Problem Set 10
Real Analysis I

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Ex 7.3 Suppose f is integrable. Prove that if either $A_n \uparrow A$ or $A_n \downarrow A$, then $\int_{A_n} f\,d\mu \to \int_A f\,d\mu$.

Proof. Let $A_n \uparrow A$ or $A_n \downarrow A$. Either way, we see that $f\chi_{A_n} \to f\chi_A$ as $n \to \infty$ and that $|f\chi_{A_n}| \leq |f|$. Since $|f|$ is integrable, it dominates the $f\chi_{A_n}$'s. By the Dominated Convergence Theorem, this proves that

$$\lim_{n\to\infty} \int f\chi_{A_n}\,d\mu = \int \lim_{n\to\infty} f\chi_{A_n}\,d\mu = \int f\chi_A\,d\mu$$

which proves the statement for both cases. □

Ex 7.4 Show that if $\sum_{n=1}^\infty \int |f_n|\,d\mu < \infty$, then $\sum_{n=1}^\infty f_n(x)$ converges absolutely almost everywhere, is integrable, and its integral is equal to $\sum_{n=1}^\infty \int f_n\,d\mu$. (NOTE: There were some typos in the original that I've corrected.)

Proof. Let $\varphi(x) = \sum_{n=1}^\infty |f_n(x)|$. Since each $|f_n|$ is positive and $|\varphi| = \varphi$, this means that by Proposition 7.6:

$$\int |\varphi|\,d\mu = \int \varphi\,d\mu = \int \sum_{n=1}^\infty |f_n|\,d\mu = \sum_{n=1}^\infty \int |f_n|\,d\mu < \infty$$

Since $\sum_{n=1}^\infty f_n(x) \leq \sum_{n=1}^\infty |f_n(x)| = \varphi$, this proves that $\sum_{n=1}^\infty f_n(x)$ is integrable. Since $\sum_{n=1}^\infty f_n(x)$ is integrable, it's finite almost everywhere, and thus it converges absolutely almost everywhere. Now let $g_k = \sum_{n=1}^k f_n$. We see that

$$|g_k| = \left|\sum_{n=1}^k f_n\right| \leq \sum_{n=1}^k |f_n| \leq \sum_{n=1}^\infty |f_n| = \varphi$$

Since $g_k \to \sum_{n=1}^\infty f_n(x)$ and g_k is dominated by φ, which is an integrable function, then by the Dominated Convergence Theorem, we see that

$$\int \sum_{n=1}^\infty f_n\,d\mu = \lim_{k\to\infty} \int g_k\,d\mu = \lim_{k\to\infty} \int \sum_{n=1}^k f_n\,d\mu = \lim_{k\to\infty} \sum_{n=1}^k \int f_n\,d\mu = \sum_{n=1}^\infty \int f_n\,d\mu$$

□

Ex 7.7 Suppose $(X, \mathcal{A}, \mu)$ is a measure space, f and each f_n is integrable and non-negative, $f_n \to f$ almost everywhere, and $\int f_n \, d\mu \to \int f \, d\mu$. Prove that for each $A \in \mathcal{A}$

$$\int_A f_n \, d\mu \to \int_A f \, d\mu$$

Proof. Since $f_n - f_n \chi_A$ is clearly positive and integrable, then by Fatou's Lemma, we see that

$$\int f - \limsup \int_A f_n = \liminf \int (f_n - f_n \chi_A) \geq \int \liminf (f_n - f_n \chi_A) = \int f - \int_A f$$

Since f is integrable, $\int f$ is finite. This means we can cancel them and get

$$\limsup \int_A f_n \, d\mu \leq \int_A f \, d\mu$$

Similarly, using $f_n + f_n \chi_A$, which is also clearly positive and integrable, we get that

$$\int_A f \, d\mu \leq \liminf \int_A f_n \, d\mu$$

Thus

$$\limsup \int_A f_n \, d\mu \leq \int_A f \, d\mu \leq \liminf \int_A f_n \, d\mu$$

Since $\limsup x_n \geq \liminf x_n$ for any sequence, this means that these inequalities are really equalities. Since liminf and limsup agree, that means

$$\int_A f_n \, d\mu \to \int_A f \, d\mu$$

$\square$

Ex 7.17 Prove that for $p > 0$

$$\sum_{k=1}^{\infty} \frac{1}{(p+k)^2} = -\int_0^1 \frac{x^p}{1-x} \log x \, dx$$

For this problem, you may use the Fundamental Theorem of Calculus.

Proof. For $0 < x < 1$, we see that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

which means that

$$-\int_0^1 \frac{x^p}{1-x} \log x \, dx = -\int_0^1 \sum_{n=0}^{\infty} x^{n+p} \log x \, dx = \int_0^1 \sum_{n=0}^{\infty} x^{n+p} \left(-\log x\right) dx$$

Since this is non-negative over $0 < x < 1$, by Proposition 7.6, we see that

$$\int_0^1 \sum_{n=0}^{\infty} x^{n+p} \left(-\log x\right) dx = -\sum_{n=0}^{\infty} \int_0^1 x^{n+p} \log x \, dx$$

Using integration by parts, letting $u = \log x$ and $dv = x^{n+p} dx$, we get that

$$\int_0^1 x^{n+p} \log x \, dx = [uv]_0^1 - \int_0^1 v \, du = \left[\log x \frac{x^{n+p+1}}{n+p+1}\right]_0^1 - \int_0^1 \frac{x^{n+p+1}}{n+p+1} \cdot \frac{1}{x} \, dx$$

This means that

$$\int_0^1 x^{n+p} \log x \, dx = \log 1 \frac{1^{n+p+1}}{n+p+1} - \lim_{x \to 0} \log x \frac{x^{n+p+1}}{n+p+1} - \frac{1}{n+p+1} \int_0^1 x^{n+p+1} \, dx$$

Since $p > 0$, then $n + p + 1 > 0$, which means that $x^{n+p+1} \log x \to 0$ as $x \to 0$. This shows that

$$\int_0^1 x^{n+p} \log x \, dx = 0 - 0 - \frac{1}{\left(n+p+1\right)^2}$$

Thus, we finally get that

$$-\int_0^1 \frac{x^p}{1-x} \log x \, dx = -\sum_{n=0}^{\infty} \int_0^1 x^{n+p} \log x \, dx = -\sum_{n=0}^{\infty} - \frac{1}{\left(n+p+1\right)^2} = \sum_{k=1}^{\infty} \frac{1}{\left(k+p\right)^2}$$

$\square$