Problem Set 8
Differential Topology

Bennett Rennier
bennett@brennier.com

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Chapter 2, Section 4

Ex 4 If $f : X \to Y$ is homotopic to a constant map, show that $I_2(f, Z) = 0$ for all complementary-dimensional closed Z in Y, except perhaps if $\dim X = 0$. [Hint: Show that if $\dim Z < \dim Y$, then f is homotopic to a constant map c_y where $y \notin Z$. If X is one point, for which Z will $I_2(f, Z) \neq 0$?]

Proof. Suppose $\dim Z < \dim Y$ and that f is homotopic to the constant map $c_y : X \to Y$ where $c_y(x) = y$. If $y \notin Z$, then

$$I_2(f, Z) = I_2(c_y, Z) = |c_y^{-1}(Z)| = |\varnothing| = 0.$$

Now suppose $y \in Z$ and let $\dim Z = n$ and $\dim Y = m$. Since $Z \hookrightarrow Y$ is an immersion, there's a chart $\phi : U \subseteq \mathbb{R}^m \to V \subseteq Y$ such that $\phi(0) = y$ and $\phi(0, \ldots, 0, x_{n+1}, \ldots, x_m) \notin Z$. We consider the path $\gamma : [0, 1] \to Y$ where $\gamma(t) = \phi(0, \ldots, 0, t, \ldots 0)$ (t is in the $(n + 1)$th place). We see that this is a path from $\gamma(0) = y$ to $\gamma(1) \notin Z$. Since c_y and $c_{\gamma(1)}$ are homotopic via the homotopy $h : X \times [0, 1] \to Y$ where $h_t(x) = \gamma(t)$, this means that

$$I_2(f, Z) = I_2(c_y, Z) = I_2(c_{\gamma(1)}, Z) = |c_{\gamma(1)}^{-1}(Z)| = |\varnothing| = 0.$$

In the case where $\dim Z = \dim Y$, things get tricker. If Z contains a path-connected component of Y and f is homotopic to a constant map whose image is in that path-connected component, then $I_2(f, Z) = 1$. However, if either of these conditions fail, then there's a $y \in Y \setminus Z$ such that f is homotopic to c_y and so $I_2(f, c_y) = 0$ by the previous argument. $\square$

Ex 5 Prove that intersection theory is vacuous in contractible manifolds: if Y is contractible and $\dim Y > 0$, then $I_2(f, Z) = 0$ for every $f : X \to Y$, X compact and Z closed, $\dim X + \dim Z = \dim Y$. In particular, interesection theory is vacuous in Euclidean space.

Proof. Being contractible means that there's some homotopy $h_t : Y \times [0, 1] \to Y$ where $h_0(y) = \mathrm{Id}_Y$ and $h_1(y) = c_{y_0}(y)$, where c_{y_0} is the constant map where $c_{y_0}(y) = y_0$. We see then that the function $H_t : X \times [0, 1] \to Y$ where $H_t = h_t \circ f$ is a homotopy from $\mathrm{Id}_Y \circ f = f$ to $c_y \circ f = c_y$. Thus, f is homotopic to a constant map. By Ex 4, this means that $I_2(f, Z) = 0$, except if $\dim(X) = 0$.

However, we we see that even if $\dim(X) = 0$, then the path $\gamma(t) = h_t(y)$ is a path from $\gamma(0) = h_0(y) = y$ to $\gamma(1) = h_1(y) = y_0$. Since any point in Y is path-connected to y_0, we see that Y is path-connected, so c_{y_0} is homotopic to any other constant map. In particular, if $Z \neq Y$, then f is homotopic to c_y for some $y \in Y \setminus Z$ and so $I_2(f, Z) = 0$. If $Z = Y$, then I'm not sure. $\square$

Ex 10 Prove that the sphere S^2 and the torus $S^1 \times S^1$ are not diffeomorphic.

Proof. Let $(p, q) \in S^1 \times S^1$, we see that the loops $\gamma_1, \gamma_2 : S^1 \to S^1 \times S^1$, where $\gamma_1(t) = (t, q)$ and $\gamma_2(t) = (p, t)$ intersection transversally at the single point (p, q). Thus, $I_2(\gamma_1, \gamma_2) = 1$.

However, we see that if $\gamma : S^1 \to S^2$ is a loop in S_2, then there's some $x \in S^2$ not in the image of γ. This means we can consider γ as a path in $S^2 \setminus \{x\}$, which is diffeomorphic to a disk. Since a disk is contractible, by the first part of Ex 5, we have that γ is homotopic to a constant map. Since S^2 is path-connected, any constant map is homotopic to any other constant map. Thus, if we have two loops γ_1, γ_2 in S^2, then they are homotopic to constant maps which we can choose to be distinct. This means that $I_2(\gamma_1, \gamma_2) = 0$ for any loops γ_1, γ_2. This proves that S^2 and $S^1 \times S^1$ cannot be diffeomorphic. $\square$

Ex 14 Two compact submanifolds X and Z in Y are *cobordant* if there exists a compact manifold with boundary, W, in $Y \times I$ such that $\partial W = X \times \{0\} \cup Z \times \{1\}$. Show that if X may be deformed into Z, then X and Z are cobordant.

Proof. Saying that X may be deformed in Z means that there's a map $h_t : X \times I \to Y$ where each h_t is an embedding and $h_0 = \mathbb{1}_X$ and h_1 embeds X onto Z. We can extends this to the function $H_t : X \times I \to Y \times I$ where $H_t(x) = (h_t(x), t)$. The image of H is then a compact manifold, call it W, in $Y \times I$ such that ∂W is the image of H where $t = 0$ and $t = 1$. That is, $\partial W = X \times \{0\} \cup Z \times \{1\}$. This proves the statement. $\square$

Ex 15 Prove that if X and Z are corbordant in Y, then for every compact manifold C in Y with dimension complementary to X and Z, $I_2(X, C) = I_2(Z, C)$.

Proof. By the definition of cobordant, there exists a compact manifold $W \subseteq Y \times I$ such that $\partial W = X \times \{0\} \cup Z \times \{1\}$. Let $f = \pi \circ i : W \to Y$ where $\pi : Y \circ I \to Y$ and $i : W \to Y \times I$ are the canonical projection and inclusion. Since $f|_{\partial W}$ is a smooth map that extends to the smooth map f on W, we have that $I_2(\partial W, C) = 0$ by the Boundary Theorem. Since

$$\begin{aligned} 0 = I_2(\partial W, C) &= |\partial W \cap C| \bmod 2 = |(X \sqcup Z) \cap C| \bmod 2 = |X \cap C| + |Z \cap C| \bmod 2 \\ &= I_2(X, C) + I_2(Z, C), \end{aligned}$$

we have that $I_2(X, C) = -I_2(Z, C) = I_2(Z, C)$ as desired. $\square$

Halloween Worksheet

Let $f : X \to \mathbb{R}^n$ be a smooth map of an $(n-1)$-dimensional manifold, where X is connected, compact without boundary. For a point $z \in \mathbb{R}^n$ not on $f(X)$, define a map $u_z : X \to S^{n-1}$ by

$$u_z(x) = \frac{f(x) - z}{|f(x) - z|},$$

so that $u_z(x)$ is the unit vector pointing from z toward $f(x)$. The *mod 2 winding number* of f with respect to z is defined to be

$$W_2(f, z) = \deg_2(u_z).$$

Assume there exists a compact manifold W with $\partial W = X$ and a smooth map $F : W \to \mathbb{R}^n$ with $F|_{\partial W} = f$ as above. Let z be a regular value of F that is not in the image of f.

Ex 1 Prove that if $z \notin \mathrm{Im}(F)$, then $W_2(f,z) = 0$.

Proof. We recall that $W_2(f,z) = \deg_2(u_z) = I_2(u_z, \{z\})$. We see then $U_z : W \to S^{n-1}$ where

$$U_z(x) = \frac{F(x) - z}{|F(x) - z|}$$

is an extension of u_z to all of W and is well-defined as $z \notin \mathrm{Im}(F)$. Since we know that u_z is a map from a $(n-1)$-dimensional manifold to a $(n-1)$-dimensional manifold and that $\partial W = X$, we can apply the Boundary Theorem to get that $I_2(u_z, \{z\}) = 0$, proving the statement. $\square$

Ex 2 Now suppose that $z \in \mathrm{Im}(F)$ and that $F^{-1}(z) = \{y_1, \ldots, y_\ell\}$. Let $B_1, \ldots, B_\ell$ be small, disjoint balls around each y_j and let $f_j : \partial B_j \to \mathbb{R}^n$ be the restriction of F to the boundary of the balls. Prove that

$$W_2(f,z) = W_2(f_1,z) + \cdots + W_2(f_\ell, z) \bmod 2.$$

Proof. Let $W' = W \setminus ((\cup_j B_j) \cup X)$ and let $g : \partial W' \to \mathbb{R}^n$ where $g(w) = f(w)$ for $w \in X$ and $g(w) = f_j(w)$ for $w \in \partial B_j$. We see then that $F|_{W'}$ is an extension of g and $z \notin (F|_{W'})$, by the previous problem we have that $W_2(g,z) = 0$. Since we see that

$$\begin{aligned}
0 = W_2(g,z) = \deg_2(U_z|_{W'}) = I_2(U_z|_{W'}, \{z'\}) = \left| U_z|_{W'}^{-1}(z') \right| = \left| U_z|_X^{-1}(z') \bigsqcup \left(\sqcup_j U_z^{-1}|_{\partial B_j}(z') \right) \right| \\
= \left| U_z|_X^{-1}(z') \right| + \sum_j \left| U_z^{-1}|_{\partial B_j}(z') \right| = I_2(U_z|_X, \{z'\}) + \sum_j I_2(U_z|_{\partial B_j}, \{z'\}) = W_2(f,z) + \sum_j W_2(f_j, z).
\end{aligned}$$

This proves that $W_2(f,z) = -\sum_j W_2(f_j,z) = \sum_j W_2(f_j,z) \bmod 2$, as desired. $\square$

Ex 3 Show that in the situation of the previous problem one can choose the balls B_j such that $W_2(f_j, z) = 1$ for each j. Conclude that for any f, F, and z as in the assumptions before problem 1, the winding number $W_2(f,z)$ is equal to $|F^{-1}(z)| \bmod 2$.

Proof. Since z is a regular value of F and regular values are open, we know there's an open ball B centered at z consisting of regular values. This means that $F^{-1}(B)$ is a submanifold and by the Stack Record Theorem (this was an optional problem on a previous homework), we know that we can make B small enough so that $F^{-1}(B)$ looks like the disjoint union of open neighborhoods around each y_j that are each diffeomorphic to an open ball. Choose these neighborhoods to be our open balls B_j. Since ∂B traverses around z exactly once, we know that $U_z(\partial B_j)$ traverses around S^{n-1} exactly once. This means that

$$W_2(f_j, z) = I_2(U_z|_{\partial B_j}, \{z'\}) = |U_z|_{\partial B_j}^{-1}(z')| = 1$$

for any $z' \in S^{n-1}$. Using this and the previous problem, we see that

$$W_2(f,z) = W_2(f_1,z) + \cdots + W_2(f_\ell, z) = \ell = |\{y_1, \ldots, y_\ell\}| = |F^{-1}(z)| \bmod 2,$$

as we wanted. $\square$

Extra Problem

Ex Prove that complex projective n-space is a smooth, compact $2n$-dimensional manifold. Also prove that $\mathbb{C}P^1$ is diffeomorphic to the 2-sphere S^2, but $\mathbb{R}P^2$ is not.

Proof. We recall that $\mathbb{C}P^n$ is the set of nonzero tuples $(z_1, \ldots, z_{n+1})$ under the equivalence relation that says two such tuples are equivalent if they are a scale multiple of the other. We see that $\mathbb{C}P^n$ can be covered by the sets $U_j = \{(z_1, \ldots, z_{n+1}) : z_j \neq 0\} = \{(z_1, \ldots, 1, \ldots, z_{n+1})\}$, where the 1 is in the jth coordinate. If we let $\phi_j : \mathbb{R}^{2n} \to U_j$ where

$$\phi_j(x_1, y_1, \ldots, x_n, y_n) = (x_1 + iy_1, \ldots, x_{j-1} + iy_{j-1}, 1, x_j + y_j, \ldots, x_n + iy_n)$$

then ϕ_j is smooth with the smooth inverse

$$\phi_j^{-1}(z_1, \ldots, z_{n+1})$$
$$= (\mathrm{Re}(z_1/z_j), \mathrm{Im}(z_1/z_j), \ldots, \mathrm{Re}(z_{j-1}/z_j), \mathrm{Im}(z_{j-1}/z_j), \mathrm{Re}(z_{j+1}/z_j), \mathrm{Im}(z_{j+1}/z_j), \ldots \mathrm{Re}(z_{n+1}/z_j), \mathrm{Im}(z_{n+1}/z_j)).$$

Since the U_j cover $\mathbb{C}P^n$ and are each diffeomorphic to $\mathbb{R}^{2n}$, we have that $\mathbb{C}P^n$ is locally diffeomorphic to $\mathbb{R}^{2n}$, proving $\mathbb{C}P^n$ is a $2n$-dimensional manifold.

We see that if $(a, b) \in \mathbb{C}P^1$, then either $a \neq 0$, so we have that $(a, b) \sim (1, a^{-1}b)$, or $a = 0$, in which case $b \neq 0$ (as $(0, 0) \notin \mathbb{C}P^1$) and $(a, b) = (0, b) \sim (0, 1)$. We see that these equivalence classes are distinct, proving that the elements of $\mathbb{C}P^1$ can be described using the representatives $(1, a)$ for $a \in \mathbb{C}$ and $(0, 1)$. The map $\phi : \mathbb{C}P^1 \to \mathbb{C} \cup \{\infty\}$ where $\phi((1, a)) = a$ and $\phi((0, 1)) = \infty$ is a diffeomorphism (I'm not sure how to prove smoothness at $(0, 1)$, though). Since $\mathbb{C} \cup \{\infty\}$ is diffeomorphic to S^2 by stereographic projection, we have that $\mathbb{C}P^1$ is diffeomorphic to S^2.

Now we need to prove that $\mathbb{R}P^2$ is not diffeomorphic to S^2. Let $g : [-\pi/2, \pi/2] \to \mathbb{R}P^2$ be the loop $\gamma(t) = (\cos(t), \sin(t), 0)$ (it's a loop as $\gamma(\pi/2) = (0, 1, 0) \sim (0, -1, 0) = \gamma(-\pi/2)$). We see we can perturb this loop slightly into $\gamma'(t) = (\cos(t), \sin(t), \varepsilon t)$ for any $\varepsilon > 0$. These two loops intersect transverally at a single point when $t = 0$. Thus, the intersection number of γ with itself is 1, that is $I_2(\gamma, \gamma) = 1$. But we know by Ex 10, that the intersection number of any two loops on the sphere is zero. This proves that $\mathbb{R}P^2$ cannot be diffeomorphic to S^2. □