Problem Set 3
Abstract Algebra II

Bennett Rennier
barennier@gmail.com

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Section 15.1

Ex 1 Prove the converse to Hilbert's Basis Theorem: if the polynomial ring $R[x]$ is Noetherian, then R is Noetherian.

Proof. Suppose that $R[x]$ is a Noetherian polynomial ring. Then the evaluation map $e : R[x] \to R$ by $e(p(x)) = p(0)$ is a surjective homomorphism. It's surjective as the constants in $R[x]$ map to R. This shows that $R \simeq R[x]/\ker e$. Since, by Proposition 1, the quotient of a Noetherian ring by any ideal is Noetherian, this proves that R is Noetherian. [Note that more explicitly, $\ker e = (x)$ in this instance.] $\square$

Ex 2 Show that each of the following rings are not Noetherian by exibiting an explicit infinite increasing chain of ideals:

a) the ring of continuous real-valued functions on $[0, 1]$.

b) the ring of all functions from any infinite set X to $\mathbb{Z}/2\mathbb{Z}$.

Proof. a) Let
$$\mathcal{I}([0,1]) \subseteq \mathcal{I}([0,1/2]) \subseteq \mathcal{I}([0,1/3]) \subseteq \dots$$
be an increasing chain of ideals in this ring, where $I_n = \mathcal{I}([0,1/n])$. We clearly see that this ideals are increasing, as any function that vanishes on $[0,1/j]$ also vanishes on $[0,1/k]$ for all $k \geq j$. It's also clear that all of these inclusions are strict, as the function $f_n(x) = \begin{cases} 0 & x \in [0,1/n] \\ x - 1/n & \text{otherwise} \end{cases}$ vanishes exactly on $[0,1/n]$ and is thus in I_n, but not in I_j for any $j < n$. This proves that this sequence of ideals never stablizes.

b) Since X is infinite, we can pick a sequence of elements in it. Call this sequence a_n. Let I_i be the ideal of all functions that send a_j to 0 for all $j \geq i$. That is, $I_i = \mathcal{I}(a_i, a_{i+1}, a_{i+2}, \dots)$. Similar to the last example, a function that vanishes on $a_i, a_{i+1}, \dots$ also vanishes on $a_{i+1}, a_{i+2}, \dots$, so this is an increasing chain of ideals. Also, the function

$f_i(x) = \begin{cases} 1 & x \in \{a_1, a_2, \dots, a_i\} \\ 0 & \text{otherwise} \end{cases}$ is in I_{i+1} but not in any of the previous ideals. This proves that the sequence of ideals is strict, and thus never stablizies.

$\square$

Ex 3 Prove that the field $k(x)$ of rational functions over k in the variable x is not a finitely generated k-algebra.

Proof. Suppose that $k(x)$ were finitely generated. Let $\frac{p_1(x)}{q_1(x)}, \frac{p_2(x)}{q_2(x)}, \dots, \frac{p_n(x)}{q_n(x)}$ be the rational functions that generate $k(x)$. We can see that the set of primes in $k(x)$ is infinite, using the same argument as Euclid's proof (Suppose $\{p_i(x)\}_{i \leq n}$ are prime, and then look at $\prod_i p_i(x) + 1$). Since $k(x)$ is an integral domain, primes are also irreducibles, meaning there are infinitely many irreducibles. Let $r(x)$ be an irreducible which is not a factor of $q_1(x)q_2(x)\dots q_n(x)$. This means that no matter how you multiply the rational functions that we've claimed to generate $k(x)$, it's impossible to get $\frac{1}{r(x)}$. This proves that $k(x)$ is not a finitely generated k-algebra. $\square$