Problem Set 8
Abstract Algebra I

Bennett Rennier
barennier@gmail.com

January 15, 2018

Section 5.5

Let H and K be groups, let φ be a homomorphism from K into $\mathrm{Aut}(H)$ and, as usual, identify H and K as subgroups of $G = H \rtimes_\varphi K$.

Ex 1 Prove that $C_K(H) = \ker\varphi$ (recall that $C_K(H) = C_G(H) \cap K$).

Ex 4 Let $p = 2$ and check that the construction of the two non-abelian groups of order p^3 is valid in this case. Prove that both resulting groups are isomorphic to D_8.

Ex 5 Let $G = \mathrm{Hol}(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

a) Prove that $G = H \rtimes K$ where $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $K \simeq S_3$. Deduce that $|G| = 24$.

b) Prove that G is isomorphic to S_4. [Obtain a homomorphism from G into S_4 by letting G act on the left cosets of K. Use Exercise 1 to show this representation is faithful.]

Ex 16 Show that there are exactly 4 distinct homomorphisms from $\mathbb{Z}_2$ into $\mathrm{Aut}(\mathbb{Z}_8)$. Prove that the resulting semidirect products are the groups: $\mathbb{Z}_8 \times \mathbb{Z}_2$, D_{16}, the quasidihedrel group QD_{16} and the modular group M.

Ex 22 Let F be a field and let n be a positive ineger and let G be the group of upper triangular matrices in $\mathrm{GL}_n(F)$.

a) Prove that G is the semidirect product $U \rtimes D$ where U is the set of upper triangular matrices with 1's down the diagonal and D is the set of diagonal matrices in $\mathrm{GL}_n(F)$.

b) Let $n = 2$. Recall that $U \simeq F$ and $D \simeq F^\times \times F^\times$. Describe the homomorphism from D into $\mathrm{Aut}(U)$ explicitly in terms of these isomorphisms (i.e., show how each element of $F^\times \times F^\times$ acts as an automorphism on F).

Ex 25 Let $H(\mathbb{F}_p)$ be the Heisenberg group over the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Prove that $H(\mathbb{F}_2) \simeq D_8$.

Additional Problems

Ex A Let $\mathbb{R}$ be the real numbers and let $\mathcal{C} = \mathbb{R}$-Vect be the collection of all $\mathbb{R}$ vector spaces. For $\mathbb{R}$-vector spaces V and W, let $\mathrm{Hom}_{\mathcal{C}}(V, W)$ be all linear functions from V to W. Please show that $\mathbb{R}$-Vect is a category.

Ex B Let $\mathbb{R}$ be the real numbers and let $\mathcal{C} = \mathbb{R}$-fdVect be the collection of all finite dimensional $\mathbb{R}$-vector spaces. For finite dimensional $\mathbb{R}$-vector spaces V and W, let $\mathrm{Hom}_{\mathcal{C}}(V, W)$ be all linear functions from V to W. Please show that $\mathbb{R}$-fdVect is a category.

Ex C Let $(G, \cdot)$ be a finite group. Let $\mathcal{C}$ have a single object, call it $*$, and let $\mathrm{Hom}_{\mathcal{C}}(*, *) = G$ with composition given by the binary operation of G. Prove that $\mathcal{C}$ is a category.

Ex D Let $(G, \cdot)$ be a finite group. Let $\mathcal{C}$ be the collection of all subgroups of G. Given two subgroups of G, H_1 and H_2, let

$$\mathrm{Hom}_{\mathcal{C}}(H_1, H_2) = \{g \in G \bigm| gH_1g^{-1} \subseteq H_2\}$$

where $gH_1g^{-1} = \{ghg^{-1} \bigm| h \in H_1\}$. Please show that $\mathcal{C}$ is a category.

Ex E Let $X = \mathbb{R} =$ be the real plane. Let $\mathcal{C}$ have as objects the points in X, and for $p_1, p_2 \in X$, let $\mathrm{Hom}_{\mathcal{C}}(p_1, p_2)$ be the collection of all paths from p_1 to p_2. For example, $\mathrm{Hom}_{\mathcal{C}}(p_1, p_1)$ is the set of all loops which begin and end at the point p_1 (including the "do nothing" loop which never leaves the point p_1). Let composition be given by concatenation of paths. Prove that $\mathcal{C}$ is a category.

Ex F Let $\mathcal{T}$ be the collection of all topological spaces and if X, Y are topological spaces, then let $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ be all continuous functions from X to Y. Please show that $\mathcal{T}$ is a category.

Ex G Let Γ be a directed graph (in the mathematical sense of a collection of points and arrows between pairs of those points). Let $\mathcal{C}$ be the collection of points in Γ. For points $x, y \in \Gamma$, let $\mathrm{Hom}_{\mathcal{C}}(x, y) = \{$all directed paths which start at x and end at $y\}$. Note that we include the "empty path" from x to x made up of no arrows at all. Please show that $\mathcal{C}$ is a category. This is the graph theory version of Problem E.