Problem Set 1
Topology II

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February 7, 2021

Ex 1 Prove that homotopy equivalence is an equivalence relation. Show all details.

Proof. We shall use the notation that $X \simeq_{f,g} Y$ if there are maps $f : X \to Y, g : Y \to X$ such that $fg \simeq \mathbb{1}_Y$ and $gf \simeq \mathbb{1}_X$. I shall also omit the composition symbol $\circ$ for simplicity. So with notation out of the way, to prove that homotopy equivalence is an equivalence relation, we see that it satifies the following:

Symmetric) We see very easily that $X \simeq_{\mathbb{1}_X,\mathbb{1}_X} X$ as $\mathbb{1}_X\mathbb{1}_X = \mathbb{1}_X$ to begin with; no homotopy necessary.

Reflexive) Suppose that $X \simeq_{f,g} Y$. Then, we see that $Y \simeq_{g,f} X$ as we already know that $fg \simeq \mathbb{1}_Y$ and that $gf \simeq \mathbb{1}_X$.

Transitive) We first note that if $f, g : X \to Y$ are homotopic via a homotopy H, then for $h_1 : U \to X$ and $h_2 : Y \to V$ we have that h_2fh_1 and h_2gh_1 are homotopic via the homotopy $(h_2 \times \mathbb{1}_I)H(h_1 \times \mathbb{1}_I)$, which is continuous as its the composition of continuous functions. Now suppose that $X \simeq_{f_1,g_1} Y$ and that $Y \simeq_{f_2,g_2} Z$. Then I claim that the $X \simeq_{f_2f_1,g_1g_2} Z$. To prove this we use our previous fact to see that

$$(f_2f_1)(g_1g_2) = f_2(f_1g_1)g_2 \simeq f_2\mathbb{1}_Yg_2 = f_2g_2 \simeq \mathbb{1}_Z$$
$$(g_1g_2)(f_2f_1) = g_1(g_2f_2)f_1 \simeq g_1\mathbb{1}_Yf_1 = g_1f_1 \simeq \mathbb{1}_X$$

as desired. $\square$

Ex 2 Do the following:

a) Define what it means for a space X to be contractible, and prove that it is equivalent to the identity map $\mathbb{1}_X$ being homotopy equivalent to the constant map const_{x_0} for any $x_0 \in X$.

b) Define what it means for X to deformation retract onto a point $x_0 \in X$.

c) Construct an example of a space X that is contractible but which does not deformation retract onto a point.

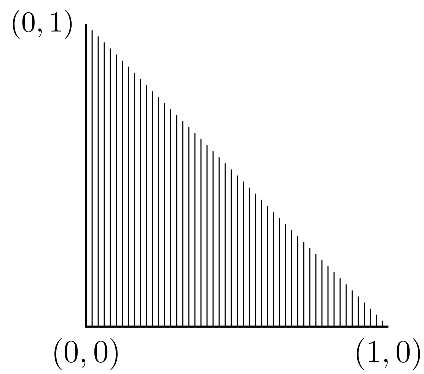
d) Give a nice condition under which these two notions coincide (and prove it).

Proof.

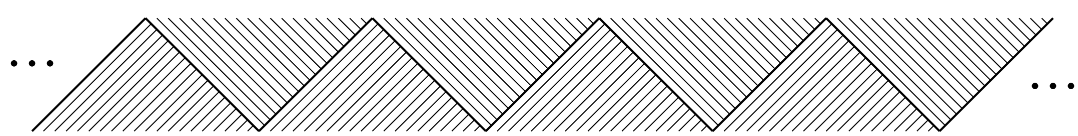
a) A space X is contractible if it is homotopy equivalent to a single point x_0 in X. We see that if $X \simeq \{x_0\}$ via (f, g), then $f : X \to \{x_0\}$, which means that $f = \text{const}_{x_0}$, and $g : \{x_0\} \to X$, which maps to a single point (we identify this point as x_0). Thus, by the definition of homotopy equivalence, we have that $gf = g\,\text{const}_{x_0} = \text{const}_{x_0} \simeq \mathbb{1}_X$, as desired. Conversely, suppose that $\text{const}_{x_0} \simeq \mathbb{1}_X$. This means that if we define $g : \{x_0\} \to X$ as the inclusion map and $f = \text{const}_{x_0}$ with its image restricted to $\{x_0\}$ then $fg = \text{const}_{x_0} g = \mathbb{1}_{\{x_0\}}$ and that $gf = g\,\text{const}_{x_0} = \text{const}_{x_0} \simeq \mathbb{1}_X$. This proves that $X \simeq \{x_0\}$.

b) We say that X deformation retracts to a point x_0 if there exists a continuous map $F : X \times [0, 1] \to X$ such that $F(x_0, t) = x_0$, $F(\_, 0) = \mathbb{1}_X$ and $F(x, 1) = x_0$. This seems to be exactly a homotopy between the identity and the constant map, but to emphasize, the difference is that a deformation retraction requires that the point stay stationary throughout the homotopy.

c) First, we observe the following space



where there is a vertical line for every rational number. This space has a deformation retraction as we can push the bristles down via the map $((x, y), t) \mapsto (x, ty)$ from $X \times I \to X$. This keeps the base still, which means we can deformation retract to any point on the base, which means the space is also contractible. However, we cannot deformation retract to any point on the bristles, as that would require us to push down the infinitely close surrounding bristles in order to be contracted up from the base, which is not a continuous map. Building on this, let's look at the following space



which is essentially infinitely many of our triangles arranged in a clever way. From the previous example, we can see that if there were a deformation retract to a point of this space, it would be to a point on the zig-zag line. However, now we run into a problem, as that would require pushing down the infinitely close bristles of the next triangle.

Thus, this space doesn't deformation retract to any point. It is contractible, though. To achieve this, as we push down the bristles of all the triangles, we move the points on the zig-zag along with them, so that when we're done, the bristles are gone and every point on the zig-zag has moved over to the same position on the next triangle. This means that the space is homotopy equivalent to just the zig-zag line, which is obviously contractible.

d) Any deformation retract is itself a homotopy from the identity map to the constant map, which means that the space must be contractible. Thus, we need a condition such that any given contractible space has a deformation retraction. There are some easy conditions that do this, such as convexity, but I feel that this is too strong. Perhaps something along the lines of a locally contractible point (i.e. "there exists a point $x \in X$ such that every neighborhood U of x is contractible.") or something similar to Problem 5 in Hatcher. However, I cannot think of a good way to prove this.

□

Ex 3 Prove the following:

a) The join of S^0 and X is the suspension of X.

b) Suspending X k times is the same as computing the join with a $(k-1)$-sphere.

c) The iterated join of three points is well-defined (i.e. independent of ordering) and is homeomorphic to the two-dimensional simplex

$$\Delta^2 = \{(t_0, t_1, t_2) \in \mathbb{R}^3 : t_i \geq 0, \sum t_i = 1\}.$$

d) The join of two circles is the three-sphere.

e) Write down the general statement for the join of two spheres and imagine writing a proof that requires essentially no extra notation from the proof that $S^1 * S^1 \approx S^3$.

Proof.

a) First, we see that the join of X and $S^0 = \{-1, 1\}$ is simply the space $(X \times \{-1, 1\} \times I)/R$, where R is the relation $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$. Since when $t \in I$ is zero, the space is a single copy of X, we can break this space up into

$$\frac{X \times \{-1\} \times I}{R} \bigcup_{(X,\_,0)} \frac{X \times \{-1\} \times I}{R}.$$

Additionally, since $\{-1\}$ is a singleton, the relation R on $X \times \{-1\} \times I$ can be reduced to simply the statement $(x_1, -1, 1) \sim (x_2, -1, 1)$. And again, since $\{-1\}$ is a singleton, we see that $(X \times \{-1\} \times I)/R \approx (X \times I)/R'$ where R' is the relation $(x_1, 1) \sim (x_2, 1)$, which is simply the cone of X. Since we can apply the same process to $(X \times \{1\} \times I)/R$, we see that the suspension of X is homeomorphic to two copies of the cone of X identified on their bases. Since this is what a suspension is, we have that $S^0 * X \approx SX$.

b) From parts (a,d), we know that $\Sigma S^n \approx S^0 * S^n \approx S^{n+1}$. We procede by induction. From part (a), we see that if we suspend X once, then that's homeomorphic to computing the join with S^0, which is our base case. Now, let $k \in \mathbb{N}$ and assume that the statement is true for $k-1$. We see then that

$$\Sigma^k X = \Sigma^{k-1}(\Sigma X) \approx \Sigma^{k-1}(S^0 * X) \approx S^{k-2} * (S^0 * X) \approx (S^{k-2} * S^0) * X \approx S^{k-1} * X$$

as desired. However, this relies on knowing that joining is associative, which I don't know how to prove.

c) We first start with two points, say 0 and 1. The join of these two points is simply the space $(\{0\} \times \{1\} \times I)/R$ where R is the relations $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$. However, since $\{0\}$ and $\{1\}$ are already singletons, these relations are trivial. Additionally, since computing the product with a singleton is homeomorphic to the original space, we see that $\{0\} \times \{1\} \times I \approx I$. Thus, the join of two points is an interval.

Now we compute the join of I with a third point, say 2. This is the space $(I \times \{2\} \times I)/R$ where R is the relations $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$. However, the first of these relations is trivial, so we can reduce R to simply the relation $(x_1, y, 1) \sim (x_2, y, 1)$. Also, similar to part (a), since $\{2\}$ is a singleton, we see that the space is homeomorphic to $(I \times I)/R'$ where R' is the relation $(x_1, 1) \sim (x_2, 1)$. This is just a square where one of its edges is quotiented into a single point, i.e. a triangle. Since the two-dimensional simplex Δ^2 is also a triangle, they are homeomorphic. Thus, if we have that $\{x, y, z\}$ are spaces each consisting of a single point, then

$$x * (y * z) \approx x * I \approx \Delta^2$$
$$(x * y) * z \approx I * z \approx \Delta^2$$

which proves the statement.

d) For this problem, we will think of all of our spheres S^n as living in $\mathbb{R}^{n+1}$. First, we will prove that $\Sigma S^n = S^{n+1}$. We unpack the definition of suspension to to get that ΣS^n is the space $(S^n \times [-1, 1])/R$ where R is the relations $(x_1, -1) \sim (x_2, -1)$ and $(x_1, 1) \sim (x_2, 1)$.

Now, we know we can embed S^n in $\mathbb{R}^{n+1}$ via $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, which means that $S^n \times I$ is the space $\{x \in \mathbb{R}^{n+2} : \sum_{i=1}^{n+1} x_i^2 = 1, x_n \in [-1, 1]\}$. Then, for each $t \in I$, we contract the sphere $S^n \times \{t\}$ by $1 - t^2$, which is a continuous function. In other words, we get the space

$$\{x \in \mathbb{R}^{n+2} : \sum_{i=1}^{n+1} x_i^2 = 1 - x_n^2, x_n \in [-1, 1]\} = \{x \in \mathbb{R}^{n+2} : |x| = 1\} = S^{n+1}$$

Since each slice $S^n \times \{t\}$ is still homeomorphic to S^n, except for when $t = -1, 1$, for which the slice is contracted to a point, we see that this exactly satisfies the conditions for how we defined ΣS^n. Thus, we have that $\Sigma S^n = S^{n+1}$. Using part (b), we get that

$$S^1 * S^1 \approx \Sigma^2 S^1 \approx \Sigma S^2 \approx S^3$$

as desired.

e) If I can assume that joining is associative, then we can fairly easily see from parts (b,d) that

$$S^n * S^m \approx \Sigma^{n+1} S^m \approx \Sigma^n S^{m+1} \approx \cdots \approx \Sigma S^{m+n} \approx S^{m+n+1}.$$

$\square$

Ex 4,5 Show that a CW complex X is path-connected if and only if its 1-skeleton X^1 is path-connected. You may assume for simplicity that X has only finitely many cells and that a path $[0,1] \to D^n$ from $x \in \partial D^n$ to $y \in \partial D^n$ is homotopic to a path in ∂D^n from x to y.

Proof. Suppose that X is path-connected. Let $x, y \in X^1$ and let $f : I \to X$ be a path connecteding them in X. Since X is a finite CW complex, we can represent X recursively, where $X_1 = X^1$, $X_n = X_{n-1} \sqcup_{\varphi_n} D^{n_j}$, and $n_j > 1$. Since X has finitely many cells, we have that $X = X_k$ for some $k \in \mathbb{N}$. We let $f_k = f : I \to X_k$. We know that $X_k = X_{k-1} \sqcup_{\varphi_{k-1}} D^{n_{k-1}}$ and $f(0), f(1) \notin \operatorname{Int}(D^{n_{k-1}})$. From this, we let D denote $D^{n_{k-1}}$ for simplicity and break into two cases:

1. Im(f) and Int(D) are disjoint) If this is the case, since the boundary of D is identified with points in X_{k-1}, we can define a path $f_{k-1} : I \to X_{k-1}$ where $f_{k-1}(x) = f_k(x)$.

2. Im(f) and Int(D) are not disjoint) Suppose that x is in the both of these sets. Now let $y \in f^{-1}(x)$. Let C_y be the connected component of $f^{-1}(D)$ that contains y. We note that C_y is clopen in $f^{-1}(D)$ (since it and its complement separate the space, meaning they are both open) and that $f^{-1}(D)$ is closed (as it's the inverse image of a closed set). Thus, since C_y is a closed set with respect to a closed subspace, it must be closed in the parent space I. As C_y is a closed, connected subset of I, it has the form $C_y = [a, b]$. We also note that $a \neq b$, as $f(y)$ is in the interior of D, which means there is a neighborhood U of $f(y)$ such that $U \subseteq D$ and $f^{-1}(U)$ is open.

   From this, we have that $f(a), f(b) \in \partial D$ (otherwise we could we could extend a and b farther and still be in D), and that f restricted to $[a, b]$ is a path in D. Thus, from our assumption, we have that f restricted to $[a, b]$ is homotopic to a function $g : [a, b] \to D$, such that $g([a, b]) \subseteq \partial D$ and that $g(a) = f(a)$ and $g(b) = f(b)$. If we let $h : I \to X_k$ be a function where

   $$h(x) = \begin{cases} f(x) & x \in [a, b]^c \\ g(x) & x \in [a, b] \end{cases}$$

   we have new path that intersects Int(D) with one less component. We repeat this procedure if Im(h) and Int(D) are still not disjoint. Thus, at the end of this process, we arrive at a function that doesn't intersect Int(D) at all, which reduces the problem to case 1.

From all of this, we have a new path $f_{k-1} : I \to X_{k-1}$. Using descent, we can repeat this process until we arrive at the path $f_1 : I \to X_1 = X^1$. Since $x, y \in X^1$ were arbitrary and we found a path from x to y in X^1, we have proved that X^1 is path-connected.

Now conversely, suppose that X^1 is path-connected and let $x, y \in X_k$. We will focus on x for now and would like to show that there's a path from x to a point in X_{k-1}. We know that

$X_k = X_{k-1} \sqcup_{\varphi_{k-1}} D^{n_{k-1}}$, so we again let D signify $D^{n_{k-1}}$ and break into two cases:

1. $x \notin \text{Int}(D)$) In this case we can just take the trivial path, since x is already in X_{k-1}.
2. $x \in \text{Int}(D)$. This case is also relatively easy. We know that D is path-connected, so take any path from x to any point in ∂D. Since ∂D gets associated with points in X_{k-1}, we have a path from x to a point in X_{k-1}.

We repeat this process for the point in X_{k-1} and via descent we eventually arrive at a point in $X_1 = X^1$. The concatentation of all of these paths gives us a path from x to a point in the 1-skeleton X^1. We can do this same produce with y to get another path to the 1-skeleton. Since X^1 is path-connected by assumption, there is a path between the these endpoints. These three paths together finally give us a path between x and y. As $x, y \in X$ were arbitrary, we have proved that X is path-connected. □

Ex 6 Show that a homotopy equivalence $X \simeq Y$ induces a bijection between the sets of path-components of X and Y. Prove the corresponding statement for connect components. Conclude that any space which is homotopy-equivalent to a connected (resp. path-connected) space is a connected (resp. path-connected) space.

Proof. Let $X \simeq Y$ be homotopy equivalent via the pair of functions (f, g). Now, we define $C(X)$ and $C(Y)$ to be the sets of connected components of X and Y respectively. We also define $C(x)$ and $C(y)$ to be the components which contain $x \in X$ and $y \in Y$ respectively.

Finally, we create functions $F : C(X) \to C(Y)$ and $G : C(Y) \to C(X)$ where $F(C(x)) = C(f(x))$ and $G(C(y)) = C(g(y))$. We will first prove that F is well-defined. To do this, let $u, v \in C(x)$. Since f is continuous and $C(x)$ is connected, we see that $f(C(x))$ is connected. This means that $f(C(x))$ is contained in some connected component $K \in C(Y)$. Thus, we have that $C(f(u)) = K = C(f(v))$, which proves well-definedness. A similar argument proves that G is well-defined as well.

Now we want to prove that F and G are inverses. We let $H : X \times I \to X$ be the homotopy of $gf \simeq \mathbb{1}_X$. We note that for any $x_0 \in X$, we have that $H(x_0, I)$ is connected, as it's a continuous function mapping a connected interval. Since $H(x_0, 1) = x_0$, we have that $H_1(x_0, I) \subseteq C(x_0)$. This means that $H(x_0, 0) = gf(x_0) \in C(x_0)$. Since this is true for any x_0, we have that $gf(x) = C(x)$. Similarly, we have that $fg(y) \in C(y)$. Thus, altogether, we get that for any $x \in X$ and $y \in Y$

$$
\begin{aligned}
FGC(x) &= FCg(x) = Cfg(x) = C(x) \\
GFC(y) &= GCf(y) = Cgf(y) = C(y)
\end{aligned}
$$

which proves that F and G are inverses. This means that F is actually a bijection between $C(X)$ and $C(Y)$ as desired. We note that this argument only used the fact that the image of a connected set is connected. Since there is a corresponding fact that the image of every path-connected set is path-connected, we see that this argument works with the words "connected" replaced with "path-connected." Lastly, we can clearly conclude that a space which is homotopy equivalent to a connected (resp. path-connected) space must be connected (resp. path-connected) itself. □