Problem Set 10
Differential Topology

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Ex 1 Let $f(z) = 1/z^m$ and $g(z) = \overline{z}^m$ on the circle of radius r around the origin in $\mathbb{C}$, where $m \in \mathbb{N}$.

a) Compute the degrees of $f/|f|$ and $g/|g|$.

b) Why does the proof of the Fundamental Theorem of Algebra not imply that $1/z^m = 0$ for some $z \in \mathbb{C}$?

Proof.

a) Using Ex 3.3.10 where we found that $\deg(u \circ v) = \deg(u)\deg(v)$, we see that z^m maps the circle m times around itself and that $1/z = \overline{z}$ on the circle, so it reflects the circle, which is orientation-reversing. Thus

$$\deg(f/|f|) = \deg(z^m/|z^m|) \cdot \deg(|z|/z) = \deg(z^m/|z^m|) \cdot \deg(\overline{z}/|\overline{z}|) = m \cdot (-1) = -m.$$

Similarly for g, z^m maps the circle m times around itself and $\overline{z}$ reflects the circle, so we have that

$$\deg(f/|f|) = \deg(z^m/|z^m|) \cdot \deg(\overline{z}/|\overline{z}|) = m \cdot (-1) = -m$$

as well.

b) In the proof of the Fundamental Theorem of Algebra, we homotope $p(z)$ to its leading coefficient z^m and used this to show that p has m zeros. However, this does not work for $1/z^m$; as $\deg(x^m) = m \neq -m = \deg(1/z^m)$, there can be no homotopy between them. $\square$

Chapter 3 Section 3

Ex 10 Suppose that $X \to^f Y \to^g Z$ are given. Prove that $\deg(g \circ f) = \deg(f) \cdot \deg(g)$.

Proof. Let $z \in Z$ be a regular value of $g \circ f$. This also means that z is a regular value of g and that

$g^{-1}(z)$ are regular values of f. We have then that

$$\begin{aligned}
\deg(g \circ f) &= \sum_{x \in (g\circ f)^{-1}(z)} \operatorname{sign}(d(g \circ f)_x) \\
&= \sum_{x \in (g\circ f)^{-1}(z)} \operatorname{sign}(dg_{f(x)} \cdot df_x) \\
&= \sum_{x \in (g\circ f)^{-1}(z)} \operatorname{sign}(dg_{f(x)}) \operatorname{sign}(df_x) \\
&= \sum_{y \in g^{-1}(z)} \sum_{x \in f^{-1}(y)} \operatorname{sign}(dg_{f(x)}) \operatorname{sign}(df_x) \\
&= \sum_{y \in g^{-1}(z)} \operatorname{sign}(dg_{f(x)}) \sum_{x \in f^{-1}(y)} \operatorname{sign}(df_x) \\
&= \deg(f) \cdot \deg(g).
\end{aligned}$$

□

Ex 13 Prove that the Euler characteristic of the product of two compact, oriented manifolds is the product of their Euler characteristics.

Incomplete. □

Chapter 3 Section 4

Ex 6 Show that the map $f(x) = 2x$ on $\mathbb{R}^k$ with a "source" at 0 has $L_0(f) = 1$. However, check that the "sink" $g(x) = \frac{1}{2}x$ has $L_0(g) = (-1)^k$.

Proof. We see that

$$\det(df_x - I) = \det(2I - I) = \det(I) = +1.$$

Since 1 is not an eigenvalue of df_x, we have that $L_0(f) = +1$. Similarly, we see that

$$\det(dg_x - I) = \det(I/2 - I) = \det(-I/2) = \frac{1}{(-2)^k}.$$

Again, as 1 is not an eigenvalue of dg_x, we get that $L_0(f) = \operatorname{sign}(1/(-2)^k) = (-1)^k$ as desired. □

Ex 8 Use the existence of Lefschetz maps for another proof that $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

Proof. Let f, g be peturbed maps of the identity maps Id_X and Id_Y respectively so that f and g are Lefschetz; that means so that $\chi(X) = L(\mathrm{Id}_X) = L(f)$ and $\chi(Y) = L(\mathrm{Id}_Y) = L(g)$. Similarly, we get that $f \times g$ is a Lefschetz map of $\mathrm{Id}_X \times \mathrm{Id}_Y$ and so $\chi(X \times Y) = L(\mathrm{Id}_{X\times Y}) = L(f \times g)$. We can now use local Lefschetz numbers to see that

$$\begin{aligned}
\chi(X \times Y) &= L(f \times g) \\
&= \sum_{(x,y)=(f(x),g(y))} L_{(x,y)}(f \times g) \\
&= \sum_{(x,y)=(f(x),g(y))} \operatorname{sign}(\det(d(f \times g)_{(x,y)} - I)) \\
&= \sum_{(x,y)=(f(x),g(y))} \operatorname{sign}(\det((df_x - I)(dg_y - I))) \\
&= \sum_{(x,y)=(f(x),g(y))} \operatorname{sign}(\det((df_x - I)))\operatorname{sign}(\det((dg_y - I))) \\
&= \sum_{x=f(x)} \operatorname{sign}(\det((df_x - I))) \sum_{y=g(y)} \operatorname{sign}(\det((dg_y - I))) \\
&= \sum_{x=f(x)} L_x(f) \sum_{y=g(y)} L_y(g) \\
&= L(f)L(g) = \chi(X)\chi(Y).
\end{aligned}$$

□

Ex 10

a) Prove that the map $z \mapsto z + z^m$ has a fixed point with local Lefschetz number m at the origin of of $\mathbb{C}$.

b) Show that for any $c \neq 0$, the homotopic map $z \mapsto z + z^m + c$ is Leftschetz, with m fixed points that are all close to zero if c is small.

c) Show that the map $z \mapsto z + \overline{z}^m$ has a fixed point with local Lefschetz number $-m$ at the origin of $\mathbb{C}$.

Incomplete. □

Chapter 3 Section 5

Ex 1 Let $\mathbf{v}$ be the vector field on $\mathbb{R}^2$ defined by $\mathbf{v}(x,y) = (x,y)$. Show that the family of diffeomorphisms $h_t : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h_t(z) = e^t z$ is the flow corresponding to $\mathbf{v}$. That is, if we fix any z, then the curve $t \mapsto h_t(z)$ is always tangent to $\mathbf{v}$; its tangent vector at any time t equals $\mathbf{v}(h_t(z))$. Draw a picture of $\mathbf{v}$ and its flow curves. Compare $\operatorname{ind}_0(\mathbf{v})$ with $L_0(h_t)$.

Proof. We see that the tangent vectors to the curve $h_t(z) = e^t z$ (with z fixed) are

$$\frac{\partial}{\partial t} h_t(z) = \frac{\partial}{\partial t} e^t z = e^t z = \mathbf{v}(e^t z) = \mathbf{v}(h_t(z))$$

This shows that the path $t \mapsto h_t(z)$ is a flow of our vector field. Since 0 is a source, we have that $\operatorname{ind}_0(\mathbf{v}) = +1$. We also have that

$$L_0(h_t) = \operatorname{sign}(\det(d(h_t)_0 - I)) = \operatorname{sign}(\det(\begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})) = \operatorname{sign}((e^t - 1)^2) = +1.$$

$\square$

Ex 2 Now let $\mathbf{v}(x,y) = (-y,x)$. Show that the flow transformations are the linear rotation maps $h_t : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix

$$\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}.$$

Draw $\mathbf{v}$ and its flow curves. Also compare $\operatorname{ind}_0(\mathbf{v})$ with $L_0(h_t)$.

Proof. We see that the tangent vectors to the curve $t \mapsto h_t(z)$ for a fixed z are

$$\begin{aligned} \frac{\partial}{\partial t} h_t(z) &= \frac{\partial}{\partial t} \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\sin(t) & -\cos(t) \\ \cos(t) & -\sin(t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x\sin(t) - y\cos(t) \\ x\cos(t) - y\sin(t) \end{bmatrix} \\ &= \mathbf{v}\left(\begin{bmatrix} x\cos(t) - y\sin(t) \\ x\sin(t) + y\cos(t) \end{bmatrix} \right) = \mathbf{v}\left(\begin{bmatrix} \cos(t) - \sin(t) \\ \sin(t) + \cos(t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) = \mathbf{v}(h_t(z)). \end{aligned}$$

Since the vector field around 0 is simply a counterclockwise circulation, we have that $\operatorname{ind}_0(\mathbf{v}) = +1$. We also see that

$$\begin{aligned} L_0(h_t) &= \operatorname{sign}(\det(d(h_t)_0 - I)) = \operatorname{sign}(\det(\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})) \\ &= \operatorname{sign}((\cos(t) - 1)^2 + \sin(t)^2) = \operatorname{sign}(\cos(t)^2 - 2\cos(t) + 1 + \sin(t)^2) = \operatorname{sign}(2 - 2\cos(t)) \\ &= +1. \end{aligned}$$

$\square$

Ex 3 Recall that the vector field $\mathbf{v}$ on a manifold X in $\mathbb{R}^N$ is a particular type of map $\mathbf{v} : X \to \mathbb{R}^N$. Show that at a zero x, the derivative $d\mathbf{v}_x : T_x(X) \to \mathbb{R}^N$ actually carries $T_x(X)$ onto itself.

Proof. Let X be a k-dimensional manifold and let $x \in X$ be a zero of the vector field $\mathbf{v}$. Let $\phi : U \subseteq \mathbb{R}^N \to X$ be a map such that when restricted to $U \cap \mathbb{R}^k$ it becomes a local parametrization of X at x. This means that $\phi^*\mathbf{v} = d\phi^{-1} \circ \mathbf{v} \circ \phi$ is a vector field on $\mathbb{R}^k$. We see that

$$d\phi^*\mathbf{v} = d\phi^{-1} \cdot d\mathbf{v} \cdot d\phi_0 \in T_0(\mathbb{R}^k) = \mathbb{R}^k.$$

Since $d\phi$ maps $\mathbb{R}^k$ to $T_x(X)$, we know that $d\phi^{-1}$ maps $T_x(X)$ to $\mathbb{R}^k$. Thus, it must be that $\mathrm{Im}(d\mathbf{v}) \subseteq T_x(X)$ at x. □

Ex 4 Let $f_t : X \to X$ be the map constructed in the proof of Poincaré-Hopf,

$$f_t(x) = \pi(x + t\mathbf{v}(x)).$$

Prove that at a zero x of $\mathbf{v}$, $d(f_t)_x = I + td\mathbf{v}_x$ as linear maps of $T_x(X)$ into itelf.

Proof. Since π restricted to X is the identity on X, then $(d\pi)_x$ restricted to $T_x(X)$ is the identity on $T_x(X)$. Let $g(x) = x + t\mathbf{v}(x)$, so that

$$d(f_t)_x = d\pi_{g(x)} \cdot dg_x = \mathrm{Id}_{T_x(X)}(I + td\mathbf{v}_x) = I + td\mathbf{v}_x$$

as we wanted. □

Ex 5 A zero x of $\mathbf{v}$ is *nondegenerate* if $d\mathbf{v}_x = T_x(X) \to T_x(X)$ is bijective. Prove that nondegenerate zeros are isolated. Furthermore, show that at a nondegenerate zero x, $\mathrm{ind}_x(\mathbf{v}) = +1$ if the isomorphism $d\mathbf{v}_x$ preverses orientation, and $\mathrm{ind}_x(\mathbf{v}) = -1$ if $d\mathbf{v}_x$ reverses orientation. [Hint: Deduce from Ex 4 that x is a nondegenerate zero of $\mathbf{v}$ if and only if it is a Lefschetz fixed point of f.]

Proof. Let $x \in X$ be a non-degenerate zero of $\mathbf{v}$. From Ex 4, we know that $d(f_t)_x = I + td\mathbf{v}_x$ at $x \in X$. Since f_t is the flow for some neighborhood of x, we have that

$$L_x(h_t) = \mathrm{sign}(\det(d(f_t)_x - I)) = \mathrm{sign}(\det(td\mathbf{v}_x)) = \mathrm{sign}(t^n \det(d\mathbf{v}_x)) = \mathrm{sign}(\det(d\mathbf{v}_x))$$

Since $\det(d\mathbf{v}_x) \neq 0$, we have that x is a Lefschetz fixed point of f. Moreover, as $\mathrm{ind}_x(\mathbf{v}) = L_x(h_t) = \mathrm{sign}(\det(d\mathbf{v}_x))$, we see that $\mathrm{ind}_x(\mathbf{v}) = +1$ if $\det(d\mathbf{v}_x) > 0$ (which means $d\mathbf{v}_x$ preverses orientation and that $\mathrm{ind}_x(\mathbf{v}) = -1$ if $\det(d\mathbf{v}_x) < 0$ (which means $d\mathbf{v}_x$ reverses orientation). □

Ex 6 A vector field $\mathbf{v}$ on X naturally defines a cross-sectional map $f_v : X \to T(X)$ by $f_v(x) = (x, \mathbf{v}(x))$.

a) Show that f_v is an embedding, so its image X_v is a submanifold of $T(X)$ diffeomorphic to X.

b) What is the tangent space of X_v are the point $(x, \mathbf{v}(x))$?

c) Note that the zeros of $\mathbf{v}$ correspond to the intersection points of X_v with $X_0 = \{(x, 0)\}$. Check that x is a nondegenerate zero of $\mathbf{v}$ if and only if $X_v \pitchfork X_0$ at $(x, 0)$.

d) If x is nondegenerate zero of $\mathbf{v}$, show that $\mathrm{ind}_x(\mathbf{v})$ is the orientation number of the point $(x, 0)$ in $X_0 \cap X_v$.

Incomplete. □