# Problem Set 1
# Differential Topology

Bennett Rennier
bennett@brennier.com

1. Let $(X,d)$ be a metric space and $U \subset X$ a subset. Recall that $U$ is said to be *open* if for every $u \in U$ there exists $r > 0$ such that the ball $B_r(u)$ is contained in $U$. Prove that if $\mathscr{T}$ is the collection of open sets (in this sense), then $(X, \mathscr{T})$ satisfies the definition of a topological space. In other words, check:

a) The empty set and the space $X$ itself are open.
b) An arbitrary union of open sets is open.
c) A finite intersection of open sets is open.

*Proof.*

a) By the definition of open ball we have that $B_r(x) \subseteq X$ for any $x \in X, r \in \mathbb{R}^{>0}$. Hence, for any $x \in X$, we can simply choose the ball $B_1(x) \subseteq X$ to satisfy the condition. The empty set satisfies the condition trivially as it has no elements.

b) Let $\{U_i\}_{i \in I}$ be an arbitrary collection of open sets. Let $U = \cup_i U_i$ and let $x \in U$. Since $x \in U$, there must be some $j \in I$ such that $x \in U_j$. Since $U_j$ is open, there is some $r$ such that $x \in B_r(x) \subseteq U_j$. Since $U_j \subseteq U$, this open ball of $x$ in $U_j$ is also an open ball of $x$ in $U$. This proves that $U$ is open.

c) Let $\{U_i\}_{i \leq n}$ be a finite collection of open sets. Let $U = \cap_i U_i$ and let $x \in U$. Since $x \in U$, it must be that $x \in U_i$ for each $i \leq n$. As each of these sets is open, there is an $r_i$ such that $x \subseteq B_{r_i}(x) \subseteq U_i$. If we let $r = \min\{r_i\}_{i \leq n}$ (which exists as there are only finitely many $r_i$'s), then we have that $B_r(x) \subseteq B_{r_i}(x) \subseteq U_i$ for each $i$. This implies that $B_r(x) \subseteq U$, proving that $U$ is open. $\square$

2. For $(X, d)$ a metric space and $S \subset X$ a subset, a *limit point* of $S$ is a point $x \in X$ such that every neighborhood of $x$ in $X$ contains at least one point of $S$ other than $x$ (the same definition can be given in a general topological space; recall that a *neighborhood* of $x$ is a set containing an open set containing $x$). Prove that $S$ is closed if and only if it contains all its limit points.

*Proof.*

$\Longrightarrow$ ) Suppose that $S$ is closed. For the sake of contradiction, let $x$ be a limit point of $S$ such that $x \in S^c$ (the complement of $S$). Since $S$ is closed, that means $S^c$ is open. Thus, there is some $r \in \mathbb{R}$ such that $B_r(x) \subseteq S^c$. This ball itself is a neighborhood of $x$ that is completely disjoint from $S$. This is a contradiction to $x$ being a limit point of $S$, proving that all limit points of $S$ must be in $S$.

$\Longleftarrow$ ) Suppose that $S$ contains all of its limit points and let $x \in S^c$. Since $x$ is not in $S$, it cannot be a limit point of $S$. This means that there must exist some neighborhood of $x$, call it $N$, such that $N$ is disjoint from $S$; meaning that $N \subseteq S^c$. This $N$ contains an open set $U$ of $x$, and this open set contains a ball $B_r(x)$. Since

$$x \in B_r(x) \subseteq U \subseteq N \subseteq S^c,$$

we have that $S^c$ must be open. This proves that $S^{cc} = S$ is closed. $\square$

3. Let $(X, d)$ be a metric space and $x, y \in X$. Prove that there exist open sets $U$ and $V$ "separating" $x$ and $y$: this means that $x \in U$, $y \in V$, and $U$ and $V$ are disjoint. A topological space in which any pair of points can be separated in this sense is called a *Hausdorff* space. Give an example of a topological space that is not Hausdorff (and hence is not a metric space).

*Proof.* Let $x, y \in X$ and let $r = d(x, y)$. I claim that $B_{r/2}(x)$ and $B_{r/2}(x)$ are disjoint open sets. To prove this, suppose otherwise and let $z \in B_{r/2}(x) \cap B_{r/2}(y)$. This means that $d(x, z) < r/2$ and that $d(y, z) < r/2$. However, this would imply that

$$d(x, z) + d(z, y) < \frac{r}{2} + \frac{r}{2} = r = d(x, y),$$

which is contradiction of the Triangle Inequality. This proves that $B_{r/2}(x)$ and $B_{r/2}(y)$ are open sets that separate $x$ and $y$.

The simplest topological space that doesn't fulfill this condition is the trivial topology on $X$, where $X$ contains at least two points. Under this topology, there is only one open set that contains any points to begin with, so there is no hope of separating any two of them. $\square$

4. Let $f : X \to Y$ be a function between topological spaces, and $x \in X$ a point. We say $f$ is *continuous at* $x$ if for every open neighborhood $V$ of $f(x)$, there is an open neighborhood $U$ of $x$ such that $f(U) \subset V$.

   a) Prove that $f$ is continuous if and only if it is continuous at each point of $X$.
   b) Now assume that $X$ and $Y$ are metric spaces (if you like, assume $X = Y = \mathbb{R}$). Prove that the definition of continuous in terms of open sets coincides with the "usual" $\epsilon$-$\delta$ definition of continuity.

*Proof.*

a) Suppose that $f$ is continuous and let $x \in X$. Let $N$ be an open neighborhood of $f(x)$ and let $V \subseteq N$ be an open set containing $f(x)$. By continuity, this means that $f^{-1}(V)$ is open as well. Since $f(x) \in V$, we know that $x \in f^{-1}(V)$, making $f^{-1}(V)$ and open neighborhood of $x$. As we also know that $f(f^{-1}(V)) \subseteq V$, this proves that $f$ is continuous at each point in $X$.

   Conversely, suppose $f$ is continuous at each point in $X$. Let $V$ be an open set of $Y$ and let $U = f^{-1}(V)$. To prove that $f$ is continuous, we need to show that $U$ is open. Now, if $U$ is the empty set, then we have nothing to show as the empty set is always considered open. So suppose $U$ is not empty. Let $x \in U$. Since $f$ is continuous at $x$ and $V$ is an open neighborhood of $f(x)$, there must be some open neighborhood $U_x$ such that $f(U_x) \subseteq V$. From this, we see that $U_x \subseteq f^{-1}(V) = U$. As $U_x$ is open, there's a ball $B_r(x) \subseteq U_x \subseteq U$, which proves that $U$ is actually open. This proves that $f$ is continuous.

b) We see that we have the follow statements are equivalent to the $\varepsilon$-$\delta$

definition of continuity:

$$\begin{array}{lll}
\forall x, \forall \varepsilon, \exists \delta, & d(x,y) < \delta \implies & d(f(x), f(y)) < \varepsilon \\
\forall x, \forall \varepsilon, \exists \delta, & y \in B_\delta(x) \implies & f(y) \in B_\varepsilon(f(x)) \\
\forall x, \forall \varepsilon, \exists \delta, & y \in B_\delta(x) \implies & y \in f^{-1}(B_\varepsilon(f(x))) \\
\forall x, \forall \varepsilon, \exists \delta, & B_\delta(x) \quad\quad\quad \subseteq & f^{-1}(B_\varepsilon(f(x))) \\
\forall x, \forall \varepsilon, \exists \delta, & f(B_\delta(x)) \quad\quad \subseteq & B_\varepsilon(f(x))
\end{array}$$

We shall refer to the last of these statements as the $\varepsilon$-$\delta$ definition of continuity and show that it's equivalent to the definition to being continuous at every point $x$ in the open neighborhood sense, which we proved was equivalent to usual open set defintion of continuity in the previous part.

Suppose then that $f$ is continuous at the point $x$ in the open neighborhood sense. If we let $\varepsilon > 0$, then $B_\varepsilon(f(x))$ is an open neighborhood of $f(x)$, which means there is an open neighborhood of $x$, call it $N$, such that $f(N) \subseteq B_\varepsilon(f(x))$. This open neighborhood contains an open ball centered at $x$, say $B_\delta(x)$. Thus, we have that $f(B_\delta(x)) \subseteq f(N) \subseteq B_\varepsilon(f(x))$, proving that $f$ is continuous in the $\varepsilon$-$\delta$ sense as desired.

Now suppose that $f$ is continuous in the $\varepsilon$-$\delta$ sense. Let $x \in X$ and let $V$ be an open neighborhood of $f(x)$. This means for some $\varepsilon > 0$, there's an open ball $B_\varepsilon(f(x))$ contained in $V$. Since $f$ is continuous in the $\varepsilon$-$\delta$ sense, there's an open ball $B_\delta(x)$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. As $B_\delta(x)$ is an open neighborhood of $x$ and $f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \subseteq V$, we see that $f$ is continous at $x$ in the open neighborhood sense. □

5. For the following, you need not prove that familiar real functions of real variables are continuous (if they really are, that is).

   a) Prove that all open intervals in $\mathbb{R}$ (including finite, semi-infinite, or infinite) are homeomorphic.
   b) Prove that the unit ball $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is homeomorphic to the open square $\{(x,y) \mid 0 < x, y < 1\}$.
   c) Prove that the punctured plane $\mathbb{R}^2 - \{(0,0)\}$ is homeomorphic to the exterior of the closed unit ball, $\mathbb{R}^2 - \{(x,y) \mid x^2 + y^2 \leq 1\}$.

*Proof.*

a) We first note that the open interval $(0,1)$ is homeomorphic to any finite open interval $(a,b)$ via the map $f(x) = (b-a)x + a$. This proves that all finite open intervals are homeomorphic to each other.

Now $(a, b)$ be any open interval, including the case where $a$ or $b$ is infinity. If we let

$$\alpha = \begin{cases} \arctan a & a > -\infty \\ -\frac{\pi}{2} & a = -\infty \end{cases}, \qquad \beta = \begin{cases} \arctan b & b < \infty \\ \frac{\pi}{2} & b = \infty \end{cases}.$$

then we see that the finite open interval $(\alpha, \beta)$ is homeomorphic to $(a, b)$ via the map $f(x) = \tan x$. This proves all all open intervals (finite or otherwise) are homeomorphic to each other.

b) Instead of giving a direct homeomorphism, we shall instead prove that both are homeomorphic to $\mathbb{R}^2$. In the case of the ball, we see that it's homeomorphic to $\mathbb{R}^2$ via the polar coordinate map $f(r, \theta) = (\tan \frac{\pi r}{2}, \theta)$, which is well-defined as $\tan \frac{\pi \cdot 0}{2} = 0$. For the square, we that it's homeomorphic to the bigger square $\{(x, y) : -1 < x, y < 1\}$ via the map $f(x, y) = (2x - 1, 2y - 1)$, which is homeomorphic to $\mathbb{R}$ via the map $f(x, y) = (\tan \frac{\pi x}{2}, \tan \frac{\pi y}{2})$.

c) We see that the punctured plane is homeomorphic to the exterior of the closed unit ball via the polar coordinate map $f(r, \theta) = (r + 1, \theta)$.

$\square$

6. Let $X$ be a topological space and $BC(X)$ the set of bounded, continuous, real-valued functions on $X$. Define the distance function on $BC(X)$ by

$$d(f, g) = \sup_{x \in X}\{|f(x) - g(x)|\}, \quad f, g \in BC(X).$$

Check that this does define a metric on $BC(X)$ (in fact on the set of all bounded functions on $X$), and prove that this distance function makes $BC(X)$ into a complete metric space. (Recall that a metric space $X$ is *complete* if every Cauchy sequence in $X$ has a limit in $X$. You may use the fact that $\mathbb{R}$, with the usual distance function $d(x, y) = |x - y|$, is complete.)

*Note:* Why are we obliged to restrict to bounded functions? Why does this result show that a sequence of (not necessarily bounded) continuous functions that converges uniformly on compact sets has a limit that is continuous?

*Proof.* First we will prove this is a valid metric. We note that $d(f, g) \geq 0$ for any functions $f, g \in BC(X)$ as $|f(x) - g(x)| \geq 0$ for any $X$ (well, this supposes that $X \neq \varnothing$, which I assume is implied). If for some $x \in X$, $f(x) \neq g(x)$ then

$$|f(x) - g(x)| > 0 \implies \sup_{x \in X}\{|f(x) - g(x)|\} > 0 \implies d(f, g) > 0.$$

Hence, by the contrapositive, $d(f, g) = 0$ implies that $f(x) = g(x)$ for all $x \in X$. We also have symmetry, as

$$d(f, g) = \sup_{x \in X}\{|f(x) - g(x)|\} = \sup_{x \in X}\{|g(x) - f(x)|\} = d(g, f).$$

And finally, we have the Triangle Inequality as

$$\begin{aligned} d(f, h) = \sup_{x \in X}\{|f(x) - h(x)|\} &\leq \sup_{x \in X}\{|f(x) - g(x)| + |g(x) + h(x)|\} \\ &\leq \sup_{x \in X}\{|f(x) - g(x)|\} + \sup_{x \in X}\{|g(x) - h(x)|\} = d(f, g) + d(g, h) \end{aligned}$$

where the first inequality comes from the Triangle Inequality on $\mathbb{R}$ and the second inequality is a property of the supremum over non-negative numbers.

Now, we need to prove that $BC(X)$ is Cauchy complete. To do this, assume that $(f_i)_{i \in \mathbb{N}}$ is a Cauchy sequence under this metric. We define $f(x)$ as $f(x) = \lim_{n \to \infty} f_n(x)$, which is well-defined as $(f_i(x))_{i \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for each $x \in X$ and thus converges for each $x$.

Now let $\varepsilon/2 > 0$ be arbitrary and let $N$ be a positive integer such that for all $n, m \geq N$, $d(f_n, f_m) < \varepsilon/2$. Fix one such $m$ and let $n \geq N$ be arbitary. Since we have that $|f_m(x) - f_n(x)| < \varepsilon/2$ for all $x \in X$, this means that

$$f_m(x) - \varepsilon/2 < f_n(x) < f_m(x) + \varepsilon/2.$$

Since this is true for any $n \geq N$, we have that

$$f_m(x) - \varepsilon/2 \leq \lim_{n \to \infty} f_n(x) \leq f_m(x) + \varepsilon/2$$

as well. As $f(x) = \lim_{n \to \infty} f_n(x)$, we get that

$$f_m(x) - \varepsilon/2 \leq f(x) \leq f_m(x) + \varepsilon/2$$

meaning that $|f_m(x) - f(x)| \leq \varepsilon/2 < \varepsilon$. Firstly, this proves that $f$ is bounded as $f_m$ is bounded and $|f_m(x) - f(x)| < \varepsilon$ holds for all $x \in X$. Secondly, since our choice of $m$ was arbitrary, this holds for all $m \geq N$, proving that $f_i \to f$ converges uniformly as well.

Finally, we need to prove that $f$ is continuous. Let $\varepsilon/3 > 0$. Since $(f_i)_{i\in\mathbb{N}}$ converges uniformly there exists an $N$ such that for all $n \geq N$ we have that $|f_n(x) - f(x)| < \varepsilon/3$. As $f_n$ is continuous, there also exists a $\delta > 0$ such that

$$|x - y| < \delta \implies |f_n(x) - f_n(y)| < \varepsilon/3.$$

Thus, for $|x - y| < \delta$ we have that

$$\begin{aligned}
|f(x) - f(y)| &= |(f(x) - f_n(x)) + (f_n(x) - f_n(y)) + (f_n(y) - f(y))| \\
&\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\end{aligned}$$

which proves that $f$ is continuous.

Thus, we have proven that every Cauchy sequence in $BC(X)$ converges to some element of $BC(X)$. We had to restrict to bounded functions to make the metric well-defined. For example, if we allowed unbounded functions, then on the interval $(-\pi/2, \pi/2)$ we'd have that $d(0, \tan x) = \sup_{x\in(-\pi/2,\pi/2)} |\tan x|$ which is greater than any real number. As for the second question, since every continuous function is bounded on a compact set (continuous functions take compact sets to compact sets, which are closed and bounded in $\mathbb{R}$), this proof shows that such a sequence of continuous functions converges uniformly to a continuous function on every compact set. Since the compact sets $\{[n, n+1]\}_{n\in\mathbb{Z}}$ cover all of $X \subseteq \mathbb{R}$, this proves that the function at the limit is continuous everywhere in $X$. $\square$