Exponentials

An <u>exponential</u> is a function of the form:

$$f(x) = b^x$$

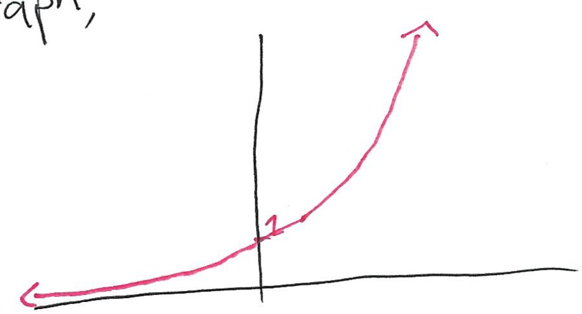
where $b > 0$ and fixed. The domain is all numbers. The range is always positive

Note: These behave distinctly from power functions, but the same exponent rules

$$b^x \cdot b^y = b^{x+y} \qquad \frac{b^x}{b^y} = b^{x-y} \qquad (b^x)^y = b^{xy}$$

As a graph,

If $b > 1$,

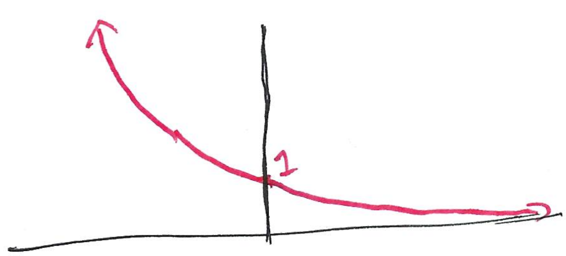


$$\lim_{x \to \infty} b^x = \infty$$

$$\lim_{x \to -\infty} b^x = 0$$

Increasing and continuous everywhere

If $b < 1$,



Note: If $b^n = b^m$, then that must mean $n = m$.

$$3^{5x-1} = 27$$

Ex $3^{5x-1} = 27$

$3^{5x-1} = 3^3$

$5x - 1 = 3$

$5x = 4$

$\boxed{x = 4/5}$

Derivative of exponentials

If $f(x) = b^x$, we see that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\overset{b^x \cdot b^h}{b^{x+h}} - b^x}{h}$$

$$= \lim_{h\to 0} \frac{b^x (b^h - 1)}{h} = b^x \underbrace{\left(\lim_{h\to 0} \frac{b^h - 1}{h}\right)}_{\text{contains no } x\text{, it's just a number}}$$

For $b=2$, we get that

$$\lim_{h\to 0} \frac{2^h - 1}{h} \approx .695$$

For $b=3$, we get that

$$\lim_{h\to 0} \frac{3^h - 1}{h} \approx 1.10$$

There is a number b such that $\lim_{h\to 0} \frac{b^h - 1}{h} = 1$.

As expected, this number is inbetween ~~to~~ 2 and 3.

We call this number "e" (for Euler/exponential).
It is irrational like π,

$$e \approx 2.718\ldots$$

This number is extremely important. By our previous derivative using limits:

$$\frac{d}{dx} e^x = e^x$$

This alone makes e^x the most important function in calculus.

Logarithms

Unlike multiplication and addition, exponentiation actually has two kinds of inverses. One kind for "undoing" a power, they are called root functions. The other kind is for undoing the base, they're called logarithms.

fixed power:	x^2	inverse: $\sqrt{x}$
fixed base:	2^x	inverse: $\log_2(x)$
	3^x	$\log_3(x)$
	e^x	$\log_e(x) = \ln(x)$

That is, $\log_2(x)$ is the function where

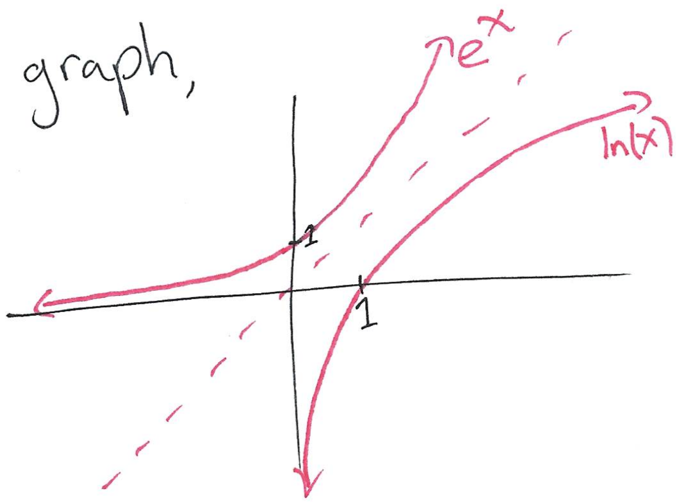
$$\log_2(2^x) = x$$

$$2^{\log_2(x)} = x$$

Similar to how things work with powers

$$\sqrt{x^2} = x \qquad (\sqrt{x})^2 = x$$

As a graph,



Notes:

$\ln(1) = 0$

$\ln(x)$ is not defined for $x \le 0$

$\ln(x)$ is always continuous and increasing

The most important fact about logarithms is that they undo exponential.

Ex 1

$\log_{16}(4) = x$

$16^{\log_{16}(4)} = 16^x$

$4 = 16^x$

$4^1 = (4^2)^x = 4^{2x}$

$1 = 2x$

$\frac{1}{2} = x$

Ex 2

$\log_x(81) = 2$

$x^{\log_x(81)} = x^2$

$81 = x^2$

$9 = x$

Ex 3

$\log_4(x) = 3$

$4^{\log_4(x)} = 4^3$

$x = 4^3 = 64$

Laws of Logarithms

Memorize these

$\ln(a \cdot b) = \ln(a) + \ln(b)$

$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

$\ln(a^b) = b \cdot \ln(a)$

$\ln(1) = 0$

$\ln(e) = 1$

Ex 1)

$$10\ln(x)+3=0$$
$$10\ln(x)=-3$$
$$\ln(x)=\frac{-3}{10}$$

$$e^{\ln(x)}=e^{-\frac{3}{10}}$$
$$x=e^{-3/10}$$
$$=(2.718\ldots)^{-.3}$$

Ex 2)

$$e^{x+2}=20$$
$$\ln(e^{x+2})=\ln(20)$$
$$(x+2)\cdot\ln(e)=\ln(20)$$

$$x+2=\ln(20)$$
$$x=\ln(20)-2$$

Ex 3)

$$\frac{1}{1+5e^x}=10$$
$$1=10(1+5e^x)$$

$$1=10+50e^x$$
$$\frac{-9}{50}=e^x \qquad \cancel{\ln\left(-\frac{9}{50}\right)=x}$$

There is no such x.

Ex 4) Expand $\ln\left(\frac{e^x}{2+e^x}\right)$

$$\ln\left(\frac{e^x}{2+e^x}\right)=\ln(e^x)-\ln(2+e^x)$$
$$=x-\underbrace{\ln(2+e^x)}$$

I can't expand addition on the inside

Ex 5) Expand

$$2\ln(x^2)+\ln(\sqrt{x}\cdot e^x)$$
$$2\cdot\ln(x)+\ln(\underbrace{\sqrt{x}}_{x^{1/2}})+\ln(e^x)$$

$$2\ln(x)+\frac{1}{2}\ln(x)+x$$
$$2.5\ln(x)+x$$

Ex 6) Let $f(x) = a + b\ln(x)$ if $f(1) = 7$ and $f(2) = 10$, find a and b.

$$7 = f(1) = a + \cancel{b\ln(1)} = a$$

because $\ln(1) = 0$

$$10 = f(2) = 7 + b\ln(2)$$

$$b\ln(2) = 3$$

$$b = \frac{3}{\ln(2)}$$

$$\boxed{f(x) = 7 + \frac{3\ln(x)}{\ln(2)}}.$$