# Final Exam
# Algebra III

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### Group I

**Ex 1.** Prove or disprove: If $M$ is an $R$-module and $N$ is a submodule of $M$ such that $N$ and $M/N$ are both semi-simple, then $M$ is also semi-simple.

*Proof.* This is false. Consider $R = k[x]/(x^2)$ for some field $k$. Then $(x) = Rx \simeq R$ and $R/(x) \simeq R$ are both simple as $R$-modules (and hence, semi-simple). However, I claim that $(x)$ is the only simple submodule of $M$, so $M$ cannot be semi-simple. In fact, $\{0\}, (x)$, and $R$ itself are the only submodules of $R$.

To prove this, let $S$ be a submodule of $R$ such that $S \neq (x)$ and $S \neq \{0\}$. Since $(x)$ is simple and $S \cap (x)$ is a submodule of $(x)$, we have that $S \cap (x) = \{0\}$ or $S \cap (x) = (x)$. If $S \cap (x) = \{0\}$, then $S$ contains a element not in $(x)$; however, all such elements are invertible, so it must be that $S = R$. If $S \cap (x) = (x)$, then $(x) \subsetneq S$. Again, this means that $S$ contains an element not in $(x)$ and so $S = R$. □

**Ex 3.**

a) If $I$ is a two-sided ideal of $R$ such that $J(R/I) = \{0\}$, prove that $J(R) \subseteq I$.

b) If $I$ is a (proper) maximal two-sided ideal of $R$, prove that $J(R) \subseteq I$.

*Proof.*

a) The conclusion is trivial in the case that $I = R$ so we shall assume that $I \neq R$. Let $x \in J(R)$. This means that $1 - rx$ is invertible for all $r \in R$. As the canonical surjection $R \to R/I$ is a non-zero homomorphism (this is where we need that $I \neq R$), it preserves units. This means $1 - rx + I \in R/I$ is invertible as well for all $r \in R$, implying that $x + I \in J(R/I)$. Since the Jacobson radical of $J(R/I)$ is trivial, it must be then that $x + I = 0 + I$, proving that $x \in I$. Thus, $J(R) \subseteq I$.

b) Let $I$ be a (proper) maximal two-sided ideal of $R$. By the Lattice Correspondence Theorem, the two-sided ideals of $R/I$ are in correspondence with the two-sided ideals of $R$ which contain $I$. As $I$ is a maximal two-sided ideal of $R$, this means that $R/I$ only has trivial two-sided ideals. Thus, as the Jacobson radical is a two-sided ideal, we know that $J(R/I) = \{0\}$ or $J(R/I) = R/I$. It cannot be the latter as 1 is never in the Jacobson radical (the element $1 - r \cdot 1$ is not invertible for $r = 1$). Thus, it must be that $J(R/I) = \{0\}$, and so $J(R) \subseteq I$ as proven in part (a). □

**Ex 4.** If $R$ is artinian, prove that $J(R)$ is the intersection of all maximal two-sided ideals of $R$.

*Proof.* First, we shall prove that for semi-simple rings, the intersection of all maximal two-sided ideals is trivial. Let $R$ be semi-simple. By Artin-Wedderburn, this means that $R \simeq \oplus_{i \leq k} M_{n_i}(D_i)$ where each $D_i$ is a skew-field. We recall that skew-fields are simple and that matrix algebras over simple rings are simple. Thus, if $k = 1$, then the only two-sided maximal ideal of $R$ is $\{0\}$. If $k > 1$, then $M_{n_1}(D_1) \oplus \{0\}$ and $\{0\} \oplus M_{n_2}(D_2)$ are two-sided ideals of $R$ that have trivial intersection. Thus, in either case, the intersection of all maximal two-sided ideals is $\{0\}$.

Now, to solve the original problem. Let $R$ be an artinian ring. Since $R$ is artinian, so is the quotient $R/J(R)$. Also, as $J(R/J(R)) = J(R)/J(R) = \{0\}$, we know that $R/J(R)$ is semi-simple. By the previous paragraph, we have that the intersection of all maximal two-sided ideals of $R/J(R)$ is trivial. Using the correspondence theorem of maximal ideals between $R$ and $R/J(R)$, we obtain that the intersection of all maximal two-sides ideals containing $J(R)$ is $J(R)$. But by Ex 3, every maximal two-sided ideal contains $J(R)$. Thus, $J(R)$ must be equal to the intersection of all maximal two-sided ideals. □

**Ex 5.** Assume that $R$ is simple as a left $R$-module. Prove or disprove: $R$ is a skew-field.

*Proof.* I claim that $R$ is indeed a skew-field. Since $R$ is simple as a $R$-module, we have that $R$ is semi-simple as a ring. By Artin-Wedderburn, this implies that $R \simeq \oplus_{i \leq k} M_{n_i}(D_i)$ where each $D_i$ is a skew-field. As $R$ is simple as an $R$-module, it must be that $k = 1$, otherwise $M_1(D_1)$ is a proper $R$-submodule. Additionally, if $R \simeq M_n(D)$ and $n > 1$, then the subset $\{(a_{ij}) \in M_n(D) : a_{ij} = 0 \text{ for } j \neq 1\}$ (i.e. the set of all matrices whose only non-zero entries are in the first column) forms a left ideal and is thus a (left) $M_n(D)$-submodule, which contradicts the simplicity of $R$ as a $R$-submodule. Thus, it must be that $n = 1$, proving that $R \simeq D$ where $D$ is a skew-field. □

## Group II

**Ex 1.** Let $G_1$ and $G_2$ be two (not necessarily finite) groups and $G = G_1 \times G_2$ be their direct product. Prove that the $k$-algebras $k[G]$ and $k[G_1] \oplus_k k[G_2]$ are isomorphic.

*Proof.* We will write the canonical bases of $k[G_1]$, $k[G_2]$, and $k[G]$ as $\{e_g : g \in G_1\}$, $\{e_h : h \in G_2\}$, and $\{e_{(g,h)} : (g,h) \in G_1 \times G_2 = G\}$ respectively. Now let $\phi_1 : k[G_1] \to k[G]$ be the map where $\phi_1(e_g) = e_{(g,1)}$ for all $g \in G_1$. Similarly, let $\phi_2 : k[G_2] \to k[G]$ be the map where $\phi_2(e_h) = e_{(1,h)}$ for all $h \in G_2$. Since $G_1$ and $G_2$ commute in $G = G_1 \times G_2$, we see that $e_{(g,1)}e_{(1,h)} = e_{(g,h)} = e_{(1,h)}e_{(1,g)}$. This means that the images of $\phi_1$ and $\phi_2$ commute, so by the universal property, $\phi : k[G_1] \oplus_k k[G_2] \to k[G]$ where $\phi(e_g \oplus e_h) = \phi_1(e_g)\phi_2(e_h) = e_{(g,1)}e_{(1,h)} = e_{(g,h)}$ is a well-defined $k$-algebra homomorphism. Additionally, as $\phi$ sends the basis $\{e_g \otimes e_h : g \in G_1, h \in G_2\}$ to the basis $\{e_{(g,h)} : g \in G_1, h \in G_2\}$, it must be that $\phi$ is an isomorphism of $k$-algebras. □

**Ex 3.**

a) Let $D$ be a finite-dimensional central $k$-division algebra and $a, b$ elements of $D$ with minimal polynomials $\mu_{a|k}(x)$, $\mu_{b|k}(x) \in k[x]$. If $\mu_{a|k}(x) = \mu_{b|k}(x)$, prove that there exists an element $d \in D^*$ with $b = dad^{-1}$.

b) Will the claim in (a) remain true if we only assume that $D$ is a central simple finite-dimensional $k$-algebra? Why or why not?

*Proof.*

a) We define the $k$-algebra homomorphisms $\phi : k[a] \to D$ and $\psi : k[b] \to D$ by requiring $\phi(a) = a$ and $\psi(b) = b$ respectively. Since $a$ and $b$ have the same minimal polynomial, then $k[a] \simeq k[b]$ via the map $f$ which sends $a$ to $b$. This means that $\phi$ and $\psi \circ f$ are both $k$-algebra homomorphisms from the simple $k$-algebra $k[a]$ to the central simple $k$-algebra $D$. Thus, by Skolem-Noether, these maps are conjugate, that is $(\psi \circ f)(x) = d\phi(x)d^{-1}$ for some $d \in D^*$. In particular when $x = a$ we obtain that

$$dad^{-1} = d\phi(a)d^{-1} = (\psi \circ f)(a) = \psi(f(a)) = \psi(b) = b$$

as we wanted.

b) Yes, the above proof still works as Skolem-Noether only requires that $D$ be a central simple algebra. $\square$

**Ex 4.** Let $D$ be a finite-dimensional $k$-division algebra and assume that $\dim_k(D)$ is square-free. Prove that $D$ is a field.

*Proof.* Let $K = Z(D)$, which is a field. This means that $D$ is a central, simple $K$-algebra. If we let $\overline{K}$ be the algebraic closure of $K$, then $D_K = \overline{K} \otimes_K D$ is also simple and so $D_{\overline{K}} \simeq M_n(\overline{K})$ by Artin-Wedderburn. This means that

$$\dim_k(D) = [k : K] \dim_K(D) = [k : K] \dim_{\overline{K}}(D_K) = [k : K] \dim_{\overline{K}}(M_n(\overline{K})) = [k : K]n^2.$$

But $\dim_k(D)$ is square-free, so it must be that $n = 1$. This implies that $\dim_K(D) = 1$ and so $D = K$, a field. $\square$

**Ex 5.**

a) Construct two inequivalent irreducible complex representations of degree 2 of $D_{10}$, the dihedral group of order 10.

b) Write down $\mathbb{C}[D_{10}]$ as a product of simple $\mathbb{C}$-algebras. Give arguments for your answer.

*Proof.*

a) We see that we can represent $D_{10} = \langle s, r : r^5 = s^2 = (sr)^2 = e \rangle$ as $\phi : D_{10} \to M_2(\mathbb{C})$ where

$$r \mapsto \begin{bmatrix} e^{2\pi i/5} & 0 \\ 0 & e^{-2\pi i/5} \end{bmatrix} \quad ; \quad s \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We can also represent $D_{10}$ as $\psi : D_{10} \to M_2(\mathbb{C})$ where

$$r \mapsto \begin{bmatrix} e^{4\pi i/5} & 0 \\ 0 & e^{-4\pi i/5} \end{bmatrix} \quad ; \quad s \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

These are indeed representations as $\phi(r)^5 = \phi(s)^2 = \phi(sr) = \mathrm{Id}_n$ and $\psi(r)^5 = \psi(s)^2 = \psi(sr) = \mathrm{Id}_n$. Since $\phi(r)$ is a rotation by an angle that is not $k\pi$ for some integer $k$, it has no invariant subspaces. This proves that $\phi$ and $\psi$ are both irreducible.

Suppose $\phi$ and $\psi$ are equivalent. That is, suppose there's an invertible matrix $M : \mathbb{C}^2 \to \mathbb{C}^2$ such that $M\phi(r) = \psi(r)M$ and $M\phi(s) = \psi(s)M$. If we let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then we see that

$$\begin{bmatrix} ae^{2\pi i/5} & be^{-2\pi i/5} \\ ce^{2\pi i/5} & de^{-2\pi i/5} \end{bmatrix} = M\phi(r) = \psi(r)M = \begin{bmatrix} ae^{4\pi i/5} & be^{4\pi i/5} \\ ce^{-4\pi i/5} & de^{-4\pi i/5}. \end{bmatrix}.$$

However, this implies that $a = ae^{2\pi i/5}$, $b = be^{\pi i/5}$, $c = ce^{-\pi i/5}$, and $d = de^{-2\pi i/5}$. Since the only complex number that is preserved under a non-identity rotation is 0, we can conclude that $a = b = c = d = 0$. This is a contradiction to our assumption that $M$ was invertible. Thus, $\phi$ and $\psi$ must be inequivalent representations.

b) As $|D_{10}| = 10$ and $\mathrm{char}(\mathbb{C}) = 0$, $\mathbb{C}[D_{10}]$ is semi-simple by Maschke's Theorem. By Artin-Wedderburn, it follows that $\mathbb{C}[D_{10}] \simeq \oplus_{i \leq \ell} M_{n_i}(D_i)$, where each $D_i$ is a $\mathbb{C}$-division algebra. However, as $\mathbb{C}$ is algebraically-closed, it must be that $D_i = \mathbb{C}$ for every $i \leq \ell$. Additionally, each of these simple $\mathbb{C}[D_{10}]$-modules corresponds to an irreducible complex representation of $D_{10}$. Since there are two representations of degree two, it must be that two of the direct summands are $M_2(\mathbb{C})$. As $\mathbb{C}[D_{10}]$ has dimension 10, our only option is that that $\ell = 4$ and that $n_3 = n_4 = 1$, i.e.

$$\mathbb{C}[D_{10}] \simeq M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus \mathbb{C}^2. \qquad \square$$