Problem Set 2
Topology II

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Ex 1 Show that a homotopy equivalence $f : X \to Y$ induces a bijection between the sets of path-components of X and Y. Prove the corresponding statement for connect components. Conclude that any space homotopy equivalent to a connected (resp. path-connected) space is connected (resp. path-connected).

Proof. Let $X \simeq Y$ be homotopy equivalent via the pair of functions (f, g). Now, we define $C(X)$ and $C(Y)$ to be the sets of connected components of X and Y respectively. We also define $C(x)$ and $C(y)$ to be the components which contain $x \in X$ and $y \in Y$ respectively.

Finally, we create functions $F : C(X) \to C(Y)$ and $G : C(Y) \to C(X)$ where $F(C(x)) = C(f(x))$ and $G(C(y)) = C(g(y))$. We will first prove that F is well-defined. To do this, let $u, v \in C(x)$. Since f is continuous and $C(x)$ is connected, we see that $f(C(x))$ is connected. This means that $f(C(x))$ is contained in some connected component $K \in C(Y)$. Thus, we have that $C(f(u)) = K = C(f(v))$, which proves well-definedness. A similar argument proves that G is well-defined as well.

Now we want to prove that F and G are inverses. We let $H : X \times I \to X$ be the homotopy of $gf \simeq \mathbb{1}_X$. We note that for any $x_0 \in X$, we have that $H(x_0, I)$ is connected, as it's a continuous function mapping a connected interval. Since $H(x_0, 1) = \mathbb{1}_X(x_0) = x_0$, we have that $H_1(x_0, I) \subseteq C(x_0)$. This means that $H(x_0, 0) = gf(x_0) \in C(x_0)$. Since this is true for any x_0, we have that $gf(x) = C(x)$. Similarly, we have that $fg(y) \in C(y)$. Thus, if we let $K \in C(X)$ be a component and let $x \in K$, then we have that

$$FGK = FGC(x) = FCg(x) = Cfg(x) = C(x) = K.$$

Similarly, we see that GF is the identity on $C(Y)$ as well, which proves that F and G are inverses. This means that F is actually a bijection between $C(X)$ and $C(Y)$ as desired. We note that this argument only used the fact that the image of a connected set is connected. Since there is a corresponding fact that the image of every path-connected set is path-connected, we see that this argument works with the words "connected" replaced with "path-connected." Lastly, we can clearly conclude that a space which is homotopy equivalent to a connected (resp. path-connected) space must be connected (resp. path-connected) itself. □

Ex 2

a) Suppose X is a space. Assume (Y, A) is a pair of spaces satisfying the homotopy extension property (HEP). Prove that $(X \times Y, X \times A)$ has the HEP.

b) For any space X, prove that $(X \times I, X \times 0)$ has the HEP.

c) Prove Exercise 13 in Chapter 0 of Hatcher: Prove that any two deformation retractions r_t^0 and r_t^1 from X onto a subspace A can be connected by the a continuous family of deformation retractions $\{r_t^s\}_{0 \leq s \leq 1}$.

Proof.

a) We let Z be the space $(Y \times \{0\}) \cup (A \times I)$. We discussed in class that (Y, A) has the HEP if and only if there's a retraction $r : Y \times I \to Z$. Using this, we construct the function
$$\mathbb{1}_X \times r : X \times Y \times I \to X \times Z.$$
If we let $i : X \times Z \to X \times Y \times I$ be the inclusion map, we see that
$$(\mathbb{1}_X \times r) \circ i = (\mathbb{1}_X \circ i_1) \times (r \circ i_{2,3}) = \mathbb{1}_X \times \mathbb{1}_Z = \mathbb{1}_{X \times Z}.$$
This proves that $\mathbb{1}_X \times r$ is also a retraction. Since $\mathbb{1}_X \times r$ is a retraction from $X \times Y \times I$ to the space
$$X \times Z = X \times ((Y \times \{0\}) \cup (A \times I)) = (X \times Y \times \{0\}) \cup (X \times A \times I)$$
this is equivalent to saying that $(X \times Y, X \times A)$ has the HEP.

b) We let
$$r : I \times I \to (I \times \{0\}) \cup (\{0\} \times I)$$
be the function which projects the points of the unit square to two of its sides via the unique line between that point and the point $(1, 1)$, except for the point $(1, 1)$ itself, which is sent to $(0, 0)$. We see that the inverse images of the basis elements are open triangles/quadrilaterals (open in the subspace topology from $\mathbb{R}^2$). This, we have that r is continous. Since it fixes elements on the two sides of the unit square, we have that r is a retraction. This proves that $(I, \{0\})$ has the HEP. Using this and part (a), we have that $(X \times I, X \times \{0\})$ has the HEP as well.

c) We first note that similar to the proof of part (b), the space $(I, \partial I)$ has the HEP as well (this time using a retraction via the projection through the point $(1/2, 1)$ and sending the point $(1/2, 1)$ itself to $(1/2, 0)$). From part (a), this means that $(X \times I, X \times \partial I)$ has the HEP. Using the HEP on this space, we have that

$$\begin{tikzcd}
(X \times \partial I_s) \times I_t \arrow[d, hook] \arrow[dr, "r_t^0{,}r_t^1"] & \\
(X \times I_s) \times I_t \arrow[r, dotted, "r_t^s"] & X \\
(X \times I_s) \times \{0\} \arrow[u, hook] \arrow[ur, "\pi"'] &
\end{tikzcd}$$

This homotopy r_t^s is, by definition, a continuous family that connects r_t^0 and r_t^1. And as per the diagram, we know that $r_0^s = \mathbb{1}_X$ for any given s. However, I have no idea how to prove that for each and every $s \in I$ that r_t^s is a deformation retraction (that is, the restriction to A is the identity and that $\mathrm{Im}(r_1^s) \subseteq A$). $\square$

Ex 3 Via one cell at a time, prove that the quotient map $X \to X/A$ is a homotopy equivalence for CW pairs (X, A) such that A is contractible.

Proof. Proof not completed. $\square$

Ex 4 Assume $\varphi_0, \varphi_1 : \partial D^n \to X$ are homotopic maps to a CW space X. Prove that $X \cup_{\varphi_0} D^n \simeq X \cup_{\varphi_1} D^n$ rel X.

Proof. Let φ_t be the homotopy between φ_0 and φ_1 and let $k : X \cup_{\varphi_0} D^n \to X \cup_{\varphi_1} D^n$ be defined by $k(x) = x$ for $x \in X$ and for $u \in D^n$,

$$k(u) = \begin{cases} 2u & |u| \leq \frac{1}{2} \\ \varphi_{2|u|-1}(u/|u|) & \frac{1}{2} \leq |u| \leq 1. \end{cases}$$

This function is obviously continuous when restricted to X and when restricted to the $u \in D^n$ such that $|u| \leq \frac{1}{2}$. We also see that it's continuous for $\frac{1}{2} \leq |u| \leq 1$ in D^n by looking at the concentric "shells" $C_r = \{u \in D^n : |u| = r\}$. Each one is mapped continuously to $\mathrm{Im}(\varphi_{2r-1})$ (since $\varphi_t(u)$ is continuous with respect to u). Additionally, since φ_{2r-1} is continuous with respect to r as well, we have that these circles are mapped continuously with respect to each other. Since we have that these restrictions of k are continuous and agree with each other on an overlap, we see that k is continuous overall.

Now we define $h : X \cup_{\varphi_1} D^n \to X \cup_{\varphi_0} D^n$ by $h(x) = x$ for $x \in X$ and for $u \in D^n$,

$$h(u) = \begin{cases} 2u & |u| \leq \frac{1}{2} \\ \varphi_{2-2|u|}(u/|u|) & \frac{1}{2} \leq |u| \leq 1. \end{cases}$$

This function is continuous for the same reasons. We see that if we do the composition $h \circ k$, we get a function that is the identity when restricted to X and for $u \in D^n$ we get that

$$(h \circ k)(u) = \begin{cases} 4u & |u| \leq \frac{1}{4} \\ \varphi_{2-4|u|}(u/|u|) & \frac{1}{4} \leq |u| \leq \frac{1}{2} \\ \varphi_{2|u|-1}(u/|u|) & \frac{1}{2} \leq |u| \leq \frac{1}{2}. \end{cases}$$

We see that for u such that $\frac{1}{4} \leq |u| \leq 1$, $h \circ k$ is a homotopy from φ_0 to φ_1 and then back to φ_0. This is homotopic to doing nothing, so after doing a reparametrization, we see that $(h \circ k)$ is homotopic to the identity on U as well, which proves that $h \circ k$ is homotopic to the identity on all of $X \cup_{\varphi_0} D^n$. By similar reasoning $k \circ h$ is homotopic to the identity on all of $X \cup_{\varphi_1} D^n$. Since both h and k are the identity on X, this proves that the two spaces are homotopy equivalent rel X. $\square$

Ex 5 Prove or disprove the converse to Ex 3.

Proof. Suppose that $q : X \to X/A$ is a homotopy equivalence. This would mean there's a $p : X/A \to X$ such that $p \circ q \simeq \mathbb{1}_X$ and $q \circ p \simeq \mathbb{1}_{X/A}$. We note that if H is a homotopy between f and g, then $H \big|_{A \times I}$ is a homotopy between $f \big|_A$ and $g \big|_A$. This means that

$$(p \circ q) \big|_A = p \circ \left(q \big|_A\right) = p \circ \mathrm{const}_{x_0} = \mathrm{const}_{p(x_0)} \simeq \mathbb{1}_X \big|_A = \mathbb{1}_A$$

where x_0 is the point that A is contracted to in X/A. If we could somehow guarantee that p sends x_0 to a point in A, then by the last homework, we'd have that A is contractible. I'm not sure how to guarantee this, though (if one can). □