Problem Set 5
Topology II

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Ex 1. Show that the free product $\mathbb{Z}_2 * \mathbb{Z}_2$ is infinite. Then show that it is, in fact, isomorphic the non-trivial semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}_2$.

Proof. Let a, b be the two non-identity letters in $\mathbb{Z}_2 * \mathbb{Z}_2$. We see these letters have the property that $a^2 = b^2 = 1$. Since $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}_2$ has the presentation $\langle s, r \mid s^2 = (sr)^2 = 1\rangle$, we can define a map $\varphi : \mathbb{Z}_2 * \mathbb{Z}_2 \to D_\infty$ by sending $a \to s$ and $b \to sr$. This is well-defined as the maps from $\mathbb{Z}_2$ to D_∞ which send $a \to s$ and $b \to sr$ are well-defined individually, and the universal property of free products induces our map φ.

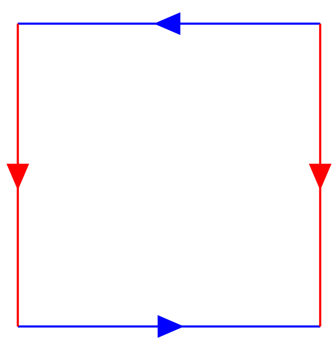
We see that φ is surjective as $\varphi(ab) = \varphi(a)\varphi(b) = ssr = s^2r = r$ and $\varphi(a) = s$, so we can create any element of D_∞. We note that any element of $\mathbb{Z}_2 * \mathbb{Z}_2$ is of the form $(ab)^n$, $(ab)^n a$, $(ba)^n$, or $(ba)^n b$ for some $n \geq 0$. Since $\varphi(ba) = \varphi(b)\varphi(a) = srs = s^2 r^{-1} = r^{-1}$, using our previous computations we see that these four types of elements map to r^n, $r^n s = sr^{-n}$, r^{-n}, and $r^{-n} sr = sr^{n+1}$ respectively. Since these images are all distinct (except for r^n and r^{-n} when $n = 0$, but in this cases the original elements were equal), we see that φ is also injective. This proves that this map φ is an isomorphism. $\square$

Ex 2. Prove the van Kampen theorem for 3 open sets assuming the theorem for 2 open sets. (In particular, point out where the condition on 3-fold intersections appears in the 3 open set version.)

Proof. Proof not completed. $\square$

Ex 3. Prove that the Klein bottle has fundamental group generated by two elements x and y subject only to the relation $x^2y^2 = 1$. Prove that this group does not have any non-trivial involutions.

Proof. We know that the Klein Bottle can be thought of as a square with its edges identified in the following way:



We see these various edge identifications imply that all four corners are actually identified as a single point and that both of the two edges are just loops from this one point to itself. Thus, as a CW complex, the Klein Bottle is just a 2-cell attached to a 1-skeleton of the form $S^1 \vee S^1$. We know that $\pi_1(S^1 \vee S^1)$ is simply the free product over two elements, so the fundamental group of the Klein bottle is simply the free group over two elements modded by the relation of the attaching map that attaches the single 2-cell.

Since all the corners are simply the same point, we can jump from corner to corner, which means there are a number of possible equivalent attaching maps. If we a be the generator for the red loop and b be the generator for the blue loop, then by using the square diagram, one possible attaching map is to start in the top left corner and traverse a, then "jumping" (since it's the same point) to the top right corner and traversing a again, then jumping to the top right corner again, traversing b, and then finally jumping to the bottom left corner and traversing b again. This means that one possible attaching map for the 2-cell is of the form a^2b^2. Thus, the fundamental group of the Klein Bottle is simply the free group generated by two elements a and b subject only to the relation $a^2b^2 = 1$. We shall refer to this group as K.

Note: I just realized there's something wrong with the above proof, as one could just as easily say that the attaching map is $abab$, which is not equivalent to $aabb$.

Unfortunately, I still have yet to prove this group does not have any non-trivial involutions. I tried doing it the straight-forward way, but there were too many cases for it to be clear. Perhaps there is some homomorphism from this group to a different group without involutions. $\square$

Ex 4. Let X be the complement of the union of n lines ℓ_i through the origin in $\mathbb{R}^3$. Compute the fundamental group of X.

Proof. If there are no lines, then $\pi(X) = \pi(\mathbb{R}^3) = 0$. Now assume that there's at least one line. This means that the origin is not in X. Thus, we can use the deformation retraction of $\mathbb{R}^3 \setminus \{0\}$ to the sphere to deformation retract X to a subset of the sphere, call this set X_S. Under this deformation retraction, our lines become pairs of antipodal points and X_S is the sphere with antipodal "holes." If we choose one of these holes, we can stereographically project onto $\mathbb{R}^2$ from this hole; call this resulting set X_R. The associated antipodal hole means that the origin is not included in X_R. Since each line in $\mathbb{R}^3$ induced antipodal holes in X_S, we see that each line induces a pair of holes in X_R, except for the line that was used for the projection which only induces the hole at the origin. This means that there are exactly $2(n-1) + 1 = 2n - 1$ holes in X_R.

Since these holes are finite, they must be isolated. We can partition the plane using a Voronoi diagram of these points and deformation retract X_R to this Voronoi 1-skeleton; let's call this 1-skeleton X_V. Using the same proof that a 1-skeleton is homotopy equivalent to the wedge of circles, we see that X_V is homotopy equivalent to the space $\bigvee_{i=1}^{2n-1} S^1$. Thus, by following these deformation retractions, we see that X is homotopy equivalent to $\bigvee_{i=1}^{2n-1} S^1$, which proves that

$$\pi_1(X) \simeq \pi_1\left(\bigvee_{i=1}^{2n-1} S^1\right) \simeq_{i=1}^{2n-1} \mathbb{Z}.$$

That is, the fundamental group of X is simply the free product over $2n - 1$ elements. $\square$

Ex 5. Let X be the union of k convex sets $X_i \subseteq \mathbb{R}^n$. Prove:

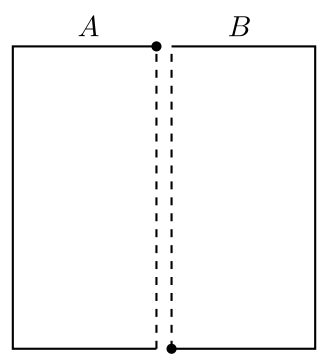
a) X is simply connected if $k \leq 2$.

b) X can be simply connected or not if $k \geq 3$.

c) Prove that X is simply connected if there exists $x_0 \in \bigcap X_i$.

d) More generally, prove that X is simply connected under the assumption that all three-fold intersections are connected.

Proof. I will assume for these proofs that $X_i \cap X_j$ is non-empty and that the X_i's are open sets; otherwise, these proofs just don't follow. The former assumption is obviously needed (since X must be path-connected to be simply connected) and the latter assumption is also needed because of examples like the following:



If we image that the two inner sides of the rectangles are on top of each other, then A and B are both convex, have a non-trivial intersection, but their union is a square with a slit in the middle which has a non-trivial fundamental group.

a) Well, if $k = 1$, then we just have that $X = X_1$. Since X is convex, it must be path-connected and all homotopies are trivial via the straight-line homotopies, so X is simply-connected. Now let $k = 2$ and let $A = X_1$ and $B = X_2$. Let $x \in A \cap B$, which is non-empty by assumption. We see that since A and B are convex, for any $a \in A$ and $b \in B$ there's a straight path from a to x and from x to b. The union of these paths is a path from a to b, which proves that $A \cup B$ is path-connected. We also note that $A \cap B$ is convex so it has trivial fundamental group. Thus, by Van Kampen's theorem with A and B being our open sets, we have that $\pi(A \cup B)$ is isomorphic to $\pi(A) * \pi(B) = 0$.

b) Let T be a triangle in $\mathbb{R}^2$. If we let X_1, X_2, X_3 be sufficiently small neighborhoods of the sides of the triangle, then we see that X_i is convex but $\pi(\cup X_i) \simeq \pi(S^1) \neq 0$.

c) This proof is very similar to the proof of (a). Since each X_i is path-connected, for any two points of X, we can connect them through x_0. Thus, X is path-connected. Additionally, since $\bigcap X_i$ is convex, we can apply Van Kampen's theorem on the open sets X_i to see that $\pi(X)$ is isomorphic to $*_i\, \pi(X_i) = 0$. This proves that X is simply-connected.

d) To prove this, we will use induction on k (the number of convex sets). For $k = 2$, three-fold intersections are just two-fold and by (a),we proved that the result held.

Let's us move on to the induction step. Suppose the result is true up to some $k \geq 2$. Let $X_1, \ldots, X_{k+1}$ be convex, open sets such that $X_i \cap X_j \cap X_\ell$ is connected. Let $Y = \bigcup_{i=1}^{k} X_i$. By the induction hypothesis, we know that Y is simply-connected. Since $X_{k+1} \cap X_1 \cap X_1$ is connected (as it's a three-fold intersection), this means that $X \cap Y$ is non-empty. Since both

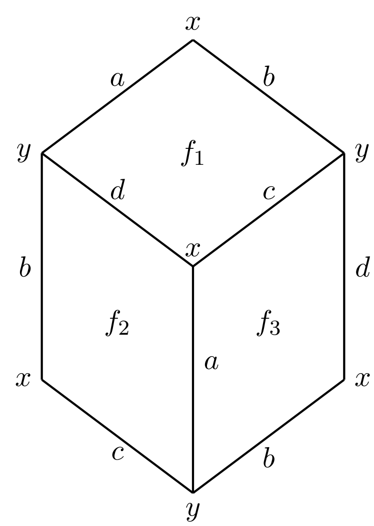
X and Y are path-connected (by convexity and the induction hypothesis respectively), this means that $X = Y \cup X_{k+1}$ must be path-connected as well using the same reasoning as in (a).

Now we wish to prove that $Y \cap X_{k+1}$ is path-connected so that we can apply Van Kampen's theorem. Let $x, y \in Y \cap X_{k+1}$. This means that $x \in X_n \cap X_{k+1}$ and that $y \in X_m \cap X_{k+1}$ for some n, m. We note that these two sets are path-connected using the same method as in (a). By assumption, $X_n \cap X_m \cap X_{k+1}$ is non-empty, so let z be an element of this set. Since $x, z \in X_n \cap X_{k+1}$ and $y, z \in X_m \cap X_{k+1}$, there's a path from x to z and from z to y. This proves that $Y \cap X_{k+1}$ is path-connected. Thus, we can apply Van Kampen's theorem to Y and X_{k+1} to get an isomorphism between $\pi_1(X_{k+1} \cup Y)$ and $\pi_1(X) * \pi_1(Y) \simeq 0$. This proves that $Y \cup X_{k+1}$ is simply-connected, concluding the induction and proving the statement.

□

Ex 6. Define a quotient X of the cube I^3 such that the induced CW structure has fundamental group Q (the quaternion group) as well as exactly two 0-cells, four 1-cells, three 2-cells, and one 3-cell.

Proof. We can quotient I^3 by identifying a face and its edges with the opposite face and edges by rotating a quarter turn counterclockwise (from the perspective of the outside). The following picture should help to visualize this:



We have done nothing with the 3-cell, so there's still only one 3-cell. Since every face is identified with its opposite face, there are three 2-cells instead of six, which means the above diagram is a complete picture. Since the above diagram is a complete picture, we see that there are indeed four 1-cells, labelled a, b, c, d, and two 0-cells, labelled x, y.

We see that the four edges a, b, c, d each connect x to y. Thus, the 1-skeleton of this CW complex is simply 4 lines connecting 2 points together. We can deformation retract one of the lines (say d), which leaves us with single point with three loops. Thus π_1 of the 1-skeleton is the free group one there elements. We see that the attaching maps φ_i for f_i can be described as $ad^{-1}cb^{-1}$, $bd^{-1}ac^{-1}$, and $cd^{-1}ba^{-1}$. Since we retracted d, these simply become acb^{-1}, bac^{-1}, and cba^{-1} respectively.

Thus, the π_1 of this structure can be presented as $\langle a, b, c \mid ac = b, ba = c, cb = a\rangle$. If we name a, b, c as i, j, k, we see that this is actually the group Q_8 as desired. $\square$