Geometric Interpretations of the second derivative

For the first derivative we used a sign chart to find intervals where f' is positive and negative. This helped us find when f is increasing/decreasing

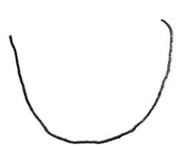
~~While~~ We'll use the sign chart again to determine when f'' is positive/negative.

$f''(x) > 0$ we'll call this "concave up"

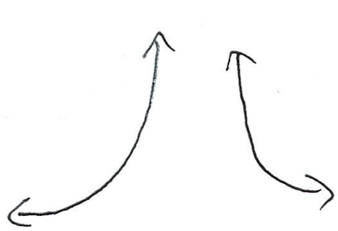
$f''(x) < 0$ we'll call this "concave down"

Geometrically, this looks like

concave up   concave down



For example



concave up

concave down

slope of tangent lines are increasing

$\Updownarrow$

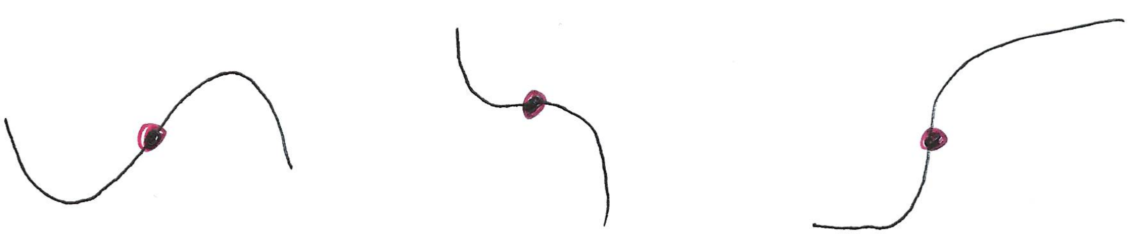
~~The~~ $f'(x)$ is increasing

$\Updownarrow$

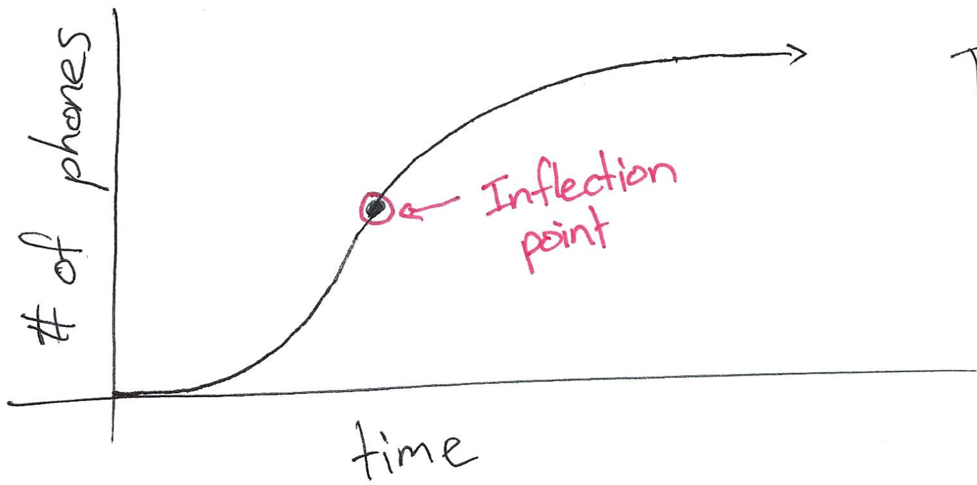
$f''(x) > 0$

	Marked points on sign chart	positive intervals	negative intervals	Change over marked points
f	zero or DNE	positive	negative	N/A
f'	critical points	increasing	decreasing	local min/max
f''	when f'' is zero or DNE	concave up	concave down	Inflection points

Inflection points are points <u>on the graph</u> of $f(x)$ where the concavity changes.



These points are inflection points, they represent when growth is speeding up or slowing down.



The inflection point represents when growth is leveling out.

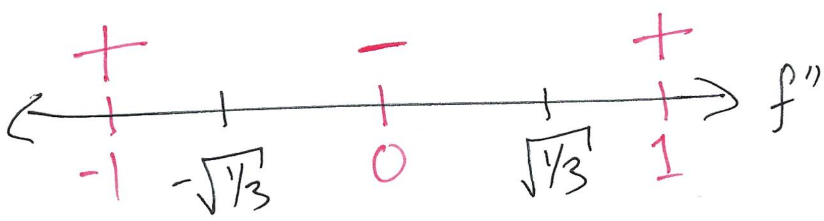
Ex $f(x) = \dfrac{1}{x^2+1}$ What are the inflection points?

$$f'(x) = \frac{(x^2+1)(0) - (1)(2x)}{(x^2+1)^2} = \frac{-2x}{(x^2+1)^2}$$

$$f''(x) = \frac{(x^2+1)^2 \cdot (-2) - (-2x)\left(2(x^2+1)^1 \cdot 2x\right)}{(x^2+1)^4}$$

$$= \frac{-2(x^2+1)^{\cancel{2}} + 8x^2\cancel{(x^2+1)}}{(x^2+1)^{\cancel{4}\,3}}$$

$$= \frac{-2x^2 - 2 + 8x^2}{(x^2+1)^3} = \frac{6x^2-2}{(x^2+1)^3}$$



$$6x^2 - 2 = 0$$
$$x^2 = 1/3$$
$$x = \pm\sqrt{1/3}$$

What are the inflection points? $(x, f(x))$

$f(x) = \frac{1}{x^2+1}$

They are

$\left(\sqrt{1/3}, f\left(\sqrt{1/3}\right)\right) = \boxed{\left(\sqrt{1/3}, 3/4\right)}$

and

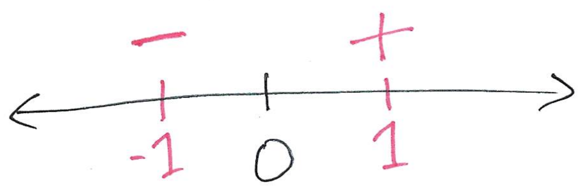
$\left(-\sqrt{1/3}, f\left(-\sqrt{1/3}\right)\right) = \boxed{\left(-\sqrt{1/3}, 3/4\right)}$

Ex 2) $f(x) = x^{5/3}$

$f'(x) = \frac{5}{3}x^{2/3}$

$f''(x) = \left(\frac{5}{3}\right)\left(\frac{2}{3}\right)x^{-1/3}$

$= \frac{10}{3\sqrt[3]{x}}$

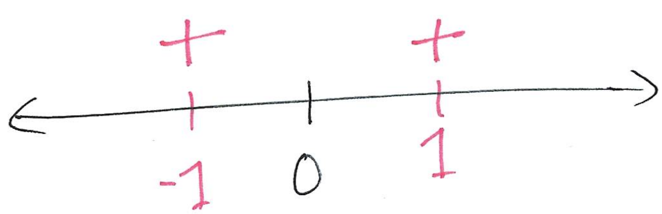


There's an inflection point at $(0, f(0)) = (0, 0)$.

Ex 3) $f(x) = x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2$



There are <u>no</u> inflection points.

$\frac{a}{b}$

The Second Derivative Test

If x is a critical number of $f(x)$, and there's a sign change at x, then there's either a relative min or relative max at x.

First Derivative Test

Alternatively, if x is a critical number of $f(x)$ and $f''(x) > 0$, then there's local minimum at x.

Second Derivative Test

In pratice, the second derivative test is almost always more difficult to apply.

Ex 1/ $f(x) = 2 + 3x - x^3$ Use the second derivative test to find relative min/max.

$$f'(x) = 3 - 3x^2 = 3(1-x^2) = 3(1-x)(1+x)$$

So critical numbers are $x = \pm 1$.

$f''(x) = -6x$

$f''(1) = -6$

$f''(-1) = 6$

So by the second derivative test, there's a relative max at $x=1$ and there's a relative min at $x=-1$.

Ex 2 $f(x) = x^3 - 9x^2 + 24x$ Use second derivative test to find relative min/max.

$f'(x) = 3x^2 - 18x + 24$

$= 3(x^2 - 6x + 8)$

$= 3(x-4)(x-2)$

So my critical numbers are $x = 2, 4$

$f''(x) = 6x - 18$

$f''(2) = 12 - 18 = -6$

$f''(4) = 24 - 18 = 6$

By the second derivative test, there's a relative max at $x=2$ and a relative min at $x=4$.

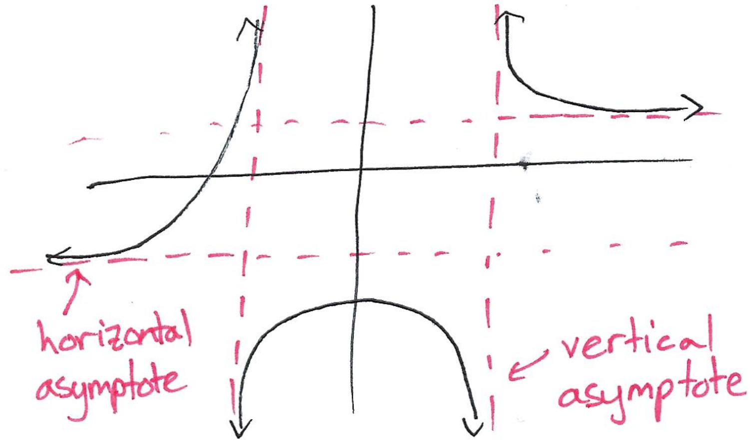
Asymptotes

The line $x=a$ is a vertical asymptote if

$$\lim_{x \to a^-} f(x) = \pm\infty$$

or

$$\lim_{x \to a^+} f(x) = \pm\infty.$$



For quotients of polynomials $\frac{p(x)}{q(x)}$, we just need to cancel and determine when the denominator is zero.

<u>Ex 1</u> What are the vertical asymptotes of $\frac{x+1}{x^2-1}$?

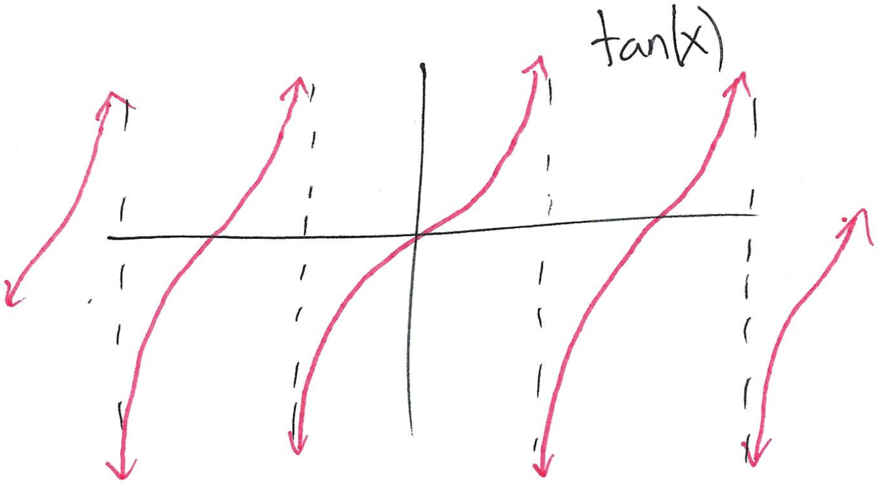
$$\frac{\cancel{x+1}}{\cancel{(x+1)}(x-1)} = \frac{1}{x-1}$$

The denominator is zero when $x=1$, so $x=1$ is a vertical asymptote.

$y=b$ is a horizontal asymptote if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

There is maximum of two horizontal asymptotes



So tan(x) has infinitely many vertical asymptotes, and no horizontal asymptotes.

Ex) $\frac{x^3-x^2}{(4-x^2)(x-1)}$ What are the vertical and horizontal asymptotes?

$$\frac{x^3-x^2}{(4-x^2)(x-1)} = \frac{x^2\cancel{(x-1)}}{(4-x^2)\cancel{(x-1)}} = \frac{x^2}{(2-x)(2+x)}$$

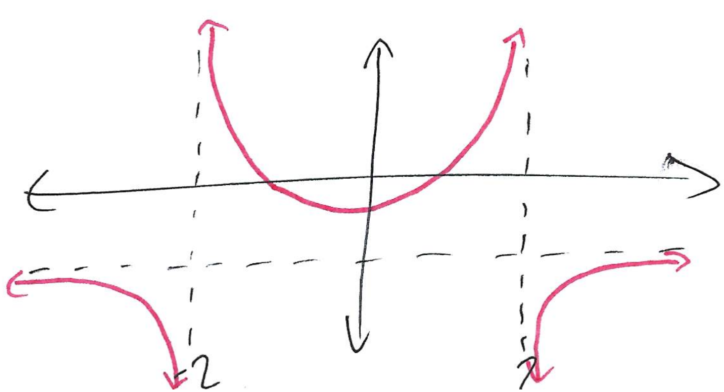
So there are ~~horizontal~~ vertical asymptotes at $x=2$ and $x=-2$.

$$\frac{x^3-x^2}{(4-x^2)(x-1)} = \frac{x^2}{4-x^2}$$

$$\lim_{x\to\infty} \frac{1x^2}{4-1x^2} = \frac{1}{-1} = -1$$

$$\lim_{x\to-\infty} \frac{x^2}{4-x^2} = \frac{1}{-1} = -1$$

So there's a horizontal asymptote at $y=-1$



Ex) Sketch $f(x) = \frac{x+1}{x-1}$

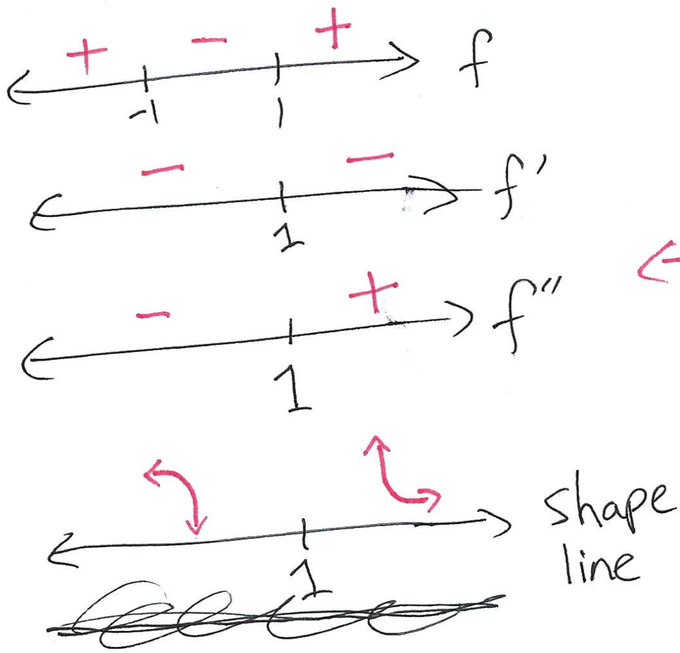
Asymptotes) There's a vertical asymptote at $x = 1$.

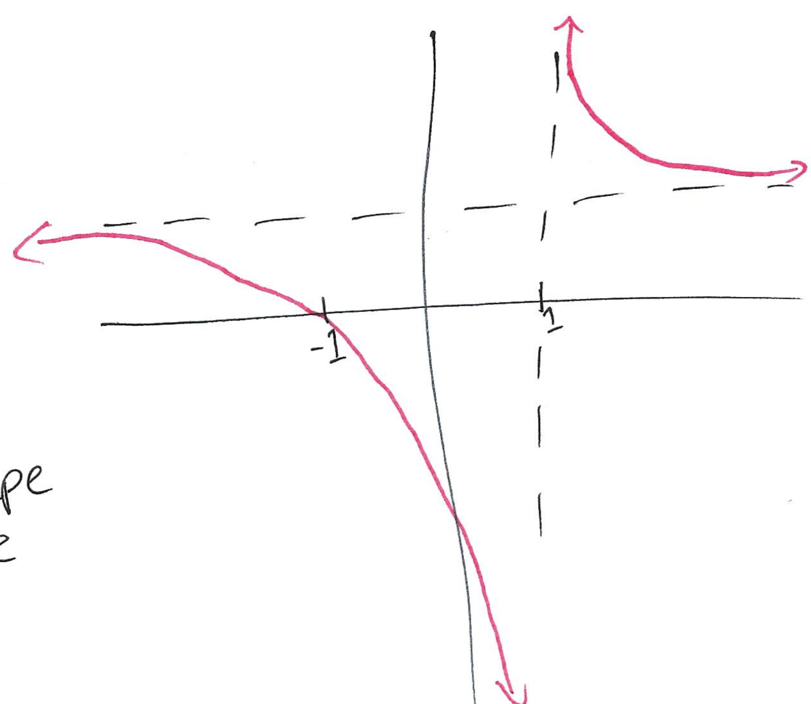
$\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to -\infty} f(x) = 1$

And so, there's a horizontal asymptote $y = 1$.

$$f'(x) = \frac{(x-1)\cdot 1 - (x+1)\cdot 1}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$$f''(x) = \frac{(x-1)^2\cdot 0 - (-2)\left(2(x-1)\cdot 1\right)}{(x-1)^4} = \frac{4(x-1)}{(x-1)^4} = \frac{4}{(x-1)^3}$$





This is the graph of $\frac{x+1}{x-1}$