Problem Set 2
Topology I

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1. Prove that if $f : X \to Y$ is continuous and $S \subset X$ is a subspace, then the restriction $f|_S : S \to Y$ is continuous (with respect to the subspace topology of S).

Proof. Let V be an open set of Y. Let V be an open set of Y. Since f is continuous, we know that $f^{-1}(V) = U$ is open in X. Since

$$f|_S^{-1}(V) = \{x \in S : f(x) \in V\} = S \cap \{x \in X : f(x) \in V\} = S \cap f^{-1}(V) = S \cap U,$$

which is open in S via the definition of the subspace topology. Thus, $f|_S$ is continuous. □

2. Let X and Y be topological spaces and $\mathscr{B}$ a base for the topology of Y. Prove that a function $f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is open for every $U \in \mathscr{B}$.

Proof.

$\implies$) Suppose f is continuous and let B be a basis element. Since B is basis element, it's also open. Thus, $f^{-1}(B)$ is open.

$\impliedby$) Suppose that $f^{-1}(B)$ is open for every basis element B of Y. Let V be an open set of Y. By the definition of a basis of a topology, this means that $V = \cup_i B_i$ for some basis elements B_i. From this we have that

$$f^{-1}(V) = f^{-1}\left(\bigcup_i B_i\right) = \bigcup_i f^{-1}(B_i),$$

which is the union of open sets and is thus open. This proves that f is continuous. □

3. Let X be a topological space and $\{x_n\}$ a sequence in X.
   a) Prove that if X is Hausdorff, then $\{x_n\}$ converges to at most one point of X (thus, in a Hausdorff space, limits of sequences are unique).
   b) Let $S \subset X$ be a subset, and assume that $\{x_n\} \subset S$. Prove that $\{x_n\}$ converges to $x_0 \in S$ if and only if it converges to x_0 when considered as a sequence in X.

Proof.

a) Suppose $(x_i)_{i\in\mathbb{N}}$ is a sequence converging to two elements, x and x'. Since X is Hausdorff, this means there are disjoint open sets U and V such that $x \in U$ and $x' \in V$. As the sequence converges to x there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$. Similarly, as the sequence converges to x' there is an $N' \in \mathbb{N}$ such that for all $n \geq N'$, $x_n \in V$. This implies that for $n \geq \max(N, N')$, $x_n \in U \cap V$. This is a contradiction, though, as U and V are disjoint. Thus, $(x_i)_{i\in\mathbb{N}}$ cannot converge to two different points in a Hausdorff space.

b) $\Longleftarrow$) Suppose $(x_i)_{i\in\mathbb{N}}$ converges to x_0 in S. Let U be an open set of x_0 in S. This means that for some open set V in X, $U = V \cap S$. Since (x_i) converges in X, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in V$. Since all the x_n are in S, though, that means that $x_n \in V \cap S = U$. Since U was an arbitrary open set of x_0, (x_n) converges to x_0 in S as well.

$\Longrightarrow$) Suppose $(x_i)_{i\in\mathbb{N}}$ is a sequence converging to x_0 in S. Let U be an open set of X containing x_0. Then $U \cap S$ is an open set of S. We note that this set is non-empty as x_0 lies in both. Since (x_i) converges in S, this means that for some $N \in \mathbb{N}$ we have that $x_n \in U \cap S$ for all $n \geq N$. Since this means that $x_n \in U$ and U was an arbitrary open set in X containing x, this proves that (x_i) converges to x_0 in X as well. □

4. Let X be a normal space, $E \subset X$ a closed subset, and $f : E \to \mathbb{R}$ a continuous function. Prove that f can be extended to a continuous function from X to $\mathbb{R}$. (See exercise 6 of section 5 for a hint.)

Proof. Let g be a homeomorphism from $\mathbb{R}$ to $(-1, 1)$. Since $g \circ f : E \to (-1, 1)$ is bounded, by Tietze's Extension Theorem there is an extension h, such that $h|_E = g \circ f$. We note that in the proof of Tietze's Extension theorem we can put a similar bound on this extension so that $|h(x)| \leq 1$ for all x; that is $h : X \to [-1, 1]$.

Now, as $\{-1, 1\}$ is a closed set, the inverse image $C = h^{-1}(\{-1, 1\})$ is closed and disjoint from E (since $h(E) \subseteq (-1, 1)$). Thus, by Urysohn's Lemma, we can construct a continuous function $u : X \to [0, 1]$ where $u(E) = 1$ and $u(C) = 0$. Using this, we see that $h'(x) = u(x) \cdot h(x)$ is a continuous function from X to $(-1, 1)$ where $h'|_E = h|_E = g \circ f$.

From this, we can use the inverse homeomorphism $g^{-1} : I \to \mathbb{R}$ to get the continuous function $g^{-1} \circ h'$ from X to $\mathbb{R}$ such that

$$(g^{-1} \circ h')|_E = g^{-1} \circ h'|_E = g^{-1} \circ h|_E = f.$$

This proves that we can extend f to the continuous function $g^{-1} \circ h'$. □

5. (Not in the text) A topological space X is *sequentially compact* if every sequence in X has a convergent subsequence. Suppose (X, d) is a metric space: prove that if X is compact then X is sequentially compact. (The converse is also true, but more difficult.)

Proof. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of points in a compact space X. We let

$$F_k = \overline{\{x_i\}_{i\geq k}},$$

which are closed by the definition of closure, and let $U_k = X \setminus F_k$, which are open. We note that $F_{k+1} \subseteq F_k$ and so $U_k \subseteq U_{k+1}$.

Now suppose that $\cap_{i\in\mathbb{N}} F_i$ is empty. That would mean $\cap_{i\in\mathbb{N}} U_i = X$, and thus the U_i form an open cover of X. This means there's a finite subcover $\{U_{n_1}, U_{n_2}, \ldots, U_{n_\ell}\}$. However, as $U_{n_i} \subseteq U_{n_\ell}$ for each $i \leq \ell$, we have that

$$X = \bigcup_{i\leq\ell} U_{n_i} = U_{n_\ell}.$$

This would imply that F_{n_ℓ} is empty, which is a contradiction, as we know that $x_{n_\ell} \in F_{n_\ell}$. Thus, there must be some $x \in X$ such that $x \in \cap_{i\in\mathbb{N}} F_i$.

Since x is in the closure of $\{x_i\}_{i\geq k}$ for any k, that means that the ball $B_{1/k}(x)$ intersects $\{x_i\}_{i\geq k}$ for each k. By choosing x_{n_k} from this intersection for each k, we create a subsequence that converges to x. This proves that X is sequentially compact. □

6. A family F of real-valued functions on a topological space X is *equicontinuous* if for each $x \in X$ and $\varepsilon > 0$, there is a neighborhood U of x such that $|f(x) - f(y)| < \varepsilon$ for each $y \in U$ and $f \in F$. Let $\{f_n\}$ be a bounded sequence of real-valued functions on a compact space X that is equicontinuous. Prove that there is a uniformly convergent subsequence of $\{f_n\}$. (See exercise 8 of section 6 for a hint. Note that "bounded sequence of functions" refers to the metric on the space of continuous functions defined in the last homework.)

Proof. Let $n \geq 1$. Since the sequence of functions is equicontinuous, for each $x \in X$ there is an open neighborhood U_x such that $|f_k(x) - f_k(y)| < 1/n$ for all $y \in U_x$ (note that this does not depend on f_k). We can easily see that $\cup_{x\in X} U_x = X$. As X is compact, there is a finite subcover. Since we have a different subcover for each n, we'll denote this subcover as $\{W_{n,j}\}_{j\leq m_n}$ where $W_{n,j}$ is a neighborhood of $x_{n,j}$.

[Incomplete] □

7. Using problem 5 and problem 4 (including the converse of the latter), prove the Arzela-Ascoli theorem: for X a compact space and $C(X)$ the space of continuous functions on X (with the topology defined by the metric you studied on the previous homework, noting $C(X) = BC(X)$ for X compact), a subset of $C(X)$ is compact if and only if it is closed, bounded, and equicontinuous.

Proof. $\Longrightarrow$) Let K be a compact subset of $C(X)$. We proved in class that in a complete metric space compact implies closed. Let consider the open cover $\{B_i(0)\}_{i\in\mathbb{N}}$ of K, where 0 here is the zero function. Since K is compact, there is a finite subcover of this open cover. As these open balls are nested, the union of all the elements in this finite subcover is simply the largest ball, say $B_k(0)$. This means that all the functions in K are contained in $B_k(0)$ and hence are bounded by the constant k.

Now, let $\varepsilon > 0$. Take the open cover $\{B_{\varepsilon/3}(f_i)\}_{f_i\in K}$ of K. As K is compact, there is a finite subcover $\{B_{\varepsilon/3}(f_1), \ldots, B_{\varepsilon/3}(f_n)\}$. We know that all the elements of this finite subcover are continuous; that means there exists some δ_i such that

$$|x - y| < \delta_i \implies |f_i(x) - f_i(y)| < \varepsilon/3$$

for each f_i in the finite subcover. If we then let $\delta = \min(\delta_i)$, we see that

$$|x - y| < \delta \implies |f_i(x) - f_i(y)| < \varepsilon/3$$

for all f_i in the finite subcover. Now let f be any element of K. We note that f must be in one of the open sets of our cover, so $f \in B_{\varepsilon/3}(f_j)$ for some f_j. This means that for $|x-y| < \delta$ (where δ is defined as before), we have that

$$|f(x) - f(y)| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Since the same δ works for any $f \in K$, we have proven that K is equicontinuous.

$\Longleftarrow$) Let K be a closed, bounded, and equicontinuous subset of $C(X)$. Let $\{f_i\}_{i\in\mathbb{N}}$ be a sequence of functions in K. Since K is bounded and equicontinuous, we can use problem 6 to see that our sequence $\{f_i\}_{i\in\mathbb{N}}$ has a subsequence $\{f_{k_i}\}_{i\in\mathbb{N}}$ that converges to some f in $C(X)$. This means that f is a limit point of our set K. As K is closed, it contains all its limit points. Thus, $\{f_{k_i}\}_{i\in\mathbb{N}}$ is a converging subsequence in K. This proves that K is sequentially compact. By the converse of problem 5, this proves that K is compact. $\square$

8. a) Prove that if X is a topological space with any of the properties of being T_1, Hausdorff, or regular, then any subspace of X also has that property.
b) Prove that a locally compact Hausdorff space is regular.

Proof.

a) For the following questions let $Y \subseteq X$, x, y be distinct points in Y, and C a closed set of Y.

   i) If X is T_1, then there is a set open in X that contains x but not y; call this set U. Under the subspace topology, $U \cap Y$ is open in Y, contains x, and does not contain y. Thus, Y is also T_1.

   ii) If X is Hausdorff, there are disjoint open sets U and V such that $x \in U$ and $y \in V$. Again, though, under the subspace topology, $Y \cap U$ and $Y \cap V$ are disjoint open sets such that $x \in Y \cap U$ and $y \in Y \cap V$. Thus, Y is Hausdorff as well.

   iii) Let C be a closed set of Y and $x \in Y \setminus C$. This means that $C = F \cap Y$ for some set F closed in X. We note that since $x \in Y$, if x where in F then, $x \in C$, which we assumed was not the case. Thus, $x \notin F$. Since F and x are closed and disjoint and X is regular, there exists open sets U, V such that $F \subseteq U$, $x \in V$, and $U \cap V = \varnothing$. We see that $C = F \cap Y \subseteq U \cap Y$ and that $x \in V \cap Y$. Since $U \cap Y$ and $V \cap Y$ are disjoint sets open in X, this proves that Y is regular.

b) We proved in classes that a locally compact Hausdorff space X can be extended to a compact Hausdorff space $X \cup \{\infty\}$. We also know that compact Hausdorff spaces are regular. By part (a), as the topology on X is simply the subspace topology, we have that X is regular as well.

$\square$