Problem Set 5
Real Analysis

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Ex 4.5 Suppose m is Lebesgue measure. Define $x+A=\{x+y \mid y\in A\}$ and $cA=\{cy \mid y\in A\}$ for $x\in\mathbb{R}$ and c a real number. Show that if A is a Lebesgue measurable set, then $m(x+A)=m(A)$ and $m(cA)=|c|m(A)$.

Proof. Let $B_i=(c_i,d_i]$ be a covering of A. Notice that since $A\subseteq\cup_{i=1}^\infty B_i$, then $x+A\subseteq\cup_{i=1}^\infty x+B_i$. We also see that $\ell(x+B_i)=\ell((x+c_i,x+d_i])=(x+d_i)-(x+c_i)=d_i-c_i=\ell((c_i,d_i])=\ell(B_i)$. Thus, $\sum_{i=1}^\infty\ell(B_i)=\sum_{i=1}^\infty\ell(x+B_i)$. Since $x+A\subseteq x+\cup_{i=1}^\infty B_i$, this shows that $m^*(x+A)\le m^*(x+\cup_{i=1}^\infty B_i)\le\sum_{i=1}^\infty m^*(x+B_i)=\sum_{i=1}^\infty\ell(x+B_i)=\sum_{i=1}^\infty\ell(B_i)$. Since $m^*(x+A)$ is less than any half-open covering of A, this means that $m^*(x+A)\le m^*(A)$. If we let $x+A=C$, we see that $A=-x+C$. Thus, we can do a similar argument on C and $-x+C$, and see that $m^*(-x+C)\le m^*(C)$, which means that $m^*(A)\le m^*(x+A)$. This proves that $m^*(A)=m^*(x+A)$.

If $c\ge 0$, then $\sum_{i=1}^\infty\ell(cB_i)=\sum_{i=1}^\infty(c\cdot c_i,c\cdot d_i]=\sum_{i=1}^\infty c(d_i-c_i)=c\sum_{i=1}^\infty d_i-c_i=c\sum_{i=1}^\infty\ell(B_i)$. If $c<0$, then $\sum_{i=1}^\infty\ell(cB_i)=\sum_{i=1}^\infty\ell(c\cdot(c_i,d_i])=\sum_{i=1}^\infty\ell((c\cdot c_i,c\cdot d_i])=\sum_{i=1}^\infty\ell([(-c)(-d_i),(-c)(-c_i)))=\sum_{i=1}^\infty -c(-c_i+d_i)=-c\sum_{i=1}^\infty\ell(B_i)$. Thus, $\sum_{i=1}^\infty\ell(cB_i)=|c|\sum_{i=1}^\infty\ell(B_i)$. Thus, using a similar argument as above, we see that $m^*(cA)=|c|m^*(A)$.

Now to prove that $x+A$ is Lebesgue measurable. Since A is Lebesgue measurable, then for any E,

$$m^*(E)=m^*(E\cap A)+m^*(E\cap A^c)$$

Let $E=-x+F$. This means that

$$m^*(-x+F)=m^*((-x+F)\cap A)+m^*((-x+F)\cap A^c)$$

Let $y\in(-x+F)\cap A$. This means that $y\in -x+F$ and $y\in A$. Thus, $y+x\in F$ and $y+x\in x+A$. This proves that $y+x\in F\cap(x+A)$ and thus $y\in -x+(F\cap(x+A))$. This is reversible, and so $m^*((-x+F)\cap A)=m^*(-x+(F\cap(x+A)))=m^*(F\cap(x+A))$. Similarly, $m^*((-x+F)\cap A^c)=m^*(F\cap(x+A^c))$. If $y\in x+A^c$, then $y-x\notin A$, which means that $y\notin x+A$, and thus $y\in(x+A)^c$. This is reversible, so $m^*((-x+F)\cap A)=m^*(F\cap(x+A^c))=m^*(F\cap(x+A)^c)$. Thus,

$$m^*(-x+F)=m^*(F)=m^*(F\cap(x+A))+m^*(F\cap(x+A)^c)$$

Since E was arbitrary, so is F, and thus, $x + A$ is measurable.

Let $E = \frac{1}{c}F$. Similarly, we get that

$$m^* \left(\frac{1}{c}F\right) = m^* \left(\frac{1}{c}F \cap A\right) + m^* \left(\frac{1}{c}F \cap A^c\right)$$

Let $y \in \frac{1}{c}F \cap A$, then $cy \in F$ and $cy \in cA$. This means that $cy \in F \cap cA$, and thus that $y \in \frac{1}{c}(F \cap cA)$. This is all reversible once again. We also see that if $y \in cA^c$, then $\frac{1}{c}y \in A^c$ which means that $\frac{1}{c}y \notin A$, and then that $y \notin cA$, and thus $y \in (cA)^c$. This is again, reversible. This shows that

$$m^* \left(\frac{1}{c}F\right) = m^* \left(\frac{1}{c}(F \cap cA)\right) + m^* \left(\frac{1}{c}(F \cap (cA)^c)\right)$$

Pulling out the $\frac{1}{c}$ as a $|\frac{1}{c}|$ and dividing by $|\frac{1}{c}|$, this proves that cA is measurable. $\square$

Ex 4.6 Let m be Lebesgue measure. Suppose for each n, A_n is a Lebesgue measurable subset of $[0, 1]$. Let B consist of those points x that are in infinitely many of the A_n.

1) Show B is Lebesgue measurable

2) If $m(A_n) > \delta > 0$ for each n, show $m(B) \geq \delta$

3) If $\sum_{n=1}^{\infty} m(A_n) < \infty$ prove that $m(B) = 0$

4) Give an example where $\sum_{n=1}^{\infty} m(A_n) = \infty$, but $m(B) = 0$

Proof. 1) We see that

$$B = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

To prove this, let $x \in B$. Then x is in infinitely many A_n. This means that $x \in \cup_{k=n}^{\infty} A_k$ for all n, which means that $x \in \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$. Now, if $x \notin B$, that is that x is only in finitely many A_n (perhaps none of them), say $A_{n_1}, \ldots, A_{n_j}$, then $x \notin \cup_{k=n_j+1}^{\infty} A_k$, which means that $x \notin \cap_{n=1}^{\infty} \cup_{k=1}^{\infty} A_k$. This proves that the two sets are equal. Since the Lebesgue measurable sets form a σ-algebra, they are closed under countable unions and countable intersections. Thus, B is Lebesgue measurable.

2) We see that $m(B) = \lim_{j\to\infty} m\left(\cap_{n=1}^{j} \cup_{k=n}^{\infty} A_k\right)$. Let $B_n = \cup_{k=n}^{\infty} A_k$. We see that $B_{n+1} \subseteq B_n$. Thus, $\cap_{n=1}^{j} B_n = B_j$. Since $A_j \subseteq B_j$, we see that $m\left(\cap_{n=1}^{j} \cup_{k=n}^{\infty} A_k\right) = m\left(\cap_{n=1}^{j} B_j\right) = m(B_j) \geq m(A_j) > \delta$ for any j. Thus, $m(B) = \lim_{j\to\infty} m\left(\cap_{n=1}^{j} \cup_{k=n}^{\infty} A_k\right) \geq \delta$.

3) Let $\varepsilon > 0$. If $\sum_{n=1}^{\infty} m(A_n) < \infty$, then there's a k such that $\sum_{n=1}^{\infty} m(A_n) - \sum_{n=1}^{k} m(A_n) < \varepsilon$. Notice that

$$\sum_{n=1}^{\infty} m(A_n) = \sum_{n=k}^{\infty} m(A_n) + \sum_{n=1}^{k} m(A_n)$$

Thus, since these are all finite, we see that $\sum_{n=k}^{\infty} m(A_k) = \sum_{n=1}^{\infty} m(A_n) - \sum_{n=1}^{k} m(A_n) < \varepsilon$. Since $B \subseteq \cup_{n=k}^{\infty} A_k$ for any k, we see that $m(B) \leq m(\cup_{n=k}^{\infty} A_n) \leq \sum_{n=k}^{\infty} A_n < \varepsilon$. Thus, $m(B) < \varepsilon$ for any $\varepsilon > 0$. This proves that $m(B) = 0$.

4) Let A_n be the Fat Cantor set on $[0,1]$, where $m(A_n) = \frac{1}{n}$. Thus, $\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$. I think this set works, but I do not know how to prove that $m(B) = 0$.

$\square$

Ex 4.7 Suppose $\varepsilon \in (0,1)$ and m is Lebesgue measure. Find a measurable set $E \subseteq [0,1]$ such that the closure of E is $[0,1]$ and $m(E) = \varepsilon$.

Proof. Let $Q = \mathbb{Q} \cap [0,1]$, and let $E = (0,\varepsilon) \cup Q$. Since $(0,\varepsilon) \subseteq [0,1]$ and $Q \subseteq [0,1]$, this means that $E \subseteq [0,1]$. Since $\bar{Q} = [0,1]$ and $Q \subseteq E$, we see that $[0,1] \subseteq \bar{E}$. Since $E \subseteq [0,1]$ and $[0,1]$ is closed, that means $\bar{E} \subseteq [0,1]$. This proves that $\bar{E} = [0,1]$.

We see that $m(E) \le m(Q) + m((0,\varepsilon)) = 0 + \varepsilon = \varepsilon$. Also, since $(0,\varepsilon) \subseteq E$, then $m((0,\varepsilon)) = \varepsilon \le m(E)$. Thus, $m(E) = \varepsilon$. This proves the statement. $\square$

Ex 4.10 Let $\varepsilon \in (0,1)$, let m be Lebesgue measure, and suppose A is a Borel measurable subset of $\mathbb{R}$. Prove that if

$$m(A \cap I) \le (1-\varepsilon)\, m(I)$$

for every interval I, then $m(A) = 0$.

Proof. Let $A_n = A \cap [-n,n]$. This means that $m(A_n) \le 2n$. Let $\{J_i\}$ be a collection of half-open/half-closed intervals covering A_n. Let $\{O_i\}$ be the same interval of $\{J_i\}$ except we remove the point on the closed side. Notice that $m(O_i) = m(J_i)$. If any of the points we removed from J_i was a point in A_n, then remove that point too from A_n and call the result A_n'. Since we're only removing at most countably many points, notice that $m(A_n') = m(A_n) \le \infty$, and that $A_n' \subseteq \cup_{i=1}^{\infty} O_i$. Now we see that,

$$m(A_n) = m(A_n') = m(A_n' \cap \cup_{i=1}^{\infty} O_i) \le m(A \cap \cup_{i=1}^{\infty} O_i) = m(\cup_{i=1}^{\infty} (A \cap O_i)) \le \sum_{i=1}^{\infty} m(A \cap O_i)$$

Since O_i is an open interval, we see that

$$m(A_n) \le \sum_{i=1}^{\infty} m(A \cap O_i) = \sum_{i=1}^{\infty} (1-\varepsilon)\, m(O_i) = (1-\varepsilon) \sum_{i=1}^{\infty} m(J_i) = (1-\varepsilon) \sum_{i=1}^{\infty} \ell(J_i)$$

If we take the infinmum over all such J_i's, we see that $m(A_n) \le (1-\varepsilon)\, m(A_n)$. This is only true if $m(A_n) = 0$. We see that

$$m(A) = m(\cup_{i=1}^{\infty} A_n) \le \sum_{i=1}^{\infty} m(A_n) = \sum_{i=1}^{\infty} 0 = 0$$

Thus, $m(A) = 0$. $\square$

Ex 4.12 Let m be Lebesgue measure. Construct a Borel subset A of $\mathbb{R}$ such that $0 < m(A \cap I) < m(I)$ for every open interval I.

Proof. Enumerate all closed intervals with rational endpoints as I_k. For I_k, construct the fat cantor set in the first half and the interval and call it A_k. Do the same for the second half of the interval and call it B_k. We see that A_k and B_k are disjoint. Let $A = \cup_{n=1}^{\infty} A_n$. If I is an open interval, then it contains at least two rationals, and thus contains an I_k for some k. Thus, $A_k \in I$ and $B_k \in I$. This means that

$$0 < m\left(A_k\right) \leq m\left(A \cap I\right) < m\left(A \cap I\right) + m\left(B_k\right) \leq m\left(I\right)$$

We see that $m\left(A \cap I\right) < m\left(A \cap I\right) + m\left(B_n\right)$ as A and B_n are disjoint. Thus $0 < m\left(A \cap I\right) < m\left(I\right)$. ☐

Ex 4.18 Suppose $A \subseteq \mathbb{R}$ has Lebesgue measure 0. Prove that there exists $c \in \mathbb{R}$ such that $A \cap (c + \mathbb{Q}) = \varnothing$, where $c + \mathbb{Q} = \{c + x \mid x \in \mathbb{Q}\}$ and $\mathbb{Q}$ is the rational numbers.

Proof. Suppose $A \subseteq \mathbb{R}$ has Lebesgue measure 0. Also, assume that for every $c \in \mathbb{R}$, $A \cap (c + \mathbb{Q}) \neq \varnothing$. For each $c \in \mathbb{R}$, let w_c be an element of $A \cap (c + \mathbb{Q})$, and let C be the collection of these w_c's. We see that $m^*\left(C\right) = m^*\left(C + q\right)$. Let $x \in [0, 1]$. Since $x \in \mathbb{R}$, then there's a $w_x = x + q$ for some rational q. Thus, $x = w_x - q$ for some rational q, which means that $x \in K - q$. This means that $[0, 1] \subseteq \cup_{q \in \mathbb{Q}} K + q$. This proves that $1 \leq m^*\left(\cup_{q \in \mathbb{Q}} K + q\right) \leq \sum_{q \in Q} m^*\left(K + q\right) = \sum_{q \in \mathbb{Q}} m^*\left(K\right)$. This implies that $m^*\left(K\right) > 0$. Since $K \subseteq A$, this means that $0 < m^*\left(K\right) \leq m^*\left(A\right) = m\left(A\right)$. However, this is a contradiction as $m\left(A\right) = 0$. This means that there must be some $c \in \mathbb{R}$ such that $A \cap (c + \mathbb{Q}) = \varnothing$. ☐