Problem Set 2
Real Analysis I

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Exercise 2.1. Find an example of a set X and a monotone class $\mathcal{M}$ consisting of subsets of X such that $\varnothing \in \mathcal{M}, X \in \mathcal{M}$, but $\mathcal{M}$ is not a σ-algebra.

Proof. Let $X = \mathbb{R}$. Consider the monotone class $\mathcal{M}$ which is the set of unbounded intervals of $\mathbb{R}$ along with the empty set. We see that this trivally includes $\varnothing$ and $\mathbb{R}$. This is definitely a monotone class. If we have a sequence of $A_i \in \mathcal{M}$, where $A_{i+1} \subseteq A_i$, look at $A_i \downarrow A$. Say that A is nonempty and bounded below by α and above by β. This means for some $A_j \in \mathcal{M}$, A_j is bounded above by β and for some A_k, A_k is bounded below by α, thus $A_{\max(j,k)}$ is bounded above by β and below by α and is nonempty. This means that it's not in $\mathcal{M}$ which is a contradiction. Thus, $A \in \mathcal{M}$.

Similarly, let's say we have a sequence of $A_i \in \mathcal{M}$, where $A_i \subseteq A_{i+1}$, and look at $A_i \uparrow A$. The union of unbounded intervals is definitely unbounded, thus $A \in \mathcal{M}$.

However, say that $a \leq b$, then this means that $(a, \infty) \cap (-\infty, b) = (a, b)$. Thus, $\mathcal{M}$ is not closed under finite intersection, which means that $\mathcal{M}$ is not a σ-algebra. □

Exercise 2.2. Find an example of a set X and two σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, each consisting of subsets of X, such that $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-algebra.

Proof. Let $X = \{1, 2, 3\}$. Then we see that $\mathcal{A}_1 = \{\varnothing, X, \{1\}, \{2, 3\}\}$ and that $\mathcal{A}_2 = \{\varnothing, X, \{2\}, \{1, 3\}\}$. We can clearly see that these are closed under complementation and contained X and $\varnothing$. It's also easy to check that they are closed under countable intersection. However, $\mathcal{A}_1 \cup \mathcal{A}_2 = \{\varnothing, X, \{1\}, \{2\}, \{2, 3\}, \{1, 3\}\}$, which is not a σ-algebra, as $\{1\} \cup \{2\} = \{1, 2\} \notin \mathcal{A}_1 \cup \mathcal{A}_2$. □

Exercise 2.3. Suppose $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \ldots$ are σ-algebras consisting of subsets of a set X. Is $\cup_{i=1}^{\infty} \mathcal{A}_i$ necessarily a σ-algebra? If not, give a counterexample.

Proof. $\cup_{i=1}^{\infty} \mathcal{A}_i$ is not necessarily a σ-algebra. For a counterexample, let $\mathcal{A}_n = \mathcal{P}(\{1, \ldots, n\})$, where $\mathcal{P}$ is the powerset function. These are trivally σ-algebras. We also see easily that $\mathcal{A}_i \subseteq \mathcal{A}_{i+1}$. Look at $\mathcal{A} = \cup_{i=1}^{\infty} A_i$. We know that $\{n\} \in \mathcal{A}_n$, as $\{n\}$ is a subset of $\{1, \ldots, n\}$. However, $\cup_{n=1}^{\infty} \{n\} = \mathbb{N}$. Since all the members of each $\mathcal{A}_i$ are finite, this means that $\mathbb{N}$ is not a member of any $\mathcal{A}_i$. Thus, $\mathbb{N} \notin \cup_{i=1}^{\infty} \mathcal{A}_i$. This shows that $\mathcal{A}$ is not closed under countable union. □

Exercise 2.5. Let $(Y, \mathcal{A})$ be a measurable space and let f map X into Y, but do not assume that f is one-to-one. Define $\mathcal{B} = \{f^{-1}(A) \mid A \in \mathcal{A}\}$. Prove that $\mathcal{B}$ is a σ-algebra of subsets of X.

Proof. Firstly, we see that since $Y \in \mathcal{A}$, then $f^{-1}(Y) = X \in \mathcal{B}$. Also, since $\varnothing \in \mathcal{A}$, then $f^{-1}(\varnothing) = \varnothing \in \mathcal{B}$.

Now, let $B \in \mathcal{B}$. This means $B = f^{-1}(A_1)$ for some $A \in \mathcal{A}$. Since $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$. Thus, $f^{-1}(A^c) \in \mathcal{B}$. We shall prove that $f^{-1}(A^c) = f^{-1}(A)^c$. Let $x \in f^{-1}(A^c)$. This means that $f(x) \in A^c$, which means that $f(x) \notin A$. This means that $x \notin f^{-1}(A)$, and finally we see that $x \in f^{-1}(A)^c$. This shows that $f^{-1}(A^c) \subseteq f^{-1}(A)^c$. Using the same reasoning in the opposite order, we conclude that $f^{-1}(A^c) = f^{-1}(A)^c$. Thus, this $f^{-1}(A^c) \in \mathcal{B}$, is actually equal to $f^{-1}(A)^c = B^c \in \mathcal{B}$. This shows that $\mathcal{B}$ is closed under complements.

Now, let $B_i \in \mathcal{B}$, where $i \in \mathbb{N}$. This means that $B_i = f^{-1}(A_i)$ for some $A_i \in \mathcal{A}$. This means that $\cup_i A_i \in \mathcal{A}$, which means $f^{-1}(\cup_i A_i) \in \mathcal{B}$. Let $x \in f^{-1}(\cup_i A_i)$. This means that $f(x) \in \cup_i A_i$. This means that $f(x) \in A_n$ for some $n \in \mathbb{N}$. Thus, $x \in f^{-1}(A_n)$, which means that $x \in \cup_i f^{-1}(A_i)$. Let $x \in \cup_i f^{-1}(A_i)$. Then this means that $x \in f^{-1}(A_n)$ for some $n \in \mathbb{N}$. Thus, $f(x) \in A_n$, which means that $f(x) \in \cup_i A_i$. Thus, $x \in f^{-1}(\cup_i A_i)$. This proves that $f^{-1}(\cup_i A_i)$ which is in $\mathcal{B}$, is equal to $\cup_i f^{-1}(A_i) = \cup_i B_i \in \mathcal{B}$. Thus, $\mathcal{B}$ is closed under countable union. This shows that $\mathcal{B}$ is indeed a σ-algebra. $\square$

Exercise 2.6. Suppose $\mathcal{A}$ is a σ-algebra with the property that whenever $A \in \mathcal{A}$ is nonempty, there exist $B, C \in \mathcal{A}$ with $B \cap C = \varnothing$. $B \cup C = A$, and neither B nor C is empty. Prove that $\mathcal{A}$ is uncountable.

Proof. First off, a simple counterexample is if $\mathcal{A} = \{\varnothing\}$ is the trival σ-algebra over $X = \varnothing$. Excluding this, though, we continue on:

Assume that $|\mathcal{A}| = n$. And let $A \in \mathcal{A}$ be any nonempty element. This means there exists sets $B, C \in \mathcal{A}$, such that they are nonempty and disjoint and their union is equal to A. Do this same thing to the set $B \in \mathcal{A}$. These two sets that union to B and are nonempty can neither be C nor A. As if one were either of these two sets, the union of such two sets would contain elements not in B, which is a contradiction as their union is exactly B. Do this n times. This gives us more than n elements which must be in $\mathcal{A}$, which is a contradiction, as $|\mathcal{A}| = n$. Thus, $\mathcal{A}$ is not finite. Since $\mathcal{A}$ is not finite, by Exe 2.8, $\mathcal{A}$ is not countable. Thus, $\mathcal{A}$ must be uncountable. $\square$

Exercise 2.7. Suppose $\mathcal{F}$ is a collection of real-valued functions on X such that the constant functions are in $\mathcal{F}$ and $f + g$, fg, and cf are in $\mathcal{F}$ whenever $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$. Suppose $f \in \mathcal{F}$ whenever $f_n \to f$ and each $f_n \in \mathcal{F}$. Define the function

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Prove that $\mathcal{A} = \{A \subseteq X \mid \chi_A \in \mathcal{F}\}$ is a σ-algebra.

Proof. We see that $\varnothing \in \mathcal{A}$ as $\chi_\varnothing = 0$, a constant function, which we assumed to be in $\mathcal{F}$. Also, $X \in \mathcal{A}$, as $\chi_X = 1$, also a constant function which we assumed to be in $\mathcal{F}$.

Let $A \in \mathcal{A}$. This means that $\chi_A \in \mathcal{F}$. Look at the function $-(\chi_A - 1)$. We see that this function is in $\mathcal{F}$, as $\mathcal{F}$ is closed under function addition and scalar multiplication. Also, one sees that this function is equivalent to χ_{A^c}. Thus, since this means $\chi_{A^c} \in \mathcal{F}$, we know that $A^c \in \mathcal{A}$. This shows closure under addition.

Let $A, B \in \mathcal{A}$. Thus $\chi_A, \chi_B \in \mathcal{F}$. We see that $\chi_A \chi_B$ is one iff $x \in A$ and $x \in B$ or, in other words, $x \in A \cap B$. Thus, $\chi_A \chi_B = \chi_{A\cap B}$. We also see that $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$. Since $\mathcal{F}$ is closed under addition, multiplication, and scalar multiplication, we see that $\chi_{A\cup B} \in \mathcal{F}$, which means that $A \cup B \in \mathcal{A}$.

Let $A_i \in \mathcal{A}$. Then let $B_n = \cup_{i=1}^n A_i$. We see that $B_2 = A_1 \cup A_2$ is in $\mathcal{A}$, as proven by the last paragraph. By induction, $B_n = A_n \cup B_{n-1}$ is in $\mathcal{A}$, as $A_n \in \mathcal{A}$ and $B_{n-1} \in \mathcal{A}$ by the inductive hypothesis. Thus, $B_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. We see that $\cup_{i=1}^\infty A_i = \lim\limits_{i\to\infty} B_i$. Since $B_i \in \mathcal{A}$, this means that $\chi_{B_i} \in \mathcal{F}$. Thus, since $\mathcal{F}$ is closed under limits, this means that $\lim\limits_{i\to\infty} \chi_{B_i} \in \mathcal{F}$. Thus, $\lim\limits_{i\to\infty} B_i \in \mathcal{F}$, which shows that $\cup_{i=1}^\infty A_i \in \mathcal{A}$. Thus, $\mathcal{A}$ is closed under countable union. This shows that $\mathcal{A}$ is a σ-algebra. $\square$

Exercise 2.8. Does there exist a σ-algebra which has countably many elements, but not finitely many?

Proof. Let's assume that $\mathcal{A}$ is an infinitely countable σ-algebra on a set X. If X were finite, then $\mathcal{P}(X)$ would be finite, and since a σ-algebra is a collection of subsets of X, this is means $\mathcal{A}$ is finite, a contradiction. Thus, X must be infinite. Define $B_x = \cap_{x\in A\in\mathcal{A}} A$. That is, the intersection of all members of $\mathcal{A}$ that contain x. We see that this is well-defined, as there are only countably infinitely many members of $\mathcal{A}$, which means the intersection is a countable intersection.

We see that $\{B_x\}_{x\in X}$ is a collection of subsets of X. Claim: This collection defines a partition of X. We see that obviously $\{B_x\}_{x\in X}$ covers X, as $x \in B_x$ for all $x \in X$. Now let $x, y \in X$, and look at the intersection of B_x and B_y. If $x \notin B_y$, then this means that $B_x \setminus B_y \in \mathcal{A}$ is a smaller set in $\mathcal{A}$ that contains x. This is a contradiction to the definition of B_x. This means that $x \in B_y$. By similar argument, we see that $y \in B_x$. Since these are the smallest sets that contain x and y respectively, we see that $x \in B_x \subseteq B_y$ and that $y \in B_y \subseteq B_x$. This proves that $B_x = B_y$, which means that $\{B_x\}_{x\in X}$ are a set of disjoint sets that union to all of X. This also shows that $\{B_x\}_{x\in X}$ is a partition.

Let $A \in \mathcal{A}$. We see obviously that $A \subseteq \cup_{x\in A} B_x$, as each B_x contains x. Now, say there is a y in $\cup_{x\in X} B_x$ that isn't in A, that is that containment is strict. This means that $y \in B_x$ for some x. Since $x \in A$ and $y \notin A$, this means that $x \in A \cap B_x$ is strictly smaller set than B_x that contains x, as it doesn't contain y. This is a contradiction to the definition of B_x, thus the containment is not strict, that is that $\cup_{x\in A} B_x = A$. Thus, every A can be written in this form.

Since every $A \in \mathcal{A}$ can be written as a union of sets of the form B_x. This means that if $\{B_x\}_{x\in X}$ were finite, then $\mathcal{A}$ would be finite as well, which is a contradiction. Thus, this partition is not finite and must be countably infinite, as it's a subset of $\mathcal{A}$. However, we know that $\mathcal{A}$ contains all possible unions of the sets in $\{B_x\}_{x\in X}$. Since all of these sets are disjoint, and there's countably infinitely many of them, the number of possible unions is equal to the number of elements in the powerset of $\{B_x\}_{x \in X}$ (each subset corresponding to which combination of elements to union together). Since the powerset must be strictly greater in cardinality than the already countably infinite set $\{B_x\}_{x \in X}$, we see that it must result in an uncountably infinite set. Since these are all in $\mathcal{A}$, as $\mathcal{A}$ is closed under countable unions, we see that $\mathcal{A}$ is uncountably infinite. $\square$