Problem Set 5
Abstract Algebra I

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January 15, 2018

Additional Problems

Ex A We call a group G cyclic if there is a single which generates the entire group. That is, if we can find a fixed $g \in G$ such that for any $x \in G$, we have $g^k = x$ for some k. Please show every subgroup of a cyclic.

Ex B Let

$$Z\left(g \in G \;\middle|\; gx = xg \text{ for all } x \in G\right)$$

Please show that $Z(G)$ is a normal subgroup of G (this is called the center of G).

Ex C1 Let G be a finite group and X a G-set. Let $x \in X$, $H = G_x = \text{Cent}_G(x)$, and $\mathcal{C} = \{g_i H\}_{i=1}^n$ be a complete, irredundant set of cosets in G. Please prove that there is a well defined bijection $\varphi : \mathcal{C} \to G.x$. Use this to show that $|G.x| = [G : H] = |G|/|H|$. This result be in your notes from class.

Ex C2 Use problem C1 to show that one has the formula

$$|X| = \sum_{i=1}^{r} [G : G_{x_i}]$$

where $x_i, \ldots, x_r$ is a complete, irredundant set of orbit representives. This result should be in your notes from class.

Ex C3 Recall that G acts on $X = G$ by the conjugation action: $g.x = gxg^{-1}$ for all $g \in G$ and all $x \in X$. Use C2 to show that we have the formula

$$|G| = |Z(G)| + \sum_{i=1}^{t} [G : C_G(g_i)]$$

where $g_1, \ldots, g_t$ is a complete, irredundant set of conjugacy class representatives which are not elements of the center of G, and where $C_G(g_i) = \text{Stab}_G(g_i) = G_{g_i}$ is the stabilizer of g_i under the conjugation action.

Ex C4 Equation (1) is called the class equation for G. Prove that each of the terms in the sum on the right hand side must evenly divide the order of G. The fact that these numbers must all divide the order of G and also sum up to the order of G makes very strong restrictions on what can happen when using the class equation. For example, please use the class equation to prove if $|G|$ is a prime, then G must be abelian.

Ex D Let G be a group with order p^2 for some prime p.

D1) Please use the Class Equation to show that $Z(G)$ is nontrivial.

D2) Please show that if $Z(G) \neq G$, then $G/Z(G)$ must be a nontrivial cyclic subgroup.

D3) Conclude that $Z(G) = G$. Therefore, every group of order p^2 for any prime p must be abelian.

Ex E Let G be a finite group with normal subgroup $N \trianglelefteq G$. Please prove or disprove: If N and G/N are cyclic, then G is cyclic.

Ex F Let A be an abelian group.

F1) Let $H \subseteq A$ be the subset of all elements of A which have finite order. Please show H is a normal subgroup of A.

F2) Please show that the only element of A/H which has finite order is the identity.

Ex G A character of a group G is a group homomorphism

$$\alpha : G \to \mathbb{C}^{\times}$$

Let $\hat{G}$ be the set of all characters of G. Please define, in a natural way, a binary operation on $\hat{G}$ and show that the binary operation makes $\hat{G}$ into a group.