Problem Set 6
Complex Analysis

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Ex 1 For the purposes of this problem

$$\limsup_{z\to iy} u(z) = \inf_{\varepsilon>0} \sup_{\substack{z\in\Omega \\ 0<|z-iy|<\varepsilon}} u(z)$$

for a function $u : \Omega \to \mathbb{R}$.

a) Let $\Omega = \{z : \mathrm{Re}(z) > 0\}$ and suppose that $f : \Omega \to \mathbb{C}$ is bounded and analytic. Suppose that $M \in [0, \infty]$ and that $\limsup_{z\to iy} |f(z)| \le M$ for all $y \in \mathbb{R}$. Show that $|f| \le M$. [Hint: Consider $f_\varepsilon(z) = \frac{f(z)}{1+\varepsilon z}$ for small $\varepsilon > 0$].

b) Prove that $f(z) = e^z$ has $\limsup_{z\to iy} |f(z)| = 1$ for all $y \in \mathbb{R}$, but f is not bounded on the right-half plane.

Proof.

a) Define $f_\varepsilon(z) = \frac{f(z)}{1+\varepsilon z}$ for $\varepsilon > 0$. We see that for any $z \in \Omega$,

$$|f_\varepsilon(z)| = \frac{|f(z)|}{|1+\varepsilon z|} < |f(z)|.$$

This means that

$$\limsup_{z\to iy} |f_\varepsilon(z)| \le \limsup_{z\to iy} |f(z)| \le M.$$

We also see since $f(z)$ is bounded, $z_n \to \infty$ implies that $f_\varepsilon(z_n) \to 0$. If we take the möbius transformation $\phi(z) = \frac{1+z}{1-z}$, we see that

$$\begin{aligned}
&\phi(-1) = 0 \\
&\phi(1) = \infty \\
&\phi(i) = \frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{2i}{2} = i \\
&\phi(0) = 1.
\end{aligned}$$

This proves that ϕ takes the unit circle to the imaginary axis and the unit disk to the right-half plane. By what we've proved so far, we see that

$$\limsup_{z\to e^{i\theta}} |f_\varepsilon(\phi(z))| = \begin{cases} \limsup_{z\to iy} |f_\varepsilon(z)| \text{ for some } y \in \mathbb{R} & \text{if } e^{i\theta} \neq 1 \\ \limsup_{z\to\infty} |f_\varepsilon(z)| & \text{if } e^{i\theta} = 1 \end{cases} \le \begin{cases} M & \text{if } e^{i\theta} \neq 1 \\ 0 & \text{if } e^{i\theta} = 1 \end{cases} \le M.$$

As $|f_\varepsilon \circ \phi|$ is bounded on the unit circle by M, by the Maximum Modulus Principle, we have that $|f_\varepsilon(\phi(z))| \leq M$ for all $z \in \mathbb{D}$. Thus, $|f_\varepsilon(z)| \leq M$ for all $z \in \Omega$. Since $f_{1/n} \to f$, we can conclude that $|f(z)| \leq M$ for all $z \in \Omega$.

b) As $e^x : \mathbb{R} \to \mathbb{R}$ is a monotonically increasing function, we see that

$$\limsup_{z \to iy} |e^z| = \limsup_{z \to iy} e^{\mathrm{Re}(z)} = \inf_{\varepsilon > 0} \sup_{\substack{z \in \Omega \\ 0 \leq |z - iy| < \varepsilon}} e^{\mathrm{Re}(z)} = \inf_{\varepsilon > 0} e^{\varepsilon} = e^0 = 1.$$

However e^z is unbounded on the right-half plane as it's unbounded on the positive real numbers. □

Ex 2 Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire and that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x + iy)| \, dx \, dy < \infty.$$

Prove that f is zero.

Proof. Let $z_0 \in \mathbb{C}$. If we integrate f over $\mathbb{C}$ using polar coordinates, we get that

$$\int_0^{2\pi} \int_0^{\infty} f(z_0 + re^{i\theta}) r dr d\theta \leq \int_0^{2\pi} \int_0^{\infty} |f(z_0 + re^{i\theta})| r dr d\theta = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x + iy)| \, dx \, dy < \infty.$$

Now, since the integral over the absolute value of f is bounded, by Fubini's Theorem, we can interchange the integrals to get that

$$\begin{aligned}\int_0^{2\pi} \int_0^{\infty} f(z_0 + re^{i\theta}) r dr d\theta &= \int_0^{\infty} \int_0^{2\pi} f(z_0 + re^{i\theta}) r d\theta dr = \int_0^{\infty} 2\pi f(z_0) r dr \\ &= 2\pi f(z_0) \int_0^{\infty} r dr = \pi f(z_0) r^2 \Big|_{r=0}^{\infty} = \lim_{r \to \infty} \pi f(z_0) r^2.\end{aligned}$$

Since this value is bounded, it must be that $f(z_0) = 0$. As z_0 was arbitrary, we have that f is the zero function. □

Ex 3

a) Let $\gamma : [a, b] \to \mathbb{C}$ be a loop such that $|\gamma(t) - 1| < 1$ for all $t \in [a, b]$. Prove that $n(\gamma; 0) = 0$.

b) Fix $w \in \mathbb{C}$. Let $\gamma_j : [a, b] \to \mathbb{C}$ be two loops such that $|\gamma_1(t) - \gamma_2(t)| < |\gamma_2(t) - w|$ for all $t \in [a, b]$. Prove that $n(\gamma_1; w) = n(\gamma_2; w)$.

Proof.

a) By Cauchy's Integral formula, we have that

$$n(\gamma; 0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{s - 0} \, ds = \frac{1}{2\pi i} \int_\gamma \frac{1}{s} \, ds$$

However, we note that $|\gamma(t) - 1| < 1$ implies that $\mathrm{Im}(\gamma)$ lies within $B_1(1)$, the open ball of radius 1 around 1. Since $1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$, it is holomorphic on the simply-connected set $B_1(1)$. Thus, by Cauchy's Integral Theorem we have that that the integral $\int_\gamma \frac{1}{s} \, ds$ is zero. This proves that $n(\gamma; 0) = 0$.

b) Let $\gamma(t) = \frac{\gamma_1(t)-w}{\gamma_2(t)-w}$. We see then that

$$|\gamma - 1| = \left|\frac{\gamma_1 - w}{\gamma_2 - w} - 1\right| = \left|\frac{\gamma_1 - w - \gamma_2 + w}{\gamma_2 - w}\right| = \frac{|\gamma_1 - \gamma_2|}{|\gamma_2 - w|} < \frac{|\gamma_2 - w|}{|\gamma_2 - w|} = 1.$$

By part (a), this means that $n(\gamma; 0) = 0$. From this we see that

$$\begin{aligned}
0 = n(\gamma; 0) &= \frac{1}{2\pi i}\int_\gamma \frac{1}{s-0}\,ds = \frac{1}{2\pi i}\int_a^b \frac{1}{\gamma(s)} \cdot \gamma'(s)\,ds \\
&= \frac{1}{2\pi i}\int_a^b \frac{\gamma_2(s) - w}{\gamma_1(s) - w} \cdot \frac{y_1'(s)(\gamma_2(s) - w) - (\gamma_1(s) - w)\gamma_2'(s)}{(\gamma_2(s) - w)^2}\,ds \\
&= \frac{1}{2\pi i}\int_a^b \frac{\gamma_1'(s)(\gamma_2(s) - w) - (\gamma_1(s) - w)\gamma_2'(s)}{(\gamma_1(s) - w)(\gamma_2(s) - w)}\,ds \\
&= \frac{1}{2\pi i}\int_a^b \frac{\gamma_1'(s)}{\gamma_1(s) - w} - \frac{\gamma_2'(s)}{\gamma_2(s) - w}\,ds \\
&= \frac{1}{2\pi i}\int_a^b \frac{\gamma_1'(s)}{\gamma_1(s) - w}\,ds - \int_a^b \frac{\gamma_2'(s)}{\gamma_2(s) - w}\,ds \\
&= \frac{1}{2\pi i}\int_{\gamma_1} \frac{1}{s-w}\,ds - \frac{1}{2\pi i}\int_{\gamma_2} \frac{1}{s-w}\,ds = n(\gamma_1; w) - n(\gamma_2; w).
\end{aligned}$$

This proves that $n(\gamma_1; w) = n(\gamma_2; w)$. □

Ex 4 Let $U \subseteq \mathbb{C}$ be open and let $f_n : U \to \mathbb{C}$ be a sequence of analytic functions. Suppose that $f : U \to \mathbb{C}$ and that $f_n \to f$ uniformly on compact subsets of U.

a) Prove that f is analytic.

b) Prove that $f_n' \to f'$ uniformly on compact subsets of U.

c) Suppose that $z_0 \in U$ and that $r > 0$ is so that $B_r(z_0) \subseteq U$. Let $f_n(z) = \sum_{k=0}^\infty a_{k,n}(z - z_0)^k$, $f(z) = \sum_{k=0}^\infty a_k(z - z_0)^k$. Prove that $a_{k,n} \to a_k$ as $n \to \infty$.

Proof.

a) Let T be a triangle (with its interior) in U, we note that T itself is compact, so we have that $f_n \to f$ uniformly on T. By Ex 7(a) on the second homework, we proved that f is continuous. Since we see that

$$\int_{\partial T} f\,dz = \int_{\partial T} \lim_{n\to\infty} f_n\,dz = \lim_{n\to\infty}\int_{\partial T} f_n\,dz = \lim_{n\to\infty} 0 = 0,$$

f is analytic by Morera's Theorem.

b) Let $K \subseteq U$ be compact set and let $\delta = \frac{d(K, \partial U)}{2}$. We see then that $K' = \{z \in U : z \in K \text{ or } d(z, K) \leq \delta\}$ is also closed and bounded and thus compact. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly on compact subsets, we have that there exists an N such that $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$ and all $z \in K'$. By construction for each $z \in K$, we have that $\overline{B_\delta(z)} \subseteq K'$. This means for $n \geq N$,

$$\begin{aligned}
|f_n'(z) - f'(z)| &= \left| \frac{1}{2\pi i} \int_{C_\delta(z)} \frac{f_n(s)}{(s-z)^2}\, ds - \frac{1}{2\pi i} \int_{C_\delta(z)} \frac{f(s)}{(s-z)^2}\, ds \right| = \left| \frac{1}{2\pi i} \int_{C_\delta(z)} \frac{f_n(s) - f_n(s)}{(s-z)^2}\, ds \right| \\
&\leq \frac{1}{2\pi} \int_{C_\delta(z)} \left| \frac{f_n(s) - f_n(s)}{(s-z)^2} \right| |ds| = \frac{1}{2\pi} \int_{C_\delta(z)} \frac{|f_n(s) - f_n(s)|}{|s-z|^2} |ds| \\
&< \frac{1}{2\pi} \int_{C_\delta(z)} \frac{\varepsilon}{|s-z|^2} |ds| = \frac{\varepsilon}{2\pi\delta^2} \int_{C_\delta(z)} |ds| = \frac{\varepsilon}{2\pi\delta^2} 2\pi\delta = \frac{\varepsilon}{\delta}.
\end{aligned}$$

Since δ is a fixed constant, we have that $f_n' \to f'$ converges uniformly on K. As K was an arbitrary compact set, we have that $f_n' \to f'$ converges uniformly on compact subsets of U.

c) We note that by part (b), $f_n' \to f'$ converges uniformly on compact subsets of U. We can then apply part (b) again to obtain that $f_n'' \to f''$ uniformly on compact subsets. Repeating this process (using induction), we get that $f_n^{(k)} \to f^{(k)}$ uniformly on compact subsets for any $k \in \mathbb{N}$. We see then that

$$|a_{k,n} - a_k| = \left| \frac{f_n^{(k)}(z_0)}{k!} - \frac{f^{(k)}(z_0)}{k!} \right| = \frac{|f_n^{(k)}(z_0) - f^{(k)}(z_0)|}{k!}$$

Since $f_n^{(k)} \to f^{(k)}$ uniformly, then for any $\varepsilon > 0$ there exists an N such that $|a_{k,n} - a_k| = |f_n^{(k)}(z_0) - f^{(k)}(z_0)| < \varepsilon$ for all $n \geq N$. This proves that $a_{k,n} \to a_k$ for any $k \in \mathbb{N}$ as desired. $\square$