# Problem Set 4
# Complex Analysis

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**Ex 1** Define $g : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ by $g(z) = \frac{z+1}{z-1}$ and let $f(z) = e^{g(z)}$.

a) Prove that $f$ is bounded on $\mathbb{D}$.

b) Fix $a \in \mathbb{D}$. Decide if $\lim_{t\to 0^+} f(t + (1-t)a)$ exists, and if so compute it.

c) Prove that $\frac{e^{it}+1}{e^{it}-1} = -i\cot(t/2)$ for all $t \in \mathbb{R}$, $t \notin \{2\pi k : k \in \mathbb{Z}\}$ [sic].

d) Decide if $\lim_{\theta\to 0^+} f(e^{i\theta})$ exists and if so, compute it.

e) Decide if $\lim_{\theta\to 0^-} f(e^{i\theta})$ exists and if so, compute it.

*Proof.*

a) We note that $g(z)$ is a Möbius Transform such that

$$\begin{aligned} g(-1) &= \frac{0}{-2} = 0 \\ g(1) &= \frac{2}{0} = \infty \\ g(i) &= \frac{i+1}{i-1} = \frac{-(i+1)^2}{(i-1)(-i-1)} = \frac{-(-1+1+2i)}{2} = -i \\ g(0) &= \frac{1}{-1} = -1, \end{aligned}$$

which means $g$ maps that unit circle onto the imaginary axis and that it maps the unit disk $\mathbb{D}$ onto the half plane defined by $\{z : \mathrm{Re}(z) \leq 0\}$. This means that for $z \in \mathbb{D}$

$$|f(z)| = |e^{g(z)}| = |e^{\mathrm{Re}(g(z))+i\,\mathrm{Im}(g(z))}| = |e^{\mathrm{Re}(g(z))}||e^{i\,\mathrm{Im}(g(z))}| = |e^{\mathrm{Re}(g(z))}| = e^{\mathrm{Re}(g(z))} \leq e^0 = 1.$$

This proves that $f$ is bounded on $\mathbb{D}$; in fact, it maps $\mathbb{D}$ back inside $\mathbb{D}$.

b) We note that $g$ is continuous on $\mathbb{C} \setminus \{1\}$, which means $f$ is also continuous on $\mathbb{C} \setminus \{1\}$. Since $t + (1-t)a \subseteq \mathbb{D}$ for $t \in [-\varepsilon, 1+\varepsilon]$ and $1 \notin \mathbb{D}$, we have by continuity that

$$\lim_{t\to 0+} f(t + (1-t)a) = f\left(\lim_{t\to 0^+} t + (1-t)a\right) = f(a) = e^{\frac{a+1}{a-1}}.$$

c) We recall that $\cos(t) = \frac{e^{it}+e^{-it}}{2}$ and $\sin(t) = \frac{e^{it}-e^{-it}}{2i}$. This means that for $t \notin \{\pi k : k \in \mathbb{Z}\}$, we have that

$$\begin{aligned} -i\cot(t) &= -i\frac{\cos(t)}{\sin(t)} = -i\frac{\left(\frac{e^{it}+e^{-it}}{2}\right)}{\left(\frac{e^{it}-e^{-it}}{2i}\right)} = -i\left(\frac{e^{it}+e^{-it}}{2}\right)\left(\frac{2i}{e^{it}-e^{-it}}\right) = \frac{e^{it}+e^{-it}}{e^{it}-e^{-it}} \\ &= \frac{e^{it}+e^{-it}}{e^{it}-e^{-it}}\frac{e^{it}}{e^{it}} = \frac{e^{2it}+1}{e^{2it}-1}. \end{aligned}$$

This proves that $\frac{e^{it}+1}{e^{it}-1} = -i\cot(t/2)$ for $t \notin \{2\pi k : k \in \mathbb{Z}\}$ as we wanted.

d) By the previous part, we know that

$$f(e^{i\theta}) = e^{\frac{e^{i\theta}+1}{e^{i\theta}-1}} = e^{-i\cot(t/2)}.$$

Since $\lim_{t\to 0^+} -\cot(t/2)$ tends to $-\infty$, $\cot(\pi/2) = \frac{\cos(\pi/2)}{\sin(\pi/2)} = 0$, and cot is continuous, there exists a monotonically decreasing sequence $t_n \in (0, \pi]$ such that $\cot(t_n) = -n \cdot \pi$. This means that

$$\lim_{n\to\infty} f(e^{it_n}) = e^{-i\cot(t_n)} = e^{in\pi} = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

Since $t_n$ is a sequence approaching 0 from the right and $f(e^{it_n})$ doesn't converge, it cannot be the case that $\lim_{t\to 0^+} f(e^{it})$ converges.

e) Similar to the previous part, so $\lim_{t\to 0^-} -\cot(t/2)$ tends to $\infty$, $\cot(-\pi/2) = 0$, and cot is continuous, there exists a monotonically increasing sequence $t_n \in [-\pi, 0)$ such that $\cot(t_n) = n \cdot \pi$. This means that

$$\lim_{n\to\infty} f(e^{it_n}) = e^{-i\cot(t_n)} = e^{-in\pi} = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

Since $t_n$ is a sequence approaching 0 from the left and $f(e^{it_n})$ doesn't converge, it cannot be the case that $\lim_{t\to 0^-} f(e^{it})$ converges. $\square$

**Ex 2** Let $f : \mathbb{C} \to \mathbb{C}$ be analytic.

a) Suppose that $f$ is nonconstant. Suppose that $z_0 \in \mathbb{C}$ and $f(z_0) \neq 0$. Show that for every $\varepsilon > 0$, there is a point $z \in B_\varepsilon(z_0)$ with $|f(z)| < |f(z_0)|$.

b) Suppose that $\lim_{z\to\infty} f(z) = \infty$. Show that there is a $z_0 \in \mathbb{C}$ so that $f(z_0) = 0$. [Hint: show that there is a point $z_0 \in \mathbb{C}$ at which $|f(z_0)| = \inf_{z\in\mathbb{C}} |f(z)|$.]

c) Explain why the previous parts imply that every nonconstant polynomial with complex coefficients has a root.

*Proof.*

a) Without loss of generality, we may assume that $z_0 = 0$ (we can simply translate $f$ if need be). Since $f$ is analytic, we have that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We know that $0 \neq f(0) = a_0$. If all the other $a_n$'s are zero, then $f$ is constant which is a contradiction. Thus, there is a least positive integer $j$ such that $a_j \neq 0$. This means that

$$f(z) = a_0 + a_j z^j + \sum_{n>j} a_n z^n.$$

Choose $r$ small enough so that $r\sum_{n>j}|a_n| < |a_j|$. If we let $\tilde{f}(z) = a_0 + a_j z^j$, we see that

$$\tilde{f}(B_r(0)) = a_0 + a_j B_r(0)^j = a_0 + a_j B_{r^j}(0) = a_0 + B_{|a_j|r^j}(0) = B_{|a_j|r^j}(a_0).$$

We want an element in this image which is as close to the origin as possible. We see that within the ball $B_{|a_j|r^j}(a_0)$, we can get arbitrarily close to an element with modulus $|a_0| - |a_j|r^j$. More precisely, for any $\varepsilon > 0$, there exists a $z_0 \in B_r(0)$ such that $|\tilde{f}(z_0)| < |a_0| - |a_j|r^j + \varepsilon$. Using this, we see that for $z_0 \in B_r(0)$

$$|f(z_0)| = \left|\tilde{f}(z_0) + \sum_{n>j} a_n z_0^n\right| = |\tilde{f}(z_0)| + \sum_{n>j}|a_n||z_0|^n < |a_0| - |a_j|r^j + \varepsilon + \sum_{n>j}|a_n|r^n$$
$$\leq |a_0| - |a_j|r^j + \varepsilon + r^{j+1}\sum_{n>j}|a_n| = |a_0| + r^j\left(r\sum_{n>j}|a_n| - |a_j|\right) + \varepsilon.$$

We note that we chose $r$ so that $r\sum_{n>j}|a_n| - |a_j| < 0$. Thus, we can choose $\varepsilon$ small enough so that $|f(z_0)| < |a_0| = |f(0)|$. We note that the claim still holds for any $r'$ such that $0 < r' \leq r$. This means that for any ball around 0, there will always be a smaller one of radius $r'$ such that there's an element $z \in B_{r'}(0)$ where $|f(z)| < |f(z_0)|$.

b) Consider the compact sets $C_n = \overline{B_n(0)}$. Since $C_n$ is compact and $|f|$ is continuous, we see that $|f|$ achieves its infinum on this set, meaning there's some $t_n \in C_n$ such that $|f(t_n)| = \inf\{|f(t)| : t \in C_n\}$. Since $C_n \subseteq C_{n+1}$, we know that either $t_{n+1} \in C_{n+1}\setminus C_n$ or that $t_{n+1} = t_n$. This means if we consider the sequence $(t_n)_{n\in\mathbb{N}}$, either the sequence goes to infinity or is eventually constant. Suppose that the sequence goes to infinity. Then since $|f(t_{n+1})| \leq |f(t_n)|$, we have that $|f(t_n)| < |f(t_1)|$ for all $n \in \mathbb{N}$. This would make $(|f(t_n)|)_{n\in\mathbb{N}}$ a bounded sequence, meaning $(t_n)_{n\in\mathbb{N}}$ goes to infinity, but $(|f(t_n)|)_{n\in\mathbb{N}}$ does not, a contradiction to our assumption. Thus, it must be that $t_n$ is eventually constant. Let $z_0$ be this constant. Since by definition $|f(z_0)| = \inf\{|f(t)| : t \in C_n\}$ for all $C_n$, we have that $|f(z_0)| = \inf\{|f(t)| : t \in \mathbb{C}\}$. By part (a), if $f(z_0) \neq 0$, there's some ball $B_\varepsilon(z_0)$ and a $z \in B_\varepsilon(z_0)$ such that $|f(z)| < |f(z_0)|$. This contradicts $|f(z_0)|$ being the infinimum of $f$. Thus, it must be that $|f(z_0)| = 0$.

c) Let $p(z) = \sum_{k=0}^n a_k x^k$ be a nonconstant polynomial with complex roots. This is a finite power series, so $p$ is analytic on $\mathbb{C}$. We see that by the Triangular inequality:

$$\lim_{z\to\infty}|p(z)| = \lim_{z\to\infty}\left|\sum_{k=0}^n a_k z^k\right| \geq \lim_{z\to\infty}|a_n z^n| - \sum_{k=0}^{n-1}|a_k z^k| = \lim_{z\to\infty}|a_n||z|^n - \sum_{k=0}^{n-1}|a_k||z|^k$$
$$= \lim_{z\to\infty}|a_n||z| - \sum_{k=0}^{n-1}|a_k||z|^{k-n+1} = \left(\lim_{z\to\infty}|a_n||z|\right) - \left(\lim_{z\to\infty}\sum_{k=0}^{n-1}|a_k||z|^{k-n+1}\right)$$
$$= \infty - |a_{n-1}| = \infty.$$

This means we can apply part (b) to show that there exists an $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$. $\square$

**Ex 3** Let $z \in \mathbb{D}$ and evaluate

$$\int_{\partial\mathbb{D}} \frac{\overline{s}}{s-z}\,ds.$$

*Proof.* We see that

$$\int_{\partial\mathbb{D}} \frac{\overline{s}}{s-z}\, ds = \int_{\partial\mathbb{D}} \frac{s}{s} \cdot \frac{\overline{s}}{s-z}\, ds = \int_{\partial\mathbb{D}} \frac{|s|^2}{s(s-z)}\, ds = \int_{\partial\mathbb{D}} \frac{1}{s(s-z)}\, ds = \int_{\partial\mathbb{D}} \frac{-1/z}{s} + \frac{1/z}{s-z}\, ds$$
$$= \frac{1}{z}\left(\int_{\partial\mathbb{D}} \frac{1}{s-z}\, ds - \int_{\partial\mathbb{D}} \frac{1}{s}\, ds\right).$$

We know that for a function $f$ holomorphic on $\overline{B_r(z_0)}$ we have for all $z \in B_r(z_0)$,

$$f(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(s)}{s-z}\, ds.$$

In this case, if we take $f$ to be the holomorphic constant function 1, then since 0 and $z$ are in $B_1(0) = \mathbb{D}$ we get that

$$\int_{\partial\mathbb{D}} \frac{1}{s-z}\, ds = \int_{\partial\mathbb{D}} \frac{1}{s}\, ds = 2\pi i.$$

This proves that

$$\int_{\partial\mathbb{D}} \frac{\overline{s}}{s-z}\, ds = \frac{1}{z}\left(\int_{\partial\mathbb{D}} \frac{1}{s-z}\, ds - \int_{\partial\mathbb{D}} \frac{1}{s}\, ds\right) = \frac{1}{z}(2\pi i - 2\pi i) = 0.$$ □

**Ex 4** Let $U, V$ be open sets, $u : V \to \mathbb{R}$ harmonic [sic], $f : U \to V$ holomorphic such that $f''$ exists and is continuous. Show that $u \circ f$ is harmonic.

*Proof.* Let $z_0 \in U$. Since $V$ is open, there is some $r > 0$, such that $B_r(f(z_0)) \subseteq V$. Since $B_r(f(z_0))$ is convex and $u$ is harmonic on this ball, we know that $u$ has some harmonic conjugate, meaning there exists some holomorphic function $g : B_r(f(z_0)) \to \mathbb{C}$ such that $g(z) = u(z) + iv(z)$.

As $f$ is continuous, $f^{-1}(B_r(f(z_0)))$ is open, so we can define the holomorphic restriction $\tilde{f} = f|_{f^{-1}(B_r(f(z_0)))}$. Since the composition of holomorphic functions is holomorphic we know that $g \circ \tilde{f}$ is holomorphic and that $\mathrm{Re}(g \circ \tilde{f}) = \mathrm{Re}((u \circ \tilde{f}) + i(v \circ \tilde{f})) = u \circ \tilde{f}$. Thus, $u \circ \tilde{f}$ is the real part of some holomorphic function, proving that $u \circ \tilde{f}$ is harmonic. Since this is the restriction of $u \circ f$ onto an open neighborhood of $z_0$ and being harmonic is a local property, this means that $u \circ f$ is harmonic at $z_0$. Since $z_0$ arbitrary, we have that $u \circ f$ is harmonic. □

**Ex 5** Let $\Omega$ be open and suppose that there is a conformal map $\phi : \mathbb{D} \to \Omega$. Prove that every harmonic $u : \Omega \to \mathbb{R}$ has a harmonic conjugate.

*Proof.* By Ex 4, we know that the composition $u \circ \phi : \mathbb{D} \to \mathbb{R}$ is harmonic. Since $\mathbb{D}$ is a convex set, this means that $u \circ \phi$ has a harmonic conjugate, that is there is a holomorphic function $g : \mathbb{D} \to \mathbb{C}$ such that $g = (u \circ \phi) + iv$. Since $\phi$ is conformal, we know $\phi^{-1}$ is well-defined and also holomorphic. Thus,

$$g \circ \phi^{-1} = (u \circ \phi \circ \phi^{-1}) + i(v \circ \phi^{-1}) = u + i(v \circ \phi^{-1})$$

is a holomorphic function. As $u$ is the real part of this holomorphic function, we have proven that $u$ has a harmonic conjugate. □

**Ex 6** Define $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $f(z) = \frac{1}{2}(z + \frac{1}{z})$.

a) Let $C_r$ denote the circle of radius $r$ centered at the origin. Show that if $r \neq 1$ then $f(C_r)$ is an ellipse. Find the center and equation of the ellipse. Show that $f(C_1) = [-1, 1]$.

b) Show that $f|_{\mathbb{C}\setminus\overline{\mathbb{D}}}$ is injective and that $f(\mathbb{C}\setminus\overline{\mathbb{D}}) = \mathbb{C}\setminus[-1,1]$.

c) Use $f$ to find a conformal map from $\mathbb{C}\setminus[-1,1]$ to $\mathbb{D}\setminus\{0\}$.

d) Fix $\theta \in \mathbb{R}$. Show that $f(\{re^{i\theta} : r > 0\})$ is a hyperbola.

*Proof.*

a) We see that in polar coordinates

$$\begin{aligned} f(re^{i\theta}) &= \frac{1}{2}\left(re^{i\theta} + \frac{1}{re^{i\theta}}\right) = \frac{r}{2}(\cos(\theta) + i\sin(\theta)) + \frac{1}{2r}(\cos(\theta) - i\sin(\theta)) \\ &= (\frac{r + r^{-1}}{2})\cos(\theta) + i(\frac{r - r^{-1}}{2})\sin(\theta). \end{aligned}$$

This means for a fixed $r \in (0,1)$, $f(C_r(0))$ is simply an ellipse centered at the origin. Furthermore, when $r = 1$, we have that $\frac{r-r^{-1}}{2} = \frac{1-1}{2} = 0$ and $\frac{r+r^{-1}}{2} = \frac{1+1}{2} = 1$, so we get that $f(e^{i\theta}) = \cos(\theta)$, which means $f(C_1) = [-1,1]$.

b) Suppose $re^{i\theta} \neq se^{i\phi}$ where $r, s > 1$. If these two numbers have different radii. Then without loss of generality, we may assume that $r > s > 1$. This would mean that

$$r - \frac{1}{r} > s - \frac{1}{r} > s - \frac{1}{s}$$

and that

$$r - s > \frac{r-s}{sr} \implies r - s > \frac{1}{s} - \frac{1}{r} \implies r + \frac{1}{r} > s + \frac{1}{s}.$$

Thus, $f(re^{i\theta})$ lies on an ellipse with strictly larger axes than $f(se^{i\phi})$. Thus proves that $f(re^{i\theta}) \neq f(se^{i\phi})$. On the other hand, if $r = s$, then it must be that $\theta \neq \phi + 2\pi k$. This means that $f(re^{i\theta})$ and $f(se^{i\phi})$ lie on the same ellipse, but with different angles. Once again, this proves that $f(re^{i\theta}) \neq f(se^{i\phi})$. Thus, $f$ is injective on $\mathbb{C}\setminus\overline{\mathbb{D}}$.

To prove that $f(\mathbb{C}\setminus\overline{\mathbb{D}}) = \mathbb{C}\setminus[-1,1]$ we will see that $f \circ g$ is the identity where $g(z) = z - \sqrt{z^2-1} = z - e^{1/2\log z^2 - 1}$ where the branch cut of log is the nonpositive real numbers and $\log(1) = 0$. We note that $g$ is defined for all $\mathbb{C}\setminus[-1,1]$ under this branch cut. Since we have that

$$\begin{aligned} (f\circ g)(z) = f(z - \sqrt{z^2-1}) &= \frac{1}{2}\left(z - \sqrt{z^2-1} + \frac{1}{z - \sqrt{z^2-1}}\right) \\ &= \frac{1}{2}\left(z - \sqrt{z^2-1} + \frac{z + \sqrt{z^2-1}}{z^2 - (z^2-1)}\right) = z. \end{aligned}$$

c) Since $f : \mathbb{C}\setminus\overline{\mathbb{D}} \to \mathbb{C}\setminus[-1,1]$ is surjective and injective, $f$ is a bijection and has an inverse. As the $g$ in the previous part is holomorphic and the right inverse of $f$, by uniqueness of inverses, $g$ is a holomorphic inverse of $f$. This proves that $f$ and $g$ are conformal. We see that the Möbius Transformation $\phi(z) = \frac{1}{z}$ inverts elements across the unit circle as $|\phi(z)| = |1/z| = 1/|z|$. This means that $\phi$ is a conformal map from $\mathbb{C}\setminus\overline{\mathbb{D}}$ to $\mathbb{D}\setminus\{0\}$. Thus, $\phi \circ g$ is a conformal map from $\mathbb{C}\setminus[-1,1]$ to $\mathbb{D}\setminus\{0\}$.

d) We see that

$$\mathrm{Re}(f(re^{i\theta}))^2 = \left(\frac{r + r^{-1}}{2}\cos(\theta)\right)^2 = \frac{r^2 + r^{-2} + 2}{4}\cos^2(\theta)$$

and that

$$\mathrm{Im}(f(re^{i\theta}))^2 = \left(\frac{r - r^{-1}}{2}\sin(\theta)\right)^2 = \frac{r^2 + r^{-2} - 2}{4}\sin^2(\theta).$$

This means that for a fixed $\theta$, we have the equation

$$\frac{\mathrm{Re}(f(re^{i\theta}))^2}{\cos(\theta)^2} - \frac{\mathrm{Im}(f(re^{i\theta}))^2}{\sin(\theta)^2} = \frac{r^2 + r^{-2} + 2}{4} - \frac{r^2 + r^{-2} - 2}{4} = 4/4 = 1$$

which proves that $f(\{re^{i\theta} : r > 0\})$ is a hyperbola for a fixed $\theta$. □

**Ex 7** Let $\gamma : [a, b] \to \mathbb{C}$ be $C^1$ and $f : \gamma([a, b]) \to \mathbb{C}$ be continuous. By completing the following outline, show that $\int_\gamma f(z)|\, dz| = \int_a^b f(\gamma(t))|\gamma'(t)|\, dt$.

a) Let $s_1 < s_2$ be elements of $[a, b]$ and set $v = \sup_{x,y\in[s_1,s_2]} |\gamma'(x) - \gamma'(y)|$. Prove that

$$\left| |\gamma(s_2) - \gamma(s_1)| - \int_{s_1}^{s_2} |\gamma'(t)|\, dt \right| \leq 2v(s_2 - s_1).$$

b) Use (a) and a modification of our proof that $\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt$ to show that $\int_\gamma f(z)|\, dz| = \int_a^b f(\gamma(t))|\gamma'(t)|\, dt$.

*Proof.*

a) We note that for a fixed $w \in [s_{1,2}]$, we have that

$$\int_{s_1}^{s_2} |\gamma'(w)|\, dt = |\gamma'(w)| \int_{s_1}^{s_2} dt = |\gamma'(w)|(s_2 - s1) = |\gamma'(w)(s_2 - s1)| = \left| \int_{s_1}^{s_2} \gamma'(w)\, dt \right|.$$

We also recall the following facts about integrals, absolute values, and the fundamental theorem of calculus:

$$\gamma(s_2) - \gamma(s_1) = \int_{s_1}^{s_2} \gamma'(t)\, dt$$
$$||u| - |w|| \leq |u - w|$$
$$\left| \int f(t)\, dt \right| \leq \int |f(t)|\, dt.$$

Using these tricks (some multiple times), we have that for an arbitrary fixed $w \in [s_1, s_2]$,

$$\begin{aligned}
&\left| |\gamma(s_2) - \gamma(s_1)| - \int_{s_1}^{s_2} |\gamma'(t)|\, dt \right| = \left| \left| \int_{s_1}^{s_2} \gamma'(t)\, dt \right| - \int_{s_1}^{s_2} |\gamma'(t)|\, dt \right| \\
&= \left| \left| \int_{s_1}^{s_2} \gamma'(t)\, dt \right| - \left| \int_{s_1}^{s_2} \gamma'(w)\, dt \right| + \int_{s_1}^{s_2} |\gamma'(w)|\, dt - \int_{s_1}^{s_2} |\gamma'(t)|\, dt \right| \\
&\leq \left| \left| \int_{s_1}^{s_2} \gamma'(t)\, dt \right| - \left| \int_{s_1}^{s_2} \gamma'(w)\, dt \right| \right| + \left| \int_{s_1}^{s_2} |\gamma'(w)|\, dt - \int_{s_1}^{s_2} |\gamma'(t)|\, dt \right| \\
&\leq \left| \int_{s_1}^{s_2} \gamma'(t)\, dt - \int_{s_1}^{s_2} \gamma'(w)\, dt \right| + \left| \int_{s_1}^{s_2} |\gamma'(w)| - |\gamma'(t)|\, dt \right| \\
&\leq \left| \int_{s_1}^{s_2} (\gamma'(t) - \gamma'(w))\, dt \right| + \int_{s_1}^{s_2} \big| |\gamma'(w)| - |\gamma'(t)| \big|\, dt \\
&\leq \int_{s_1}^{s_2} \big|\gamma'(t) - \gamma'(w)\big|\, dt + \int_{s_1}^{s_2} \big|\gamma'(w) - \gamma'(t)\big|\, dt \\
&\leq \int_{s_1}^{s_2} v\, dt + \int_{s_1}^{s_2} v\, dt = 2v(s_2 - s_1)
\end{aligned}$$

as we wanted.

b) Let $M_1 = \sup_{t\in[a,b]} |\gamma'(t)|$, $M_2 = \sup_{t\in[a,b]} |f(\gamma(t))|$, and let $\varepsilon > 0$. By the continuity of $f \circ \gamma$ and of $\gamma'$, there exists a $\delta > 0$ such that

$$|t_1 - t_2| < \delta \implies |f(\gamma(t_1)) - f(\gamma(t_2))| < \varepsilon$$

and

$$|t_1 - t_2| < \delta \implies |\gamma'(t_1) - \gamma'(t_2)| < \varepsilon.$$

Now fix a partition $p = \{s_k\}_{0\leq k\leq n}$ such that $|s_k - s_{k-1}| < \delta$. We note that any refinement of $P$ also has the property that it's step sizes are less than $\delta$. We also let $v_k = \sup_{x,y\in[s_{k-1},s_k]} |\gamma'(x) - \gamma'(y)|$ and note that $v_k < \varepsilon$ by our choice of partition. With all this, we get that

$$\begin{aligned}
&\left| \int_a^b f(\gamma(t))|\gamma'(t)|\, dt - \sum_{k=1}^n f(\gamma(\tau_k))|\gamma(s_k) - \gamma(s_{k-1})| \right| \\
&= \left| \sum_{k=1}^n \int_{s_{k-1}}^{s_k} f(\gamma(t))|\gamma'(t)|\, dt - f(\gamma(\tau_k))|\gamma(s_k) - \gamma(s_{k-1})| \right| \\
&\leq \sum_{k=1}^n \left| \int_{s_{k-1}}^{s_k} f(\gamma(t))|\gamma'(t)|\, dt - f(\gamma(\tau_k)) \int_{s_{k-1}}^{s_k} |\gamma'(t)|\, dt \right| \\
&\quad + \left| f(\gamma(\tau_k)) \int_{s_{k-1}}^{s_k} |\gamma'(t)|\, dt - f(\gamma(\tau_k))|\gamma(s_k) - \gamma(s_{k-1})| \right| \\
&= \sum_{k=1}^n \left| \int_{s_{k-1}}^{s_k} (f(\gamma(t)) - f(\gamma(t_k)))|\gamma'(t)|\, dt \right| + \left| f(\gamma(\tau_k)) \left( \int_{s_{k-1}}^{s_k} |\gamma'(t)|\, dt - |\gamma(s_k) - \gamma(s_{k-1})| \right) \right| \\
&\leq \sum_{k=1}^n \int_{s_{k-1}}^{s_k} |f(\gamma(t)) - f(\gamma(t_k))||\gamma'(t)|\, dt + |f(\gamma(\tau_k))|\, |2v_k(s_k - s_{k-1})| \\
&\leq \sum_{k=1}^n \int_{s_{k-1}}^{s_k} \varepsilon M_1\, dt + 2M_2\varepsilon(s_k - s_{k-1}) = \sum_{k=1}^n (M_1 + 2M_2)(s_k - s_{k-1})\varepsilon = (M_1 + 2M_2)(b - a)\varepsilon
\end{aligned}$$

Since this same reasoning works for any refinement of $P$, it must be that

$$\left| \int_a^b f(\gamma(t))|\gamma'(t)|\,dt - \int_\gamma f(z)|dz| \right| < (M_1 + 2M_2)(b-a)\varepsilon.$$

Since $\varepsilon$ was arbitrary, we have that

$$\int_a^b f(\gamma(t))|\gamma'(t)|\,dt = \int_\gamma f(z)|dz|$$

as desired. $\square$