Study Guide for the Math 1210 Oral Final Exam

Overview:

What to expect:

- solve problems that require a little bit of computation ($<$ 1.5 minutes);
- explain how to do something (e.g. how to find the absolute maximum of a function, how to draw a sign line, how to find inflection points, etc.);
- identify concepts visually given a graph (e.g. where is a graph discontinuous, where is a graph differentiable, what is the limit based on the graph, etc.);
- provide definitions of key concepts (e.g. what is a critical number, what is the limit definition of a derivative, etc.).

For problems on the exam that require short calculations, you should have a notepad or notebook to write on. You are encouraged to share your thinking as you are writing. If necessary, you can show your work to the camera. **Significant credit will be awarded for sharing thoughts that are relevant to solving a problem, even if your final solution is incorrect.** If necessary, you can show your work to the camera.

It's fine to have notes (on paper) and/or your book nearby but relying heavily on notes to solve a problem will waste time and might result in a loss of some credit (e.g., an A response dropped to B). The questions will be displayed on screen through Zoom. You will not need to interact with Zoom during exam (that is, you will not have to type your answers into chat).

Below you will find descriptions of the skills, knowledge, and level of understanding you will need to earn a high score on the Oral Final Exam. You should focus on being able to explain your thinking, not on memorizing solutions.

Week 1

(1) **Pre-Calculus**

(a) Finding the domain of a function

(b) Basic algebra: factorization of polynomials, properties of exponents, etc.

(c) Properties of lines (slope, intercept, parallel, perpendicular); finding equations of a line

(d) The Pythagorean Theorem and the distance formula

(e) The quadratic formula, which states that the solutions of the equation $ax^2+bx+c=0$ are $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

***Q1** Sample Question*: Where will a laser beam following the line through $(1,2)$ and $(4,3)$ strike the line $x=7$?

***Q2** Sample Question*: If $A(t)=24\cdot 2^{-t/8}$ is the amount of radioactive iodine, in mg, that a hospital will have in storage after t days, then how much will the hospital after 16 days?

***Q3** Sample Question*: Find the domain of $f(x)=\frac{\sqrt{x}}{x-1}$, expressing your answer in interval notation.

(2) **Limits**

(a) You should have a intuitive understanding of limits: $\lim_{x\to a} f(x)=L$ means that if we plug-in x closer and closer to a, then $f(x)$ becomes closer and closer to L.

(b) You should be able to find limits visually/graphically and algebraically.

(c) The algebraic approach to finding $\lim_{x\to a} f(x)$ often requires simplifying $f(x)$ before evaluating the limit (e.g. factoring and canceling; moving the square-root to the numerator; finding a common denominator)

(d) You should understand what one-sided limits are and how to use them (useful for functions that are piecewise-defined or contain absolute values).

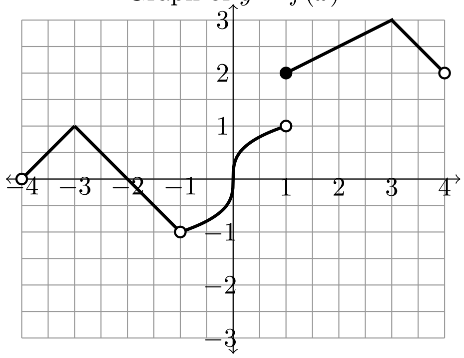
***Q4** Sample Question*: Find the following limits. If a limit does not exist, explain why.

(a) $\displaystyle\lim_{x\to 2}\frac{x^2-4}{x^2-3x+2}$ (b) $\displaystyle\lim_{x\to 2}\frac{x^2-4}{|x-2|}$ (c) $\displaystyle\lim_{t\to\infty}\frac{t^3+2t+2}{5-2t^2+4t^3}$ (d) $\displaystyle\lim_{x\to 2}\frac{x-2}{\sqrt{x-2}}$.

***Q5** Sample Question*: For the function whose graph appears to the below, find or explain why the limit DNE.

(a) $\lim_{x\to -1} f(x)$ (b) $\lim_{x\to -4^+} f(x)$ (c) $\lim_{x\to 1} f(x)$

Graph of $y = f(x)$



(3) **Continuity and Differentiability**

(a) A function f is *continuous* at a if $\lim_{x\to a} f(x) = f(a)$.

***Q6** Sample Question*: Let

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 2x+1 & \text{if } x \geq 1 \end{cases}$$

Is f continuous at 1? Justify your answer using the definition of continuity of f at 1.

(b) The *derivative* of f at a, denoted $f'(a)$, is given by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \quad \text{provided the limit exists.}$$

When the derivative of f at a exists, we say f is *differentiable at a*.

(c) You should feel comfortable analyzing continuity and differentiability graphically:

- Continuity at a means there is no hole or jump in the graph of f at a;
- Differentiability at a means that there is no corner, discontinuity, or infinite slope at $x = a$.

(d) You should also keep in mind that if f is differentiable at $x = a$, then f is continuous at a. The opposite is not true, for example, $f(x) = |x|$ is continuous at 0, but not differentiable at 0.

***Q7** Sample Question*: If $f'(5) = 3$, must $\lim_{x\to 5} f(x)$ exist? Explain.

(e) Recall that many functions, such as polynomials, are continuous everywhere that they're defined.

(f) If a question asks you *if there exists* an x that satisfies an equation, then you should consider the Intermediate Value Theorem.

***Q8** Sample Question*: Is there a number x such that $x = x^3 + 1$?

Week 2 and 3

(1) **Differentiation Rules, Tangent Lines, and Rates of Change.**

(a) You should feel comfortable differentiating using differentiation rules. The most common rules are the power rule, the product rule, the quotient rule, and the chain rule. It may also be beneficial to memorize common applications of the chain rule, such as:

$$\frac{d}{dx} e^{f(x)} = f'(x)e^{f(x)} \quad \text{and} \quad \frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}.$$

***Q9** Sample Question*: (a) If $f(x) = x^3 + 2x^4 + e^x + x^e + \ln(5)$, then what is $f'(x)$?

(b) If $f(x) = \frac{e^{2x}}{x^2+1}$, what is $f'(x)$?

(c) If $f(x) = x^x$, what is $f'(x)$?

(d) if $f(x) = \left(e^{2x} + 7\right)^6$, then what is $f'(0)$?

(b) You should know that $f'(a)$ yields the slope of the line tangent to the graph of f at $(a, f(a))$.

***Q10** Sample Question*: Find an equation of the line tangent to the graph of $f(x) = \ln(2x+1) + 3$ at $(0, 3)$.

(c) You should know that $f'(a)$ may be interpreted as the instantaneous rate of change in f at a. For example, suppose $s(t)$ denotes the position of an object. Then $s'(t)$ is the object's (instanteous) velocity at time t and $s''(t)$ is the object's (instanteous) acceleration at time t.

***Q11** Sample Question*: (a) The position of an object traveling along a line is given by $s(t) = t^3 + 2t + 1$, where s is measured in feet and t is measured in seconds. What is the object's velocity at $t = 1$? (Include units.) What is the object's acceleration at $t = 1$? (Include units). Is there a time t when the object's velocity is 29 ft/sec?

(b) Let $P(t) = 100e^{0.1t}$ be the population of bacteria in a culture at time t (measured in hours). What is the rate of change in the the culture's population at time $t = 0$? Include units.

(2) **Relative Extrema, Concavity, Asymptotes**

(a) A *critical number* (or *critical point*) of a function f is any number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist. You should be able to find critical numbers of functions.

***Q12** Sample Question*: (a) Find the critical numbers, if any, of $f(x) = x^{2/3}$.

(b) The critical numbers of $f(x) = x + \dfrac{1}{x}$ are ____________________.

(b) You should be able to assess whether a critical number yields a relative maximum value of a function, a relative minimum value of a function, or neither a relative maximum nor relative minimum.

- If c is a critical number at which f is continuous and the sign of f' changes from negative to the left of c to positive to the right, then f has a relative minimum at c.

$$\searrow \qquad \text{minimum} \qquad \nearrow$$

- If c is a critical number at which f is continuous and the sign of f' changes from positive to the left of c to negative to the right, then c has a relative maximum at c.

$$\nearrow \qquad \text{maximum} \qquad \searrow$$

- If c is a critical number at which f is continuous and the sign of f' does not change at c (that is, it's the same on either side of c), then f has neither a relative maximum nor a relative minimum at c.

***Q13** Sample Question*: Find and classify the critical numbers of $f(x) = x^3 - 3x$.

(c) You should be able to determine the intervals on which a function is increasing and those on which it is decreasing using the f' sign line.

***Q14** Sample Question*: On what interval(s) is the function $f(x) = 1 - (x - 2)^2$ increasing? decreasing?

(d) You should be able to determine the intervals on which the graph of a function is concave up and those on which it is concave down using the f'' sign line.

***Q15** Sample Question*: On what intervals is the graph of the function $f(x) = x + \frac{1}{x}$ concave up? concave down?

(e) A point $(a, f(a))$ on the graph of f at which the concavity of the graph changes is a *point of inflection*. You should be able to find points of inflection on the graph of f.

***Q16** Sample Question*: Explain how to use an appropriate sign line to locate points of inflection on the graph of a function f.

(f) You should be able to find horizontal and vertical asymptotes of the graph of a function.

***Q17** Sample Question*: Find the vertical and horizontal asymptotes of $f(x) = \frac{x(x-1)}{x^2-1}$.

(g) You should be able to find the absolute maximum value and absolute minimum value of a continuous function on a closed interval.

***Q18** Sample Question*: Find the absolute maximum value and absolute minimum value of $f(x) = \ln(1 + x^2)$ on $[-1, 1]$.

(3) **Exponentials and Logarithms**

(a) You should be able to use laws of exponents and laws of logarithms.

***Q19** Sample Question*: Simplify $\frac{2^{1/2}2^{3/2}}{27^{-1/3}} + \ln\left(\frac{1}{e^3}\right)$.

(b) You should be able to solve equations involving exponential functions and be able to solve real-world problems involving exponential and logarithmic functions.

***Q20** Sample Question*: The population of bacteria in a culture t hours after 1 p.m. is given by

$$P(t) = 1000e^{0.1t}.$$

When will the population reach 2000?

(c) Be familiar with the graphs of exponential and logarithmic functions

***Q21** Sample Question*: Sketch the graph of $g(x) = \ln(x)$.

(d) You should be able to differentiate functions involving exponentials and logarithms (this is mentioned earlier under Differentiation Rules).

Week 4

(1) **Antiderivatives (aka Indefinite Integrals)**

(a) You should be able to find antiderivatives. If $f(x)$ is a function, all possible antiderivatives of $f(x)$ are represented by the symbol $\int f(x)dx$. Using this notation, we might say that

$$\int f(x)dx = \text{ the family of all functions having derivative } f(x);$$

We might also say,

$$\int f(x)dx = \text{the general antiderivative of } f(x).$$

For example, $\int xdx = \frac{x^2}{2} + C$ means that all functions having derivative $f(x) = x$ are of the form $\frac{x^2}{2} + C$ where C is any constant.

Remark: In general, finding antiderivatives is not as straightforward as finding derivatives. There are simple-looking functions such as $f(x) = e^{x^2}$ that do not have "nice antiderivatives". On the final exam, if you are asked to find an antiderivative of a function f, then you will be dealing with a function f that *does* have a nice antiderivative, which you'll be able to find using the antidifferentiation rules and techniques you have learned in this course.

(b) Some common antidifferentiation/integration patterns that derive directly from the derivative rules:

Rule	Notation
Integral of a constant	$\int kdx = kx + C$
Constants slide through integrals	$\int kf(x)dx = k\int f(x)dx$
The integral of a sum is the sum of the integrals	$\int (f(x) + g(x))\,dx = \int f(x)dx + \int g(x)dx$
Integral of a power of x	$\int x^n dx = \frac{1}{n+1}x^{n+1} + C \qquad n \neq -1$
Integral of $\frac{1}{x}$	$\int \frac{dx}{x} = \ln(x) + C$
Integral of an exponential	$\int e^x dx = e^x + C$

***Q22** Sample Question*: Evaluate the following indefinite integrals.

(a) $\int (3x^2 + 4)\,dx$ (b) $\int (3e^x + \sqrt{x})\,dx$ (c) $\int (3x+1)^2\,dx$ (d) $\displaystyle\int \frac{1+x}{x^2}\,dx$

(c) You should feel comfortable with the following natural application of antidifferentation: the velocity of an object as a function of time $v(t)$ is an antiderivative of its acceleration $a(t)$ and the position of an object as a function of time $s(t)$ is an antiderivative of the velocity $v(t)$, that is,

$$v(t) = \int a(t)dt \quad \text{and} \quad s(t) = \int v(t)dt,$$

where we interpret the preceding equations to specify the forms of v and s, with constants of integration being determined by appropriate‘initial conditions.

For example if the acceleration as a function of time is $a(t) = t^2 - 7$ then its velocity will have the form $v(t) = \int a(t)dt = \int (t^2 - 7)\, dt = \frac{t^3}{3} - 7t + C$ where the constant C is determined after an initial condition is specified for v.

(d) You should feel comfortable solving first-order initial-value problems.

Example. Find a function f satisfying $f'(x) = \frac{1}{x} + 2x$ and $f(1) = 3$. The first step in the solution process is to integrate both sides of $f'(x) = \frac{1}{x} + 2x$ obtaining a general solution

$$f(x) = \ln(x) + x^2 + C.$$

The second step is to substitute $x = 1$ into both sides of the general solution and use $f(1) = 3$ to determine the constant C in the general solution: $3 = \ln(1) + 1^2 + C$, so that $C = 2$. Thus the solution to our problem is $f(x) = \ln|x| + x^2 + 2$.

***Q23** Sample Question*: Find f given that $f'(x) = \frac{1}{x^2} + x$ and $f(1) = 2$.

(e) You should be prepared to find indefinite integrals via "u-substitution."

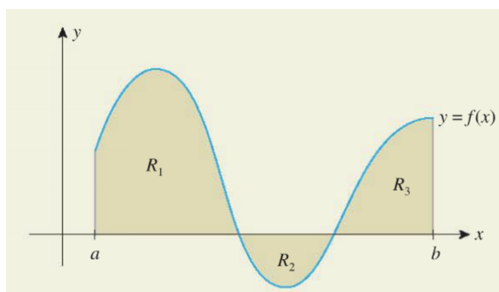
***Q24** Sample Question*: (a) Find $\int \frac{\sqrt{\ln(x)}}{x}\, dx$

(b) Find $\int x^2 e^{x^3}\, dx$.

(2) **Definite Integrals**

(a) You should be able to find a definite integral $\int_a^b f(x)\, dx$ using the following techniques

- Interpretation of the definite integral as an area: $\int_a^b f(x)\, dx$ is the *signed* area between $f(x)$ and the x-axis from $x = a$ to $x = b$. Recall that *signed* area means that the area below the x-axis is counted as negative. Thus if the graph of f is as pictured below, then $\int_a^b f(x)\, dx = \text{Area}(R_1) - \text{Area}(R_3) + \text{Area}(R_3)$.



- The Fundamental Theorem of Calculus: Let f be a continuous function on $[a, b]$. Then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f; that is, $F'(x) = f(x)$.

- Properties of the definite integral:

  (i) $\int_a^a f(x)dx = 0$

  (ii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

  (iii) $\int_a^b (f(x) + kg(x))\, dx = \int_a^b f(x)dx + k\int_a^b g(x)dx$

  (iv) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

***Q25** Sample Question*: Find (a) $\int_0^2 (e^x + x)\, dx$ (b) $\int_0^1 |x^2 - 1|\, dx$

(c) True/False: If $a < b$ and if $\int_a^b f(x)\, dx > 0$, then $f(x) > 0$ for every x in (a, b).

(d) Assume f is integrable over $[-1, 5]$, satisfying $\displaystyle\int_{-1}^{5} f(x)\, dx = -3$ and $\displaystyle\int_{1}^{5} f(x)\, dx = 7$. Then $\displaystyle\int_{1}^{-1} f(x)\, dx =$ ____________________.

(e) State the Fundamental Theorem of Calculus.

(b) You should feel comfortable evaluating definite integrals using u-substitution. When you use u-substitution to evaluate an integral, be sure to define u and compute du. This will yield partial credit even if you don't arrive at the correct answer. For example, in order to find $\int_0^{\sqrt{2}} xe^{x^2}\,dx$ we use again the substitution $u = x^2$, $du = 2xdx$. Be sure the keep in mind that the limits of integration change, we see that

$$x = 0 \implies u = 0 \quad \text{and} \quad x = \sqrt{2} \implies u = 2$$

so the definite integral becomes

$$\int_0^{\sqrt{2}} xe^{x^2}\,dx = \frac{1}{2}\int_0^2 e^u du = \frac{1}{2}\left(e^u\right)\Big|_0^2 = \frac{1}{2}\left(e^2 - e^0\right) = \frac{1}{2}\left(e^2 - 1\right).$$

***Q26** Sample Question*: (a) Find $\int_0^1 \sqrt{3x+1}\,dx$ (b) Find $\int_1^e \frac{\ln(x)}{x}\,dx$.

(c) You should know how to find the average value of a function over $[a, b]$. Suppose that $f(x)$ is a function integrable on $[a, b]$. Then its **average value over** $[a, b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)dx$$

***Q27** Sample Question*: The temperature of a 3-centimeter-long horizontal metal wire x centimeters from its left endpoint is $T(x) = 1 + x^2$ degrees Celsius. Find the average temperature of the wire.