# Problem Set 6
# Graph Theory

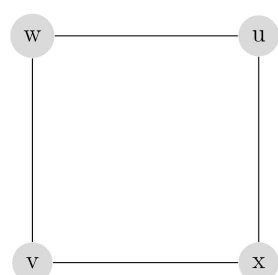
Bennett Rennier
barennier@gmail.com

October 4, 2017

**Ex 2.1.10** Let $u$ and $v$ be vertices in a connected $n$-vertex simple graph. Prove that if $d(u,x) > 2$, then $d(u) + d(v) \leq n + 1 - d(u,v)$. Construct an example to show that this can fail whenever $n \geq 3$ and $d(u,v,) \leq 2$.

*Proof.* As $d(u,v) > 2$, it must be that $u$ and $v$ have no common neighbors. This means that $d(u) + d(v)$ is simply the number of neighbors of either $u$ or $v$. Let $P = (x_i)_{i \in \{0,\ldots,d(u,v)\}}$ be the shortest path from $u$ to $v$. Since $P$ is the shortest $u,v$-path, it cannot be that $x_i$ is a neighbor of $u$ or $v$ for $i \neq 1, d(u,v)-1$. Thus, there are at least $d(u,v)-1$ vertices not in the neighbor of either $u$ or $v$. This means that the number of vertices in either neighborhood is at most $n - (d(u,v) - 1) = n + 1 - d(u,v)$. Thus, $d(u) + d(v) \leq n + 1 - d(u,v)$.

We see that for the following graph, $d(u) + d(v) = 4 > 3 = n + 1 - d(u,v)$:




□

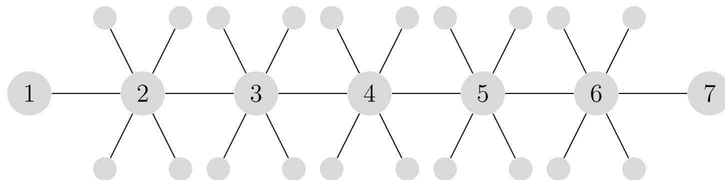
**Ex 2.1.23** Let $T$ be a tree in which every vertex has degree 1 or degree $k$. Determine the possible values of $n(T)$.

*Proof.* Let $m$ be the number of vertices with degree $k$. Using the degree-sum formula, we see then that

$$\sum_{v \in G} d(v) = mk + (n(T) - m) = 2e(G) = 2(n(T) - 1) = 2n - 2.$$

which means that $n(T) = m(k-1) + 2$. Since $m \geq 0$, we have that $n(T)$ is two more than a multiple of $k-1$.

For any such $n, m, k \in \mathbb{N}$ where $n = m(k-1) + 2$, we have such a tree by creating a path of $m+2$ vertices and then adding $k-2$ leaves to each inner vertex. An illustration for when $k = 6$ and $m = 5$ is seen below:




□

**Ex 2.1.26** For $n \geq 3$, let $G$ be a $n$-vertex graph such that every graph obtained by deleting one vertex is a tree. Determine $e(G)$, and use this to determine $G$ itself.

*Proof.* Let $\{v_1, \ldots, v_n\}$ be the vertices of $G$ and let $G_i = G - v_i$. Since $G_i$ is a tree and has $n-1$ vertices, we have that $e(G_i) = n-2$. This means that $\sum_i e(G_i) = n(n-2)$. For each $e \in e(G)$, we have that $e \in G_i$ for $n-2$ such $i$'s. Thus,

$$e(G) = \frac{\sum_{i=1}^{n} e(G_i)}{n-2} = \frac{n(n-2)}{n-2} = n.$$

Since $G$ has $n$ vertices and $n$ edges, $G$ contains a cycle. Since $G_i$ is a tree for any $1 \leq i \leq n$, it must be that any cycle in $G$ contains $v_i$ for each $i$. Since $G$ has a cycle and that cycle must contain all vertices and since $e(G) = n(G) = n$, it must be that $G = C_n$. □

**Ex 2.1.27** Let $d_1, \ldots, d_n$ be possible integers, with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum_i d_i = 2n - 2$.

*Proof.* Let $T$ be a $n$-vertex tree. This means that $e(T) = n-1$. Thus, by the degree-sum formula, we have that $\sum_i d_i = 2e(T) = 2n - 2$.

We will prove the converse by induction. For $n = 2$, the only such list is $(1, 1)$, which is the degree list of the tree $P_2$. Now, assume that the induction hypothesis works for $n$ and we will prove that it works for $n+1$. Let $d = (d_i)_{1 \leq i \leq n}$ be a list of integers where $n \geq 2$ and $\sum_i d_i = 2n - 2$. Since $\sum_i d_i < 2n$, there is a $d_j$ such that $d_j \leq 1$. Additionally, since $\sum_i d_i > n$, there is a $d_k$ such that $d_k > 1$. We let $d'$ be the list where we remove $d_j$ and replace $d_k$ with $d_k - 1$. We see then that $d'$ is a list of $n$ elements and that the sum of the elements of $d'$ is $2(n-1) - 2 = 2(n-2)$. Thus, by the induction hypothesis, there is a tree on $n-1$ vertices with $d'$ as its vertex degrees. If we add a vertex and an edge to the vertex whose degree we had previously substracted from, then we obtain a tree with our desired vertex degrees. □

**Ex 2.1.31** Prove that a simple connected graph having exactly two vertices that are not cut-vertices is a path.

*Proof.* Let $u, v \in V(G)$ be the only non-cut vertices in $G$. Let $P$ be the shortest $x, y$-path. If $V(P) \neq V(G)$, then let $w$ be the vertex the furthest distance from $P$. For any $w' \in G - P - w$, it must be that $d(P, w') \leq d(P, w)$. This, there is a path from $P$ to $w'$ that doesn't not go through $w$. Since this is true for every such $w'$, it must be that $w$ is a non-cut vertex. This is a contradiction, which means that $V(P) = V(G)$. Since $P$ was the shortest $x, y$-path and $G$ is simple, there cannot be any other edges. Thus, $G = P$ is a path. $\square$