Antidifferentiation / Integration

$$\begin{tikzcd} \text{Position} \arrow[r, bend left, "\text{differentiate}"] & \text{Velocity} \arrow[l, bend left, "\text{antidifferentiation}"] \arrow[r, bend left, "\text{differentiate}"] & \text{Acceleration} \arrow[l, bend left, "\text{antidifferentiation}"] \end{tikzcd}$$

However, there are multiple antiderivatives for a single function. To find the original function from its derivative, you need to know what the function is at one point.

Ex | Find $f(x)$ given that

$$f'(x) = 3x^2 - 4x + 8 \quad \text{and} \quad f(1) = 9.$$

$$\int f'(x)\,dx = \int (3x^2 - 4x + 8)\,dx = x^3 - 2x^2 + 8x + C$$

$$f(x) = x^3 - 2x^2 + 8x + C$$

$$9 = f(1) = 1 - 2 + 8 + C$$

$$9 - 7 = C$$

$$2 = C$$

$$\boxed{f(x) = x^3 - 2x^2 + 8x + 2}$$

Ex] Find a function f such that

$$f'(x) = \frac{1}{x} + 2x \qquad f(1) = 3.$$

$$\int f'(x)\,dx = \int \left(\frac{1}{x} + 2x\right)dx = \ln(x) + x^2 + C$$

$$f(x) = \ln(x) + x^2 + C$$

$$f(1) = \ln(1) + 1^2 + C = 3 \qquad \boxed{f(x) = \ln(x) + x^2 + 2}$$

$$1 + C = 3$$

$$C = 2$$

Ex] A model rocket (starting at rest) launches off of a platform 3ft above the ground. Let $h(t)$ be the height after t seconds. We know that the acceleration $h''(t) = 12t$ for $0 \le t \le 3$.

(a) What is the velocity after 3 seconds?

$$v(t) = \int a(t)\,dt = \int h''(t)\,dt = \int 12t\,dt = 6t^2 + C$$

Since the rocket started at rest, we know

$$v(0) = 6(0)^2 + C = 0 \implies C = 0.$$

So $v(t) = 6t^2$, and $v(3) = 6 \cdot 3^2 = \boxed{54 \text{ ft/sec}}$

(b) What is the height of the rocket at $t=3$ secs?

$$h(t) = \int v(t)\,dt = \int 6t^2\,dt = 2t^3 + C$$

Since the rocket starts 3ft off the ground, we know that

$$h(0) = 2(0)^3 + C = 3$$

$$C = 3$$

$$h(t) = 2t^3 + 3$$

So the height at $t=3$ secs is $h(3) = 2 \cdot 3^3 + 3$

$$= \boxed{57 \text{ ft}}$$

<u>Integration by Substitution</u> (u-substitution)

For example, what if we wanted to integrate

$$\int \frac{(\ln(x))^2}{x}\,dx \;?$$

Well, we have ~~a~~ a trick up our sleeves: variable substitution. You'll need to practice a little, but I ~~th~~ think it's more straight-forward than the chain rule.

Ex

$$\int (2x+1)^7\,dx \qquad u = 2x+1 \qquad \frac{du}{dx} = 2 \qquad du = 2\,dx \qquad \frac{du}{2} = dx$$

$$= \int u^7 \frac{du}{2}$$

$$= \int \frac{u^7}{2}\,du = \frac{u^8}{16} + C = \boxed{\frac{(2x+1)^8}{16} + C}$$

Ex

$$\int \frac{x}{3x^2+1}\,dx \qquad u = 3x^2+1 \qquad \frac{du}{dx} = 6x \qquad dx = \frac{du}{6x}$$

$$= \int \frac{\cancel{x}}{u}\,\frac{du}{6\cancel{x}} = \int \frac{1}{6u}\,du = \frac{1}{6}\ln(u) + C$$

$$= \boxed{\frac{1}{6}\ln(3x^2+1) + C}$$

Ex

$$\int x\sqrt{x^2+1}\,dx \qquad u = x^2+1 \qquad \frac{du}{dx} = 2x \qquad dx = \frac{du}{2x}$$

$$= \int \cancel{x}\sqrt{u}\,\frac{du}{2\cancel{x}}$$

$$= \int \frac{\sqrt{u}}{2}\,du = \int \frac{u^{1/2}}{2}\,du = \frac{1}{2}\,\frac{u^{3/2}}{3/2} + C = \frac{u^{3/2}}{3} + C$$

$$= \boxed{\frac{(x^2+1)^{3/2}}{3} + C}$$

Ex

$$\int e^{-3x}\,dx \qquad u=-3x \qquad \frac{du}{dx}=-3 \qquad dx=\frac{du}{-3}$$

$$=$$

$$\int e^{u}\,\frac{du}{-3} = -\frac{1}{3}\int e^{u}\,du = -\frac{1}{3}e^{u}+C$$

$$= -\frac{1}{3}e^{-3x}+C$$

Ex

$$\int \frac{(\ln(x))^2}{2x}\,dx \qquad u=\ln(x) \qquad \frac{du}{dx}=\frac{1}{x} \qquad dx = x\,du$$

$$=$$

$$\int \frac{u^2}{2\cancel{x}}\,\cancel{x}\,du = \int \frac{u^2}{2}\,du = \frac{u^3}{6}+C$$

$$= \frac{\ln(x)^3}{6}+C$$

Ex

$$\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})^3}\,dx \qquad u=1+\sqrt{x} \qquad \frac{du}{dx}=\frac{1}{2\sqrt{x}} \qquad dx = 2\sqrt{x}\,du$$

$$=$$

$$\int \frac{1}{\cancel{\sqrt{x}}\,u^3}\,2\cancel{\sqrt{x}}\,du$$

$$=$$

$$\int \frac{2}{u^3}\,du = \int 2u^{-3}\,du = \frac{2u^{-2}}{-2}+C = -\frac{1}{u^2}+C$$

$$=$$

$$\boxed{-\frac{1}{1+\sqrt{x}}+C}$$

Ex $\displaystyle\int \frac{e^{-x}-1}{e^{-x}+x}\,dx$

$u = e^{-x}+x$

$\dfrac{du}{dx} = -e^{-x}+1 \qquad dx = \dfrac{du}{-e^{-x}+1}$

$$\int \frac{e^{-x}-1}{u}\,\frac{du}{-e^{-x}+1} = \int \frac{\cancel{e^{-x}-1}}{u}\,\frac{du}{-(\cancel{e^{-x}-1})} = \int -\frac{1}{u}\,du$$

$$= -\ln(u) + C = \boxed{-\ln(e^{-x}+x) + C}$$

To verify:

$$\frac{d}{dx}\,-\ln(e^{-x}+x) + C = -\frac{(e^{-x}+x)'}{e^{-x}+x} = \frac{-(-e^{-x}+1)}{e^{-x}+x}$$

$$= \frac{e^{-x}-1}{e^{-x}+x}$$

Ex $\displaystyle\int \left(x^2e^{x^3}+\frac{1}{x}\right)dx = \int x^2e^{x^3}\,dx + \int \frac{1}{x}\,dx$

$u = x^3$

$\dfrac{du}{dx} = 3x^2$

$dx = \dfrac{du}{3x^2}$

$$= \int \cancel{x^2}\,e^u\,\frac{du}{3\cancel{x^2}} + \ln(x) + C$$

$$= \frac{1}{3}\int e^u\,du + \ln(x) + C$$

$$= \frac{1}{3}e^u + \ln(x) + C$$

$$= \boxed{\frac{1}{3}e^{x^3} + \ln(x) + C}$$