Problem Set 3
Topology II

Bennett Rennier
bennett@brennier.com

February 7, 2021

Ex Find the gap in the proof in class that the fundamental group of S^n is trivial for $n > 1$.

Proof. The gap is that assumption that a compact subset of $[0, 1]$ has only finitely many components. This is untrue as the set

$$\bigcup_{\substack{n\in\mathbb{N}\\ n \text{ is even}}} \left[\frac{1}{n+1}, \frac{1}{n}\right] \cup \{0\}$$

is compact. This is easy to verify as its bounded and the complement (in $[0, 1]$) is

$$\bigcup_{\substack{n\in\mathbb{N}\\ n \text{ is odd}}} \left(\frac{1}{n+1}, \frac{1}{n}\right)$$

which is open as its the union of open intervals. □

The following exercises are from Hatcher Section 1.1.

Ex 3 For a path-connected space X, show that $\pi_1(X)$ is abelian if and only if all basepoint-change homomorphisms β_h depend only on the endpoints of the path h.

Proof. Suppose that β_h depends only on the endpoints of h for any path h. Let $x_0 \in X$ be fixed and let $[g]$ be in $\pi_1(X, x_0)$. Since g is a loop with both endpoints being x_0, it has the same endpoints as the constant path const_{x_0}. By our assumption, this means that β_g is the same homomorphism as $\beta_{\text{const}_{x_0}}$. It's clear, though, that $\beta_{\text{const}_{x_0}}$ is simply the identity homomorphism. Thus, we see that for any $[f] \in \pi_1(X, x_0)$ we have that

$$[f] = \beta_{\text{const}_{x_0}}[f] = \beta_g[f] = [gfg^{-1}] = [g][f][g]^{-1}.$$

If we then multiply on the right by $[g]$, we have that $[f][g] = [g][f]$. Since $[f]$ and $[g]$ were arbitrary, we know that $\pi_1(X, x_0)$ is indeed abelian.

Now suppose that $\pi_1(X, x_0)$ is abelian for any $x_0 \in X$. We let f, g be paths from x_0 to x_1 and let $[h] \in \pi_1(X, x_0)$. We note that $f^{-1}g$ is loop from x_0 to x_0 so $[f^{-1}g] \in \pi_1(X, x_0)$ as well. Since $\pi_1(X, x_0)$ is abelian, we know that

$$[hf^{-1}g] = [h][f^{-1}g] = [f^{-1}g][h] = [f^{-1}gh].$$

In other words, $hf^{-1}g \simeq f^{-1}gh$. This implies that $hf^{-1} \simeq f^{-1}ghg^{-1}$. Thus, we have that

$$\beta_f[h] = [fhf^{-1}] = [ff^{-1}ghg^{-1}] = [ghg^{-1}] = \beta_g[h].$$

Since f, g, h were arbitrary, this proves that basepoint-change homomorphisms depend only on the endpoints. $\square$

Ex 5 Show that for a space X, the following three conditions are equivalent:

a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point.

b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space X is simply-connected if and only if all maps $S^1 \to X$ are homotopic (without regard to basepoints).

Proof.

$(a) \implies (b)$: Let $\varphi : S^1 \to X$ be a map. By assumption, there is a homotopy $h : S^1 \times [0, 1] \to X$ such that $h(e^{i\theta}, 0) = \varphi(e^{i\theta})$ and $h(e^{i\theta}, 1) = x_0$ for some $x_0 \in X$. Let $\psi : D^2 \to X$ be defined as $\psi(re^{i\theta}) = h(e^{i\theta}, 1 - r)$. This is well-defined since, even though $0 \cdot e^{i\theta} = 0$ for any θ, we have that

$$\varphi(0 \cdot e^{i\theta}) = h(e^{i\theta}, 1) = x_0$$

for any θ. Since h is continuous on the product $S^1 \times [0, 1]$ and $1 - r$ is a continuous function of r, we see that ψ is also continuous. Finally, as $\psi(1 \cdot e^{i\theta}) = h(e^{i\theta}, 0) = \varphi(e^{i\theta})$, ψ is an extension of φ to a map on D^2, as desired.

$(b) \implies (c)$: Let $\varphi : (S^1, 1) \to (X, x_0)$ be a loop with a fixed basepoint of $\varphi(e^{i0}) = \varphi(1) = x_0$. By assumption, this map φ can be extended to a continuous function $\psi : D^2 \to X$. This gives us the commutative diagram:

$$\begin{tikzcd}
(D^2, 1) \arrow[dr, "\psi"] & \\
(S^1, 1) \arrow[u, hook, "\iota"] \arrow[r, "\varphi"] & (X, x_0)
\end{tikzcd}$$

If we then apply the π_1 functor, we get following diagram in the category of groups:

$$\begin{tikzcd}
\pi_1(D^2, 1) \arrow[dr, "\psi_*"] & \\
\pi_1(S^1, 1) \arrow[u, hook, "\iota_*"] \arrow[r, "\varphi_*"] & \pi_1(X, x_0)
\end{tikzcd}$$

We know that $\pi_1(D^2, 1)$ is trivial as it's convex, so any loop can be straight-lined homotopied to its basepoint. This means that ψ_* must be the map from the identity in $\pi_1(D^2, 1)$ to the one in $\pi_1(X, x_0)$. Since the identities of these groups are simply the the homotopy class of the constant loop in their respective spaces, we get that

$$[\varphi] = \varphi_*[\mathrm{const}_1] = \psi_*\iota_*[\mathrm{const}_1] = [\mathrm{const}_{x_0}].$$

Thus, any loop with fixed basepoint $x_0 \in X$ is homotopic to the constant loop, which proves that $\pi_1(X, x_0)$ is trivial for any fixed x_0.

$(c) \implies (a)$: Let $\varphi : S^1 \to X$ be a map and let x_0 be the point $\varphi(e^{i0}) = \varphi(1)$. Since $\pi_1(X, x_0)$ is trivial, this means that $[f]$ and $[\mathrm{const}_{x_0}]$ must be the same element in $\pi_1(X, x_0)$. This means that f is homotopic to the constant map const_{x_0} as desired. (We even get that $f \simeq \mathrm{const}_{x_0}$ relative to the basepoint.)

Now, suppose that X is simply-connected. This means that X is path-connected and that $\pi_1(X, x_0)$ is trivial for any $x_0 \in X$. By $(c) \implies (a)$, this means that any map $S^1 \to X$ is homotopic to a constant map. Since X is path-connected, all constant maps are homotopic to each other. Thus, all maps $S^1 \to X$ are homotopic to each other.

Conversely, assume that all maps $S^1 \to X$ are homotopic. Particularly, we have that all constant maps $S^1 \to X$ are homotopic. A homotopy between constant maps const_{x_0} and const_{x_1} is simply a path from x_0 to x_1. This proves that X is path-connected. Additionally, since any map $S^1 \to X$ is homotopic to a constant map $S^1 \to X$, we see from $(a) \implies (c)$ that $\pi_1(X, x_0)$ is trivial for all $x_0 \in X$. This proves that X must be simply-connected. $\square$

Ex 6 We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$, with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$ when X is path-connected.

Proof. Let X be path-connected. We will first prove that Φ is onto. Let $x_0 \in X$ be fixed and let $f : S^1 \to X$ be any map. We would like to prove that f is homotopic to some map $(S^1, 1) \to (X, x_0)$. Since X is path-connected, there exists a path p from $f(1)$ to x_0. I proved in the last homework that $(I, \partial I)$ has the HEP. Thus if we identify ∂I as a single point, we see that $(S^1, 1)$ also has the HEP. Using this we obtain a homotopy h that makes the following diagram commute:

$$\begin{tikzcd}
\{1\} \times I \arrow[d, hook] \arrow[dr, "p"] & \\
S^1 \times I \arrow[r, dotted, "h"] & X \\
S^1 \times \{0\} \arrow[u, hook'] \arrow[ur, "f"'] &
\end{tikzcd}$$

By the diagram we see that this homotopy has the properties that $h_0 = f$ and that $h_t(1) = p(t)$. Thus, $f = h_0$ is homotopic to the map $h_1 : (S^1, 1) \to (X, x_0)$. This means we have that $\Phi([h_1]) = [f]$, which proves that Φ is surjective.

Now we will prove that

$$\Phi([f]) = \Phi([g]) \iff \exists h \in \pi_1(X, x_0),\ [f] = [h][g][h]^{-1}.$$

First, let $f, g : (S^1, 1) \to (X, x_0)$ be maps and assume that $\Phi([f]) = \Phi([g])$. This means f, g are in the same homotopy class when ignoring basepoints. Thus, there is some non-basepoint-preserving homotopy $h : S^1 \times I \to X$ where $h_0 = f$ and $h_1 = g$. Using this, we define a new homotopy h' by $h'_t = p_t h_t p_t^{-1}$ under path composition where $p_t(s) = h_{st}(1)$, which is the path from x_0 to $h_t(1)$. We see that this new homotopy preserves the basepoint and has the properties that $h'_0 = \mathrm{const}_{h_0(1)}\, h_0 \,\mathrm{const}_{h_0(1)} = h_0 = f$ and that $h'_1 = p_1 h_1 p_1^{-1} = p_1 g p_1^{-1}$ (remember, the multiplication is path-concatenation). Thus, we have that $f \simeq p_1 g p_1^{-1}$. Since $p_1(s) = h_s(1)$, we see that p_1 is a path from x_0 to $h_1(1) = x_0$. This proves that

$$[f] = [p_1 g p_1^{-1}] = [p_1][g][p_1]^{-1}$$

for some element $[p_1] \in \pi_1(X, x_0)$.

Now, let $f, g : (S^1, 1) \to (X, x_0)$ be loops and suppose there there's an element $[h] \in \pi_1(X, x_0)$ such that $[f] = [h][g][h]^{-1} = [hgh^{-1}]$. This means that $f \simeq hgh^{-1}$ (basepoint-preserving, but it doesn't matter). Now we wish to prove that there's a homotopy from hgh^{-1} to g (not necessarily basepoint-preserving). We let $H : S^1 \times I \to X$ be the homotopy defined by $H_t = p_t g p_t^{-1}$ under path composition where $p_t(s) = h((1-s)t + s)$, which is a path from $h(t)$ to $h(1)$. We see then that $H_0 = hgh^{-1}$ and that $H(1) = \mathrm{const}_{x_0}\, g \,\mathrm{const}_{x_0} = g$. This proves that $f \simeq hgh^{-1} \simeq g$ without preserving basepoints, which means that $\Phi([f]) = \Phi([g])$ as desired. □

Ex 8 Does the Borsuk-Ulam theorem hold for the torus? In other words, for every map $f : S^1 \times S^1 \to \mathbb{R}^2$ must there exist $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$.

Proof. First, we embed the torus in $\mathbb{R}^3$ so that it rests like a donut on top of the x, y-plane. If we then take the function π which projects down onto the x, y-plane, we see that it's continuous as it's a projection. Via this projection, the only points with the same image are those that lie on the same vertical line. Since (x, y) and $(-x, -y)$ are never on the same vertical line (they are on opposite sides of the origin and no point on the torus is above the origin), this means that they are always projected onto distinct points. Thus, there are no points such that $\pi(x, y) = \pi(-x, -y)$ which means that π is a counterexample to the proposed claim. □

Ex 10 From the isomorphism $\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$ it follows that loops in $X \times \{y_0\}$ and $\{x_0\} \times Y$ represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy demonstrating this.

Proof. Let $f : I_t \to X \times \{y_0\}$ and $g : I_t \to \{x_0\} \times Y$ be loops with a fixed basepoint of $x_0 \times y_0$. Via the definition of path concatenation, we see that

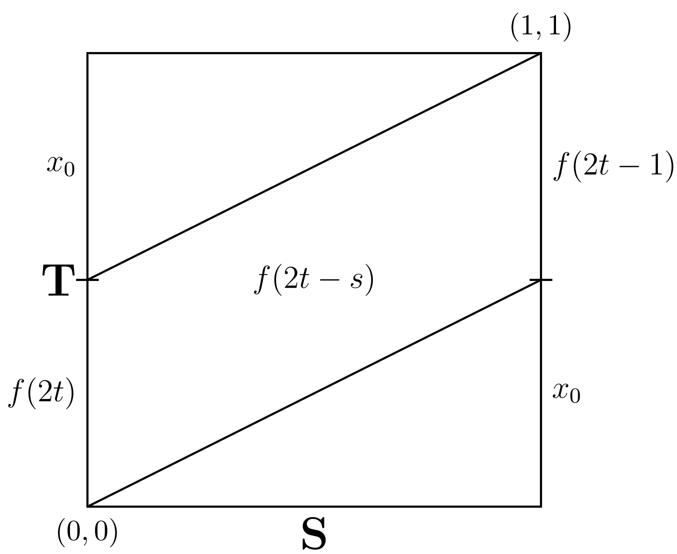
$$(fg)(t) = \begin{cases} f(2t) & t \in [0, \frac{1}{2}] \\ g(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

and that

$$(gf)(t) = \begin{cases} g(2t) & t \in [0, \frac{1}{2}] \\ f(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

We would like to define an explicit homotopy $h : I_t \times I_s \to X \times Y$ where $h(0, s) = h(1, s) = x_0 \times y_0$, $h(t, 0) = (fg)(t)$ and that $h(t, 1) = (gf)(t)$. Constructing such a homotopy is easier if we break h into components as $h(s, t) = x(s, t) \times y(s, t)$.

Now, if we look at the x component of (fg) and (gf), we see that the former path does $f_1(2t)$ for $t \in [0, \frac{1}{2}]$ and then stays at x_0 for $t \in [\frac{1}{2}, 1]$ while the latter path stays at x_0 for $t \in [0, \frac{1}{2}]$ and then does $f_1(2t-1)$ for $t \in [\frac{1}{2}, 1]$. A homotopy $x : I_t \times I_s \to X$ that does this is easier to visualize on the square $I_t \times I_s$ (I apologize in advance for my photoshop skills):



This is simply the function

$$x(t, s) = \begin{cases} x_0 & t \in [0, s/2] \\ f(2t-s) & t \in [s/2, (s+1)/2] \\ x_0 & t \in [(s+1)/2, 1]. \end{cases}$$

(It's easier to tell that this represents the picture faithfully by looking at the lines in the square as a function of s.) Using the same method but for the y component, we can define $y(t, s)$ as

$$y(t, s) = \begin{cases} y_0 & t \in [0, (1-s)/2] \\ g(2t+s-1) & t \in [(1-s)/2, (2-s)/2] \\ y_0 & t \in [(2-s)/2, 1]. \end{cases}$$

By how we've defined $x(t, s)$ and $y(t, s)$, if we let $h(t, s) = (x(t, s), y(t, s))$, then h is a homotopy from (fg) to (gf). Additionally, one can see that $h(0, s) = h(1, s) = x_0 \times y_0$, which means that h preserves basepoints. Thus, h is an explicit homotopy of paths from (fg) to (gf). Since via this homotopy we know that $fg \simeq gf$, we have that $[f][g] = [g][f]$ in $\pi_1(X \times Y, x_0 \times y_0)$ as desired. $\square$