Problem Set 9
Topology II

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Ex 1.

a) Explain what an abelian covering space is. Show that there's an abelian covering space that is a covering space of every other abelian covering space and that such a "universal" abelian covering space is unique up to isomorphism. Describe the space explicitly for $S^1 \vee S^1$ and $S^1 \vee S^1 \vee S^1$.

b) Repeat the previous exercise for nilpotent covering spaces.

Proof.

a) An abelian covering space is a normal covering space whose deck transformation group is abelian. Let X be a nicely connected space and let $G = \pi_1(X, x_0)$. Recall that $[G, G]$, the commutator subgroup of G, is the unique smallest normal subgroup of G such that $G/[G, G]$ is abelian. Let $p : \tilde{X} \to X$ be a covering such that $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = [G, G]$. Since $[G, G]$ is a normal subgroup of G, we know that p is a normal covering. As p is normal, we also know that $\mathrm{Deck}(\tilde{X}) = \pi_1(\tilde{X}, \tilde{x}_0)/[G, G]$, which is abelian by the properties of $[G, G]$. This proves that p is an abelian cover.

   Now let $q : \bar{X} \to X$ be any other abelian cover with $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Since q is an abelian cover, its deck group is the abelian group $\pi_1(\tilde{X}, \tilde{x}_0)/H$. Since $[G, G]$ is the smallest normal subgroup such that $G/[G, G]$ is abelian, it must be that $H \leq [G, G]$. This proves that $\tilde{X}$ covers $\bar{X}$. We see that $\tilde{X}$ is unique up to isomorphism because $[G, G]$ is the unique subgroup with these required properties.

   Let $S^1 \vee S^1$ be generated by a and b and let $\tilde{X}$ be the space of integer grid lines in $\mathbb{R}^2$. If we project the vertices of the grid to the wedge point and the horizontal and vertical sides to a and b respectively, then we see that $\tilde{X}$ is a covering space of $S^1 \vee S^1$. We see the deck group of $\tilde{X}$ is simply the group of integer translations $\mathbb{Z} \oplus \mathbb{Z}$. As any vertex can be translated to any other vertex via translation, this covering is normal. Since

$$\mathbb{Z} * \mathbb{Z}/[\mathbb{Z} * \mathbb{Z}, \mathbb{Z} * \mathbb{Z}] = (\mathbb{Z} * \mathbb{Z})_{\text{ab}} = \mathbb{Z} \oplus \mathbb{Z},$$

   we see that this is indeed the univeral abelian covering space of $S^1 \vee S^1$. Similarly the universal abelian covering space of $S^1 \vee S^1 \vee S^1$ is simply the integer grid residing in $\mathbb{R}^3$.

b) Assume G is finite. Similar to how $[G, G]$ is the smallest normal subgroup of G such that $G/[G, G]$ is abelian (which is really just a group with nilpotent class 1), if we let $G_0 = G$ and $G_i = [G_{i-1}, G]$ for $i \geq 1$, then G_i is the smallest normal subgroup of G such that G/G_i is nilpotent with class i. Since G is finite, these G_i must converge after some finite number of steps. Let $H = \cap_i G_i$ be this group. We see that H is the smallest normal subgroup of G such that G/H is nilpotent. The existence of a universal nilpotent covering space follows similarly from the previous exercise.

I'm not sure how to prove that such an H exists if G is infinite, though. And unfortunately, I also have no idea how to describe the universal nilpotent covering space of $S^1 \vee S^1$. $\square$

Ex 2. Prove that a closed, orientable surface M_g^2 of genus g has a connected normal covering space with deck group $\mathbb{Z}^n$ if and only if $n \leq 2g$. Draw pictures and understand for $g \in \{0, 1, 2\}$ and all possible n.

Proof. From the first few pages of the book (specifically page 5), a closed orientable surface M_g^2 can be constructing as a CW complex involving one 0-cell, 2g 1-cells, and 1 2-cell. This means that the fundamental group of M_g^2 is the free group on 2g elements modded by a single relation, specifically

$$\pi_1(M_g^2) = \langle a_1, b_1, a_2, b_2 \ldots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \ldots [a_g, b_g] \rangle.$$

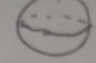
By Exercise 1, the universal abelian cover of M_g^2 has deck group $\pi_1(M_g^2)_{\text{ab}} \simeq \mathbb{Z}^n$ (as the modded relation becomes trivial in the abelianization). Since this covers all over abelian covers, there is no cover with deck group $\mathbb{Z}^m$ for $m > 2g$.

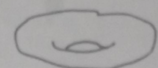
Since the group

$$G = \langle a_1, b_1, a_2, b_2, \ldots, a_g | [a_1, b_1][a_2, b_2][a_{g-1}, b_{g-1}] \rangle$$

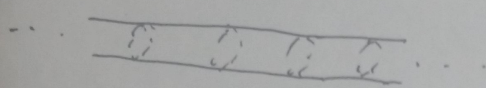
is a subgroup of $\pi_1(M_g^2)$, there is a covering of M_g^2 with fundamental group G. Similar to the previous paragraph, this covering has its own universal abelian covering space, which has deck group $\pi_1(G)_{\text{ab}} \simeq \mathbb{Z}^{n-1}$. Since this space covers a space covering M_g^2, it is a covering of M_g^2 as well. Note that since normality can be defined purely in terms of the automorphisms of the covering space, this means that the cover is still normal over M_g^2. We can continue removing generators from G in this fashion to get abelian covering spaces with deck group $\mathbb{Z}^n$ for all $n \leq 2g$.

Here are the pictures for $g \in \{0, 1, 2\}$ and all possible n (found on the next page):

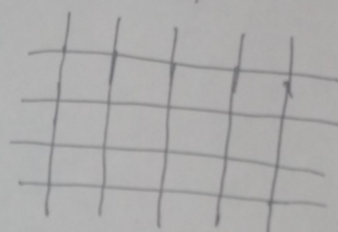
$g=0$: M_0^2 looks like the sphere . This only has one covering space which is itself. The deck group is $\pi_1(M_0^2)=0\simeq\mathbb{Z}^0$ as desired.

$g=1$: M_1^2 looks like the torus . This has the normal covering spaces:

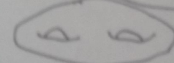
(1) $S^1\times\mathbb{R}$



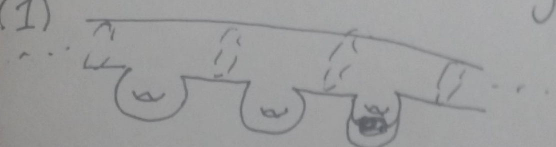
(2) $\mathbb{R}^2$



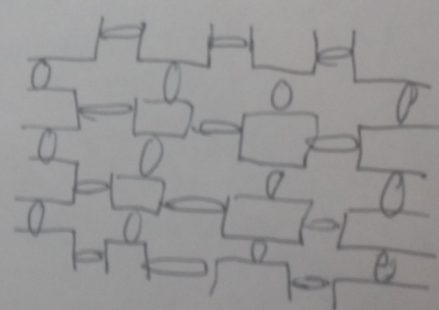
which have deck groups $\mathbb{Z}$ and $\mathbb{Z}^2$ respectively via integer translations.

$g=2$: M_2^2 looks like the double torus . This has the normal covering spaces:

(1)

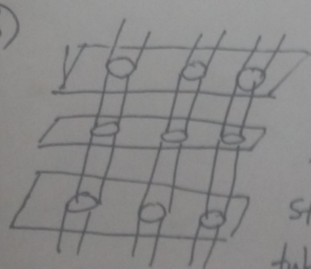


(2)



Supposed to be grid of tubes

(3)



Err, I'm getting worse at drawing these. It's an infinite stack of planes with tubes connecting them at integer grid points

(4) $\mathbb{R}^4$

(Drawing omitted)

which have deck groups $\mathbb{Z}$, $\mathbb{Z}^2$, $\mathbb{Z}^3$, and $\mathbb{Z}^4$ respectively via integer translations

□

Ex 3. Recall that each fiber $p^{-1}(x)$ of the universal cover $p : \tilde{X} \to X$ admits a left action by the deck group and a right action by $\pi_1(X, x)$. Remind yourself why these actions are both free and transitive and that they commute. When are they the same action?

Proof. Since $\tilde{X}$ is a universal cover, let $\varphi : \mathrm{Deck}(\tilde{X}) \to \pi_1(X, x_0)$ be the usual isomorphism. The question is does this isomorphism agree across the two actions. To prove this, let d and b be deck transformations and α and β be loops that lift to paths on the fibers such that

$$\varphi(d) = \alpha, \qquad \varphi(b) = \beta.$$

Simply converting the right action $\tilde{x}.\alpha$ by $\pi_1(X, x_0)$ into a left action $\alpha.\tilde{x}$ gives use the same left action by $\mathrm{Deck}(\tilde{X})$ if and only if

$$\tilde{x}.(\alpha\beta) = (\alpha\beta).\tilde{x} = (db).\tilde{x} = d.(b.\tilde{x}) = d.(\beta.\tilde{x}) = \alpha.(\beta.\tilde{x}) = (\beta\alpha).\tilde{x} = \tilde{x}.(\beta\alpha)$$

As $\pi_1(X, x_0)$ acts transitively, this is equivalent to saying that $\alpha\beta = \beta\alpha$. Thus the two actions are the same when $\pi_1(X, x_0)$ is abelian. $\square$

Ex 4. (Hatcher 1.3.32) When a space X is a CW complex, we often require maps to and from X to map cells to cells. Moreover, when $p : \tilde{X} \to X$ is a covering map between CW complexes, we further insist that the cells are sufficiently small so that cells are mapped homeomorphically to cells.

a) Prove that the restriction to the 1-skeleton is a covering, which is normal if and only if the original covering is normal. Moreover, the groups of deck transformations of the cover and its restriction are the same.

b) Prove that two such covers are isomorphic if and only if their restrictions to the 1-skeleton are.

Proof. Proof not completed. $\square$