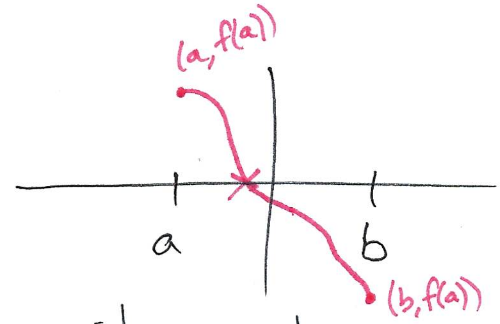
Sign chart

By IVT, if f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ have different signs, then $f(c)=0$ where $a \le c \le b$.



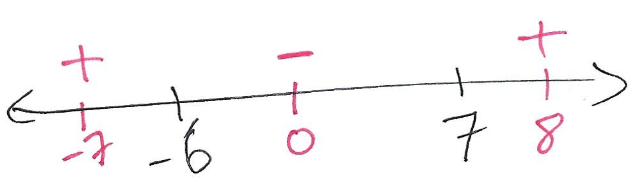
Ex) $f(x) = x^2 - x - 42$

When is $f(x)$ positive and when is it negative?

1) Identify when the function is zero or undefined.

$$x^2 - x - 42 = 0$$
$$(x-7)(x+6) = 0$$
$$x = 7, -6$$

2) Mark these points on a line



$f(0) = -42$

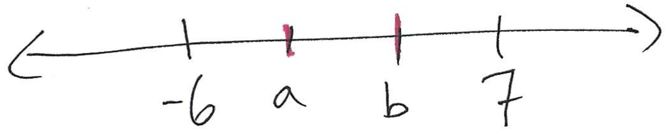
$f(-7) = 49 + 7 - 42$
$= 14$

$f(8) = 64 - 8 - 42 > 0$

Our conclusion is that

$f(x) > 0$ on $(-\infty, -6)$ and $(7, \infty)$

$f(x) < 0$ on $(-6, 7)$

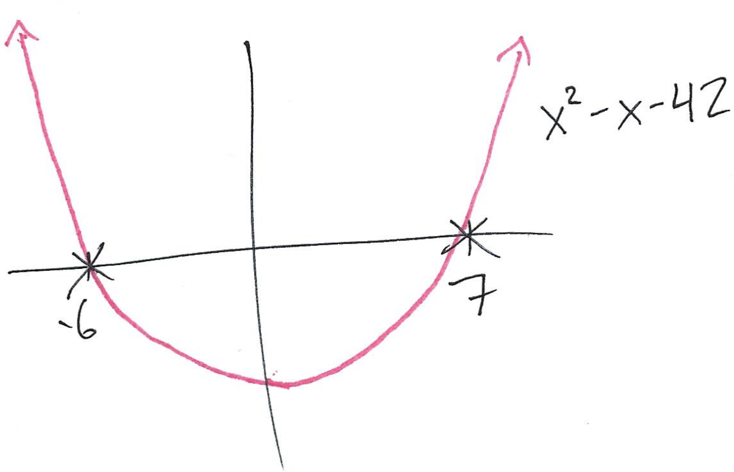


Why do $f(a)$ and $f(b)$ have the same sign?

Answer: If $f(a)$ and $f(b)$ had different signs, then by IVT, there'd be a c where $f(c)=0$ and $a<c<b$. However, the only points where $f(x)=0$ is when $x=-6, 7$. So $f(a)$ and $f(b)$ must have the same sign.

Usually, we use IVT as $f(a)$ and $f(b)$ have different signs, ~~therefore~~ therefore there's root inbetween.

but we're saying there's not a root inbetween, therefore $f(a)$ and $f(b)$ have the same sign.

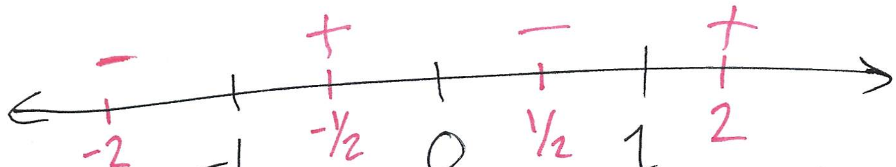


Ex] $f(x) = \frac{x^2-1}{x}$ when is $f(x) > 0$ and $f(x) < 0$?

1) $\frac{x^2-1}{x} = 0$ $\quad x^2-1=0$ $\quad (x+1)(x-1)=0$

$x = \pm 1$

and it's undefined at $x=0$

2) 

$f(-2) = \frac{3}{-2} < 0$

$f(-\tfrac{1}{2}) = \frac{-\frac{3}{4}}{-\frac{1}{2}} > 0$

$f(\tfrac{1}{2}) = \frac{-\frac{3}{4}}{\frac{1}{2}} < 0$

$f(2) = \frac{3}{2} > 0$

We can apply this technique to the derivative to find when the derivative is positive/negative.

When $f'(x) > 0$, $f(x)$ is increasing.

When $f'(x) < 0$, $f(x)$ is decreasing.

Ex] When is $f(x) = x^3 - 3x^2 - 24x + 1000$ increasing/decreasing?

$$f'(x) = 3x^2 - 6x - 24$$

$f'(x)$ is always defined.

$f'(x) = 3(x^2 - 2x - 8) = 3(x-4)(x+2)$,

So $f'(x) = 0$ when $x = -2, 4$

$$f'(-3) = 3 \cdot 9 + 6 \cdot 3 - 24 = 27 + 18 - 24 > 0$$
$$f'(0) = -24$$
$$f'(5) = 3 \cdot 25 - 6 \cdot 5 - 24 = 75 - 30 - 24 > 0$$

So $f(x)$ is increasing on $(-\infty, -2)$ and $(4, \infty)$

is decreasing on $(-2, 4)$

Note: Remember to plug test values into f'.

Ex) $f(x) = x + \frac{1}{x}$ when is $f(x)$ increasing/decreasing?

$$f'(x) = 1 + (-1)x^{-2} = 1 - \frac{1}{x^2}$$

$$= \frac{x^2-1}{x^2} = \frac{(x+1)(x-1)}{x^2}$$

Our marked points are $x = -1, 0, 1$

$+$	-1	$-$	0	$-$	1	$+$

$$f'(-2) = \frac{(-1)(-3)}{(-2)^2} = \frac{3}{4}$$

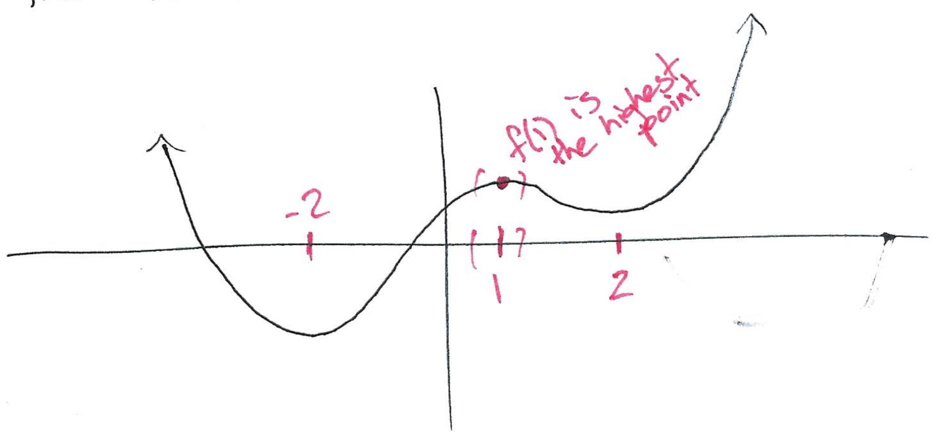
$$f'\left(-\tfrac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\left(-\frac{1}{2}\right)^2} = \frac{\left(\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\left(\frac{1}{4}\right)}$$

$f'\left(\tfrac{1}{2}\right) < 0$

$f'(2) > 0$

Relative minimum/maximum

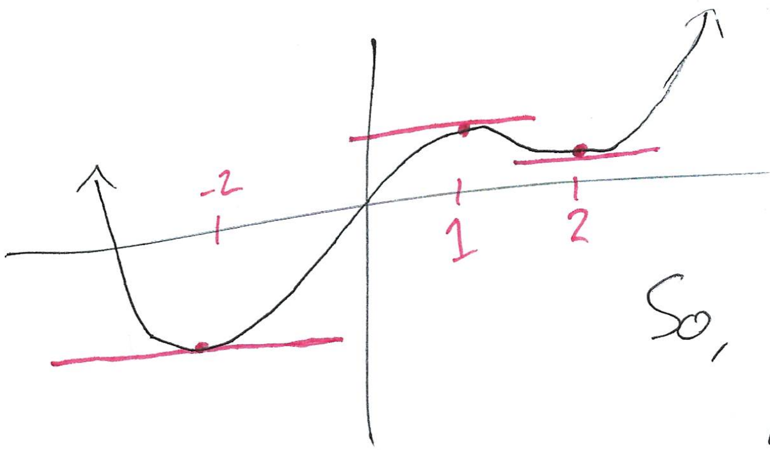
A relative minimum/maximum is a point c such that $f(c)$ is less/greater than $f(x)$ for the x around c.



There's a relative min at $x = -2, 2$

relative max at $x = 1$

We call x values where $f'(x) = 0$ or $f'(x)$ DNE critical points. (These are the marked points)

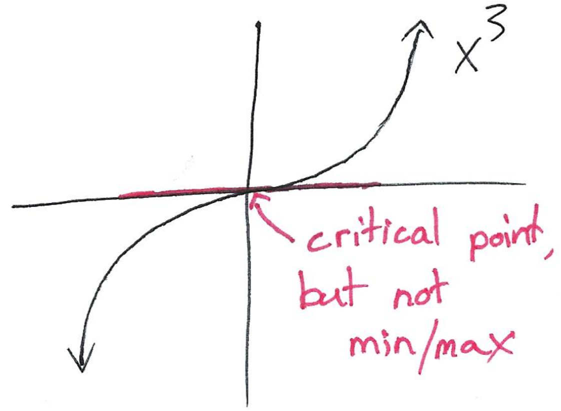


So, relative min/max are critical points.

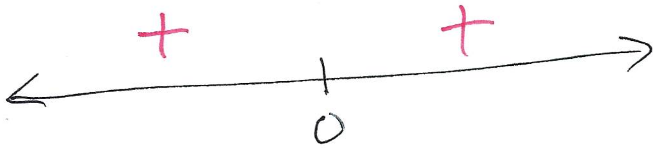
~~However~~

<u>However</u>, not all critical points are relative min/max.

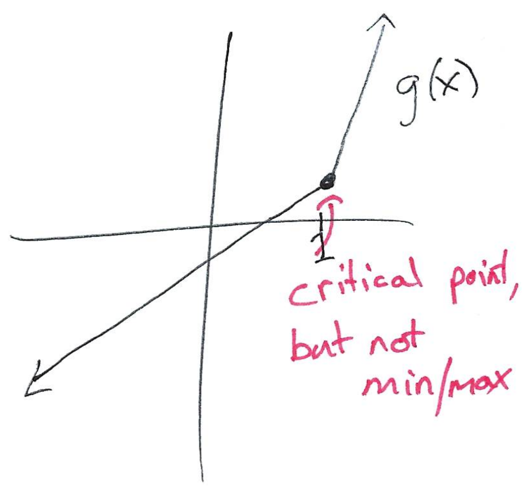
For example: $f(x) = x^3$



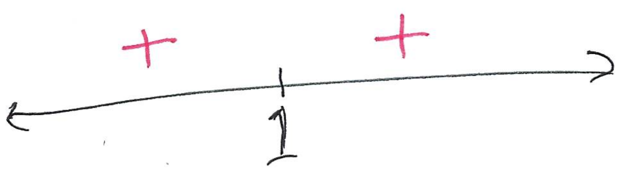
$f'(x) = 3x^2$



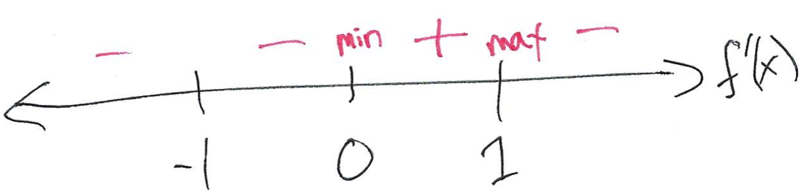
Second example:



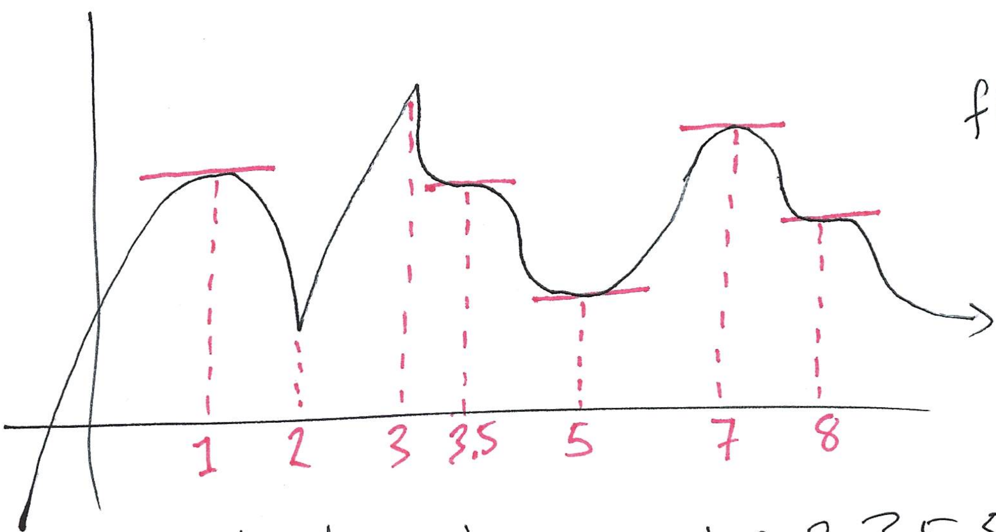
So the sign chart for $g'(x)$



If $f'(x)$ changes sign at a critical point, then there's a relative min/max at that point.



In this example, there's local min at $x=0$ and there's a local max at $x=1$



Critical points are 1, 2, 3, 3.5, 5, 7, 8

Relative mins are 2, 5

Relative maxs are 1, 3, 7

Ex What are the relative min/max of $x(x+1)^4$?

$$\frac{d}{dx} x\cdot(x+1)^4 = 1\cdot(x+1)^4 + x\left(\left(\boxed{x+1}\right)^4\right)'$$

$$= (x+1)^4 + x\,4(x+1)^3\cdot 1$$

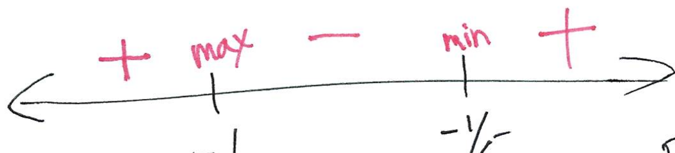
$$= (x+1)^4 + 4x(x+1)^3$$

$$= (x+1)^3\left((x+1)^1 + 4x\right)$$

$$= (x+1)^3(5x+1)$$

~~The deri~~ The critical points $x = -1, -1/5$

$$f(-1/2) = \left(\tfrac{1}{2}\right)^3\left(-\tfrac{5}{2}+1\right) < 0$$



Relative max at $x=-1$

Relative min at $x=-1/5$