Principles of Mathematical Analysis
Chapter 1

Bennett Rennier
barennier@gmail.com

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Exercise 1.1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Proof. Assume $r+x$ is rational, so that $r+x = q \in \mathbb{Q}$. Then $x = q + (-r) \in \mathbb{Q}$. This is a contradiction, as $x \notin \mathbb{Q}$. Thus, $r + x \notin \mathbb{Q}$.

Assume rx is rational, so that $rx = q \in \mathbb{Q}$. Then $x = qr^{-1} \in \mathbb{Q}$, which is possible, as $r \neq 0$. This is a contradiction, though, as $x \notin \mathbb{Q}$. Thus, $rx \notin \mathbb{Q}$. □

Exercise 1.2. Prove that there is no rational number whose square is 12.

Proof. We see that $\sqrt{12} = 2 \cdot \sqrt{3}$. Assume $\sqrt{3}$ is rational. Then $\sqrt{3} = \frac{a}{b}$ for some $a, b \in \mathbb{N}$ and $b \neq 0$. This means that $a^2 = 3b^2$. There is no solution to this equation by argument of the parity of 3 as a prime factor. Thus, $\sqrt{3}$ is irrational. By Execerise (1.1), we see that a irrational multiplied with a rational is irrationl. Thus, $\sqrt{12}$ is irrational. □

Exercise 1.3. Prove the following axioms:

a) If $x \neq 0$ and $xy = xz$ then $y = z$.

b) If $x \neq 0$ and $xy = x$ then $y = 1$.

c) If $x \neq 0$ and $xy = 1$ then $y = \frac{1}{x}$.

d) If $x \neq 0$ then $1/(1/x) = x$.

Proof.

a) $xy = xz \implies x^{-1}xy = x^{-1}xz \implies 1y = 1z \implies y = z$.

b) Special case of part (a) when $z = 1$.

c) $xy = 1 \implies x^{-1}xy = x^{-1}1 \implies 1y = x^{-1} \implies y = \frac{1}{x}$.

d) $(1/x)x = 1$ implies that x is the unique inverse of $\frac{1}{x}$. Thus, the inverse of $\frac{1}{x}$, that is $1/(1/x)$, must be equal to x.

□

Exercise 1.4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof. Since E is nonempty, let $x \in E$. Since α is a lower bound of E and that $x \in E$, we know that $\alpha \leq x$. Likewise, since β is an upper bound of E and that $x \in E$, we know that $x \leq \beta$. By transitivity, we can thus see that $\alpha \leq x \leq \beta$. □

Exercise 1.5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf(A) = -\sup(-A).$$

Proof. Let $\alpha = \inf(A)$. This means that $\alpha \leq x$ for all $x \in A$. Thus, $-\alpha \geq -x$ for all $x \in A$, in other words, $-\alpha \geq x$ for all $x \in -A$. This proves that $-\alpha$ is an upper bound of $-A$.

Let β be an upper bound of $-A$ that is less than $-\alpha$. Thus, $-\alpha > \beta \geq x$ for all $x \in -A$, or in other words, $-\alpha > \beta \geq -x$ for all $x \in A$. This means that $\alpha < -\beta \leq x$ for all $x \in A$. This is a contradiction, as α is the greatest lower bound of A. Thus, no such β exists, and there is no upper bound of $-A$ that is less than $-\alpha$. This proves that $-\alpha$ is the least upper bound of $-A$. Thus, $\inf(A) = \alpha = -\sup(-A)$.

□

Exercise 1.6. Fix $b > 1$.

1) If m, n, p, q are integers, $n, q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence, it makes sense to define $b^r = (b^m)^{1/n}$

2) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

3) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup(B(r))$$

when r is rational. Hence it makes sense to define

$$b^x = \sup(B(x))$$

for every real x.

4) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Proof. TODO $\square$

Exercise 1.7. TODO

Proof. TODO $\square$

Exercise 1.8. Prove that no order can be defined in the complex field that turns it into an ordered field. (Hint: -1 is a square)

Proof. Suppose $\mathbb{C}$ is an ordered field. Then $\mathbb{C}$ must be an ordered set. This means that $i = 0$, $i < 0$, or $i > 0$. We definitely know that $i \neq 0$. Say $i > 0$. Then, to be an ordered field, $x > 0$ and $y > 0$ implies that $xy > 0$. But $i > 0$, and $i \cdot i = -1 < 0$. This is a contradiction. So let's say $i < 0$. Then $-i > 0$. But $(-i) \cdot (-i) = i \cdot i = -1 < 0$. Another contradiction. Thus, i does not follow the trichotomy law, which means $\mathbb{C}$ cannot be an ordered field. $\square$

Exercise 1.9. Suppose $z = a+bi$, $w = c+di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This is called a dictionary order or a lexicographic order.) Does this ordered set have the least-upper-bound property?

Proof. Let $A = \{w : \mathrm{Re}(w) = 0\}$. Let α be a least upper bound of A. If $\mathrm{Re}(\alpha) < 0$, then $\alpha < w$ for all $w \in A$. This cannot be true, as α is an upper bound of A. If $\mathrm{Re}(\alpha) = 0$, then $\alpha \in A$. Also then, $\mathrm{Re}(2\alpha) = 0$, which means $\alpha < 2\alpha \in A$. This is a contradiction that α is an upper bound. If $\mathrm{Re}(\alpha) > 0$, then $\mathrm{Re}(\frac{\alpha}{2}) > 0$. Thus $w < \frac{\alpha}{2} < \alpha$ for all $w \in A$. This is also a contradiction, as α is the least upper bound. Thus, α cannot exist. $\square$

Exercise 1.10. Suppose that $z = a+bi$, $w = u+iv$, and

$$a = \frac{w+u}{2}^{1/2}, b = \frac{w-u}{2}^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

Proof. $\square$

Exercise 1.11. If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

Proof. Let $w = \frac{z}{|z|}$ and $r = |z|$. Then $|w| = \left|\frac{z}{|z|}\right| = \frac{|z|}{|z|} = 1$ and $r = |z| \geq 0$. If $|z| = 0$, then $0 = |z| = |rw| = r|w| = r$. In this case, w can be anything. Thus, w and r are not always uniquely determined. $\square$

Exercise 1.12. If $z_1, ..., z_n$ are complex, prove that

$$z_1 + z_2 + ... + z_n \leq z_1 + z_2 + ... + z_n.$$

Proof. This will be a proof by Induction. For $n = 1$ case, we see trivally that $|z_1| \leq |z_1|$. Let's assume the n case and try to prove it for $n+1$. Let $z' = |z_1, z_2, ..., z_n|$. Then, $|z_1, z_2, ..., z_{n+1}| = |z' + z_{n+1}| \leq |z'| + |z_{n+1}|$. By the inductive assumption, we know that $|z'| \leq |z_1| + |z_2| + ... + |z_n|$. Thus, $|z_1 + z_2 + ... + z_{n+1}| \leq |z'| + |z_{n+1}| \leq |z_1| + |z_2| + ... + |z_n| + |z_{n+1}|$. This proves the general case. $\square$

Exercise 1.13. If x, y are complex, prove that

$$x - y \leq x - y.$$

Proof. $\square$

Exercise 1.14. If z is a complex number such that $z = 1$, that is, such that $z\bar{z} = 1$, compute

$$1 + z^2 + 1 - z^2.$$

Proof. $\square$

Exercise 1.15. Under what condition does equality hold in the Schwarz inequality?

Proof. $\square$

Exercise 1.16. TODO

Proof. $\square$

Exercise 1.17. Prove that

$$\boldsymbol{x} + \boldsymbol{y}^2 + \boldsymbol{x} - \boldsymbol{y}^2 = 2\boldsymbol{x}^2 + \boldsymbol{y}^2$$

if $\boldsymbol{x} \in \mathbb{R}^k$ and $\boldsymbol{y} \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

Proof. $\square$

Exercise 1.18. If $k \geq 2$ and $\boldsymbol{x} \in \mathbb{R}^k$, prove that there exists $\boldsymbol{y} \in R^k$ such that $\boldsymbol{y} \neq 0$ but $\boldsymbol{x} \cdot \boldsymbol{y} = 0$. Is this also true if $k = 1$?

Proof. Let $k \geq 2$ and $\boldsymbol{x} \in \mathbb{R}^k$, where $\boldsymbol{x} = (x_1, x_2, ..., x_k)$. Let $y_1, y_2, ..., y_{k-1} \in \mathbb{R}$, where at least one is non-zero. Now let

$$\boldsymbol{y} = (y_1, ..., y_{k-1}, -\frac{1}{x_k} \cdot \sum_{i=1}^{k-1} x_i y_i)$$

Then,

$$\boldsymbol{x} \cdot \boldsymbol{y} = x_k y_k + \sum_{i=1}^{k-1} x_i y_i = -\frac{x_k}{x_k} \sum_{i=1}^{k-1} x_i y_i + \sum_{i=1}^{k-1} x_i y_i = 0$$

This, however, depends on $k \geq 2$. If $k = 1$, then $\boldsymbol{xy} = xy$. When $xy = 0$, either $x = 0$ or $y = 0$. Thus, this does not hold in the case of $k = 1$ unless $\boldsymbol{x} = 0$ to begin with. □

Exercise 1.19. Suppose that $\boldsymbol{a}, \boldsymbol{b} \in R^k$. Find $\boldsymbol{c} \in R^K$ and $r > 0$ such that

$$\boldsymbol{x} - \boldsymbol{a} = 2\boldsymbol{x} - \boldsymbol{b}$$

if and only if $\boldsymbol{x} - \boldsymbol{c} = \boldsymbol{r}$. (Solution: $3\boldsymbol{c} = 4\boldsymbol{b} - \boldsymbol{a}, 3r = 2\boldsymbol{b} - \boldsymbol{a}$.)

Proof. □

Exercise 1.20. With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition staisfies axioms (A1) to (A4) (with a slightly different zero-element!) but that (A5) fails.

Proof. □