More things about definite integrals

Recall that $\int_a^b f(x)\,dx$ is the signed area under the graph of $f(x)$, between $x=a$ and $x=b$. Under this interpretation, the following patterns should be apparent:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

These are the same rules as for indefinite integrals

$$\int_a^b c\cdot f(x)\,dx = c\cdot\int_a^b f(x)\,dx$$

$$\int_a^b \left[f(x)+g(x)\right] dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Ex) $$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx = 0$$

Ex) We know that

$$\int_{-5}^{10} f(x)\,dx = 12 \qquad \int_{0}^{1} f(x)\,dx = -1 \qquad \int_{1}^{10} f(x) = 5$$

What is $\int_{-5}^{1} f(x)\,dx$?

$$\int_{-5}^{1} f(x)\,dx = \int_{-5}^{10} f(x)\,dx - \int_{1}^{10} f(x)\,dx = 12 - 5 = \boxed{7}$$

What is $\int_{0}^{10} f(x)\,dx$?

$$\int_{0}^{10} f(x)\,dx = \int_{0}^{1} f(x)\,dx + \int_{1}^{10} f(x)\,dx = -1 + 5 = \boxed{4}$$

What is $\int_{-5}^{0} f(x)\,dx$?

$$\int_{-5}^{0} f(x)\,dx = \int_{-5}^{10} f(x)\,dx - \int_{0}^{10} f(x)\,dx = 12 - 4 = 8$$

or $$= \int_{-5}^{1} f(x)\,dx - \int_{0}^{1} f(x)\,dx = 7 - (-1) = 8$$

U-substitution with Definite Integrals

Ex 1

$$\int_0^2 x e^{2x^2}\,dx \qquad u = 2x^2 \qquad \frac{du}{dx} = 4x \qquad dx = \frac{du}{4x}$$

$$x=2 \implies u=8$$
$$x=0 \implies u=0$$

$$= \int_0^8 \cancel{x} e^u \frac{du}{4\cancel{x}}$$

$$= \frac{1}{4}\int_0^8 e^u\,du = \frac{1}{4}e^u \Big|_{u=0}^{8} = \frac{1}{4}e^8 - \frac{1}{4}e^0$$

$$= \boxed{\frac{1}{4}\left(e^8 - 1\right)}$$

Ex 2

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx \qquad u = 1/x \qquad \frac{du}{dx} = -1/x^2 \qquad dx = -x^2\,du$$

$$x=2 \implies u=1/2$$
$$x=1 \implies u=1$$

$$= \int_1^{1/2} \frac{e^u}{\cancel{x^2}}\left(-\cancel{x^2}\right)du$$

$$= -\int_1^{1/2} e^u\,du = -e^u \Big|_1^{1/2} = -e^{1/2} - \left(-e^1\right)$$

$$= \boxed{e - \sqrt{e}}$$

Ex 3 We know $\int_0^2 f(x)\,dx = 3$ and $\int_2^4 f(x)\,dx = 5$,

what is $\int_0^2 f(2x)\,dx$?

$$u = 2x$$

$$\int_0^2 f(2x)\,dx \qquad \frac{du}{dx} = 2 \qquad dx = \frac{du}{2}$$

$$x = 2 \Longrightarrow u = 4$$
$$x = 0 \qquad\quad u = 0$$

$$= \int_0^4 f(u)\,\frac{du}{2} = \frac{1}{2}\int_0^4 f(u)\,du$$

$$= \frac{1}{2}\int_0^2 f(u)\,du + \frac{1}{2}\int_2^4 f(u)\,du = \frac{3+5}{2} = \boxed{\cancel{8}}\ \boxed{4}$$

Ex 4

$$\int_1^e \frac{\ln(x)}{x}\,dx \qquad u = \ln(x)$$

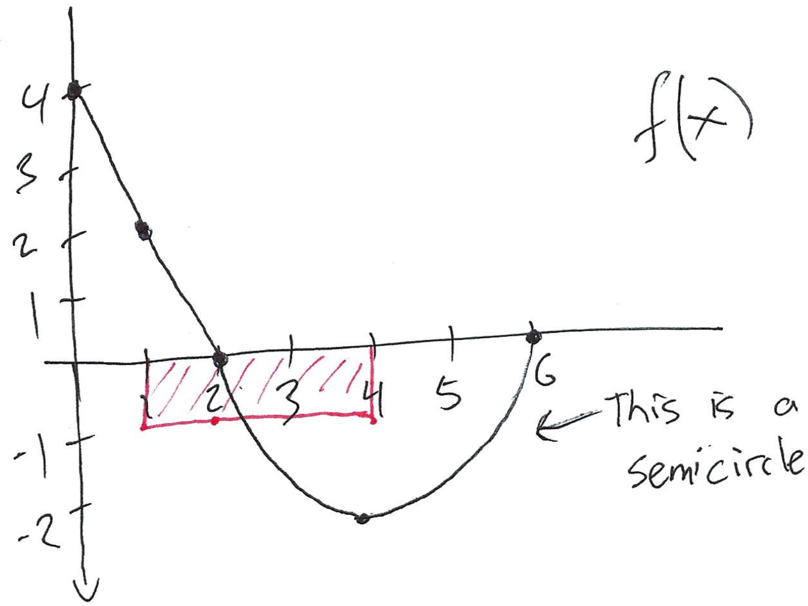
$$\frac{du}{dx} = \frac{1}{x} \qquad dx = x\,du$$

$$x = e \Longrightarrow u = 1$$
$$x = 1 \qquad\quad u = 0$$

$$= \int_0^1 \frac{u}{\cancel{x}}\,\cancel{x}\,du$$

$$= \frac{u^2}{2}\Big|_0^1 = \frac{1}{2} - \frac{0}{2} = \boxed{\frac{1}{2}}$$

Average Value of a Function



What is the average of $f(x)$ between $x=1$ and $x=4$?

The formula for average value of $f(x)$ between $x=a$ and $x=b$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

So in this case, it's $\frac{1}{4-1}\int_1^4 f(x)\,dx$

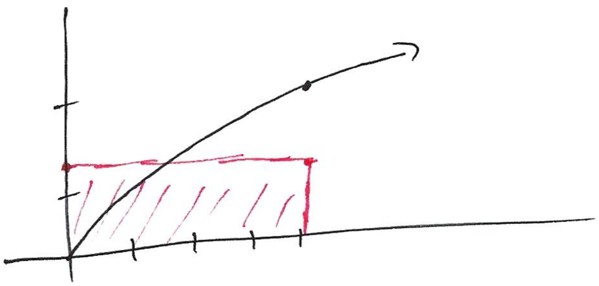
$$\frac{1}{3}\left(\frac{1\cdot 2}{2}-\frac{\pi\cdot 2^2}{4}\right)=\frac{1-\pi}{3}$$

Ex 1) What is the average value of $\sqrt{x}$ over $[0,4]$?

$$\frac{1}{4-0}\int_0^4 \sqrt{x}\,dx = \frac{1}{4}\int_0^4 x^{1/2}\,dx = \frac{1}{4}\,\frac{x^{3/2}}{3/2}\Big|_0^4$$

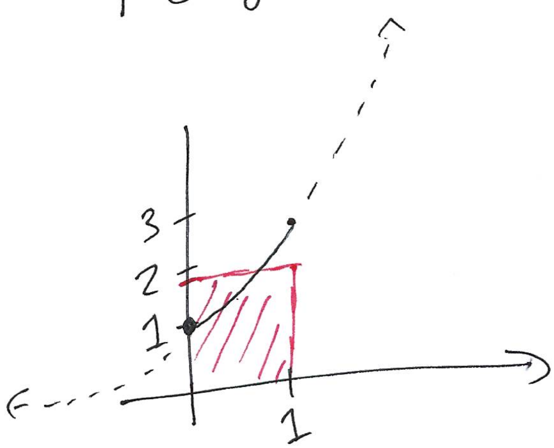
$$= \frac{1}{4}\,\frac{4^{3/2}}{3/2} - \frac{1}{4}\,\frac{0^{3/2}}{3/2} = \frac{1}{4}\cdot\frac{2}{3}\left(\sqrt{4}\right)^3$$

$$= \frac{1}{6}\,8 = \boxed{\frac{4}{3}}$$



Ex 2) Average value of e^x over $[0,1]$?

$$\frac{1}{1-0}\int_0^1 e^x\,dx = e^x\Big|_0^1 = e^1 - e^0 = \boxed{e-1}$$

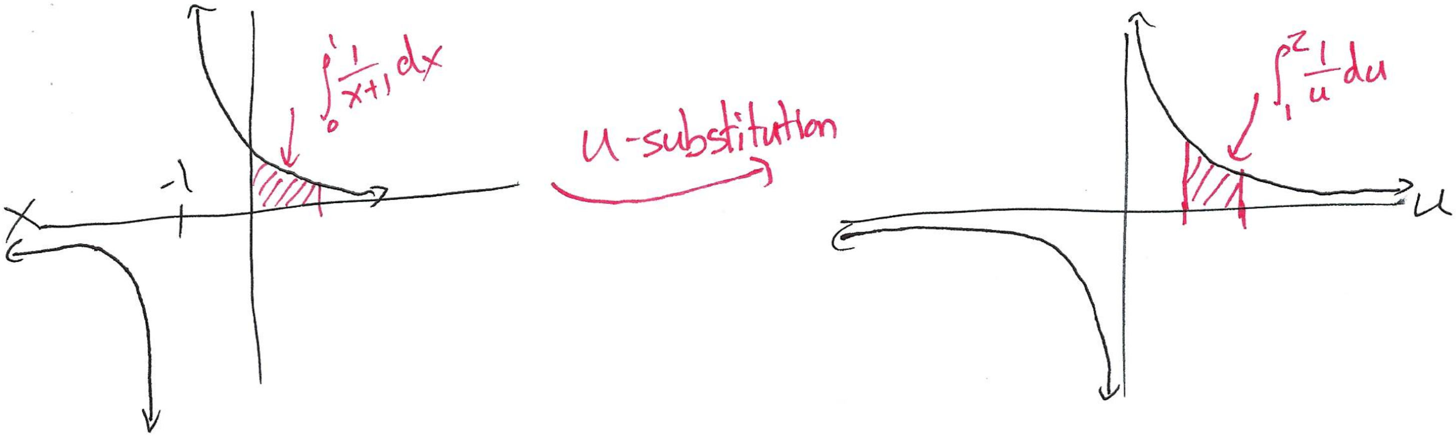


Ex 3 What is average value of $\frac{1}{x+1}$ over $[0,1]$?

$$\frac{1}{1-0}\int_0^1 \frac{1}{x+1}\,dx \qquad u=x+1 \qquad \frac{du}{dx}=1 \qquad dx=du$$

$$x=1 \Rightarrow u=2$$
$$x=0 \Rightarrow u=1$$

$$= \int_1^2 \frac{1}{u}\,du = \ln(u)\Big|_1^2 = \ln(2)-\ln(1) = \boxed{\ln(2)}$$



Ex 4 Average Value of $\frac{e^x}{1+e^x}$ over $[0,4]$?

$$\frac{1}{4-0}\int_0^4 \frac{e^x}{1+e^x}\,dx \qquad u=1+e^x \qquad \frac{du}{dx}=e^x \qquad dx=\frac{du}{e^x}$$

$$x=4 \Rightarrow u=1+e^4$$
$$x=0 \Rightarrow u=2$$

$$= \frac{1}{4}\int_2^{1+e^4} \frac{\cancel{e^x}}{u}\,\frac{du}{\cancel{e^x}} = \frac{1}{4}\ln(u)\Big|_2^{1+e^4} = \frac{1}{4}\left(\ln(1+e^4)-\ln(2)\right)$$

$$= \boxed{\frac{1}{4}\ln\left(\frac{1+e^4}{2}\right)}$$