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Problem Set 4

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Ex 1.3.1 Prove or disprove: If u and v are the only vertices of odd degree in a graph G, then G contains a u, v-path.

Proof. The proposition is true. To prove this, suppose that u and v are in distinct components U and V. Then, if we take the component U as a subgraph, it only has one vertex of odd degree. This is a contradiction, though, as the sum of the degrees of the vertices of a graph is always even. Thus, it must be that u and v are in the same component, which proves that there is a u, v-path. $\square$

Ex 1.3.3 Let u and v be adjacent vertices in a simple graph G. Prove that uv belongs to at least $d(u) + d(v) - n(G)$ triangles in G.

Proof. Suppose that there are less than $d(u) + d(v) - n(G)$ vertices that are neighbors to both u and v. Then, the number of neighbors to either u or v is greater than $d(u) + d(v) - (d(u) + d(v) - n(G)) = n(G)$, which is a contradiction, as there are only $n(G)$ vertices in G. Thus, there are at least $d(u) + d(v) - n(G)$ vertices that are neighbors to both u and v. Each such neighbor gives rise to the triangle $uwvu$ which includes the edge uv. This proves that are at least $d(u) + d(v) - n(G)$ triangles that include uv. $\square$

Ex 1.3.7 Determine the maximum number of edges in a bipartite subgraph of P_n, of C_n, and of K_n.

Proof.

a) P_n is itself bipartite, so the maximum number of edges in a bipartite subgraph is the number of edges in P_n itself, which is n.

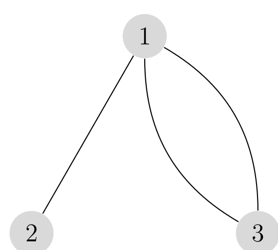
b) If n is even, then C_n is bipartite, which means that the maximum is $E(C_n) = n$. If n is odd, then C_n is not bipartite, so the maximum is strictly less than n. If one removes an edge in C_n, one gets the graph P_{n-1}, which is a bipartite subgraph with $n-1$ edges. Thus, $n - 1$ is the maximum in this case.

c) Let B be a bipartite subgraph of K_n that decomposes into the independent sets U and V, where $n(U) = k$ and $n(V) = \ell$. Since K_n is complete, we can add all the edges between U and V and see that $B \subseteq K_{k,\ell} \subseteq K_n$. We see that to maximize the number of edges of $K_{k,\ell}$, we want that $k + \ell = n$ and that k, ℓ are as close as possible. Thus, if n is even, our maximum bipartite subgraph is $K_{n/2,n/2}$ and if n is odd, our maximum bipartite subgraph is $K_{(n+1)/2,(n-1)/2}$.

□

Ex 1.3.14 Prove that every simple graph with at least two vertices has two vertices of equal degree. Is the conclusion true for loopless graphs?

Proof. Let G be a simple graph on n vertices and suppose that no two vertices have equal degree. This for any $v \in G$, $0 \le d(v) \le n - 1$, there are only n possible degrees for a given vertex. Since the degree of every vertex is different, it must be that all of these possibilities are realized. Thus, there are vertices u and v such that $d(u) = 0$ and $d(v) = n-1$. However, since $d(v) = n - 1$, v must be adajecent to every other vertex of G, including the vertex u. This is a contradiction, as u has no neighbors. Thus, every simple graph must have at least two vertices with the same degree. The same is not true for loopless graphs as demonstrated by the graph below:



□

Ex 1.3.20 Count the cycles of length n in K_n and the cycles of length $2n$ in $K_{n,n}$.

Proof.

a) We see that an n-cycle must go through all the vertices of K_n. We then choose a vertex of K_n. Then, there are $n - 1$ edges to the other $n - 1$ vertices to choose from. Once we choose our next edge, we repeat with $n - 2$ choices. In total, there are $(n-1)(n-2)\dots 1 = (n-1)!$ possibilities. However, there are two ways to go around every cycle, which means we've double-counted. Thus, there are $\frac{(n-1)!}{2}$ cycles in K_n for $n > 2$, and no cycles in K_1 or K_2.

b) Again, $2n$-cycle must go through all the vertices of $K_{n,n}$. If we choose a vertex of $K_{n,n}$, then there are n edges going to the vertices in the other independent set. After choosing one of these edges, we have $n - 1$ edges to choose from and so on. In total, there are $n(n-1)(n-1)(n-2)(n-2)\dots 2 \cdot 2 \cdot 1 \cdot 1 = n(n-1)!^2$ possibilities. Again, though, we've double-counted, which means that there are $\frac{n(n-1)^2}{2}$ cycles in $K_{n,n}$ for $n > 1$, and no cycles in $K_{1,1}$. □