Problem Set 1
Complex Analysis

Bennett Rennier
bennett@brennier.com

Problem 1.

Prove that if $|a| = 1$ or $|b| = 1$, and $a \neq b$, then $\left|\frac{a-b}{1-\overline{a}b}\right| = 1$.

Proof. Suppose $|a| = 1$ and that $a \neq b$. We note that this means

$$a\overline{a} = |a|^2 = 1^2 = 1.$$

From this we can see that

$$\left|\frac{a-b}{1-\overline{a}b}\right| = \frac{|a-b|}{|1-\overline{a}b|} = \frac{|a-b|}{|a||1-\overline{a}b|} = \frac{|a-b|}{|a-(a\overline{a})b|} = \frac{|a-b|}{|a-b|} = 1,$$

as desired. Now suppose that $|b| = 1$ and that $a \neq b$. For this, we note that $b\overline{b} = 1$ as in the first case and also that

$$|\overline{z}| = (\overline{z}z)^2 = \overline{z}z\overline{z}z = z\overline{z}z\overline{z} = (z\overline{z})^2 = |z|.$$

From these two facts, we get that

$$\left|\frac{a-b}{1-\overline{a}b}\right| = \frac{|a-b|}{|1-\overline{a}b|} = \frac{|a-b|}{|\overline{b}||1-\overline{a}b|} = \frac{|a-b|}{|\overline{b}-\overline{a}(\overline{b}b)|} = \frac{|a-b|}{|\overline{b}-\overline{a}|} = \frac{|a-b|}{|\overline{\overline{b}-\overline{a}}|} = \frac{|a-b|}{|b-a|} = \frac{|a-b|}{|a-b|} = 1.$$

These two cases prove the problem. $\square$

Problem 2 (Continuous Cauchy-Schwartz)**.**

Let $a < b$ be real numbers and $f, g\colon [a,b] \to \mathbb{C}$ be continuous functions.

(a) Prove that

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq \left(\int_a^b |f(x)|^2\,dx\right)^{1/2} \left(\int_a^b |g(x)|^2\,dx\right)^{1/2}.$$

(b) Prove that if $\left|\int_a^b f(x)g(x)\,dx\right| = \left(\int_a^b |f(x)|^2\,dx\right)^{1/2} \left(\int_a^b |g(x)|^2\,dx\right)^{1/2}$, and if $g \neq 0$, then there is a $\lambda \in \mathbb{C}$ so that $f = \lambda g$.

(Hint: if you use Riemann sums to prove (a), then you will not be able to deduce (b) from (a). Find a proof of (a) that doesn't use Riemann sums by following our proof of the discrete Cauchy-Schwartz inequality from class).

Proof.

a) We note that the continuous functions from $[a,b]$ to $\mathbb{C}$ form an inner product space where

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx.$$

This product can easily be shown to be linear in the first argument, conjugate symmetric, and that

$$\langle f, f\rangle = \int_a^b f(x)\overline{f(x)}\,dx = \int_a^b |f(x)|^2\,dx \geq 0.$$

Note that if $f(c) \neq 0$ for some $c \in [a,b]$, then by continuity $f(x) \neq 0$ for $x \in (c-\delta, c+\delta) \cap [a,b]$ for some $\delta > 0$, which means that

$$\langle f, f\rangle = \int_a^b |f(x)|^2\,dx \geq \int_{x\in(c-\delta,c+\delta)\cap[a,b]} |f(x)|^2\,dx > 0.$$

By the contrapositive, if $\langle f, f\rangle = 0$, then $f(x) = 0$ for all $[a,b]$. Thus, $\langle \cdot, \cdot\rangle$ is semi-definite as well, proving the last axiom to be an inner product. Note that this also gives us a norm where $||f|| = \sqrt{\langle f, f\rangle}$.

Now I will prove the Cauchy-Schwarz Inequality for inner product spaces in general. That is that

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle \cdot \langle v, v\rangle$$

for u, v in any (complex) inner product space. To prove this, note first that if $v = 0$, then the proof is trivially true. Thus, we can assume that $v \neq 0$. Now let $\lambda = \frac{\langle u,v\rangle}{||v||^2}$, then we have that

$$\begin{aligned}0 \leq ||u-\lambda v||^2 &= \langle u-\lambda v, u-\lambda v\rangle = \langle u,u\rangle - \langle \lambda v, u\rangle - \langle u, \lambda v\rangle + \langle \lambda v, \lambda v\rangle \\ &= ||u||^2 - \lambda\overline{\langle u,v\rangle} - \overline{\lambda}\langle u,v\rangle + \lambda\overline{\lambda}||v||^2 \\ &= ||u||^2 - \frac{\langle u,v\rangle\overline{\langle u,v\rangle}}{||v||^2} - \frac{\overline{\langle u,v\rangle}\langle u,v\rangle}{||v||^2} + \frac{\langle u,v\rangle\overline{\langle u,v\rangle}}{||v||^4}||v||^2 \\ &= ||u||^2 - \frac{|\langle u,v\rangle|^2}{||v||^2} - \frac{|\langle u,v\rangle|^2}{||v||^2} + \frac{|\langle u,v\rangle|^2}{||v||^2} \\ &= ||u||^2 - \frac{|\langle u,v\rangle|^2}{||v||^2}.\end{aligned}$$

Thus, we have that $|\langle u,v\rangle|^2 \leq ||u||^2||v||^2 = \langle u,u\rangle \cdot \langle v,v\rangle$ as desired. Note that this also means that if we take the square root of both sides we get that $|\langle u,v\rangle| \leq ||u|| \cdot ||v||$. For the inner product space we defined above, this translates to

$$\begin{aligned}\left|\int_a^b f(x)\overline{g(x)}\,dx\right| &= |\langle f,g\rangle| \leq ||f|| \cdot ||g|| = \sqrt{\langle f,f\rangle}\sqrt{\langle g,g\rangle} \\ &= \left(\int_a^b |f(x)|^2\,dx\right)^{\frac{1}{2}} \left(\int_a^b |g(x)|^2\,dx\right)^{\frac{1}{2}}\end{aligned}$$

as we intended to prove.

b) Using the same λ as in the previous part, that is $\lambda = \frac{\langle f,g\rangle}{||g||}^2$, assume that $f \neq \lambda g$ and that $g \neq 0$. This would mean that $f - \lambda g \neq 0$ and thus that $||f-\lambda g||^2 > 0$, giving us a strict inequality in the proof of the Cauchy-Schwarz Inequality in the previous part. Thus, by the contrapositive, if we have equality, it must be that $f = \lambda g$ as we wanted to prove.

$\square$

Problem 3 (Continuous Triangle Inequality)**.**

Let $a < b$ be real numbers, and $f\colon [a,b] \to \mathbb{C}$ a continuous function.

(a) Prove that

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

(b) Prove that if

$$\left|\int_a^b f(x)\,dx\right| = \int_a^b |f(x)|\,dx,$$

then there is a constant $\beta \in \mathbb{C}$ so that βf is non-negative valued.

Hint: find an $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ so that $\alpha\int_a^b f(x)\,dx = \left|\int_a^b f(x)\,dx\right|$, and use that $\left|\int_a^b f(x)\,dx\right| = \int_a^b \mathrm{Re}(\alpha f(x))\,dx$, to reduce the problem to a well known property of integrals of continuous (real-valued) functions.

Proof.

a) Let $\alpha \in \mathbb{C}$ be such that $|\alpha| = 1$ and

$$\left|\int_a^b f(x)\,dx\right| = \alpha \int_a^b f(x)\,dx.$$

Using this α, we can see that

$$\begin{aligned}\left|\int_a^b f(x)\,dx\right| &= \alpha \int_a^b f(x)\,dx = \int_a^b \alpha f(x)\,dx = \int_a^b \mathrm{Re}(\alpha f(x))\,dx + i\int_a^b \mathrm{Im}(\alpha f(x))\,dx \\ &= \int_a^b \mathrm{Re}(\alpha f(x))\,dx \le \int_a^b |\alpha f(x)|\,dx = \int_a^b |\alpha||f(x)|\,dx = \int_a^b |f(x)|\,dx\end{aligned}$$

as desired.

b) Suppose that αf were not non-negative valued, where α is the same as in part (a). This means there's some $c \in [a, b]$ such that $\alpha f(c)$ is not non-negative. By the continuity of f, there must be some open set of c, call it U, such that $\alpha f(x)$ is not non-negative for all $x \in U$. Then we would have that for any $x \in U$

$$|\alpha f(x)|^2 = \mathrm{Re}(\alpha f(x))^2 + \mathrm{Im}(\alpha f(x))^2 > \mathrm{Re}(\alpha f(x))^2$$

which means that $|\alpha f(x)| > \mathrm{Re}(\alpha f(x))$ on U. Thus, we have that

$$\begin{aligned}\int_a^b \mathrm{Re}(\alpha f(x))\,dx &= \int_U \mathrm{Re}(\alpha f(x))\,dx + \int_{U^c} \mathrm{Re}(\alpha f(x))\,dx \\ &< \int_U |\alpha f(x)|\,dx + \int_{U^c} |\alpha f(x)|\,dx = \int_a^b |\alpha f(x)|\,dx,\end{aligned}$$

which means the inequality in the proof of part (a) would be a strict inequality. By the contrapositive, if these were equal, then αf would be a non-negative valued function.

$\square$

Problem 4 (Stereographic Projection)**.**

Given $z = x + iy \in \mathbb{C}$, we define $z^* = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$ it is not hard to show that $z^* \in \mathbb{S}^2 = \{(a, b, c) \in \mathbb{R}^3 : a^2 + b^2 + c^2 = 1\}$ (you do not have to prove this).

Pictorally: view $\mathbb{C} \subseteq \mathbb{R}^3$ by identifying $z = x + iy$ with $(x, y, 0)$. Draw a straight line from $(x, y, 0)$ to $(0, 0, 1)$. This line intersects $\mathbb{S}^2$ in exactly one point, and this point is z^*.

(1) Prove that if $p \in \mathbb{S}^2$ and $p \neq (0, 0, 1)$ then there is a unique $z \in \mathbb{C}$ so that $z^* = p$. (Hint: Given p, it might be useful to use the pictorial description above to find z so that $p = z^*$. Remember: a picture is not a proof, but it can guide a proof).

(2) A circle C in $\mathbb{S}^2$ is the intersection of a plane in $\mathbb{R}^3$ with $\mathbb{S}^2$ provided this intersection is nonempty (take this as a definition if you want). Prove that if C is a circle in $\mathbb{S}^2$, then there is a $\widetilde{C} \subseteq \mathbb{C}$ so that

$$C \setminus \{(0,0,1)\} = \{z^* : z \in \widetilde{C}\}$$

where $\widetilde{C}$ is either a circle in $\mathbb{C}$ or a line in $\mathbb{C}$.

Hint: Suppose C is a circle in $\mathbb{S}^2$ which is the intersection of the plane $ax_1 + bx_2 + cx_3 = d$ and $\mathbb{S}^2$. If $p \in C$ and $p \neq (0,0,1)$ write $p = z^*$, plug the formula for z^* into the equation of the plane and expand to find the equation of either a line or a circle.

Proof.

a) Let $p \in S^2$ and let $p = (a,b,c)$ such that $a^2+b^2+c^2 = 1$. We see that the line equation through p and $(0,0,1)$ is

$$\gamma(t) = tp+(1-t)(0,0,1) = (ta,tb,tc)+(0,0,1-t) = (ta,tb,tc+1-t) = (ta,tb,t(c-1)+1).$$

This line goes through the x-y plane when $t(c-1)+1 = 0$, that is when $t = \frac{1}{1-c}$ (note that this is the unique such t). Plugging this into γ, we get that

$$\gamma\left(\frac{1}{1-c}\right) = \left(\frac{a}{1-c}, \frac{b}{1-c}, 0\right)$$

as our proposed z. Let $P(x,y,z) = \frac{a}{1-c} + i\frac{b}{1-c}$ be this projection map from $S^2 \setminus (0,0,1)$ to $\mathbb{C}$. To prove that this is the only element of $\mathbb{C}$ that maps to p under $*$, we can show that P and $*$ are inverse operations.

First, to prove that $P(p)^* = p$ we note that

$$\begin{aligned}\frac{1}{\left(\frac{x}{1-c}\right)^2 + \left(\frac{b}{1-c}\right)^2 + 1} &= \frac{1}{\frac{a^2+b^2}{(1-c)^2} + 1} = \frac{1}{\frac{a^2+b^2+(c^2-2c+1)}{(1-c)^2}} = \frac{(1-c)^2}{(a^2+b^2+c^2)-2c+1} \\ &= \frac{(1-c)^2}{2-2c} = \frac{(1-c)^2}{2(1-c)} = \frac{1-c}{2},\end{aligned}$$

and that

$$\begin{aligned}\left(\frac{a}{1-c}\right)^2 + \left(\frac{b}{1-c}\right)^2 - 1 &= \frac{a^2+b^2}{(1-c)^2} - 1 = \frac{a^2+b^2-(c^2-2c+1)}{(1-c)^2} \\ &= \frac{(1-c^2)-(c^2-2c+1)}{(1-c)^2} = \frac{-2c^2+2c}{(1-c)^2} \\ &= \frac{2c(1-c)}{(1-c)^2} = \frac{2c}{(1-c)}\end{aligned}$$

Thus, we get

$$\begin{aligned}P(p)^* &= \left(\frac{a}{1-c} + i\frac{b}{1-c}\right)^* = \left(\frac{2a}{1-c}\cdot\frac{1-c}{2}, \frac{2b}{1-c}\cdot\frac{1-c}{2}, \frac{2c}{1-c}\cdot\frac{1-c}{2}\right) \\ &= (x,y,z) = p\end{aligned}$$

as we wanted. The fact that $P(z^*) = z$ can be worked out in a similar computational manner. This proves that P and $*$ are inverses maps that define a bijection between $C \setminus (0,0,1)$ and $\mathbb{C}$, meaning that for $p \in C$, there is exactly one $z \in \mathbb{C}$ such that $z^* = p$.

b) Let C be a circle in S^2, that is C is the set of points $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that $ax_1 + bx_2 + cx_3 = d$ where $x_1^2 + x_2^2 + x_3^2 = 1$ for some constants a, b, c, d. Let $z = x + iy$ be such that $z^* \in C$; that is, z^* satisfies the plane equation as well as the usual sphere equation. This means

$$a\left(\frac{2x}{x^2+y^2+1}\right) + b\left(\frac{2y}{x^2+y^2+1}\right) + c\left(\frac{x^2+y^2-1}{x^2+y^2+1}\right) = d$$

which simplifies to

$$2xa + 2yb + c(x^2+y^2-1) - d(x^2+y^2+1) = 0.$$

We note that if $c = d$, then the plane equation intersects the sphere at $(0,0,1)$ and we get that

$$2xa + 2yb - c - d = 0$$

which means the z's that map into C satisfy a linear equation, i.e. a line. Now, if $c \neq d$, we can complete the squares and rearrange to get that

$$\left(x + \frac{a}{c-d}\right)^2 + \left(y + \frac{b}{c-d}\right)^2 = \frac{c+d}{c-d} + \frac{a^2+b^2}{(c-d)^3}$$

which means that the z's that map into C satisfy the equation for a circle. Since the projection and inverse projection maps are bijections between $S^2 \setminus (0,0,1)$ and $\mathbb{C}$, then $\tilde{C} = \{z \in \mathbb{C} : z^* \in C \setminus (0,0,1)\}$ (which is either a line or a circle, possibly degenerate) satisfies $C \setminus (0,0,1) = \{z^* : z \in \tilde{C}\}$.

$\square$

Problem 5.

(a) Prove that the complex series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely. We define $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.
(b) Prove that $e^z e^w = e^{z+w}$ for all $z, w \in \mathbb{C}$.
(c) Prove that $e^{it} = \cos(t) + i\sin(t)$, where $\cos(t), \sin(t)$ for all real t, where $\cos(t), \sin(t)$ are defined to be given by their power series representations.

Proof.

a) We see that if $z = re^{i\theta}$, then

$$|z^n| = |(re^{i\theta})^n| = |r^n e^{i(n\theta)}| = |r^n||e^{i(n\theta)}| = |r^n| = |r|^n = |r|^n|e^{i\theta}|^n = |re^{i\theta}|^n = |z|^n.$$

With this, we see that

$$\sum_{n=0}^{\infty} \left|\frac{z^n}{n!}\right| = \sum_{n=0}^{\infty} \frac{|z^n|}{|n!|} = \sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|} < \infty,$$

which proves that $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely.

b) The Binomial Theorem says that

$$(z+w)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k}.$$

Thus, using the Cauchy product formula, we get that

$$\begin{aligned} e^z e^w &= \left(\sum_{i=0}^{\infty} \frac{z^i}{i!}\right)\left(\sum_{j=0}^{\infty} \frac{z^j}{j!}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k}\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(z+w)^n = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = e^{z+w} \end{aligned}$$

as desired.

c) We recall that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

From these power series representations, we see that

$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty}\left(\frac{x^n}{n!} + \frac{ix^{n+1}}{(n+1)!} + \frac{-x^{n+2}}{(n+2)!} + \frac{-ix^{n+3}}{(n+3)!}\right) \\ &= \sum_{n=0}^{\infty}\left(\frac{x^n}{n!} + \frac{-x^{n+2}}{(n+2)!}\right) + \sum_{n=0}^{\infty}\left(i\frac{x^{n+1}}{(n+1)!} + \frac{-ix^{n+3}}{(n+3)!}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \cos x + i \sin x, \end{aligned}$$

as desired.

$\square$