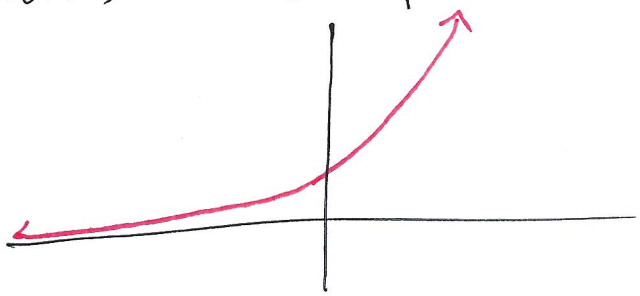
~~Midte~~ Final Exam

- Oral Exam
- Mostly verbal, but can use paper
- Around 7 questions on average (each worth 10)
  - questions 8 and 9 are worth less, and don't bring down your grade (each worth 3)
  - Topics with a lot of writing are not going to appear: Optimization, Exponential models
  - Topics definitely include: differentiating, relative min/max, concavity, integration
- Study Guide available tonight
  - 1st question on the exam, you can choose a question from the study ~~gu~~ guide
  - 2nd and 3rd questions will be from the study guide
- Thirty ~~minute~~ minute time-slot, but the exam should take 20 minutes.
- E-mail sent out for you to indicate your availability.

Exponential Models

A function like $f(x) = Ae^{kx}$ for some constants A and k is a model for <u>exponential growth</u> when $k>0$ and a model for <u>exponential decay</u> when $k<0$.

Exponential growth models things like bacteria, population growth, and compound interest. The graph looks like



Exponential decay models things like the decay of chemicals, and the dissipation of medicine.

A problem might simply say "exponential model" instead of saying a function of the form Ae^{kx}

Ex 1) Exponential growth: Suppose 10,000 bacteria become 60,000 bacteria after two hours.

a) how many bacteria are there after 4 hours?

b) What is the growth rate at 4 hours?

We know the population function looks like $P(t) = Ae^{kt}$. We know that $P(0) = 10{,}000 = Ae^{0} = A$.

We also know that $P(2) = 60{,}000 = 10{,}000e^{k2}$.

So, $6 = e^{2k} \Longrightarrow \ln(6) = \ln(e^{2k}) \Longrightarrow \ln(6) = 2k$

$$\Longrightarrow k = \frac{\ln(6)}{2}.$$

Altogether, our function is

$$P(t) = 10{,}000\, e^{\frac{\ln(6)}{2}t}$$

a) $$P(4) = 10{,}000e^{2\ln(6)} = 10{,}000e^{\ln(6^2)}$$

$$= 10{,}000 \cdot 6^2 = 36{,}000 \text{ bacteria}$$

b) $$P'(t) = \frac{\ln(6)}{2} \cdot 10{,}000e^{\frac{\ln(6)}{2}t}$$

$$P'(4) = \frac{\ln(6)}{2} 10{,}000e^{2\ln(6)} = \frac{\ln(6)}{2} 10{,}000e^{\ln(6^2)}$$

$$= \frac{\ln(6)}{2} 10{,}000 \cdot 6^2 = 180{,}000 \ln(6)$$

$$\approx 322{,}000 \text{ bacteria/hour}$$

Ex 2) Suppose Carbon-14 has a half-life of 5730 years. How much of 200mg is left after 2,865 years?

We can model the situation as $C(t) = Ae^{kt}$.

$C(t)$ = the amount of ~~carbon-14~~ Carbon-14

$$C(0) = 200 = Ae^0 = A$$

$$C(5730) = \underbrace{100}_{\text{half has decayed}} = 200e^{k \cdot 5730}$$

$$\Rightarrow \frac{1}{2} = e^{5730k} \Rightarrow \ln\left(\frac{1}{2}\right) = \ln\left(e^{5730k}\right)$$

$$\Rightarrow \ln\left(\tfrac{1}{2}\right) = 5730k \Rightarrow k = \frac{\ln\left(\frac{1}{2}\right)}{5730}$$

$$C(2865) = 200e^{\frac{\ln(\frac{1}{2})}{5730} \cdot 2865} = 200e^{\frac{1}{2} \cdot \ln(\frac{1}{2})}$$

$$= 200e^{\ln\left(\sqrt{\frac{1}{2}}\right)} = 200\sqrt{\tfrac{1}{2}} = \frac{200}{\sqrt{2}} \text{ mg.}$$

Ex 3) If a skull has $\frac{1}{10}$th the amount of Carbon-14 that it originally had, how old is it?

$$C(t) = Ae^{kt}$$

$$C(5730) = \frac{A}{2} = Ae^{5730k}$$

$$\Rightarrow \frac{1}{2} = e^{5730k} \Rightarrow \ln\left(\tfrac{1}{2}\right) = 5730k$$

$$\Rightarrow k = \frac{\ln\left(\frac{1}{2}\right)}{5730}$$

$$C(t) = \frac{A}{10} = Ae^{\frac{\ln(\frac{1}{2})}{5730}t}$$

$$\frac{1}{10} = e^{\frac{\ln(\frac{1}{2})}{5730}t}$$

$$\ln\left(\frac{1}{10}\right) = \frac{\ln(\frac{1}{2})}{5730}t$$

$$t = \frac{\ln(\frac{1}{10}) \cdot 5730}{\ln(\frac{1}{2})} \approx \boxed{19{,}030 \text{ years}}$$

Integration

An Antiderivative of a function $f(x)$ is a function $F(x)$ where $F'(x) = f(x)$.

Ex] What is an antiderivative of x^2?

An example is the function $\frac{x^3}{3}$.

Another example is the function $\frac{x^3}{3} + 2$.

These are both antiderivatives of x^2.

The formula for any antiderivative of x^2 is $\frac{x^3}{3} + C$

some constant

We can reverse some of our simple derivative rules to get antiderivative rules:

notation for finding antiderivative

$$\int x^2 dx = \frac{x^3}{3} + C \qquad \int 5 dx = 5x + C$$

$$\int 0 dx = C \qquad \int (x^2+5) dx = \int x^2 dx + \int 5 dx$$

$$= \frac{x^3}{3} + 5x + C$$

$$\int e^x dx = e^x + C$$

$$\int \frac{1}{x} dx = \ln(x) + C \qquad \int x^n dx = \frac{x^{n+1}}{n+1} + C$$

"Power Rule for antiderivatives"

Ex

$$\int \frac{1+x+x^2}{\sqrt{x}}\,dx = \int \frac{1}{\sqrt{x}} + \frac{x}{\sqrt{x}} + \frac{x^2}{\sqrt{x}}\,dx$$

$$= \int x^{-1/2}\,dx + \int x^{1/2}\,dx + \int x^{3/2}\,dx$$

$$= \boxed{\frac{x^{1/2}}{1/2} + \frac{x^{3/2}}{3/2} + \frac{x^{5/2}}{5/2} + C}$$

$$= \frac{1}{2}\sqrt{x} + \frac{3}{2}x^{3/2} + \frac{5}{2}x^{5/2} + C$$

Ex

$$\int \left(3e^x + \frac{1}{x} - 8\right)dx$$

$$= \int 3e^x\,dx + \int \frac{1}{x}\,dx - \int 8\,dx$$

$$= 3e^x + \ln(x) - 8x + C$$

Ex

$$\int x\left(\sqrt[3]{x^2} + \sqrt{x}\right)dx$$

$$= \int x\sqrt[3]{x^2} + x\sqrt{x}\,dx$$

$$= \int x\,\sqrt[3]{x^2}\,dx + \int x\sqrt{x}\,dx$$

$$= \int x^{5/3}\,dx + \int x^{3/2}\,dx$$

$$= \boxed{\frac{x^{8/3}}{8/3} + \frac{x^{5/2}}{5/2} + C}$$