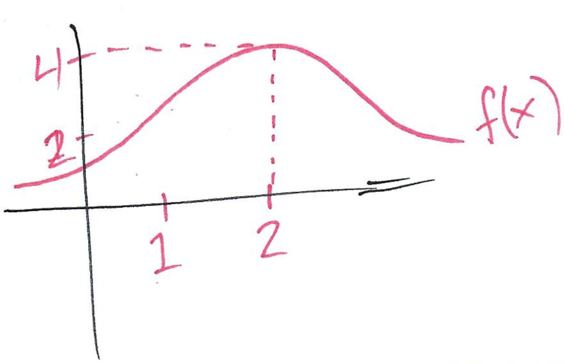
Optimization

There's an absolute maximum occurs at $x = c$ if $f(c) \geq f(x)$ for all x in the domain.

The absolute maximum is the number $f(c)$.



$f(x)$ has an absolute maximum of 4 over $[0, 3]$; it occurs at $x = 2$.

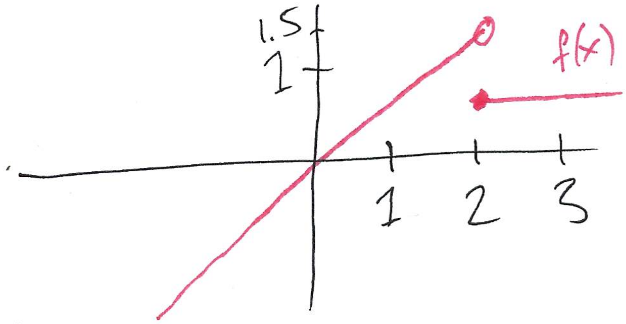
Extreme Value Theorem: If f is continuous on an interval $[a, b]$, then there exists an absolute maximum over $[a, b]$.

Why is EVT impotant? There are some edge cases we need to exclude:

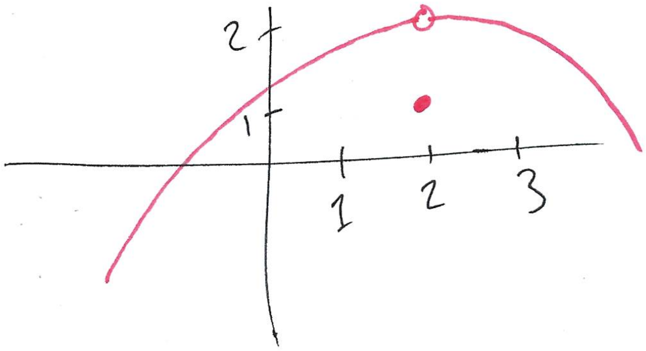
Let's say we consider an open interval (a, b)

This function has no absolute maximum over $(-1, 1)$

If f was not continuous:



$f(x)$ does not have an absolute max over $[1,3]$.



$f(x)$ does not have an absolute max over $[1,3]$.

These same edge cases occur for finding absolute minimums.

Ex 1 | Find the absolute maximum/minimum of $f(x) = x^3 - 2x^2 - 4x + 4$ on $[0,3]$.

$$f'(x) = 3x^2 - 4x - 4$$

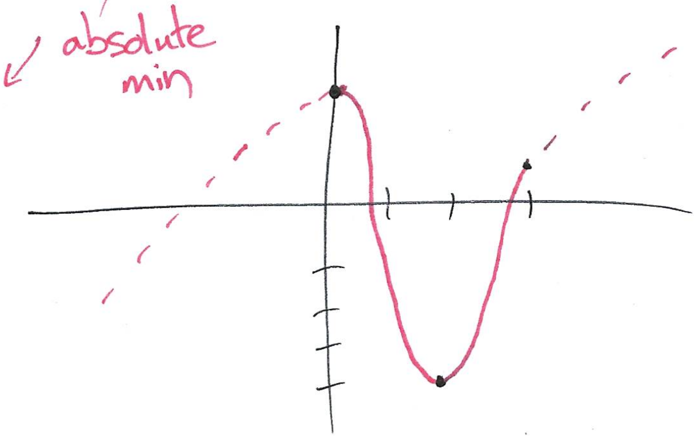
$$= (3x+2)(x-2)$$

The critical numbers are $x = \cancel{-2/3}, 2$

$f(2) = 8 - 8 - 8 + 4 = \boxed{-4}$ ← absolute min

$f(0) = \boxed{4}$ ← absolute max

$f(3) = 27 - 18 - 12 + 4$
$= 1$



Ex 2 Find absolute min/max of $f(x) = x^{2/3}$ on $[-1, 8]$.

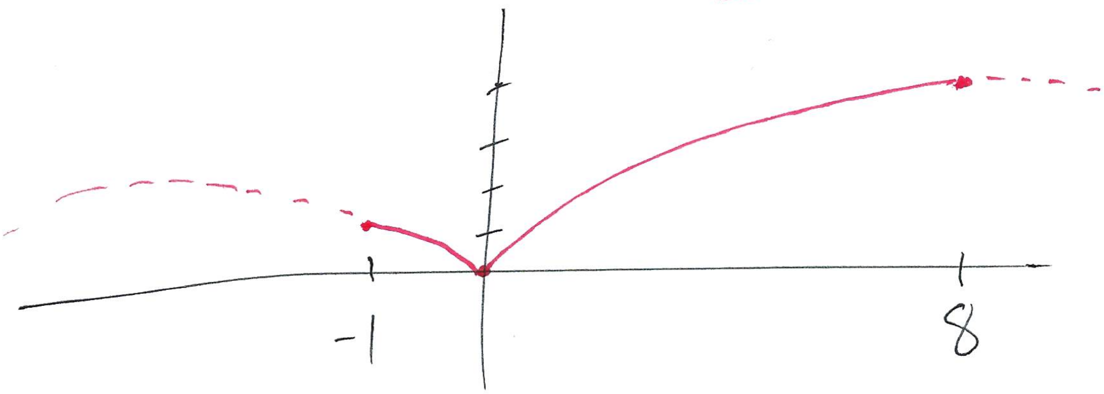
$$f'(x) = \frac{2}{3}x^{-1/3} = \cancel{\frac{2}{3\sqrt[3]{x}}} = \frac{2}{3\cdot\sqrt[3]{x}}$$

There is a critical number at $x = 0$.

$f(0) = \boxed{0}$ ← absolute min

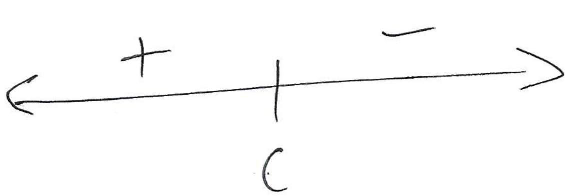
$f(-1) = (-1)^{2/3} = \left(\sqrt[3]{-1}\right)^2 = (-1)^2 = 1$

$f(8) = 8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = \boxed{4}$ ← absolute max



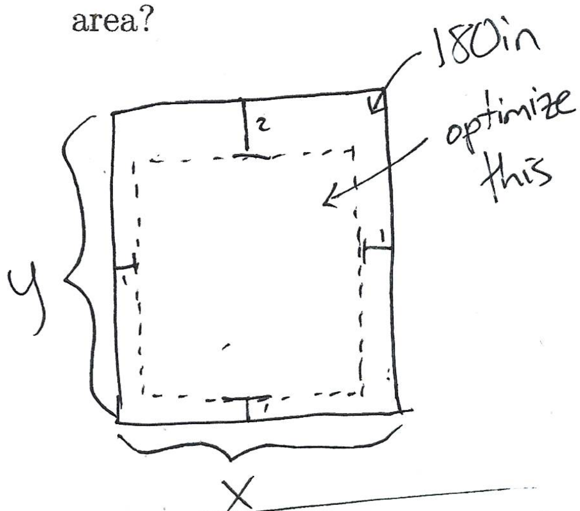
Alternative method to find absolute min/max:

If there's only one critical number and the sign chart looks like



Then $f(c)$ has to be the absolute maximum.

Exercise 3. A poster is to have an area of 180 in^2 with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printable area?



1. There's always two variables to identify.

2. Write the function you want to optimize

$$f(x) = (x-2)(y-3)$$

= area of the printable region

3. Identify the constraint

$$xy = 180$$

4. Rewrite our function in part (2)

$$y = \frac{180}{x}$$

$$f(x) = (x-2)\left(\frac{180}{x} - 3\right)$$

5. Actually optimize the function

$$f'(x) = 1 \cdot \left(\frac{180}{x} - 3\right) + (x-2)\left(\frac{-180}{x^2}\right)$$

$$= \cancel{\frac{180}{x}} - 3 + \left(\cancel{-\frac{180}{x}}\right) + \frac{360}{x^2}$$

$$= \frac{360 - 3x^2}{x^2} = \frac{3(120 - x^2)}{x^2}$$

Critical numbers are $x = \cancel{0}, \cancel{\pm}\sqrt{120}$

we can exclude because we need margins

The only critical number that makes sense is $x = \sqrt{120}$

$+ \quad -$

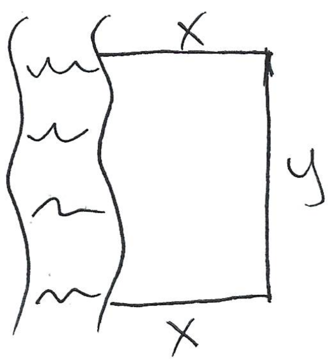
$0 \quad \sqrt{120}$

So $\sqrt{120}$ is where the absolute max occurs.

$x = \sqrt{120}$ inches

$y = \frac{180}{\sqrt{120}}$ inches

Ex. The US Military is planning to build a rectangular containment facility (i.e. a fenced off area) near Area 51 in order to contain classified extraterrestrials. Since the budget given by the Department of Defense only allows for 2960 yards of fencing, it is planned to build the prison adjacent to a straight river so that fencing is only needed on three sides (everyone knows aliens dissolve in water). If we let x be the length of fencing perpendicular to the river, find a function f in the variable x that gives the area of the enclosed land if all the fencing is used. Optimize this function to find the maximum possible area the facility can contain.



$f(x) = xy \leftarrow$ we want to optimize this

The constrait is amount of fencing

we have: $2x + y = 2960$

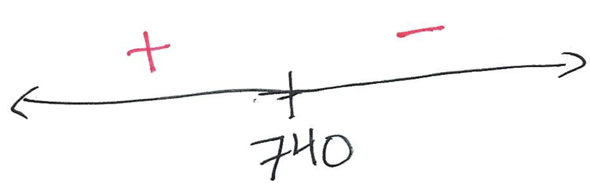
$y = 2960 - 2x$

The function we want to optimize becomes:

$$f(x) = x(2960 - 2x) = 2960x - 2x^2$$

$$f'(x) = 2960 - 4x$$

We have one critical number at $x = \frac{2960}{4} = 740$

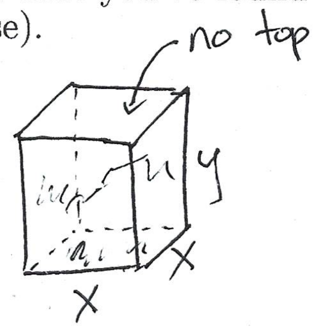


So the absolute occurs when $x = 740$.

$$f(740) = 2960(740) - 2(740)^2$$

$$= 1{,}095{,}200 \text{ yards}^2$$

Exercise 1. Based on customer trials and surveys, CV8 Theater decides that a small serving of popcorn should be 256 in^3. CV8 will serve the popcorn in rectangular boxes with square base and no top. Assuming that it is to hold 256 in^3, what dimensions for the box should the theater choose in order to minimize the box's surface area (thereby minimizing the cost of the cardboard forming the box)? You must fully justify your claim that you've found the dimensions minimizing the surface area (using calculus, of course).



Our variables are x = sidelength of the base

y = height

We want the absolute of

$f(x)$ = Surface area

$$= \underbrace{x^2}_{\text{area of bottom}} + \underbrace{xy + xy + xy + xy}_{\text{area of the sides}}$$

Our constraint:

$$x^2 y = 256$$

$$y = \frac{256}{x^2}$$

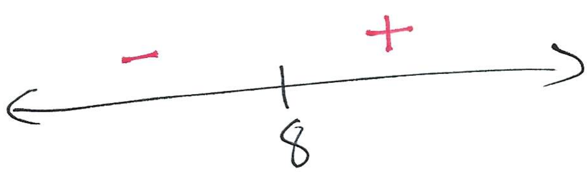
$$= x^2 + 4xy$$

$$= x^2 + 4x\left(\frac{256}{x^2}\right) = x^2 + \frac{1024}{x}$$

$$f'(x) = 2x - \frac{1024}{x^2} = \frac{2x^3}{x^2} - \frac{1024}{x^2}$$

$$= \frac{2(x^3 - 512)}{x^2}$$

The critical numbers are: $x = \cancel{0}, 8$



So the minimum occurs at

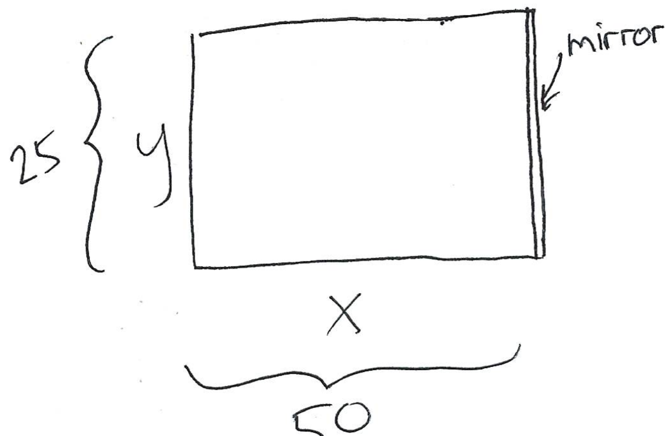
$x = 8$.

$y = \frac{256}{8^2} = 4$.

So the dimensions of the box should be.

$8 \times 8 \times 4$ inches.

Ex. A farmer has \$300 to spend on building a rectangular fence to protect his rambunctious sheep. Three sides of the fence will be constructed with chain-link at a cost of \$1 per foot. The fourth side is facing the neighbor, who the farmer hates with a passion, so that side of the fence will be constructed as a giant mirror facing outward at a cost of \$5 per foot. Find the dimensions of the largest area the fence can enclose.



We want to optimize

$$f(x) = xy.$$

Budget Constraint gives us:

$$\$300 = \$1 \cdot x + \$1 \cdot y + \$1 \cdot x + \$5 \cdot y$$

$$300 = 2x + 6y$$

$$50 - \frac{1}{3}x = \frac{300 - 2x}{6} = y$$

$$f(x) = x\left(50 - \frac{1}{3}x\right) = 50x - \frac{1}{3}x^2$$

$$f'(x) = 50 - \frac{2}{3}x$$

So our critical number is $x = \frac{3}{2} \cdot 50 = 75\text{ft}$

$$\overset{+}{\longleftarrow}\underset{75}{|}\overset{-}{\longrightarrow}$$

So the absolute max occurs when $x = 75\text{ft}$

$$y = 50 - \frac{1}{3} \cdot 75 = 25\text{ft}.$$