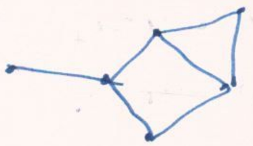
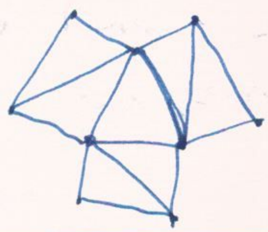
Bennett Rennier

Ex 1.1.41 The graph



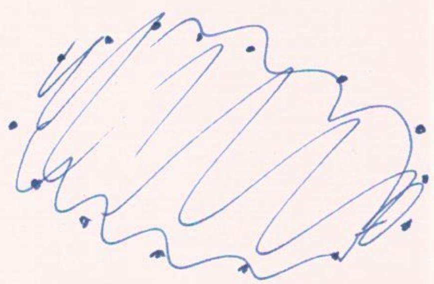
is a six-vertex graph with only the identity automorphism.

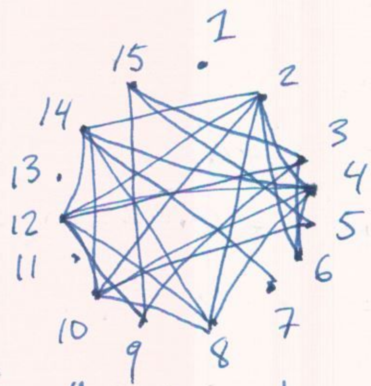
and the graph



has exactly three automorphisms.

Ex 1.2.3





# of components = 4

Since 1, 11, 13 are isolated vertices, the longest path is ≤ 12. We see that 2, 4, 6, 3, 9, 15, 5, 10, 8, 12, 14, 7 is a path which achieves this. Thus, the maximum length of a path in G is 12.

Ex 1.2.5 Let C be a component of $G \setminus v$ and let $u \in C$. Since G is connected, there's a u,v-path in G. If we remove v from the u,v-path, we have a new path, say a u,w-path, in $G \setminus v$, where w is a neighbor of v. Since there is a u,w-path, it must be that u and w are in the same component. Thus, $w \in C$ and w is a neighbor of v. Since C was arbitrary, this proves the statement.

Suppose $v \in G$ is a cut-vertex of degree 1. Let w be the only vertex ~~neighboring~~ neighboring v. By our previous result, every component of $G \setminus v$ contains a neighbor of v. Since w is the only neighbor of v, all the components of $G \setminus v$ contain w. This proves that w is path-connected to every vertex of $G \setminus v$ and thus that $G \setminus v$ is connected. This is a contradiction, as v is a cut-vertex. Thus, no such v exist.

Ex 1.2.~~8~~20

First, we note that if we remove a vertex and then take a graph's complement, it is the same as taking the graph's complement and then removing the vertex. Thus, $\overline{G-v} = \overline{G} - v$. Since v is a cut-vertex, we have that $G-v$ is disconnected (>1 component). By Ex 1.1.10, ~~we see that it must~~ it follows that $\overline{G-v} = \overline{G} - v$ is connected

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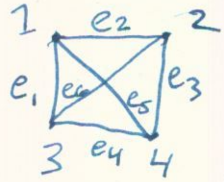
Ex 1.2.2 If we label K_4 as



then

(a) $1, e_1, 3, e_1, 1$ is a walk but not a trail

(b) $3, e_1, 1, e_5, 4, e_4, 3, e_6, 2$ is a non-closed trail that

(c) ~~Suppose there were a closed~~ trail T ~~that wasn't a cycle.~~ isn't a path

~~We see that if the vertices of T are distinct (except for the first and last vertices), then T is a cycle. Thus, T has a repeated non-endpoint vertex. However, since T is a trail, its edges are distinct, and thus there must be two "incoming" and two "outgoing" edges incident to our repeated vertex. Since every vertex has degree ≤ 3, this is impossible. Thus, no such T exists.~~

(c). Suppose there were a closed trail T that wasn't a cycle. We see that if the vertices of T are distinct (except for the first and last vertices), then T would be a cycle. Thus, T has a repeated vertex that isn't an endpoint. However, since T is a trail the two "incoming" edges and the two "outgoing" edges incident to this repeated vertex must be distinct. However, we see that the degree of every vertex is ≤ 3, which means this is impossible. Thus, no such T exists.