Problem Set 6
Abstract Algebra II

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Section 10.4

Ex 2 Show that the element "$2 \otimes 1$" is 0 in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ but is nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.

Proof. In $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, we get that $2 \otimes 1 = (1 \cdot 2) \otimes 1 = 1 \otimes (2 \cdot 1) = 1 \otimes 0 = 0$, since $2 = 0$ in $\mathbb{Z}/2\mathbb{Z}$.

To prove that it's nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. We construct the bilinear set map $f : 2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ by $f(x, y) = x$. We see that this induces a module homomorphism $\tilde{f}$ from $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$ which agrees with f. Thus, $\tilde{f}(2 \otimes 1) = f(2, 1) = 2$. Since every homomorphism sends 0 to 0, this proves that $2 \otimes 1 \neq 0$. $\square$

Ex 3 Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are both left $\mathbb{R}$-modules, but are not isomorphic as $\mathbb{R}$-modules.

Ex 4 Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic left $\mathbb{Q}$-modules. [Show they are both 1-dimensional vector spaces over $\mathbb{Q}$.]

Ex 5 Let A be a finite abelian group of order n and let p^k be the largest power of the prime p dividing n. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A.

Ex 6 If R is any integral domain with quotient field Q, prove that $(Q/R) \otimes_R (Q/R) = 0$.

Ex 11 Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.

Ex 13 Prove that the usual dot product of vectors defined by letting $(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n)$ be $a_1 b_1 + \cdots + a_n b_n$ is a bilinear map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$.

Ex 14 Let I be an arbitrary nonempty index set and for each $i \in I$ let N_i be a left R-module. Let M be a right R-module. Prove the group isomorphism: $M \otimes (\oplus_{i \in I} N_i) \simeq \oplus_{i \in I}(M \otimes N_i)$.

Ex 24 Prove that the extension of scalars from $\mathbb{Z}$ to the Gaussian integers $\mathbb{Z}[i]$ of the ring $\mathbb{R}$ is isomorphic to $\mathbb{C}$ as a ring: $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{C}$.

Ex 25 Let R be a subring of the commutative ring S and let x be an indeterminate over S. Prove that $S[x]$ and $S \otimes_R R[x]$ are isomorphic as S-algebras.

Additional Problems

Ex A Let M be a fixed right R-module. Let R-mod denote the category of left R-modules and AbGroups denote the category of Abelian groups. Define T : R-mod $\to$ AbGroups on objects by $T(N) = M \otimes_R N$. Define T on morphisms and show that it is a functor. You don't have to do it, but notice that we can also use the tensor product to define a functor from right R-modules to AbGroups.

Ex B Now assume that M is a fixed (S, R)-bimodule. Show that $M \otimes_R N$ is a left S-module and that T from part A defines a functor to the category of left S-modules. Even more generally, please verify in this case that we can use T to define a functor from (R, T)-bimodules to (S, T)-bimodules.

Ex C Let R be a commutative ring and let M and N be left R-modules viewed as bimodules in the standard way. Show that $M \otimes_R N$ is an R-module and use the universal property to show that $M \otimes_R N \simeq N \otimes_R M$.

Ex D Let M be a right R-module, N a (R, S)-bimodule, and U a left S-module. Use the universal property to show that

$$(M \otimes_R N) \otimes_S U \simeq M \otimes_R (N \otimes_S U)$$

as abelian groups. You don't have to write it up, but do notive that if M is a left module for some ring and/or U is a right module for some ring, then this isomorphism preserves these structures.