Differentiation

Recall that $\frac{d}{dx} e^x = e^x$. Any exponential can rewritten as exponential with base e as follows:

$$2^x = \left(e^{\ln(2)}\right)^x = e^{\overbrace{\ln(2)}^{\text{just some number}} x}$$

We can apply the chain rule to $e^{f(x)}$ to get

$$\frac{d}{dx} e^{f(x)} = \frac{d}{dx} \exp(\underbrace{f(x)}_{\text{inside}})$$

$$\exp(x) = e^x$$
$$\exp'(x) = e^x$$

$$= \exp'(f(x)) \cdot f'(x)$$

$$= f'(x) e^{f(x)}$$

For example,

Ex 1)

$$\frac{d}{dx} 2^x = \frac{d}{dx} e^{\ln(2)x} = (\ln(2)x)' e^{\ln(2)x}$$

$$= \boxed{\ln(2) e^{\ln(2)x}}$$

Ex 2)

$$\frac{d}{dx} e^{2x} = (2x)' e^{2x} = 2e^{2x}$$

Ex 3)

$$\frac{d}{dx} e^{2x^2+x+8} = (2x^2+x+8)' e^{2x^2+x+8}$$

$$= (4x+1) e^{2x^2+x+8}$$

Ex 4 ~~x^2e^{3x}~~

$$\frac{d}{dx}x^2 \cdot e^{3x} = 2xe^{3x} + x^2\left(3e^{3x}\right)$$
$$= 3x^2e^{3x} + 2xe^{3x} = xe^{3x}(3x+2)$$

Keep in mind that $e^{f(x)} > 0$.

Ex 5 $$\frac{d}{dt}\frac{e^t}{e^t+e^{-t}} = \frac{\left(e^t+e^{-t}\right)e^t - e^t\left(e^t-e^{-t}\right)}{\left(e^t+e^{-t}\right)^2}$$

$$= \frac{\cancel{e^{t+t}} + \underbrace{e^{-t+t}}_{=1} - \cancel{e^{t+t}} + \underbrace{e^{t-t}}_{=1}}{\left(e^t+e^{-t}\right)^2}$$

$$= \frac{2}{\left(e^t+e^{-t}\right)^2}.$$

Differentiating Logarithms

Recall that $e^{\ln(x)} = x$ and $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$.

$$e^{\ln(x)} = x$$
$$\frac{d}{dx}e^{\ln(x)} = \frac{d}{dx}x$$
$$(\ln(x))'e^{\ln(x)} = 1$$

$$(\ln(x))' = \frac{1}{e^{\ln(x)}}$$

$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}.$$

Ex 1)

$$f(x) = x \cdot \ln(x)$$

$$f'(x) = 1 \cdot \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$$

Ex 2)

$$h(x) = \frac{\ln(x)}{x}$$

$$h'(x) = \frac{x \cdot \frac{1}{x} - \ln(x) \cdot 1}{x^2} = \frac{1 - \ln(x)}{x^2}$$

Similar to how $\frac{d}{dx} e^{f(x)} = f'(x) e^{f(x)}$, we can use the chain rule to get

$$\frac{d}{dx} \ln(f(x)) = \ln'(f(x)) \cdot f'(x)$$

$$= \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

Ex 1)

$$\frac{d}{dx} \ln(x^2 + 1) = \frac{2x}{x^2 + 1}$$

Sometimes it's easier to use log laws before differentiating.

Ex 2) $\frac{d}{dx}\ln\left((x^3-5)^{10}\right) = \frac{\frac{d}{dx}(x^3-5)^{10}}{(x^3-5)^{10}} = \frac{10(x^3-5)^9\cdot 3x^2}{(x^3-5)^{10}}$

$\frac{d}{dx}10\ln(x^3-5) = 10\,\frac{3x^2}{x^3-5} = \frac{30x^2}{x^3-5}$

$= \frac{30x^2}{x^3-5}$

Ex 3) $h(x) = \ln\left((x^2+1)\cdot(x^3+2)^6\right)$

If you try to compute $h'(x)$ now, you'll have to do a chain rule, a product rule, and then another chain rule.

$h(x) = \ln(x^2+1) + \ln\left((x^3+2)^6\right)$

$= \ln(x^2+1) + 6\ln(x^3+2)$

$h'(x) = \frac{2x}{x^2+1} + \frac{6(3x^2)}{x^3+2}$

Ex 4) $g(t) = \ln\left(t^2 \cdot e^{-t^2}\right)$

$= \ln(t^2) + \ln(e^{-t^2})$

$= 2\ln(t) + (-t^2)\cdot\underbrace{\ln(e)}_{1}$

$= 2\ln(t) - t^2$

$g'(t) = \frac{2}{t} - 2t$

Logarithmic Differentiation

We can introduce logarithms where there are none in order to simplify the calculation of derivatives.

Ex 1 $f(x) = x(x+1)(x^2+1)^6$

We can introduce logarithms by rewriting as

$$f(x) = e^{\ln(x\cdot(x+1)\cdot(x^2+1)^6)}$$

$$= e^{\ln(x) + \ln(x+1) + \ln((x^2+1)^6)}$$

$$= e^{\ln(x) + \ln(x+1) + 6\ln(x^2+1)}$$

If we then differentiate

$$f'(x) = \left(\ln(x) + \ln(x+1) + 6\ln(x^2+1)\right)' e^{\ln(x)+\ln(x+1)+6\ln(x^2+1}$$

$$= \left(\frac{1}{x} + \frac{1}{x+1} + 6\cdot\frac{2x}{x^2+1}\right) e^{\ln(x(x+1)(x^2+1)^6)}$$

$$= \left(\frac{1}{x} + \frac{1}{x+1} + \frac{12x}{x^2+1}\right) x(x+1)(x^2+1)^6$$

$$= (x+1)(x^2+1)^6 + x(x^2+1)^6 + 12x(x)(x+1)(x^2+1)^5$$

Not only does logarithmic differentiation make some derivatives easier, it can make some previously impossible derivatives possible.

Ex] $q(x) = x^x$ ← this is neither an exponential nor a power function

Incorrect approach: $\cancel{q'(x) = x \cdot x^{x-1} = x^{1+x-1}} = x^x$

$$q(x) = e^{\ln(x^x)} = e^{x\ln(x)}$$

$$q'(x) = (x \cdot \ln(x))' e^{x\ln(x)}$$

$$= \left(\ln(x) + x\frac{1}{x}\right) e^{\ln(x^x)}$$

$$= (\ln(x) + 1) x^x$$

Ex] $h(x) = x^x + 5$ Incorrect approach: $h(x) = e^{\ln(x^x+5)}$ You can't break this up

Correct approach:

$$h(x) = e^{\ln(x^x)} + 5 = e^{x\ln(x)} + 5$$

$$h'(x) = (x\ln(x))' \cdot e^{x\ln(x)} + 0$$

$$= (\ln(x) + 1) x^x$$

Ex $g(x) = 2^{2x+x^2}$

$$g(x) = e^{\ln(2^{2x+x^2})} = e^{(2x+x^2)\ln(2)}$$

$$g'(x) = \left[(2x+x^2)\ln(2)\right]' e^{(2x+x^2)\ln(2)}$$

$$= (2\ln(2) + 2\ln(2)x)\, e^{\ln(2^{2x+x^2})}$$

$$= (2\ln(2) + 2\ln(2)x)\, 2^{2x+x^2}$$

$$= 2\ln(2)(x+1)\, 2^{2x+x^2}$$

Ex $f(x) = x^{\ln(x)}$

$$= e^{\ln(x^{\ln(x)})} = e^{\ln(x)\cdot\ln(x)} = e^{(\ln(x)^2)}$$

$$f'(x) = \left(\ln(x)^2\right)' e^{(\ln(x)^2)}$$

You can't use log law because the exponent is on the outside

$$= 2(\ln(x)) \cdot \frac{1}{x} \cdot x^{\ln(x)}$$

$$= 2\ln(x)\, x^{\ln(x)-1}$$

Note: For the last problem think about the graph of e^x