# Problem Set 8
# Complex Analysis

Bennett Rennier
bennett@brennier.com

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**Ex 1**

a) Let $U$ be a simply-connected domain and $f : U \to \mathbb{C} \setminus \{0\}$ be analytic. Let $g : U \to \mathbb{C}$ be an analytic function with $e^g = f$. Let $z, w \in U$ with $f(z) = f(w)$. let $\gamma : [a, b] \to U$ be a piecewise $C^1$ curve with $\gamma(a) = z$, $\gamma(b) = w$. Show that $g(z) - g(w)$ is the $2\pi i$ times the winding number of $f \circ \gamma$ about 0.

b) Let $V$ be a simply-connected domain, $a_1, \ldots, a_k \in V$ distinct points, and set $U = V \setminus \{a_1, \ldots, a_k\}$. Let $\delta > 0$ be small enough so that $\overline{B_\delta(a_j)}$ are disjoint. For $j = 1, \ldots, k$, let $\gamma_j$ be $\partial B_\delta(a_j)$ traversed once counterclockwise. Given an analytic $f : U \to \mathbb{C} \setminus \{0\}$, show that $f$ has a logarithm if and only if $n(f \circ \gamma_j, 0) = 0$ for $j = 1, \ldots, k$. [Hint: It suffices to show that for any cycle $\Gamma$ in $U$, we have that $\int_\Gamma \frac{f'}{f}\, dz = 0$. Reduce to the case of a general cycle $\Gamma$ to a cycle which is the sum of the $\gamma_j$.]

*Proof.*

a) We see that

$$n(f \circ \gamma; 0) = \frac{1}{2\pi i} \int_{f\circ\gamma} \frac{1}{s}\, ds = \frac{1}{2\pi i} \int_0^1 \frac{f'(\gamma(s))\gamma'(s)}{f(\gamma(s))}\, ds = \frac{1}{2\pi i} \int_\gamma \frac{f'(s)}{f(s)}\, ds = \frac{1}{2\pi i}(g(z) - g(w))$$

which proves that $g(z) - g(w)$ is $2\pi i$ times $n(f \circ \gamma; 0)$ as desired.

b) Suppose $f$ has a logarithm, that is, a holomorphic function $g$ such that $e^g = f$. Since each $\gamma_j$ is a closed curve, we can apply part (a) to get that

$$n(f \circ \gamma_j; 0) = \frac{1}{2\pi i}(g(\gamma_j(0)) - g(\gamma_j(1))) = 0$$

for each $\gamma_j$. Now conversely, suppose that $n(f \circ \gamma_j; 0) = 0$ for $j = 1, \ldots, k$ and let $\gamma$ be an arbitrary cycle in $U$. If we let $\Gamma = \gamma - \sum_i n(\gamma; a_i)\gamma_i$, we see that

$$n(\Gamma; a_j) = n(\gamma; a_j) - \sum_i n(\gamma; a_j) n(\gamma_i; a_j) = n(\gamma; a_j) - \sum_i n(\gamma; a_j)\delta_{ij} = n(\gamma; a_j) - n(\gamma; a_j) = 0.$$

Since $\Gamma$ is contained in $U$ and its winding number around each $a_j$ is zero, this means that $\Gamma$

is homologous to zero on $U$. As $f$ is never zero, we get by the Cauchy Integral Theorem that

$$\begin{aligned}0 = \int_\Gamma \frac{f'}{f}\,dz &= \int_\gamma \frac{f'}{f}\,dz - \sum_i n(\gamma;a_i)\int_{\gamma_i}\frac{f'}{f}\,dz \\ &= \int_\gamma \frac{f'}{f}\,dz - \sum_i n(\gamma;a_i)\int_{f\circ\gamma_i}\frac{1}{s}\,ds \\ &= \int_\gamma \frac{f'}{f}\,dz - \sum_i n(\gamma;a_j)n(f\circ\gamma;0) = \int_\gamma \frac{f'}{f}\,dz.\end{aligned}$$

Since $\int_\gamma \frac{f'}{f}\,dz = 0$ for any cycle $\gamma$ in $U$, path integrals in $U$ are path-indepedent. After fixing some $a \in U$, this allows use to define the logarithm

$$g(z) = \int_{\gamma_{a,z}} \frac{f'(s)}{f(s)}\,ds$$

where $\gamma_{a,z}$ is any path from $a$ to $z$. □

**Ex 2**

a) Suppose that $U, V \subseteq \mathbb{C}$ are open and that $f : U \to V$ is a conformal map. Let $z_n$ be a sequence in $U$. Show that $z_n \to \partial U$ if and only if $f(z_n) \to \partial V$.

b) Let $A = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and $B = \{z \in \mathbb{C} : 1 < |z| < 2\}$. Show that there is no conformal map $f : A \to B$.

*Proof.*

a) Let $z_n \to \partial U$; that is $z_n$ eventually escapes any compact subset of $U$. Let $K \subseteq V$ be a compact set. Since $f^{-1}$ is continuous, we have that $f^{-1}(K)$ is a compact subset of $U$. Thus, for some $N$, we have that for all $n \geq N$, $z_n \notin f^{-1}(K)$. But this means that for all $n \geq N$, $f(z_n) \notin K$. Since $K$ was an arbitrary compact set, we have that $f(z_n) \to \partial V$.

   The converse is proved similarly. Suppose that $f(z_n) \to \partial V$. Let $K \subseteq U$ be a compact set. As $f$ is continuous, we know that $f(K)$ is a compact subset of $V$. Thus, for some $N$, for all $n \geq N$, $f(z_n) \notin f(K)$. This means that for all $n \geq N$, we have that $z_n \notin K$. Since $K$ was an arbitrary compact set, we have that $z_n \to \partial U$.

b) Suppose that $f : A \to B$ was a conformal map. Since $\mathrm{Im}(f) = B$, we know that $f$ is bounded. In particular, $f$ is bounded near 0, which means that 0 is a removable singularity of $f$. This proves that we can extend $f$ holomorphically to a function $\tilde{f} : \mathbb{D} \to B$ . By part (a), if $z_n \to 0 \in \partial A$, then $f(z_n) \to \partial B$. This proves that $\tilde{f}(0) \in \partial B$. However, this means that $\mathrm{Im}(\tilde{f}) = B \cup \{\tilde{f}(0)\}$, where $\tilde{f}(0) \in \partial B$. However, this image cannot be open, contradicting the open mapping theorem. Thus, there can be no conformal map $f : A \to B$. □

**Ex 3** Suppose that $f$ is entire and satisfies $\lim_{z\to\infty} f(z) = \infty$. Show that $f$ is a polynomial.

*Proof.* Let $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be given by $g(z) = f(1/z)$. We see that if $g$ had an essential singularity at 0, then by Casorati–Weierstrass, for any $z \in \mathbb{C}$ there is a sequence $z_n \to 0$ such that $g(z_n) \to z$. However, we see that $\lim_{z\to 0} g(z) = \lim_{z\to\infty} f(z) = \infty$, contradicating Casorati-Weierstrass. Thus, $g$ does not have an essential singularity at 0, meaning $g(z) = z^{-n}h(z)$ for some holomorphic function

$h$. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $h(z) = \sum_{k=0}^{\infty} b_k z^k$ be the analytic representations of $f$ and $h$. This gives us that

$$\sum_{k=-\infty}^{0} a_{-k} x^k = \sum_{k=0}^{\infty} a_k x^{-k} = f(1/z) = g(z) = \frac{h(z)}{z^n} = \sum_{k=0}^{\infty} b_k x^{k-n} = \sum_{k=-n}^{\infty} b_{k+n} x^k.$$

By Exercise 5, Laurant series are unique, so it must be that $a_{-k} = b_{k+n}$ for all $k$. In particular, this means that $a_k = 0$ for all $k \geq n$. Thus, $f(z) = \sum_{k=0}^{n} a_k z^k$, proving that $f$ is a polynomial. $\square$

**Ex 4** Suppose that $f : \mathbb{C} \to \mathbb{C}$ is injective and entire. Define $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $g(z) = f(1/z)$.

a) Show that the singularity of $g$ at 0 is not an essential singularity.

b) Show that $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

c) Show that if $h : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is injective and analytic, then $h$ is a linear fractional transformation.

*Proof.*

a) Let $D$ be the punctured disk and let $E$ be the set $\{z : |z| > 1\}$. Suppose that 0 was an essential singularity of $g$. By Cassorati-Weierstrass, we have that $g(D)$ is dense in $\mathbb{C}$. This means that $f(1/D) = f(E)$ is dense in $\mathbb{C}$. But $f(D)$ is open by the Open Mapping Theorem, so it must be that $f(D) \cap f(E) \neq \varnothing$ contradicting the injectivity of $f$.

b) By part (a), we know that 0 is either a removable singularity or a pole. If 0 were a removable singularity, then we could extend $g$ to be defined at 0, but this is equivalent to extending $f$ at infinity to be some complex number. But this implies that $f$ is bounded around infinity, meaning $f$ must be constant by Louiville's Theorem, contradicing injectivity of $f$. Thus, 0 must be a pole of $f$. This means that $\lim_{z\to\infty} f(z) = \lim_{z\to 0} g(z) = \infty$. By Ex 3, this implies that $f$ is a polynomial. But polynomials of degree greater than 2 are not injective, so it must be that $f(z) = az + b$.

c) If $\infty \notin \operatorname{Im}(f)$, then $f|_{\mathbb{C}}$ is an entire function such that $\lim_{z\to\infty} f(z) = c \in \mathbb{C}$. This implies that $f$ is bounded around infinity, meaning $f$ must be constant by Louiville's Theorem, contradicting the injectivity of $f$.

   If $h(\infty) = \infty$, then by injectivity, $h|_{\mathbb{C}}: \mathbb{C} \to \mathbb{C}$ is an injective, entire function such that $\lim_{z\to\infty} h|_{\mathbb{C}} = h(\infty) = \infty$. By part (b), it must be that $h|_{\mathbb{C}}(z) = az + b$. By continuity, we can conclude that $h(z) = az + b$.

   Now suppose that $h(c) = \infty$ where $c \neq \infty$. Let $\phi(z) = \frac{cz}{z-c}$ which is a mobius transformation such that $c \mapsto \infty$ and $\infty \mapsto c$. If we then define $g = h \circ \phi$, then $g$ is injective (as its the composition of injective functions) and $g(\infty) = h(\phi(\infty)) = h(c) = \infty$. By the previous paragraph, this means that $g(z) = az + b$. Since the inverse of $\phi$ is $\phi^{-1}(z) = \frac{-cz}{z-c}$, we get that

$$h(z) = (h \circ \phi \circ \phi^{-1})(z) = (g \circ \phi^{-1})(z) = g(\frac{-cz}{z-c}) = a\frac{-cz}{z-c} + b = \frac{(b-ac)z - bc}{z-c}$$

   which shows that $h$ is a linear fractional as desired. $\square$

**Ex 5** Let $r, R \in [0, +\infty]$ and let $A_{r,R} = \{z \in \mathbb{C} : r < |z| < R\}$. Suppose that $(a_n)$ and $(b_n)$ are sequences of complex numbers such that

$$\sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n=-\infty}^{\infty} b_n z^n \quad \text{for all } z \in A_{r,R}.$$

Prove that $a_n = b_n$ for all $n$.

*Proof.* Combining the terms to one side we get that

$$\sum_{n=-\infty}^{\infty} (a_n - b_n) z^n = 0 \quad \text{for all } z \in A_{r,R}.$$

Since this Laurent series converges to the holomorphic function 0 on $A_{r,R}$, we can use the Cauchy Integral Formula to find the coefficents of this power series as

$$a_n - b_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}}\, dz = \frac{1}{2\pi i} \int_\gamma \frac{0}{z^{n+1}}\, dz = 0$$

where $\gamma$ is a path in $A_{r,R}$ that traverses around 0 once counterclockwise. This proves that $a_n = b_n$ for all $n$. $\square$

**Ex 6**

a) For each of the following functions $f$ has an isolated singularity at $z = 0$. Determine the type of singularity.

   i) $f(z) = \frac{\cos(z)}{z}$

   ii) $f(z) = \frac{\log(z+1)}{z^2}$

   iii) $f(z) = z \sin(1/z)$

   iv) $f(z) = \frac{1}{1-e^z}$

b) Let $f(z) = \frac{1}{z(z-1)(z-2)}$. Give the Laurent expansion of $f$ in each of the following annuli: $A_{0,1}$, $A_{1,2}$, $A_{2,\infty}$.

c) Show that $f(z) = \tan(z)$ is analytic in $\mathbb{C}$ except for simple poles at $z = \frac{\pi}{2} + n\pi$ for each integer $n$. Determine the singular part of $f$ at each of these poles.

*Proof.*

a) i) Using the Taylor series for cosine, we get that

$$f(z) = \cos(z)/z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \cdot \frac{1}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k-1}}{(2k)!}.$$

We see then that $f$ has a pole of order one at 0 and that the singular part of $f$ is $x^{-1}$.

ii) Using the Taylor series for log, we get that

$$f(z) = \frac{\log z + 1}{z^2} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^k}{k z^2} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^{k-2}}{k} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{k-1}}{k+1}.$$

We see then that $f$ has a pole of order one at 0 and that the singular part of $f$ is $x^{-1}$.

iii) Using the Taylor series for sin, we get that

$$f(z) = \sin(1/z) \cdot z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{-(2k+1)}}{(2k+1)!} \cdot z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{-2k}}{(2k+1)!}.$$

Since this Laurent series has infinitely many non-zero terms in its singular part, we get that the singularity of $f$ at 0 is an essential singularity.

iv) Since $1 - e^z = \sum_{n=1}^{\infty} -\frac{z^n}{n!}$ has a zero of order 1 at $z = 0$, we know that $f$ must have a pole of order 1 at $z = 0$. Since this pole is simple, we can calculate the residue as

$$\operatorname{res}_0(f) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{z}{1 - e^z} = \lim_{z \to 0} -\frac{1}{e^z} = -1.$$

This means the singular part of $f$ is $-x^{-1}$.

b) First, we use partial fractions to obtain that

$$\frac{1}{z(z-1)(z-2)} = \frac{1/2}{z} + \frac{-1}{z-1} + \frac{1/2}{z-2} = \frac{1/2}{z} + \frac{1}{1-z} - \frac{1/2}{2-z}$$

Now we see that for $0 < |z| < 1$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

and for $0 < |z| < 2$ we have that

$$-\frac{1/2}{2-z} = -\frac{1}{4}\left(\frac{1}{1-z/2}\right) = -\frac{1}{4}\sum_{n=0}^{\infty} \frac{z^n}{2^n} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+2}}.$$

This means the Laurent series for $A_{0,1}$ is

$$f(z) = \frac{1/2}{z} + \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+2}} = \frac{1/2}{z} + \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+2}}\right) z^n = \sum_{n=-1}^{\infty}\left(1 - \frac{1}{2^{n+2}}\right) z^n.$$

Now for $|z| > 1$, we have that

$$\frac{1}{1-z} = -\frac{1}{z-1} = -\frac{1}{z}\frac{1}{(1-1/z)} = -\frac{1}{z}\sum_{n=0}^{\infty} z^{-n} = -\frac{1}{z}\sum_{n=-\infty}^{0} z^n = -\sum_{n=-\infty}^{0} z^{n-1} = -\sum_{n=-\infty}^{-1} z^n$$

This means the Laurent series for $A_{1,2}$ is

$$f(z) = \frac{1/2}{z} - \sum_{n=-\infty}^{-1} z^n - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+2}}.$$

And finally, for $|z| > 2$, we have that

$$-\frac{1/2}{2-z} = \frac{1}{2z} \cdot \frac{1}{1-2/z} = \frac{1}{2z}\sum_{n=0}^{\infty} 2^n z^{-n} = \frac{1}{2z}\sum_{n=\infty}^{0} 2^{-n} z^n = \sum_{n=\infty}^{0} 2^{-n-1} z^{n-1} = \sum_{n=\infty}^{-1} 2^{-n-2} z^n.$$

This means the Laurent series for $A_{2,\infty}$ is

$$f(z) = \frac{1/2}{z} - \sum_{n=-\infty}^{-1} z^n - \sum_{n=-\infty}^{-1} 2^{-n-2} z^n = \frac{1/2}{z} - \sum_{n=-\infty}^{-1}\left(1 - \frac{1}{2^{n+2}}\right) z^n.$$

c) The zeros of $\cos(z)$ are the points $z = \frac{\pi}{2} + n\pi$ for each integer $n$. Since $\frac{d}{dz}\cos(z) = -\sin(z)$ and the zeros of sine (which are $n\pi$ for each integer $n$) are distinct from our original zeros, it must be that the zeros of $\cos(z)$ are of order 1. Again, as the zeros of sine are distinct from cosine, they don't cancel out in $f(z) = \frac{\sin(z)}{\cos(z)}$. This implies that the poles of $\tan(z)$ are at the points $\frac{\pi}{2} + n\pi$ and that these poles are simple. Since the poles are simple, we can calculate the residue as

$$\mathrm{res}_{\frac{\pi}{2}+n\pi}(f) = \lim_{z\to\pi/2+n\pi}(z - \frac{\pi}{2} - n\pi)\frac{\sin(z)}{\cos(z)} = \lim_{z\to\pi/2+n\pi}\frac{\sin(z) + (z - \pi/2 - n\pi)\cos(z)}{-\sin(z)} = -1.$$

This means that the singular part of each of these poles is simple $-x^{-1}$. $\square$