Problem Set 2
Real Analysis II

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Ex 12.2 Let μ be a signed measure. Define

$$\int f\,d\mu = \int f\,d\mu^+ - \int f\,d\mu^-$$

Prove that

$$\left|\int f\,d\mu\right| \le \int |f|\,d\,|\mu|$$

Proof. Firstly, we see that in general,

$$\int f\,d(\mu+\nu) = \int f\,d\mu + \int f\,d\nu$$

where $(\mu+\nu)(A) = \mu(A)+\nu(A)$. This can be proven easily using simple functions and then building up to arbitrary functions. Using this, we see that

$$\begin{aligned}\left|\int f\,d\mu\right| &= \left|\int f\,d\mu^+ - \int f\,d\mu^-\right| \le \left|\int f\,d\mu^+\right| + \left|\int f\,d\mu^-\right| \\ &\le \int |f|\,d\mu^+ + \int |f|\,d\mu^- = \int |f|\,d(\mu^+ + \mu^-) = \int |f|\,d\,|\mu|\end{aligned}$$

□

Ex 12.3 Let μ be a finite signed measure on $(X,\mathcal{A})$. Prove that

$$|\mu|(A) = \sup\left\{\left|\int_A f\,d\mu\right| : |f| \le 1\right\}$$

Proof. In the last exercise, we proved that

$$\left|\int f\,d\mu\right| \le \int |f|\,d|\mu|$$

This means that

$$\left|\int_A f\,d\mu\right| = \left|\int f\chi_A\,d\mu\right| \le \int |f\chi_A|\,d|\mu| = \int_A |f|\,d|\mu| \le \int_A 1\,d|\mu| = |\mu|(A)$$

if $|f| \le 1$. This proves one inequality, that is that

$$\sup\left\{\left|\int_A f\,d\mu\right| : |f| \le 1\right\} \le |\mu|(A).$$

To prove the other inequality, we use the Hahns Decomposition Theorem. Let P and N be the positive and negative sets of μ in this decomposition, and let $f = \chi_P - \chi_N$. Since P and N are disjoint, we have that $|f| \le 1$. We see that

$$\begin{aligned}\int_A f\,d\mu = \int \chi_A(\chi_P - \chi_N)\,d\mu = \int \chi_{A\cap P}\,d\mu + \int \chi_{A\cap N}\,d\mu &= \mu(A\cap P) + \mu(A\cap N)\\ &= \mu^+(A) + \mu^-(A) = |\mu|(A)\end{aligned}$$

This proves that reverse inequality and thus the statement. □

Ex 12.7 Suppose that μ is a signed measure on $(X, \mathcal{A})$. Prove that if $A \in \mathcal{A}$, then

$$|\mu|(A) = \sup\left\{\sum_{j=1}^{n} |\mu(B_j)| : \text{each } B_j \in \mathcal{A}, \text{the } B_j \text{ are disjoint}, \cup_{j=1}^{n} B_j = A\right\}$$

Proof. Use the Hahns Decomposition to decompose X into P and N. Suppose $B_1 = A \cap P$ and $B_2 = A \cap N$. We see that $B_1 \cup B_2 = A$, that $B_1, B_2 \in \mathcal{A}$, and that B_1 and B_2 are disjoint. Thus, the supremum on the right-hand side must be bigger than or equal to $|\mu(B_1)| + |m(B_2)| = |\mu(A\cap P)| + |\mu(A\cap N)| = \mu^+(A) + \mu^-(A) = |\mu|(A)$. This proves $\le$.

For the reverse inequality, let $\{B_j\}_{j\le n}$ be an finite arbitrary disjoint collection of measurable sets that union up to A. We see that

$$|\mu(B_j)| = |\mu(B_j\cap N) + \mu(B_j\cap P)| \le |\mu(B_j\cap N)| + |\mu(B_j\cap P)| = \mu^+(B_j) + \mu^-(B_j) = |\mu|(B_j).$$

This proves that

$$\sum_{j=1}^{n} |\mu(B_j)| \le \sum_{j=1}^{n} |\mu|(B_j) = |\mu|(\cup_j B_j) = |\mu|(A)$$

as the B_j's were assumed to be disjoint. This proves $\ge$, and thus the statement. □

Ex 14.2 If f is integrable and real-valued, $a \in \mathbb{R}$, and

$$F(x) = \int_a^x f(y)\,dy$$

prove that F is of bounded variation and is absolutely continuous.

Proof. Recall from Lemma 14.14 that if a function is absolutely continuous, then it is of bounded variation. Thus, we only need to prove that $F(x)$ is absolutely continuous. Suppose that $F(x)$ is not absolutely continuous. This means that we can find a finite collection of disjoint intervals $\{(a_i, b_i)\}$ such that $\sum_{i=1}^{k} |b_i - a_i|$ can be as small as we want, but $\sum_{i=1}^{k} |f(b_i) - f(a_i)| > M$ for some $M \in \mathbb{R}$. We see that

$$|F(b_i) - F(a_i)| = \left| \int_a^{b_i} f(y)\, dy - \int_a^{a_i} f(y)\, dy \right| = \left| \int_{a_i}^{b_i} f(y)\, dy \right| \leq \int_{a_i}^{b_i} |f(y)|\, dy$$

which means that

$$\sum_{i=1}^{k} |f(b_i) - f(a_i)| \leq \sum_{i=1}^{k} \int_{a_i}^{b_i} |f(y)|\, dy = \int_{\cup_i (a_i, b_i)} |f(y)|\, dy$$

We see though that if $\sum_{i=1}^{k} |b_i - a_i| \to 0$, then this implies that $\sum_{i=1}^{k} m((a_i, b_i)) = m(\cup_i (a_i, b_i)) \to$ 0. This means that

$$\lim_{m(\cup_i (a_i, b_i)) \to 0} \int_{\cup_i (a_i, b_i)} |f(y)|\, dy = \int \lim_{m(\cup_i (a_i, b_i)) \to 0} \chi_{\cup_i (a_i, b_i)} |f(y)|\, dy = 0$$

by the Dominated Convergence Theorem. Therefore, there can be no such M, which proves that $F(x)$ must be absolutely continuous. $\square$

Ex 14.3 Suppose that f is a real-valued continuous function on $[0, 1]$ and that $\varepsilon > 0$. Prove that there exists a continuous function g such that $g'(x)$ exists and equals 0 for almost every x and

$$\sup_{x \in [0,1]} |f(x) - g(x)| < \varepsilon$$

Proof. Since f is continuous on a compact set, this means that f is uniformly continuous. Using this, let $\delta > 0$ such that

$$|x - y| \leq \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}.$$

We are going to use δ to partition $[0, 1]$. Suppose $x \in [n\delta, (n + 1)\delta]$. We let $g(x) = f(n\delta) + C((x - n\delta)/\delta)(f((n + 1)\delta) - f(n\delta))$, where $C(x)$ is the Cantor-Lebesgue function. Intuitively, we are breaking f into pieces small enough that they can be approximated by a scaled version of the Cantor-Lebesgue function.

We see that for each interval, that $g(x)$ is continuous as $C(x)$ is continuous. We also see that at each endpoint $n\delta$, the two definitions agree, which means that $g(x)$ is well-defined and continuous as a whole.

Also, on each interval, We see that $g'(x) = 0$ almost everywhere, as the only nonconstant part of $g(x)$ is a scaled version of $C(x)$. This proves that $g'(x) = 0$ almost everywhere on $[0, 1]$ as well.

Finally, we see that for $x \in [n\delta, (n + 1)\delta]$,

$$|f(x) - g(x)| = |f(x) - f(n\delta) + C((x - n\delta)\delta)(f((n + 1)\delta) - f(n\delta))|$$

Since $x \in [n\delta, (n+1)\delta]$, we see that $|x - n\delta| \leq \delta$ and that $|(n+1)\delta - n\delta| \leq \delta$, which means that $|f(x) - f(n\delta)| < \frac{\varepsilon}{2}$ and that $|f((n+1)\delta) - f(n\delta)| < \frac{\varepsilon}{2}$. This proves that

$$|f(x) - g(x)| < \frac{\varepsilon}{2} + C((x - n\delta)\delta)\frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

since $0 \leq C(x) \leq 1$. This proves the statement. $\square$