Complex Analysis I
Problem Set 2

Bennett Rennier
barennier@gmail.com

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Ex 2 Let f be holomorphic on an open set U which is the interior of a disc or a rectangle. Let $p, q \in U$. Let $\gamma_j : [a, b] \to U$, $j = 1, 2$, be C^1 curves such that $\gamma_j(a) = p$, $\gamma_j(b) = q$, $j = 1, 2$. Show that

$$\oint_{\gamma_1} f\,dz = \oint_{\gamma_2} f\,dz$$

Proof. Let $\gamma : [a, 2b + a] \to U$ be defined as the following:

$$\gamma(t) = \begin{cases} \gamma_1(t) & a \leq t \leq b \\ \gamma_2(2b - t) & b \leq t \leq 2b + a. \end{cases}$$

More intuitively, γ can be seen as the curve that travels the path of γ_1 and then travels back along the path γ_2 in the opposite direction. By Theorem 1.5.3, since f is holomorphic on an open disc or an open rectangle U, there is a holomorphic function H such that $f = H'$ on U. Using this, we see that

$$\oint_{\gamma_1} f\,dz - \oint_{\gamma_2} f\,dz = \oint_{\gamma} f\,dz = \oint_{\gamma} H'\,dz = H(\gamma(2b+a)) - H(\gamma(a)) = H(\gamma_2(a)) - H(\gamma_1(a)) = 0$$

which proves that

$$\oint_{\gamma_1} f\,dz = \oint_{\gamma_2} f\,dz$$

as desired. $\square$

Ex 4b Compute the following complex line integral:

$$\oint_{\gamma} \overline{z} + z^2\overline{z}\,dz$$

where γ is the unit square with clockwise orientation.

Proof. We define $\gamma(t)$ as specified in the following way

$$\gamma(t) = \begin{cases} it & t \in [0,1] \\ (t-1)+i & t \in [1,2] \\ 1+i(3-t) & t \in [2,3] \\ 4-t & t \in [3,4] \end{cases}$$

Using the definition of a line integral (Definition 2.1.5), we have that

$$\begin{aligned}
&\oint_\gamma F(z)\,dz = \int_0^4 F(\gamma(t))\gamma'(t)\,dt \\
&= \int_0^1 F(it)i\,dt + \int_1^2 F((t-1)+i)\,dt + \int_2^3 F(1+i(3-t))(-i)\,dt + \int_3^4 F(4-t)(-1)\,dt \\
&= \int_0^1 iF(it)\,dt + F(t+i) - iF(1+i(1-t)) - F(1-t)\,dt \\
&= \int_0^1 i(-it)(1-t^2) + (t-i)(1+(t+i)^2) \\
&\qquad - i(1-i(1-t))(1+(1+i(1-t))^2) - (1-t)(1+(1-t)^2)\,dt \\
&= \int_0^1 t(4+2i) - (1+3i)\,dt = \frac{4+2i}{2} - (1+3i) = 1-2i.
\end{aligned}$$

□

Ex 17 Give an example to show that Lemma 2.3.1 is false if F is not assumed to be continuous at p.

Proof. Take the functions $F = \operatorname{sign}(x)$ and $H = |x|$ on $(-1,1)$ and the point $p = 0$. We clearly see that $H'(x) = F(x)$ on $(-1,1) \setminus \{0\}$. However, we also know that $H'(0)$ does not exist and that there's no way to extend $H'(x)$ to 0 as it's not a removable discontinuity. Thus, Lemma 2.3.1 is clearly false in this case. □

Ex 18abcf Compute each of the following complex line integrals:

a) $\oint_\gamma \frac{s^2}{s-1}\,ds$ where γ describes the circle of radius 3 with center 0 and counterclockwise orientation;

b) $\oint_\gamma \frac{s}{(s+4)(s-1+i)}\,ds$ where γ describes the circle of radius 1 with center 0 and counterclockwise orientation;

c) $\oint_\gamma \frac{1}{s+2}\,ds$ where γ is a circle, centered at 0, with radius 5, oriented clockwise;

f) $\oint_\gamma \frac{s(s+3)}{(s+i)(s-8)}\,ds$ where γ is the circle with center $2+i$ and radius 3 with clockwise orientation.

Proof.

a) Since $f(z) = z^2$ is holomorphic, we have by the Cauchy Integral Formula that

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(s)}{s-z}\, ds$$

for z inside γ. This means that

$$1 = 1^2 = f(1) = \frac{1}{2\pi i} \oint_\gamma \frac{s^2}{s-1}\, ds$$

which proves that our integral is equal to $2\pi i$.

b) We see that $f(z) = \frac{z}{(z+4)(z-1+i)}$ is itself holomorphic over an open disk containing γ. Thus, $\oint_\gamma f(z)\, dz = 0$.

c) Let γ' be γ but oriented counterclockwise. Since $f(z) = 1$ is holomorphic everywhere, we have by the Cauchy Integral Formula that

$$1 = f(z) = \frac{1}{2\pi i} \oint_{\gamma'} \frac{f(s)}{s-z}\, ds = \frac{1}{2\pi i} \oint_{\gamma'} \frac{1}{s-z}\, ds$$

for z inside γ. This means that

$$1 = f(-2) = \frac{1}{2\pi i} \oint_{\gamma'} \frac{1}{s+2}\, ds$$

proving that our integral over γ' is equal to $2\pi i$. This means that our integral over γ is equal to $-2\pi i$.

d) Again, we let γ' be γ but oriented counterclockwise. Since $f(z) = \frac{z(z+3)}{z-8}$ is holomorphic over an open disk containing γ', we have that

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma'} \frac{f(s)}{s-z}\, ds$$

for z inside of γ'. If we let $z = -i$, then we have that

$$\frac{1}{2\pi i} \oint_{\gamma'} \frac{s(s+3)}{(s-8)(s+i)}\, ds = f(-i) = \frac{-i(3-i)}{-(8+i)} = \frac{1+3i}{8+i} = \frac{(1+3i)(8-i)}{64-1} = \frac{11}{63} + i\frac{23}{63}.$$

This proves that our integral over γ' is equal to $\left(\frac{11}{63} + i\frac{23}{63}\right) 2\pi i = -\frac{46\pi}{63} + i\frac{22\pi}{63}$ and thus our original integral over γ is equal to $\frac{46\pi}{63} - i\frac{22\pi}{63}$.

$\square$