Problem Set 7
Complex Analysis

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Ex 1 We say that any entire $f : \mathbb{C} \to \mathbb{C}$ is of *exponential type* if there are constants $c_1, c_2 > 0$ such that $|f(z)| \leq c_1 e^{c_2|z|}$. Show that f is of expontential type if and only if f' is of exponential type.

Proof. Suppose $|f(z)| \leq c_1 e^{c_2|z|}$. Then by the Cauchy Estimates we have that for any $z \in \mathbb{C}$

$$|f'(z)| \leq \frac{c_1 e^{c_2 r}}{r} \leq c_1 e^{c_2 r}$$

for sufficiently large r. This proves that f' is of expontential type. Conversely, suppose $|f'(z)| \leq c_1 e^{c_2|z|}$. We see then that for any $z \in \mathbb{C}$, if we let $\gamma : [0,1] \to \mathbb{C}$ be the path $\gamma(t) = zt$ then

$$||f(z)| - |f(0)|| \leq |f(z) - f(0)| = \left| \int_\gamma f'(z)\, dz \right| = \left| \int_0^1 f'(\gamma(t))\gamma'(t)\, dt \right| \leq \int_0^1 |f'(\gamma(t))||\gamma'(t)|\, dt$$
$$\leq \int_0^1 c_1 e^{c_2|\gamma(t)|}|\gamma'(t)|\, dt \leq c_1 e^{c_2|z|} \int_0^1 |\gamma'(t)|\, dt \leq c_1 e^{c_2|z|}|z| \leq c_1 e^{c_2|z|} e^{|z|} = c_1 e^{(c_2+1)|z|}.$$

This means that

$$|f(z)| \leq c_1 e^{(c_2+1)|z|} + |f(0)| \leq c_1 e^{(c_2+1)|z|} + |f(0)| e^{(c_2+1)|z|} = (|f(0)| + c_1) e^{(c_2+1)|z|}$$

which proves that f is of expontential type. □

Ex 2

a) Find a nonzero harmonic function $u : \mathbb{D} \to [0, \infty)$ with the following property. For every $w \in \partial\mathbb{D}$ with $w \neq 1$ and for every $z_n \in \mathbb{D}$ with $z_n \to w$, we have that $\lim_{n\to\infty} u(z_n) = 0$.

b) Suppose that u is as in part (a). Show that there must be a sequence $z_n \in \mathbb{D}$ with $z_n \to 1$ so that $\limsup u(z_n) > 0$.

Proof.

a) Take the möbius transformation $\phi(z) = \frac{1+z}{1-z}$. We see that

$$\begin{aligned}
&\phi(-1) = 0 \\
&\phi(1) = \infty \\
&\phi(i) = \frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{2i}{2} = i \\
&\phi(0) = 1.
\end{aligned}$$

This proves that ϕ takes the unit circle to the imaginary axis and the unit disk to the right-half plane. If we write $\phi = u + iv$ where u, v are non-zero real-valued functions, then we know that u is harmonic and that u is nonnegative on the unit disk. I claim that u restricted to the unit disk is our desired function. Let $w \in \partial\mathbb{D}$ with $w \neq 1$ and let $z_n \in \mathbb{D}$ be a sequence approaching w. We see then that f maps w into the imaginary axis so that

$$\lim_{n\to\infty} u(z_n) = \lim_{n\to\infty} \operatorname{Re} f(z_n) = \operatorname{Re} f(\lim_{n\to\infty} z_n) = \operatorname{Re} f(w) = 0$$

as desired.

b) Suppose that for every sequence $z_n \to 1$ in $\mathbb{D}$ has the property that $\limsup u(z_n) = 0$. Since u is nonnegative, this means that $u(z_n)$ converges to 0. With part (a), we have then that any sequence z_n approaching any point on the boundary of the disk has the property that $u(z_n)$ converges to 0. Thus, we can extend u to be a continuous function on $\overline{\mathbb{D}}$ where $u(\partial\mathbb{D}) = 0$. Since u is harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, by the maximum modulus principle, $|u(z)|$ achieves its maximum on ∂D. But, $|u(z)| = 0$ for any $z \in \partial D$. This proves that u must be the zero function, contradicting our assumption in part (a). Thus, we have that there exists a sequence $z_n \to 1$ in $\mathbb{D}$ such that $\limsup u(z_n) \neq 0$. □

Ex 3

a) Let $p_1, \ldots, p_n \in \partial\mathbb{D}$. Show that there is a $z \in \partial\mathbb{D}$ such that $\prod_{j=1}^n |z - p_j| \geq 1$.

b) Show that there is a $z \in \partial\mathbb{D}$ such that $\prod_{j=1}^n |z - p_j| = 1$.

Proof.

a) Let $p(z) = \prod_{j=1}^n (z - p_j)$ which is a polynomial with all roots on the unit circle. For sake of contradiction, suppose that $|p(z)| = \prod_{j=1}^n |z - p_j| < 1$ for all $z \in \partial\mathbb{D}$. However, we see that $|p(0)| = \prod_{j=1}^n |p_j| = 1$. By the maximum modulus principle, though, $|p(z)|$ achieves its maximum over $\overline{\mathbb{D}}$ on the boundary. Thus, we have have a contradiction, proving there must be some $z \in \partial\mathbb{D}$ such that $p(z) = \prod_{j=1}^n |z - p_j| \geq 1$.

b) Let $z_0 \in \partial\mathbb{D}$ such that $|p(z_0)| \geq 1$ as in part (a) and $\gamma : [0, 1] \to \mathbb{C}$ be the path on $\partial\mathbb{D}$ from z_0 to the closest root of $p(z)$. Thus, we know that $|p(\gamma(z))|$ is a continuous function from $[0, 1]$ to $\mathbb{R}$ such that $|p(\gamma(0))| = |p(z_0)| \geq 1$ and $|p(\gamma(1))| = |0|$. By the Intermediate Value Theorem, there is some $c \in [0, 1]$ such that

$$1 = |p(\gamma(c))| = |\prod_{j=1}^n (\gamma(c) - p_j)| = \prod_{j=1}^n |\gamma(c) - p_j|$$

Thus, $\gamma(c) \in \mathbb{D}$ is our desired element. □

Ex 4

a) Evaluate the integral

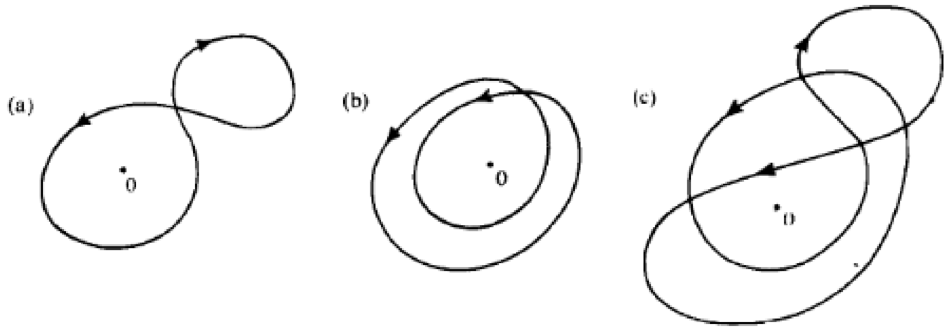
$$\int_\gamma \frac{dz}{z^2 + 1}$$

where $\gamma(\theta) = 2|\cos(2\theta)|e^{i\theta}$ for $0 \leq \theta \leq 2\pi$.

b) Let $\gamma(\theta) = \theta e^{i\theta}$ for $0 \leq \theta \leq 2\pi$ and $\gamma(\theta) = 4\pi - \theta$ for $2\pi \leq \theta \leq 4\pi$. Evaluate

$$\int_\gamma \frac{dz}{z^2 + \pi^2}.$$

c) Evaluate $\int_\gamma \frac{e^z - e^{-z}}{z^4}\, dz$ where γ is one of the curves depicted below:



Proof.

a) We see that via partial fraction decomposition

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{-\frac{1}{2i}}{z+i} + \frac{\frac{1}{2i}}{z-i}.$$

This proves that

$$\begin{aligned}\int_\gamma \frac{dz}{z^2+1} &= \frac{1}{2i}\left(\int_\gamma \frac{dz}{z-i} - \int_\gamma \frac{dz}{z+i}\right)\\ &= \frac{1}{2i}\left(n(\gamma;i)2\pi i - n(\gamma;-i)2\pi i\right)\\ &= (n(\gamma;i) - n(\gamma;-i))\pi.\end{aligned}$$

Since $n(\gamma;i) = n(\gamma;-i) = 1$, we can conclude that the integral is zero.

b) We see that via partial fraction decomposition

$$\frac{1}{z^2+\pi^2} = \frac{1}{(z+\pi i)(z-\pi i)} = \frac{-\frac{1}{2\pi i}}{z+\pi i} + \frac{\frac{1}{2\pi i}}{z-\pi i}.$$

This proves that

$$\begin{aligned}\int_\gamma \frac{dz}{z^2+\pi^2} &= \frac{1}{2\pi i}\left(\int_\gamma \frac{dz}{z-\pi i} - \int_\gamma \frac{dz}{z+\pi i}\right)\\ &= \frac{1}{2\pi i}\left(n(\gamma;\pi i)2\pi i - n(\gamma;-\pi i)2\pi i\right)\\ &= n(\gamma;\pi i) - n(\gamma;-\pi i).\end{aligned}$$

Since $n(\gamma;\pi i) = 0$ and $n(\gamma;-\pi i) = 1$, we can conclude that the integral is -1.

c) We see that by the Cauchy Integral Formula

$$\int_\gamma \frac{e^z - e^{-z}}{z^4}\,dz = n(\gamma;0)2\pi i f^{(3)}(0) = n(\gamma;0)2\pi i \left(\frac{e^z - e^{-z}}{3!}\right)^{(3)}|_{z=0}$$
$$= n(\gamma;0)2\pi i \frac{e^z + e^{-z}}{3!}|_{z=0} = \frac{2\pi}{3} n(\gamma;0).$$

Simply by drawing a line from 0 to infinity and counting the number of times it crosses γ_i with sign, we have that

(a) $n(\gamma_1;0) = 1$ so $\int_\gamma \frac{e^z-e^{-z}}{z^4}\,dz = \frac{2\pi i}{3}$

(b) $n(\gamma_2;0) = 2$ so $\int_\gamma \frac{e^z-e^{-z}}{z^4}\,dz = \frac{4\pi i}{3}$

(c) $n(\gamma_3;0) = 2$ so $\int_\gamma \frac{e^z-e^{-z}}{z^4}\,dz = \frac{4\pi i}{3}$

□

Ex 5 Let Ω be a bounded domain. Suppose that $\mathbb{C}\setminus\Omega = \cup_{j=0}^n \overline{D_j}$, where $D_1, \ldots, D_n$ are connected, bounded open sets, and D_0 is unbounded. Suppose that $\{\overline{D_j}\}_{j=1}^n$ is a disjoint family. Suppose additionally that there are piecewise C^2 simple loops $\gamma_j : [a_j, b_j] \to \mathbb{C}$, $0 \le j \le n$ with $\gamma_j([a_j,b_j]) = \partial D_j$ (here simple means not self-intersecting), and that $\partial\Omega = \cup_{j=1}^n \partial D_j$.

Lastly, assume that

1) For all $1 \le j \le n$ we have that $n(\gamma_j;z) = 1$ for all $z \in D_j$ and $n(\gamma_j;z) = 0$ for all $z \in D_j^c$ while $n(\gamma_0;z) = 0$ for all $z \in D_0$ and $n(\gamma_0;z) = 1$ for all $z \in \Omega \cup \bigcup_{j=1}^n \overline{D_j}$.

2) For all $0 \le j \le n$ and all $t \in [a_j,b_j]$, there is an $\varepsilon > 0$ such that $\gamma_j(t) + \kappa i \frac{\gamma_j'(t)}{\gamma_j(t)} \in \Omega$ for all $\kappa \in (0,\varepsilon)$.

Let $f : \overline{\Omega} \to \mathbb{C}$ be continuous and suppose that $f|_\Omega$ is analytic. Show that $f = \sum_{j=0}^n f_j$ where for each $1 \le j \le n$, f_j extends analytically to $\mathbb{C}\setminus\overline{D_j}$ and where f_0 extends analytically to $\Omega \cup \bigcup_{j=1}^n \overline{D_j}$.

Proof. Using formal path summation, define

$$\Gamma = \gamma_0 - \sum_{i=1}^n \gamma_i.$$

Since $n(\Gamma;z) = n(\gamma_0;z) - \sum_{i=1}^n n(\gamma_i;z)$, which by construction is 0 for all $z \in \Omega\circ$ and is 1 for all $z \in \Omega$. This means that Γ is homologous to zero, so we can use Cauchy's Integral formula to deduce that

$$f(z) = \frac{1}{2\pi i n(\Gamma;z)} \int_\Gamma \frac{f(s)}{s-z}\,ds = \frac{1}{2\pi i}\left(\int_{\gamma_0} \frac{f(s)}{s-z}\,ds - \sum_{i=1}^n \int \frac{f(s)}{s-z}\,ds\right).$$

We define

$$f_0(z) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(s)}{s-z}\,ds \quad ; \quad f_j(z) = \frac{1}{2\pi i}\int_{\gamma_j} -\frac{f(s)}{s-z}\,ds \text{ for } j > 0,$$

which means $f = \sum_j f_j$. Since γ_0 is homologous to zero on the open set $\Omega \cup \bigcup_j \overline{D_j}$, we see that f_0 has a sensible extension to $\Omega \cup \bigcup_j \overline{D_j}$. It is holomorphic extension as it is the integral of the quotient of two holomorphic functions (and the denominator is never zero). Similarly, as γ_j is homologous ot zero on the set $\mathbb{C} \setminus \overline{D_j}$, f_j can be holomorphically extended to this set. These f_j are the desired functions. $\square$

Ex 6 Let f be an entire function with only finitely many zeros. Let $m(r) = \inf_{|z|=r} |f(z)|$. Show that if f is not a polynomial, then $\lim_{r\to\infty} m(r) = 0$.

Proof. Let $p_1, \dots, p_n$ be the roots of f, some repeated to account for multiplicity. This means that there exists some $g : \mathbb{C} \to \mathbb{C}$ which is never zero such that $f(z) = (\prod_{i=1}^n (z - p_i))g(z)$. Thus, the function $1/g$ is an entire holomorphic function. We see via Cauchy Estimates that

$$\begin{aligned}\left|\left(\frac{1}{g}\right)^{(k)}(z)\right| &\le \frac{n!}{r^k}\sup_{|z|=r}\left|\frac{1}{g(z)}\right| = \frac{n!}{r^k}\sup_{|z|=r}\left|\frac{\prod_{i=1}^n (z-p_i)}{f(z)}\right| = \frac{n!}{r^k}\sup_{|z|=r}\frac{\prod_{i=1}^n |z-p_i|}{|f(z)|}\\ &\le \frac{n!}{r^k}\sup_{|z|=r}\frac{\prod_{i=1}^n (r+|p_i|)}{|f(z)|} = \frac{n!\prod_{i=1}^n (r+|p_i|)}{r^k}\frac{1}{\inf_{|z|=r}|f(z)|} = \frac{n!\prod_{i=1}^n (r+|p_i|)}{m(r)r^k}.\end{aligned}$$

Thus, if $m(r)$ does not tend to zero, then for $k > n$, $|(1/g)^{(k)}| = \lim_{r\to\infty} \frac{n!\prod_{i=1}^n r+|p_i|}{m(r)r^k} = 0$. As the kth derivative of $1/g$ is proportional to the a_k's in the analytic expansion of $1/g$, we can conclude that $1/g$ is a polynomial. However, $1/g$ doesn't have any zeros on $\mathbb{C}$. This means that $1/g(z) = C$ for some constant $0 \ne C \in \mathbb{C}$. Thus, $g(z) = 1/C$ and we have that $f(z) = (\prod_{i=1}^n (z - p_i)) \cdot 1/C$, proving that f is a polynomial. Contrapositively, if f is not a polynomial, then it must be that $\lim_{r\to\infty} m(r) = 0$. $\square$