Problem Set 2
Homological Algebra

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Ex 1 Let I be a directed set and $(D_i, (f_{ij}))$ be a compatible system of (left) R-modules. Let D' be the disjoint union of the D_i and define a (symmetric) equivalence relation on D' as follows: For $d_i \in D_i$ and $d_j \in D_j$, $d_i \sim d_j$ if and only if there exists a $k \in I$ with $i \leq k$, $j \leq k$ such that $f_{ik}(d_i) = f_{jk}(d_j)$. We denote by $[d_i]$ the equivalence class of d_i and we set $D'_i := \{[d_i] : d_i \in D_i\}$ and $D := D'/\sim = \cup_i D'_i$. We also denote $f_i : D_i \to D$ the map which sends d_i to $[d_i]$.

a) Show that for any $x, y \in D$, there exists a $k \in I$ such that $x, y \in D'_k$.

b) For $x, y \in D$ and $r \in R$, define $x + y = [a + b]$ and $rx = [ra]$ whenever $x, y \in D'_k$ and $x = [a]$, $y = [b]$, and $a, b \in D_k$. Show that these operations are well-defined and turn D into an R-module. Note that f_i then becomes an R-module homomorphism.

c) Prove that the system $(D, (f_i))$ is the direct limit of the D_i.

d) If $d_i \in D_i$ and $f_i(d_i) = 0$, show that there exists a $j \in I$ with $i \leq j$ and $f_{ij}(d_i) = 0$.

Proof.

a) Let $x = [a]$ and $y = [b]$ be elements of D where $a \in D_i$ and $b \in D_j$. Since I is a directed set, there is some $k \in I$ such that $k \geq i, j$. This means that $f_{ik}(a)$ and $f_{jk}(b)$ are elements of D_k. Since $f_{ik}(a) \sim a$ and $f_{jk}(b) \sim b$, we have that $x = [a] = [f_{ik}(x)]$ and $y = [b] = [f_{jk}(b)]$ are elements of D'_k.

b) Let $x = [a] = [a']$ and $y = [b] = [b']$ be elements of D, where $a, b \in D_i$ and $a', b' \in D_j$. Since $[a] = [a']$ there is some $k \geq i, j$ such that $f_{ik}(a) = f_{jk}(a')$. Similarly, as $[b] = [b']$, there is some $\ell \geq i, j$ such that $f_{i\ell}(b) = f_{j\ell}(b')$. Without loss of generality, assume that $k \geq \ell$. This means that $f_{\ell k}(f_{i\ell}(b)) = f_{\ell k}(f_{j\ell}(b'))$, which means that $f_{ik}(b) = f_{jk}(b')$. This proves that

$$[a + b] = [f_{ik}(a + b)] = [f_{ik}(a) + f_{ik}(b)] = [f_{jk}(a') + f_{jk}(b')] = [f_{jk}(a' + b')] = [a' + b'].$$

Additionally, if $x = [a] = [a']$ where $a \in D_i$ and $a' \in D_j$, then there is some $k \geq i, j$ such that $f_{ik}(a) = f_{jk}(a')$. This means that

$$[ra] = [f_{ik}(ra)] = [rf_{ik}(a)] = [rf_{jk}(a')] = [f_{jk}(ra')] = [ra'].$$

This proves that the operations are well-defined, turning D into an R-module.

c) Let $(E, (e_i))$ be an object E with compatible morphisms $e_i : D_i \to E$. Now, if we ignore the compatiblity relations, then by the universal property of coproducts, there is some map

$\phi : \sqcup_i D_i \to E$ such that $\phi \circ f_i = e_i$. Let d_i and d_j be elements such that there exists a $k \geq i, j$ where $f_{ik}(d_i) = f_{jk}(d_j)$. If we apply f_k to both sides, we get that $f_i(d_i) = f_j(d_j)$. If we then compose with ϕ to both sides, we obtain that $e_i(d_i) = e_j(d_j)$. This means that d_i and d_j are identified in E. By the universal property of quotients, the map $\phi : \sqcup_i D_i \to E$ must factor through D. This means that $(D, (f_i)) \to (E, (e_i))$, proving that $(D, (f_i))$ is universal.

d) If $f_i(d_i) = [d_i] = 0$ in D, then this means that there exists a $k \geq i, j$ such that $f_{ik}(d_i) = f_{jk}(0) = 0$. This relation $f_{ik}(d_i) = 0$ is exactly what we want. □

Ex 2 [Continue from Ex 1] Assume that $(E_i, (g_{ij}))$ is a second compatible system with corresponding direct limit $(E, (g_i),)$. Also assume there are homomorphisms $h_i : D_i \to E_i$ satisfying $g_{ij}h_i = h_j f_{ij}$ for all $i < j$. Reprove, in the current setup, the unique existence of a homomorphism $h : D \to E$ satisfying $g_i h_i = h f_i$ and describe h explicitly.

Proof. Since we need h to satisfy $g_i h_i = h f_i$, we see for $x = [d_i] \in D$ where $d_i \in D_i$, we must have that

$$g_i(h_i(d_i)) = h(f_i(d_i)) = h([d_i]).$$

Thus, we have no choice for how to define h. All that we must show is that such an h is well-defined. To do so, suppose that $[d_i] = [d_j]$ in D, where $d_i \in D_i$ and $d_j \in D_j$. This means that there exists a $k \geq i, j$ such that $f_{ik}(d_i) = f_{jk}(d_j)$. If we apply h_k to both sides we get $h_k(f_{ik}(d_i) = h_k(f_{jk}(d_j))$. By our assumptions on h_i, this means that $g_{jk}(h_i(d_i)) = g_{jk}(h_j(d_j))$. If we then apply g_k to both sides, we obtain that $g_i(h_i(d_i)) = g_j(h_j(d_i))$. Thus, $h([d_i]) = h([d_j])$, proving that h is well-defined. □

Ex 3 A sequence of direct systems and homomorphisms

$$M \to N \to P$$

is *exact* if the corresponding sequence of modules and module homomorphisms is exact for each $i \in I$. Show that the sequence $M \to N \to P$ of direct limits is then exact.

Proof. Let (M_{ii}, μ_{ij}), (N_i, ν_{ij}), and (P_i, p_{ij}) be direct systems with direct limits (M, μ_i), (N, ν_i), and (P, p_i) respectively. Additionally, let $\phi_i : M_i \to N_i$ and $\psi_i : N_i \to P_i$ be maps which induce the homomorphisms of direct systems $\phi : M \to N$ and $\psi : N \to P$. We note that this means the following diagram commutes for all $i \in I$:

$$\begin{array}{ccccc} M_i & \xrightarrow{\phi_i} & N_i & \xrightarrow{\psi_i} & P_i \\ \big\downarrow{\scriptstyle \mu_i} & & \big\downarrow{\scriptstyle \nu_i} & & \big\downarrow{\scriptstyle p_i} \\ M & \xrightarrow{\phi} & N & \xrightarrow{\psi} & P \end{array}$$

Now to prove that $\mathrm{Im}(\phi) \subseteq \ker(\psi)$, we need only to prove that $\psi \circ \phi = 0$. To do this, let $x \in M$. By problem 2.15 (done on the previous homework), we know that for some $i \in I$ and some $x_i \in M_i$, $x = \mu_i(x_i)$. Using the diagram above and the fact that $\psi_i \circ \psi_i = 0$ (since the sequence is exact), we see that

$$(\phi \circ \psi)(x) = (\phi \circ \psi \circ \mu_i)(x_i) = (p_i \circ \psi_i \circ \phi_i)(x_i) = (p_i \circ 0)(x_i) = p_i(0) = 0.$$

This proves that $\psi \circ \psi = 0$ and so that $\mathrm{Im}(\phi) \subseteq \ker(\psi)$.

Proving that $\ker(\psi) \subseteq \mathrm{Im}(\phi)$ can be a little complicated, so here is a commutative diagram and a diagram tracking a specific element to help understand the proof:

$$
\begin{tikzcd}
 & N_i \arrow[r, "\psi_i"] \arrow[d, "\nu_{ij}"] \arrow[dd, bend right=90, "\nu_i"'] & P_i \arrow[d, "p_{ij}"] \arrow[dd, bend left=90, "p_i"] \\
M_j \arrow[r, "\phi_j"] \arrow[d, "\mu_j"] & N_j \arrow[r, "\psi_j"] \arrow[d, "\nu_j"] & P_j \arrow[d, "p_j"] \\
M \arrow[r, "\phi"] & N \arrow[r, "\psi"] & P
\end{tikzcd}
\qquad
\begin{tikzcd}
 & y_i \arrow[r, mapsto, "\psi_i"] \arrow[d, mapsto, "\nu_{ij}"] & \psi_i(y_i) \arrow[d, mapsto, "p_{ij}"] \\
x_j \arrow[r, mapsto, "\phi_j"] \arrow[d, mapsto, "\mu_j"] & \nu_{ij}(y_i) \arrow[r, mapsto, "\psi_j"] \arrow[d, mapsto, "\nu_j"] & 0 \arrow[d, mapsto, "p_j"] \\
x \arrow[r, mapsto, "\phi"] & y \arrow[r, mapsto, "\psi"] & 0
\end{tikzcd}
$$

Now assume that $y \in \ker(\psi)$. Using problem 2.15 again, there is some $i \in I$ and $y_i \in N_i$ such that $\nu_i(y_i) = y$. Now since $(\psi \circ \nu_i)(y_i) = \psi(y) = 0$, we have that $(p_i \circ \psi_i)(y_i) = p_i(\psi_i(y_i)) = 0$. By Ex 1 (d), there exists a $j \in I$ with $i \leq j$ where $p_{ij}(\psi_i(y_i)) = \psi_j(\nu_{ij}(y_i)) = 0$. This proves that $\nu_{ij}(y_i) \in \ker(\psi_j)$. Since the sequence $M_j \to N_j \to P_j$ is exact, there is some $x_j \in M_j$ such that $\phi_j(x_j) = \nu_{ij}(y_i)$. Let $x = \mu_j(x_j)$. We see then that

$$\phi(x) = \phi(\mu_j(x_j)) = \nu_j(\phi_j(x_j)) = \nu_j(\nu_{ij}(y_i)) = \nu_i(y_i) = y.$$

This proves that $y \in \mathrm{Im}(\phi)$, meaning $\ker(\psi) \subseteq \mathrm{Im}(\phi)$. Thus, the sequence $M \to N \to P$ is exact. $\square$

Ex 4

a) If $(S, s : S \to D)$ is the equalizer of two morphisms $f, g : D \to E$, then s is a monomorphism.

b) If $(S, s : E \to S)$ is the coequalizer of two morphisms $f, g : D \to E$, then s is a epimorphism.

Proof.

a) Let $\phi, \psi : A \to S$ such that $s \circ \phi = s \circ \psi$. This gives use the commutative diagram

$$
\begin{tikzcd}
A \arrow[r, shift left, "\psi"] \arrow[r, shift right, "\phi"'] & S \arrow[r, "s"] & D \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & E.
\end{tikzcd}
$$

Since $s \circ \phi : A \to D$ is a homomorphism such that $f \circ (s \circ \phi) = g \circ (s \circ \phi)$, by the universal property of the equalizer, there exists a unique homomorphism h from A to S that makes the above diagram commute. Since this homomorphism is unique, it must be that $\phi = \psi$. This proves that s is a monomorphism.

b) Let $\phi, \psi : S \to A$ be morphisms such that $\phi \circ s = \psi \circ s$. This gives use the following commutative diagram:

$$
\begin{tikzcd}
D \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & E \arrow[r, "s"] & S \arrow[r, shift left, "\psi"] \arrow[r, shift right, "\phi"'] & A.
\end{tikzcd}
$$

Since $\phi \circ s : E \to A$ satisfies the equation $(\phi \circ s) \circ f = (\phi \circ s) \circ g$, we have by the universal property of the co-equalizer that there exists a unique homomorphism from S to A that makes the above diagram commute. Since this homomorphism is unique, it follows that $\phi = \psi$. This proves that s is an epimorphism. $\square$

Ex 5 Dually to the push out in 1.3.15(a), construct the pull back in 1.3.15(b), provided that the category $\mathcal{D}$ has products and equalizers.

Proof. Let $X, Y, Z \in \mathcal{D}$ such that $f : X \to Z$ and $g : Y \to Z$ are morphisms in $\mathcal{D}$. Since $\mathcal{D}$ has products, we let P with the morphisms $p_1 : P \to X$ and $p_2 \to P \to Y$ be the product of X and Y. Now, this gives us maps gp_2 and fp_1 from P to Z. Since we know $\mathcal{D}$ also has equalizers, there exists some E with a morphism $e : E \to P$ such that the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "e"] & P \arrow[r, shift left, "gp_2"] \arrow[r, shift right, "fp_1"'] & Z \end{tikzcd}$$

However, we see that this means that the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "p_2e"] \arrow[d, "p_1e"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

I claim that this E with the morphisms p_1e and p_2e is the desired pushout. All we need to do is prove that it satisfies the universal property. As such, suppose Q is in $\mathcal{D}$ with morphisms $q_1 : Q \to X$ and $q_2 : Q \to Y$ such that the following diagram commutes

$$\begin{tikzcd} Q \arrow[r, "q_2"] \arrow[d, "q_1"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

If we just ignore Z, f, and g for now, we see that by the universal property of products, there exists some unique $q : Q \to P$ such that $q_1 = p_1q$ and $q_2 = p_2q$. If we apply f to the first equation and g to the second we get $fq_1 = fp_1q$ and $gq_2 = gp_2q$. Since the above diagram shows that $fq_1 = gq_2$, we have that $fp_1q = gp_2q$. But this means that Q with the morphism q equalize the maps fp_1 and gp_2. By the universal property of equalizers, we finally obtain a unique morphism $q' : Q \to E$ such that $gp_2eq' = fp_1eq'$. But this exactly means that the following diagram commutes

$$\begin{tikzcd} Q \arrow[dr, "q'"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & E \arrow[r, "p_2e"] \arrow[d, "p_1e"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

Since q' is unique, this exactly proves that E satisfies the universal property of a pullback. Thus, the category $\mathcal{D}$ has pullbacks. $\square$

Ex 6 Show that the transformation $t = (t_{MA})$ introduced in Example 1.4.2(b) is natural.

Proof. Recall that $t_{MA} : \mathrm{Hom}_{\mathbb{Z}}(M_{\text{grp}}, A) \to \mathrm{Hom}_R(M, A^{\#})$ such that $t_{MA}(\phi) = \phi'$, where $\phi'(m)(r) = \phi(r.m)$. We want to prove that this is a natural transformation between the functors $F : {}_R\mathbf{Mod} \to \mathbf{Ab}$ and $G : \mathbf{Ab} \to {}_R\mathbf{Mod}$, where $F(M) = M_{\text{grp}}$ (that is, we forget the R-module structure) and $G(A) = A^{\#} = \mathrm{Hom}_R(R, A)$. To prove this, we need to show that for any R-module homomorphism $f : N \to M$ and any abelian group homomorphism $g : A \to B$, the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{Z}}(M_{\mathrm{grp}}, A) & \xrightarrow{t_{MA}} & \mathrm{Hom}_R(M, A^{\#}) \\
\Big\downarrow{\scriptstyle (F(f),g)} & & \Big\downarrow{\scriptstyle (f,G(g))} \\
\mathrm{Hom}_{\mathbb{Z}}(N_{\mathrm{grp}}, B) & \xrightarrow{t_{NB}} & \mathrm{Hom}_R(N, B^{\#})
\end{array}
$$

where $(F(f), g)(\phi) = g \circ \phi \circ F(f)$ and $(f, G(g))(\phi) = G(g) \circ \phi \circ f$. We note that $F(f)$ is f considered as a group homomorphism and $G(g) : A^{\#} \to B^{\#}$ is defined by $G(g)(\phi)(r) = g(\phi(r))$. This means we need only to show that $(f, G(g)) \circ t_{MA} = t_{NB} \circ (F(f), g)$. To do this, let $\phi : M_{\mathrm{grp}} \to A$ be some group homomorphism. We see that

$$
\begin{aligned}
((f, G(g)) \circ t_{MA})(\phi)(n)(r) = (f, G(g))(t_{MA}(\phi))(n)(r) = (f, G(g))(\phi')(n)(r) = (G(g)(\phi'(f(n))))(r) \\
= g(\phi'(f(n))(r)) = g(\phi(r.f(n))) = g(\phi(f(r.n)))
\end{aligned}
$$

and that

$$
\begin{aligned}
(t_{NB} \circ (F(f), g))(\phi)(n)(r) = t_{NB}((F(t), g)(\phi))(n)(r) = t_{NB}(g \circ \phi \circ F(t))(n)(r) = (g \circ \phi \circ F(t))(r.n) \\
= g(\phi(F(f)(r.n))) = g(\phi(f(r.n))),
\end{aligned}
$$

which proves the statement. $\square$

Ex 7 Show that the maps t_{NA} and u_{NA} introduced in the proof of Proposition 1.4.3 are inverse to each other.

Proof. Recall that

$$
\begin{aligned}
t_{NA} &: \mathrm{Hom}_{\mathbb{Z}}(N \otimes M, A) \to \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, A)); \\
u_{NA} &: \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, A)) \to \mathrm{Hom}_{\mathbb{Z}}(N \otimes M, A)
\end{aligned}
$$

where $t_{NA}(\phi)(n)(m) = \phi(n \otimes m)$ and $u_{NA}(\psi)(n \otimes m) = \psi(n)(m)$. We see then that for any $\phi \in \mathrm{Hom}_{\mathbb{Z}}(N \otimes M, A)$,

$$u_{NA}(t_{NA}(\phi))(n \otimes m) = t_{NA}(\phi)(n)(m) = \phi(n \otimes m)$$

and that for any $\psi \in \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, A))$ we have that

$$t_{NA}(u_{NA}(\psi))(n)(m) = u_{NA}(\psi)(n \otimes m) = \psi(n)(m).$$

This shows that $u_{NA}(t_{NA}(\phi)) = \phi$ and that $t_{NA}(u_{NA}(\psi)) = \psi$, proving that t_{NA} and u_{NA} are inverses to each other. $\square$