Problem Set 6
Real Analysis I

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Ex 11 Suppose m is Lebesgue measure and A is a Borel measurable subset of $\mathbb{R}$ with $m(A) > 0$. Prove that if

$$B = \{x - y \mid x, y \in A\}$$

then B contains a non-empty open interval centered at the origin. This is known as the Steinhaus theorem.

Proof. Suppose there was no nonempty open interval centered at the origin contained in B. Then there is some element in $\left(\frac{1}{-n}, \frac{1}{n}\right)$ that's not in B. Let this element be x_n. Suppose $(x_n + A) \cap A \neq \varnothing$, then let w be in this set. This means that $w \in A$ and $w \in x_n + A$, and thus $w - x_n \in A$. However, this would mean that $w - (w - x_n) = x_n \in A - A$, a contradiction. This shows that $(x_n + A) \cap A = \varnothing$, which implies that $m((x_n + A) \cap A) = m(x_n + A) + m(A) = 2m(A)$. However, $m(A) = \lim_{n\to 0} m((x_n + A) \cap A) = \lim_{n\to 0} 2m(A) = 2m(A)$. This means that $2m(A) = m(A)$, which is a contradiction, as $m(A) > 0$. Thus, there must be some open interval centered at the origin contained in B. $\square$

Ex 13 Let N be the non-measurable set defined in Section 4.4. Prove that if $A \subseteq N$ and A is Lebesgue measurable, then $m(A) = 0$.

Proof. Let $Q = \mathbb{Q} \cap [0, 1]$. Recall that $\cup_{q\in Q} (q + N) \subseteq [-1, 2]$, and that $q + N$ is disjoint for all q. Thus, $A \subseteq [-1, 2]$, and $q + A$ is disjoint for all q. Since A is measurable, this means that $\sum_{q\in Q} m(q + A) = m\left(\cup_{q\in Q} (q + A)\right) \leq m([-1, 2]) = 3$. Since $m(q + A) = m(A)$, this means that $\sum_{q\in Q} m(A) < 3$. This can only happen if $m(A) = 0$. This proves the statement. $\square$

Ex 14 Let m be Lebesgue measure. Prove that if A is a Lebesgue measurable subset of $\mathbb{R}$ and $m(A) > 0$, then there is a subset of A that is non-measurable.

Proof. Note that $\cup_{q\in\mathbb{Q}} (q + N) = \mathbb{R}$, where N is the set from Section 4.4. Thus we see that $\left(\cup_{q\in\mathbb{Q}} (q + N)\right) \cap A = A$. This means that

$$m(A) = m\left(\left(\bigcup_{q\in\mathbb{Q}} q + N\right) \bigcap A\right) = m\left(\bigcup_{q\in\mathbb{Q}} ((q + N) \cap A)\right) \leq \sum_{q\in\mathbb{Q}} m((q + N) \cap A)$$

However, $(q+N)\cap A\subseteq q+N$. Using a slight variation of Exercise 13, this means that $(q+N)\cap A$ is either nonmeasurable or has measure zero. Suppose $(q+N)\cap A$ has measure zero for all $q\in\mathbb{Q}$, then $m(A)\leq\sum_{q\in\mathbb{Q}}m((q+N)\cap A)=\sum_{q\in\mathbb{Q}}0=0$ This is a contradiction, as $m(A)>0$. Thus, $(q+N)\cap A\subseteq A$ must be nonmeasurable for some $q\in\mathbb{Q}$. $\square$