Problem Set 1
Real Analysis I

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Ex 1

a) For sequences (x_n) and y_n of real numbers, prove that

$$\limsup\{x_n\} + \liminf\{y_n\} \leq \limsup\{x_n + y_n\} \leq \limsup\{x_n\} + \limsup\{y_n\}.$$

b) Give a specific example where both inequalities are strict.

Proof.

a) Let $\alpha = \limsup\{x_n\}$ and let $\varepsilon > 0$. We see then that there are an infinite number of j's such that $\alpha - \frac{\varepsilon}{2} < x_j \leq \alpha$. Now let $\beta = \liminf\{y_n\}$. This means there is an N such that for all $n \geq N$,

$$\beta - \frac{\varepsilon}{2} \leq \inf_{k \geq n}\{y_k\} \leq y_n.$$

This means there are an infinite number of j's where $x_j > \alpha - \frac{\varepsilon}{2}$ and $y_n \geq \beta - \frac{\varepsilon}{2}$. Thus, for an N, there are an infinitely number of $n \geq N$ such that $x_n + y_n \geq \alpha + \beta - \varepsilon$. This proves that $\sup\{x_n + y_n\} \geq \alpha + \beta - \varepsilon$. Since ε was arbitrary, we have that

$$\limsup\{x_n\} + \liminf\{y_n\} \leq \limsup\{x_n + y_n\}$$

which proves the first inequality.

Let $x_j \in \{x_k\}_{k \geq n}$ and $y_j \in \{y_k\}_{k \geq n}$. Then we have that $x_j \leq \sup_{k \geq n}\{x_k\}$ and $y_j \leq \sup_{k \geq n}\{y_k\}$, which means that

$$x_j + y_j \leq \sup_{k \geq n}\{x_k\} + \sup_{k \geq n}\{y_k\}.$$

We then take the supremum of all possible j's. Since the RHS is a constant, it remains the same. Thus, we have that

$$\sup_{j \geq n}\{x_j + y_j\} \leq \sup_{k \geq n}\{x_k\} + \sup_{k \geq n}\{y_k\}.$$

Taking the limit as $n \to \infty$, we arrive at the second inequality.

b) We define (x_n) and (y_n) as follows:

$$y_n = \begin{cases} n & \text{if } n \text{ is odd} \\ -n & \text{if } n \text{ is even} \end{cases} \qquad x_n = \begin{cases} -n & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

If we compute the limsups and liminfs, we obtain that

$$\begin{aligned} &\limsup\{x_n\} + \liminf\{y_n\} = 1 - \infty = -\infty \\ &\limsup\{x_n + y_n\} = 0 \text{ as } x_n + y_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1-n & \text{if } n \text{ is odd} \end{cases} \\ &\limsup\{x_n\} + \limsup\{y_n\} = 1 + \infty = \infty. \end{aligned}$$

As can be seen, the inequalities are strict for these x_n and y_n. □

Ex 2 Let f be a mapping from a set X to the set of subsets of X. Showe that there is a subset of X which is not in the range of f.

Proof. First, we let $f : X \to \mathcal{P}(X)$ and let $E = \{x \in X : x \notin f(x)\}$. Suppose that f were surjective. Since $E \in \mathcal{P}(X)$, there exists a x' such that $f(x') = E$. Now, x' must either be in E or not in E. Suppose that $x' \in E = f(x')$. However, by the definition of E, we also have that $x' \notin f(x')$. This is a contradiction. Now suppose that $x' \notin E = f(x')$. Again, though, by the definition of E, this would mean that $x' \in E$ as well. Thus, either way we arrive at a contradiction. This proves that there is no $x \in X$ such that $f(x') = E$, which proves that f is not sujective. □

Ex 3 If $S \subseteq X$ is uncountable and $A \subseteq X$ is countable, show that $S \cap A^c$ is uncountable.

Proof. Assume that $S \cap A^c$ is countable. Then $(S \cap A^c) \cup A$ is the union of two countable sets, and is thus countable. By distributivity we see that

$$(S \cap A^c) \cup A = (S \cup A) \cap (A \cup A^c) = S \cup A.$$

However, since S is uncountable, clearly $S \cup A$ must be uncountable. Thus, we have a contradiction, which proves that $S \cap A^c$ is uncountable. □

Ex 4

a) Is the set of rationals open or closed in the set of real numbers?

b) Which sets of real numbers are both open and closed?

Proof.

a) Every open interval on the real line contains both rationals and irrationals. Therefore, we let $q \in \mathbb{Q}$, then any open neighborhood containing $\mathbb{Q}$ contains an irrational numbers. This proves that $\mathbb{Q}$ is not open. Similarly, if $\mathbb{Q}$ were closed, then that would mean $\mathbb{R} \setminus \mathbb{Q}$ would be open. However, if we let $i \in \mathbb{R} \setminus \mathbb{Q}$, then any open neighborhood of i contains a rational number. Thus, $\mathbb{R} \setminus \mathbb{Q}$ is not open, which means that $\mathbb{Q}$ is neither closed nor open.

b) We note that $\varnothing$ is trivally open and closed, which means that its complement, $\mathbb{R}$, is also open and closed. Let assume there is an additional open and closed set A. Again, this would mean that A^c also open and closed. Now fix $a \in A$ and $b \in A^c$. Without loss of generality, suppose $a < b$. Let $C = \{x \in \mathbb{R} : [a, x] \subseteq A\}$. We note that C is non-empty as $a \in C$. Additionally, C is bounded above by b as $b \notin A$, which proves that C must have a least upper bound. Let's call this least upper bound α.

   Suppose $\alpha \in A$. Then, since A is open, there is a ball of radius $r > 0$ such that $(\alpha - r, \alpha + r) \subseteq B$. Since $[\alpha, \alpha + \frac{r}{2}] \subseteq (\alpha - r, \alpha + r)$, we have that $[a, \alpha] \cup [\alpha, \alpha + \frac{r}{2}] = [a, \alpha + \frac{r}{2}] \subseteq S$. This proves that $\alpha + \frac{r}{2} \in C$, which is a contradiction as α was the least upper bound of C. Thus, $\alpha \notin A$, a contradition.

   Suppose $\alpha \in A^c$. Then, since A^c is open, there is a ball of radius $r > 0$ such that $(\alpha - r, \alpha + r) \subseteq A^c$. Since $[\alpha - \frac{r}{2}, \alpha] \subseteq (\alpha - r, \alpha + r) \subseteq A^c$, we have that $\alpha - \frac{r}{2} \notin A$. This would mean that $\alpha - \frac{r}{2} \notin C$, which is a contradiction as $\alpha - \frac{r}{2} < \alpha$ and α is the least upper bound of C. Thus, $\alpha \notin A^c$, a contradition.

   This means that either way we arrive at a contradiction, which proves that there can be no such set A. Hence, $\varnothing$ and $\mathbb{R}$ are the only two open and closed sets of $\mathbb{R}$.

□

Ex 5 Prove that a set X is infinite if and only if there is a proper subset of X of the same cardinality as X.

Proof. Suppose X is infinite. This means there exists an injective function φ from $\mathbb{N}$ to X. Using this injective function, we can define $\psi : X \to X$ where

$$\psi = \begin{cases} \varphi(n+1) & x = \varphi(n) \text{ for some } n \in \mathbb{N} \\ x & x \notin \mathrm{Im}(\varphi). \end{cases}$$

We see that ψ is injective as φ is injective; however, since $\varphi(0) \notin \mathrm{Im}(\psi)$, ψ is not surjective. Thus, ψ is a bijection between X and $\mathrm{Im}(\psi) \subsetneq X$ as desired.

Conversely, let A be a proper subset of X with the same cardinality and let $\varphi : X \to A$ be bijection. We construct a function $\psi : \mathbb{N} \to X$ as follows: we let $\psi(0) = a$ and let $\psi(i+1) = \varphi(\psi(i))$ for all $0 \neq n \in \mathbb{N}$. Since φ is injective and $a \notin \mathrm{Im}(\varphi)$, we see that ψ is also injective. If the cardinality of X were finite, then there'd be a bijection $f : X \to \{1, \ldots, n\}$ for some $n \in \mathbb{N}$. Thus, the composition, $f \circ \psi : \mathbb{N} \to \{1, \ldots, n\}$ would be injective. We note that the inclusion map from $\{1, \ldots, n\}$ to $\mathbb{N}$ is also injective. Thus, by the Schroder-Bernstein theorem, there would be a bijection between $\mathbb{N}$ and $\{1, \ldots, n\}$, which is a contradiction. Thus, X must be infinite. □