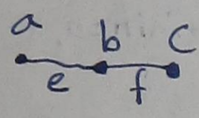
Bennett Rennier

# Ex 1.1.2




a) <u>Adjacency matrices</u>

$$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \quad \begin{array}{c} \\ a \\ c \\ b \end{array}\begin{array}{c} \begin{array}{ccc} a & c & b \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \quad \begin{array}{c} \\ b \\ a \\ c \end{array}\begin{array}{c} \begin{array}{ccc} b & a & c \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

By symmetry (since vertex a + c are symmetric), these are all the possible adjancey matrices.

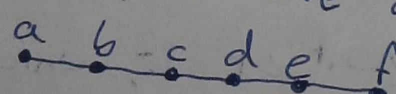
<u>Incidence Matrices</u>

$$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{cc} e & f \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array} \quad \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{cc} f & e \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \quad \begin{array}{c} \\ a \\ c \\ b \end{array}\begin{array}{c} \begin{array}{cc} e & f \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \quad \begin{array}{c} \\ a \\ c \\ b \end{array}\begin{array}{c} \begin{array}{cc} f & e \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}$$

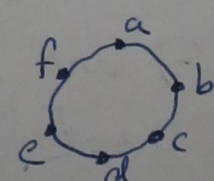
$$\begin{array}{c} \\ b \\ a \\ c \end{array}\begin{array}{c} \begin{array}{cc} e & f \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \quad \begin{array}{c} \\ b \\ a \\ c \end{array}\begin{array}{c} \begin{array}{cc} f & e \end{array} \\ \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}.$$

Again, by ~~sy~~ the symmetry of a and c, these are all possible incidence matrices

b)




$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$




$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Ex 4) Let $f: V(G) \to V(H)$ be an isomorphism of graphs. This means that $uv \in E(G) \iff f(u)f(v) \in E(H)$. By definition, $uv \in E(G) \iff uv \notin E(\overline{G})$. Thus, we have that

$$uv \notin E(\overline{G}) \iff uv \in E(G) \iff f(u)f(v) \in E(H) \iff f(u)f(v) \notin E(\overline{H})$$

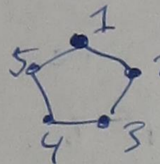
Since $uv \notin E(\overline{G}) \iff f(u)f(v) \notin E(\overline{H})$, this proves that

$$uv \in E(\overline{G}) \iff f(u)f(v) \in E(\overline{H})$$

Since $V(\overline{G}) = V(G)$, we have that $f$ is also an isomorphism of $\overline{G}$ and $\overline{H}$. Since all of our implications are reversible, this also proves that $\overline{G} \simeq \overline{H} \iff G \simeq H$.

Ex 10) Let $G$ be a simple disconnected graph and let $x, y \in V(G)$. If $xy \notin E(G)$, then $xy \in E(\overline{G})$ and thus have a path between them in $\overline{G}$. If $xy \in E(G)$, then $x$ and $y$ must be in the same component of $G$. Since $G$ is disconnected, $\exists z \in V(G)$ in another component of $G$, which means that $xz \notin E(G)$ and $yz \notin E(G)$. However, this means that $xz \in E(\overline{G})$ and $yz \in E(\overline{G})$. Thus, $xzy$ is a path in $\overline{G}$ between $x$ and $y$. This proves that for any $x, y \in V(G)$ there is a path between them in $\overline{G}$. Thus, $\overline{G}$ must be connected.

Ex 12 | Since $P$ has girth 5, this means that $C_5 \subseteq P$. Since $|E(C_5)| \geq 1$, $\chi(C_5) \neq 1$. Suppose $\chi(C_5) = 2$, and take




to be a representation of $C_5$. Let $R, G$ be our two colors and $i \in C$ to mean $i$ is the color $C$.

Since $C_5$ is vertex-transitive, wlog, we let $1 \in R$. This means that $2, 5 \in G$, and then that $3, 4 \in R$, which is a contradiction as $34 \in E(C_5)$. Thus $\chi(C_5) \neq 2$. Since $C_5 \subseteq P$, we have that $\chi(P) \geq 3$, which proves that $P$ is not bipartite.

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First, we notice that $\{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}\}$ is an independent set of $P$. Suppose $S$ is an independent set of $P$ such that $|S| \geq 5$. Let $G$ be the graph where $V(G) = \{1,2,3,4,5\}$ and $ij \in E(G) \iff \{i,j\} \in S$. This means that $|E(G)| \geq 5$.

Claim: Every edge in $G$ shares an endpoint with every other edge.

Proof: ~~Wlog, let~~ Suppose $ij, kl \in E(G)$ where $i, j, k, l$ are distinct. Then $\{i,j\}, \{k,l\} \in S$, which is a contradiction as since $\{i,j\} \cap \{k,l\} = \emptyset$, they are connected in $P$.

Claim: There is a vertex which is the endpoint of every edge in $G$.

Proof: Suppose otherwise. Wlog, let $12 \in E(G)$. ~~the next~~ Let $ij \in E(G)$ be a different edge. Then $\{1,2\} \cap \{i,j\} \neq \emptyset$. Wlog, let $ij = 23$. ~~Let kl be a third edge. It must share a vertex with both 12 and 23, which means that kl = 13. Finally, let~~ Since we supposed that no vertex is the endpoint of every edge, there must be an edge $kl$ where $2, k, l$ are distinct. Since $kl$ must share a vertex with $12$ and $23$, it must be that $kl = 13$. Finally, let $xy$ be a fourth edge. Since $xy$ must share an edge with $12$, $23$, and $13$, it must be that one of the endpoints of $xy$ is $1, 2,$ or $3$. Wlog, let $x = 1$. Then $y \neq 2, 3$, as $xy$ is a distinct edge. Thus, $xy = 14$ or $xy = 15$. ~~Either way, xy~~

Either way, $xy$ does not share a vertex with $23$, which is a contradiction. Since $|E(G)| \geq 5$, it must be that there is a vertex common to all edges.

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~~Let~~ Wlog, let $1$ be a vertex common to all edges in $G$. Since $\deg(1) \leq 4$, this means that $|E(G)| \leq 4$, and thus that $|S| \leq 4$. This is a contradiction, which means there is no such set $S$. This proves that the ~~maximal independent set of $P$ is of cardin cardinality~~ size of ~~the~~ the largest independent set of $P$ is $4$.

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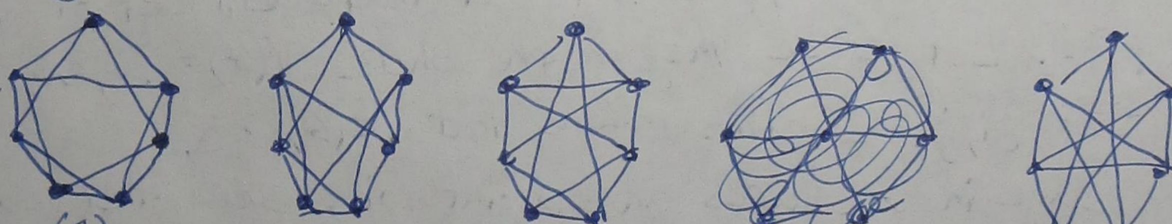
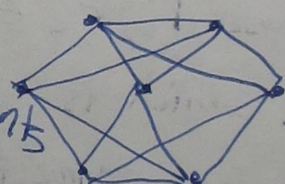
Ex 22) (I apologize for the terrible artwork)

The graphs are

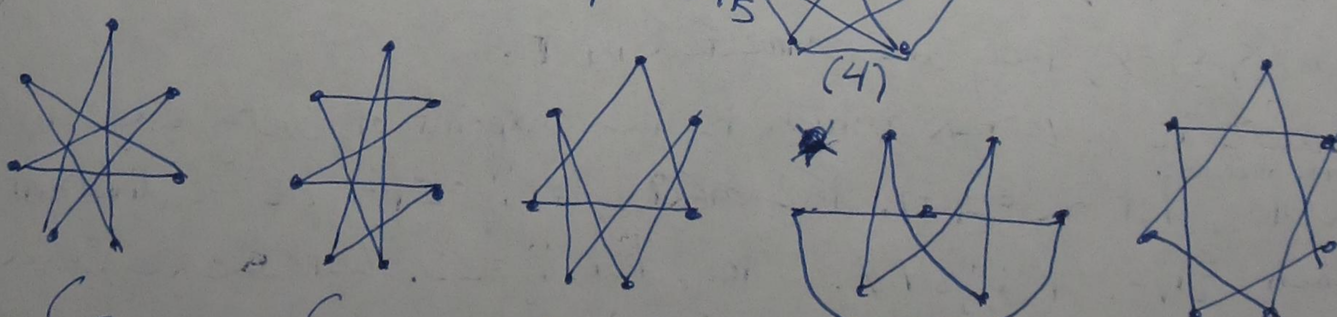


(1) (2) (3) (5)

which have the complements



(4)



$C_7$ $\quad$ $C_7$ $\quad$ $C_4 \times C_3$ $\quad$ $C_4 \times C_3$ $\quad$ $C_7$

Since $G \simeq H \Leftrightarrow \overline{G} \simeq \overline{H}$, this proves that ~~graphs (1), (2), (5) are pairwise isomorphic and that (3) and (4) are isomorphic and that these isomorphism classes~~ $\{(1), (2), (5)\}$ and $\{(3), (4)\}$ are the distinct isomorphism classes.