proof. Since $n \geq 1$, by the Fundamental Theorem of Algebra (proved on the last homework), we know that $p(z)$ has a root r_1. This means that $z - r_1$ divides $p(z)$, so that $p(z) = (z - r_1)\tilde{p}(z)$. By comparing degrees of polynomials, we see that the degree of $\tilde{p}$ must be $n - 1$. If $n - 1 \geq 1$, we can continue this process by applying the Fundamental Theorem of Algebra to $\tilde{p}$ to obtain a root r_2 of $\tilde{p}$ (and of p). Repeating this process, we get that $p(z) = c(z - r_1)\cdots(z - r_n)$ for some constant $c \in \mathbb{C}$. We see that if we expand this out cz^n is the highest power term and $cr_1 \cdots r_n$ is the constant term. By comparing with our original form of $p(z)$, we see that $cz^n = z^n$, meaning $c = 1$ and that $r_1 \cdots r_n = a_0$. Suppose that for $|r_i| \leq 1$ for all $i \leq n$. Then $|a_0| = |r_1 \cdots r_n| = |r_1| \cdots |r_n| \leq 1$, which is a contradiction to our assumption. Thus, there is at least one r_k such that $|r_k| > 1$ and $p(r_k) = 0$, which proves the claim. □

Ex 5 Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire and that there is a $\theta \in [0, 2\pi)$ such that $f(\mathbb{C}) \cap \{re^{i\theta} : r \geq 0\} = \varnothing$. Show that f is constant.

Proof. Let $R(z) = ze^{i(\pi-\theta)}$, which rotates $\mathbb{C}$ so that $(R \circ f)(\mathbb{C}) \cap \{re^{i\pi} : r \geq 0\} = \varnothing$. We note from homework 3, question 2, there is a conformal map ϕ from $\Omega_\pi = \{r^{i\theta} : r > 0, -\pi < \theta < \pi\}$ (which is the complement of $\{re^{i\pi} : r \geq 0\}$) to $\mathbb{D}$. Thus, the composition $\phi \circ R \circ f$ is a holomorphic function from $\mathbb{C}$ to $\mathbb{D}$. By Liouville's Theorem, $\phi \circ R \circ f$ must be constant. Since ϕ and R are conformal, they are bijections, so it must be that f is a constant. □

Ex 6

a) Suppose that f, g are entire functions and that $|f| \leq |g|$. Prove that there is a $c \in \overline{\mathbb{D}}$ such that $f(z) = cg(z)$.

b) Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire. Suppose that there exists $R \in (0, \infty)$ and $\alpha \in \mathbb{N}$ such that $|f(z)| \leq R|z|^\alpha$. Prove that f is a polynomial.

Proof.

a) If f is identically zero on $\mathbb{C}$, then we have that $f = 0 \cdot g$, so the result is satisfied. Now suppose that f is not identically zero. This means that g cannot be identically zero either. Suppose that $g(z_0) = 0$ for some $z_0 \in \mathbb{C}$. Since g is analytic, there is a power series representation of g around z_0; that is $g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. Since g is not identically zero, there is a least positive integer k such that $a_k \neq 0$. We have then that

$$\tilde{g}(z) = (z - z_0)^{-k} g(z) = (z - z_0)^{-k} \sum_{n=k}^{\infty} a_n(z - z_0)^n = \sum_{n=k}^{\infty} a_n(z - z_0)^{n-k} = \sum_{n=0}^{\infty} a_{n+k}(z - z_0)^n$$

is a holomorphic function and that $\tilde{g}(z_0) = a_k \neq 0$. Now, since $g(z_0) = 0$, we know that $f(z_0) = 0$. Similar as with g, let $f(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$ and let m be the least positive integer such that $b_m \neq 0$. Again, we define $\tilde{f}(z) = (z - z_0)^{-m} f(z)$ and note that $\tilde{f}(z_0) = b_m \neq 0$. Now it must be that $m \geq k$, otherwise we have that

$$|(z - z_0)^m \tilde{f}(z)| = |f(z)| \leq |g(z)| = |(z - z_0)^k \tilde{g}(z)|.$$

This gives that $|\tilde{f}(z)| \leq |\tilde{g}(z)||z - z_0|^{k-m}$. Plugging in z_0, we get that

$$|b_m| = |f(z_0)| \leq |\tilde{g}(z_0)||z_0 - z_0|^{k-m} = |a_k| 0^{k-m} = 0,$$

which is a contradiction. Thus, $f(z) = (z - z_0)^m \tilde{f}(z)$ and $g(z) = (z - z_0)\tilde{g}(z)$ where $m \geq k$. This means that

$$\frac{f(z)}{g(z)} = \frac{(z - z_0)^m \tilde{f}(z)}{(z - z_0)^k \tilde{g}(z)} = \frac{(z - z_0)^{m-k} \tilde{f}(z)}{\tilde{g}(z)}.$$

Since $\tilde{g}(z_0) = a_k \neq 0$, this proves that f/g can be defined at z_0. As z_0 was an arbitrary zero of g, we have that f/g is actually an entire function. Since $|f(z)/g(z)| = |f(z)|/|g(z)| \leq |g(z)|/|g(z)| = 1$, by Liouville's Theorem, $f/g = c$ for some constant $c \in \mathbb{C}$. We see that $|c| = |f/g| \leq 1$, so the constant must lie in $\overline{\mathbb{D}}$. This proves that $f(z) = cg(z)$ for some constant $c \in \overline{\mathbb{D}}$ as we desired.

b) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Cauchy's Integral formula gives us that for any circle C_r of radius r,

$$a_n = f^{(n)}(0) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(s)}{s^{n+1}} \, ds.$$

We see then that

$$\begin{aligned} |a_n| &= \left| \frac{n!}{2\pi i} \int_{C_r} \frac{f(s)}{s^{n+1}} \, ds \right| \leq \frac{n!}{2\pi} \int_{C_r} \frac{|f(s)|}{|s^{n+1}|} |ds| \leq \frac{n!}{2\pi} \int_{C_r} \frac{R|s|^\alpha}{|s|^{n+1}} |ds| \leq \frac{n!}{2\pi} \int_{C_r} Rr^{\alpha-(n+1)} |ds| \\ &\leq \frac{n!}{2\pi} \int_{C_r} Rr^{\alpha-(n+1)} |ds| = \frac{n!}{2\pi} Rr^{\alpha-(n+1)} \int_{C_r} |ds| \leq \frac{n!}{2\pi} Rr^{\alpha-(n+1)} \ell(C_r) \leq n! Rr^{\alpha-n}. \end{aligned}$$

Thus, if $n \geq \alpha$, we have that

$$|a_n| = \lim_{r \to \infty} n! Rr^{\alpha-n} = n! R \lim_{r \to \infty} r^{\alpha-n} = n! R \cdot 0 = 0.$$

This proves that $f(z) = \sum_{n=0}^{\lfloor \alpha \rfloor} a_n z^n$, which is a polynomial. $\square$

Ex 7 Let $S = \{x + iy : |x| \leq 1, |y| \leq 1\}$. Suppose that $f : \overline{S} \to \mathbb{C}$ is continuous and that $f\,|_S$ is holomorphic. Let the four sides of the square be denoted by S_1, S_2, S_3, S_4. Suppose that $M_1, M_2, M_3, M_4 \in [0, \infty)$ and $|f|_{S_i}| \leq M$. Prove that $|f(0)| \leq (M_1 M_2 M_3 M_4)^{1/4}$.

Proof. Consider the function $g : \overline{S} \to \mathbb{C}$ where $g(z) = f(z)f(iz)f(-z)f(-iz)$. We note that g is holomorphic on S and that for $z \in \partial S$,

$$|g(z)| = |f(z)||f(iz)||f(-z)||f(-iz)| \leq M_1 M_2 M_3 M_4$$

since z, iz, $-z$, and $-iz$ are on the four sides of S. Since $\overline{S}$ is compact, it attains its maximum and by the maximum modulus principle, this maximum is attained somewhere on the boundary (Note: this works even if g is constant). Thus,

$$|f(0)|^4 = |f(0)f(0)f(0)f(0)| = |g(0)| \leq \sup_{z \in S} |g(z)| \leq \sup_{z \in \partial S} |g(z)| \leq M_1 M_2 M_3 M_4.$$

After taking fourth roots, we have the result. $\square$