# Problem Set 6
# Differential Topology

Bennett Rennier
bennett@brennier.com

October 19, 2018

**Chapter 1, Section 5**

**Ex 5** More generally, let $f : X \to Y$ be a map transversal to a submanifold $Z$ in $Y$. Then $W = f^{-1}(Z)$ is a submanifold of $X$. Prove that $T_x(W)$ is the preimage of $T_{f(x)}(Z)$ under the linear map $df_x : T_x(X) \to T_{f(x)}(Y)$.

*Proof.* Let the manifolds be $X^n$, $Y^m$, $W^\ell$, and $Z^k$. We have a map $\phi : Y^m \to \mathbb{R}^m$ where locally points $z \in Z$ look like $\phi(z) = \phi(0, \dots, 0, z_1, \dots, _k)$ where there are $m - k$ zeros. Thus, we can consider the projection $\pi : \mathbb{R}^m \to \mathbb{R}^{m-k}$ and see that $\ker(\pi \circ \phi) = Z$. Taking the differentials of these maps we get that

$$T_x(X^n) \xrightarrow{df_x} T_{f(x)}(Y^m) \xrightarrow{d\phi_{f(x)}} \mathbb{R}^m \xrightarrow{d\pi_{\phi(f(x))}} \mathbb{R}^{m-k}.$$

Since $f \pitchfork W$ implies that $\operatorname{Im}(df_x) + T_{f_x(Z)} = T_{f(x)}(Y)$ for $x \in W$, we have that $d\phi$ and $d\pi \circ d\phi$ are onto for $x$ in $W$. This means that the differential of our composition $d(\pi \circ \phi \circ f) = d\pi \circ d\phi \circ df$ is surjective for any $x \in W$. Thus, 0 is a regular value of $\pi \circ \phi \circ f$ and that $W = (\pi \circ \phi \circ f)^{-1}(0)$. This gives us

$$T_x(W) = d(\pi \circ \phi \circ f)_x^{-1}(0) = df_x^{-1} \circ d(\pi \circ \phi)_{f(x)}^{-1}(0) = df_x^{-1}(\ker(d(\pi \circ \phi)_{f(x)})) = df_x^{-1}(T_{f(x)}(Z))$$

as desired. □

**Ex 6** Suppose that $X$ and $Z$ do not intersect transversally in $Y$. May $X \cap Z$ still be a manifold? If so, must its codimension still be $\operatorname{codim} X + \operatorname{codim} Z$. Answer with drawings.

*Proof.* Yes, consider the following drawings. In both, the manifolds intersect at a single point, which is a 0-dimensional manifold. In the first instance, taking place in $\mathbb{R}^2$, we have that $\operatorname{codim} X + \operatorname{codim} Z = 1 + 1 = 2$, which is indeed the codimension of a 0-dimensional manifold in $\mathbb{R}^2$. In the second example, taking place in $\mathbb{R}^3$, we have that $\operatorname{codim} X + \operatorname{codim} Z = 1 + 1 = 2$, which is not the the codimension of a 0-dimensional manifold in $\mathbb{R}^3$.

□

**Ex 7** Let $X \to^f Y \to^g \to Z$ be a sequence of smooth maps of manifolds, and assume that $g$ is transversal to a submanifold $W$ of $Z$. Show $f \pitchfork g^{-1}(W)$ if and only if $g \circ f \pitchfork W$.

*Proof.* (Incomplete. I spent way too much time on the Whitney's Immersion Theorem question.) □

**Ex 8** For which values of $a$ does the hyperboloid defined by $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversally? What does the intersection look like for different values of $a$?

*Proof.* We see that if the two surfaces intersect then we can substitute $x^2 + y^2 = 1 + z^2$ from the first equation into $x^2 + y^2 + z^2 = a$, to get that $2z^2 + 1 = a$, i.e. $z = \pm\sqrt{\frac{a-1}{2}}$. Plugging this back into the first equation, we see that $x^2 + y^2 = 1 + \frac{a-1}{2} = \frac{a+1}{2}$. We see that these equations only make sense for $a \geq 1$; otherwise $z$ would be an imaginary number. Thus, the two surfaces only intersect for $a \geq 1$. At $a = 1$, we have that they intersect when $z = 0$ and when $x^2 + y^2 = 1$, that is, the unit circle in the $xy$-plane. For $a > 1$, then $\sqrt{\frac{a-1}{2}}$ and $-\sqrt{\frac{a-1}{2}}$ are distinct possiblities for $z$, that means the surfaces intersect along two different circles, both parallel to the $xy$-plane.

By homework 4, exercise 8, we proved that the tangent plane to $x^2 + y^2 - z^2 = 1$ at $(1, 0, 0)$ is simply the vector space $\{(0, \alpha, \beta) : \alpha, \beta \in \mathbb{R}\}$. Since this is the same for the sphere when $a = 1$, we have that the two surfaces are not transverse when $a = 1$. The following picture shows that when $a > 1$, the surfaces are in fact transverse.

□

**Ex 9** Let $V$ be a vector space and let $\Delta$ be the diagonal of $V \times V$. For the linear map $A : V \to V$, consider the graph $W = \{(v, Av) : v \in V\}$. Show that $W \pitchfork \Delta$ if and only if $+1$ is not an eigenvalue of $A$.

*Proof.* We note that $\Delta$, $W$, and $V \times V$ are all vector spaces of dimension $n$, $n$, and $2n$ respectively. This means that their tangent spaces at any point are simply themselves, so the question becomes when does $\Delta + W = V \times V$ hold. Consider the set

$$\Delta \cap W = \{(v, v) : v \in V\} \cap \{(v, Av) : v \in V\} = \{(v, v) : Av = v\}.$$

If $A$ does not have $+1$ as an eigenvalue, then this set is simply $\{0\}$. Thus, $\Delta + W$ is actually a direct sum, meaning

$$\dim(\Delta + W) = \dim(\Delta \oplus W) = \dim(\Delta) + \dim(W) = 2n = \dim(V \times V).$$

This proves that $V \times V = \Delta + W$. Now if $A$ does have $+1$ as an eigenvalue, then we have that

$$\dim(\Delta + W) = \dim(\Delta) + \dim(W) - \dim(\Delta \cap W) = 2n - \dim(\Delta \cap W) < 2n = \dim(V \times V).$$

This proves that $\Delta + W$ does not span all of $V \times V$, proving that $\Delta$ and $W$ are not transverse. ☐

**Ex 10** Let $f : X \to X$ be a map with fixed point $x$; that is $f(x) = x$. If $+1$ is not an eigenvaule of $df_x : T_x(X) \to T_x(X)$, then $x$ is called a *Lefschetz fixed point of* $f$. $f$ is called a *Lefschetz map* if all its fixed points are Lefschetz. Prove that if $X$ is compact and $f$ is Lefschetz, then $f$ has only finitely many fixed points.

*Proof.* Let $X$ be $n$-dimensional. Let $\Gamma$ be the graph of $f$ and let $\Delta$ be the diagonal of $X \times X$. Let $x \in \Gamma \cap \Delta$. By a previous homework, the tangent space of $\Delta$ at $x$ is the diagonal of the vector space $T_x(X) \times T_x(X)$ and and the tangent space of $\Gamma$ at $x$ is the vector space $\{(v, df_x(v)) : v \in T_x(X)\}$. Since $f$ is Lefschetz, $df_x$ has $+1$ as an eigenvalue. By the Ex 9, we have that these two vector spaces are tranverse, which implies that their sum is all of $\mathbb{R}^{2n}$. This further implies that $\Gamma \pitchfork \Delta$.

We see then that $\Gamma \cap \Delta$ is a submanifold of $X \times X$. Since $\Gamma$ and $\Delta$ both have the same dimension as $X$, we have that

$$\operatorname{codim}(\Gamma \cap \Delta) = \operatorname{codim}(\Gamma) + \operatorname{codim}(\Delta) = 2n = \dim(X \times X)$$

This proves that $\Gamma \cap \Delta$ is zero-dimensional submanifold of $X \times X$, i.e. a collection of points. Since $X$ is compact so is $X \times X$. As $\Gamma$ and $\Delta$ are both closed in $X \times X$ (this is a standard point-set proof), we have that the submanifold $\Gamma \cap \Delta$ is compact. This proves that $\Gamma \cap \Delta$ must actually be a finite number of points. ☐

**Chapter 1, Section 7**

**Ex 5** Exhibit a smooth map $f : \mathbb{R} \to \mathbb{R}$ whose set of critical values is dense.

*Proof.* From the third homework in exercise 18, we can construct a smooth bump function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(0) = 0$, $f(x) = 1$ for $|x| < \frac{1}{4}$, $f(x) = 0$ for $|x| > \frac{1}{3}$ and $0 < f(x) < 1$ for $\frac{1}{4} < |x| < \frac{1}{3}$. Since $\mathbb{Q}$ is countable, we can enumerate all the rationals as $\{q_i : i \in \mathbb{N}\}$. We then construct the function $f(x) = \sum_i q_i \cdot f(x + i)$. This sum is well-defined and smooth as the bump functions are spread far enough apart that only one bump function is nonzero for any particular point $x \in \mathbb{R}$ and there is an interval of length $1/3$ inbetween each bump such that the sum is identically zero. Since for any $q_i \in \mathbb{Q}$, the point $-i$ has the property that $f(-i) = q_i$ and $f'(-i) = 0$, we see that all of $\mathbb{Q}$ are critical values of $f$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, the set of all critical values of $f$ is dense in $\mathbb{R}$. ☐

**Chapter 1, Section 8**

**Ex 10** Prove that every $k$-dimensional manifold $X$ may be immersed in $\mathbb{R}^{2k}$.

*Proof.* Consider the maps

$$g(x,y,t) = t(f(x) - f(y)) \qquad ; \qquad h(x,v) = df_x(v),$$

as found in the proof of Whitney's Immersion Theorem. If we are mapping into $\mathbb{R}^{2k+1}$ instead of $\mathbb{R}^{2k+2}$, we can no longer assume that there's an $a$ not in the image of either of these maps; however, we can assume that there's an $a$ not in the image of $h$. We then follow the rest of the proof of Whitney's Immersion theorem and embed $X$ into $\mathbb{R}^{2k}$ with potentially (i.e. most likely) self-intersections, which we fix after the fact. In particular, if one looks at the self-intersections that occur after the projection, by how we chose $a$, these self-intersections are transverse and locally they look like a $k$-dimensional manifold intersecting with a $k$-dimensional manifold. Since this is taking place in $\mathbb{R}^{2k}$, their intersection is a 0-dimensional manifold, meaning the intersections are just a collection of points. I suspect there's some way to "wiggle" (i.e. use homotopies) that makes the manifolds not intersect anymore; though, I'm not quite sure how to make this notion rigorous. After doing this, $X$ will be immersed in $\mathbb{R}^{2k}$ as we wanted. $\square$

**Other Problems**

**Ex 4** Recall the stereographic projection maps from $S^2$ minus a pole to $\mathbb{R}^2$; they are $f_+ : S^2 \setminus (0,0,1) \to \mathbb{R}^2$ and $f_- : S^2 - (0,0,-1) \to \mathbb{R}^2$ where

$$f_+(x_1,x_2,x_3) = \frac{1}{1-x_3}(x_1,x_2)$$
$$f_-(x_1,x_2,x_3) = \frac{1}{1+x_3}(x_1,x_2)$$

with respective inverses

$$f_+^{-1}(u_1,u_2) = \frac{1}{1+|u|^2}(2u_1, 2u_2, -1+|u|^2)$$
$$f_-^{-1}(u_1,u_2) = \frac{1}{1+|u|^2}(2u_1, 2u_2, 1-|u|^2),$$

where $|u|^2 = u_1^2 + u_2^2$.

a) Thinking of $\mathbb{R}^2$ as $\mathbb{C}$, prove that $(f_+ \circ f_-^{-1})(z) = \frac{1}{z}$.

b) Let $p(z) = \sum_{j=0}^n a_j z^j$ be a complex polynomial, thought of as a smooth map $\mathbb{C} \to \mathbb{C}$. Define $\tilde{p} : S^2 \setminus (0,0,1) \to S^2 \setminus (0,0,1)$ by

$$\tilde{p}(s) = (f_+^{-1} \circ p \circ f_+)(s).$$

Prove that $\tilde{p}$ extends uniquely to a smooth map $S^2 \to S^2$.

c) Show that if $p$ is not constant then $\tilde{p}(0,0,1) = (0,0,1)$ and if $p$ has degree at least 2 then $(0,0,1)$ is a critical point of $\tilde{p}$.

*Proof.*

a) We see that for $z = x + iy$

$$
\begin{aligned}
(f_+ \circ f_-^{-1})(z) = f_+(f_-^{-1}(z)) &= f_+\left(\frac{1}{1+|z|^2}(2x, 2y, 1-|z|^2)\right) = f_+\left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2}\right) \\
&= \frac{1}{1-\frac{1-|z|^2}{1+|z|^2}}\left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}\right) = \frac{1}{\frac{2|z|^2}{1+|z|^2}}\left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}\right) \\
&= \frac{1+|z|^2}{2|z|^2}\left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}\right) = \frac{1}{|z|^2}(x, y) = \frac{z}{|z|^2} = \frac{z}{z\bar{z}} = \frac{1}{\bar{z}}.
\end{aligned}
$$

b) If $p$ is constant, then the map $\tilde{p}$ is constant as well, so we can easily patch it at $(0, 0, 1)$ by letting it be that constant. Now assume that $p$ is not constant, that is that $n > 0$. Consider the map $f_- \circ \tilde{p} \circ f_-^{-1}$, we see that trying to extend $\tilde{p}$ to (0,0,1) is equivalent to trying to extend $f_- \circ \tilde{p} \circ f_-^{-1}$ at 0. Since $\phi(z) = \frac{1}{\bar{z}}$ is its own inverse, we can use part (a) to see that

$$
\begin{aligned}
f_- \circ \tilde{p} \circ f_-^{-1} &= f_- \circ \left(f_+^{-1} \circ p \circ f_+\right) \circ f_-^{-1} = \left(f_- \circ f_+^{-1}\right) \circ p \left(\circ f_+ \circ f_-^{-1}\right) \\
&= \left(f_- \circ f_+^{-1}\right) \circ p \left(\circ f_+ \circ f_-^{-1}\right) = \phi \circ p \circ \phi.
\end{aligned}
$$

Since conjugation distributes over addtion and multiplication and $p$ is a polynomial, we can rewrite this function

$$
(\phi \circ p \circ \phi)(z) = \frac{1}{\overline{p(\frac{1}{\bar{z}})}} = \frac{1}{p(\frac{1}{z})} = \frac{1}{\sum_{j=0}^{n} a_j z^{-j}} = \frac{z^n}{\sum_{j=0}^{n} a_j z^{n-j}}.
$$

We see that at $z = 0$, this function is $0/a_n$ which is well-defined to be 0 as the leading coefficient of a polynomial can be assumed to be non-zero. We also see that $\sum_{j=0}^{n} a_j z^{n-j}$ has only $n$ roots, so there's some open ball around 0 such that this function is well-defined. Using this smooth function as a patch, we can extend $f_- \circ \tilde{p} \circ f_-^{-1}$ to be defined at 0, and thus, we can patch $\tilde{p}$ to be $(0, 0, 1)$ at the point $(0, 0, 1)$.

c) The first part of this was proved in part (b). The rest of this is incomplete for now. □