Class notes:

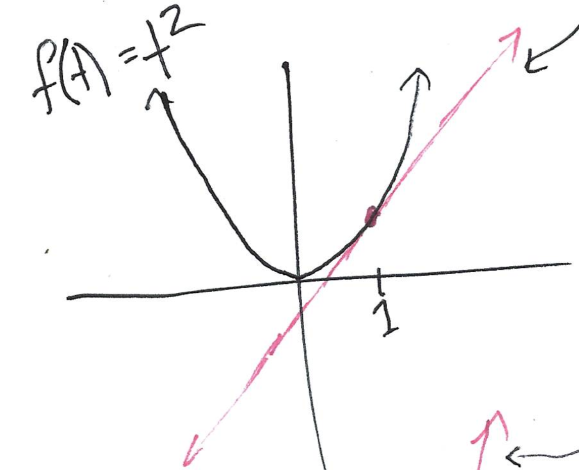
- No office hours today
- Homework today is due Monday
- Notes from yesterday and today will uploaded after class

The derivative

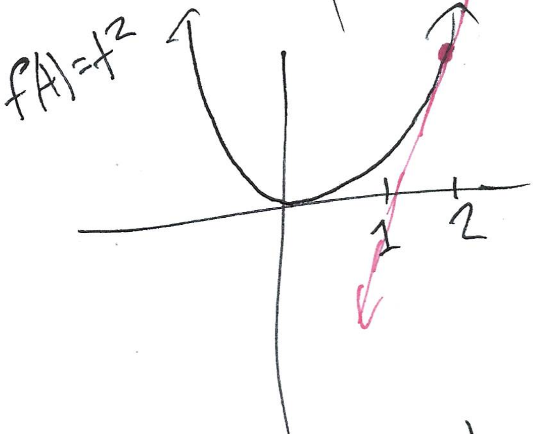
The derivative of a function f at x is a measurement of how the function is changing.

What if the function is a line?
Then the derivative is the slope.

We're going to extend the concept of slope. For any x in the domain of f, we want the line tangent to the graph at $(x, f(x))$. The slope of this line is our derivative of f at x.



this is the tangent line.
The slope of this line is
the derivative of t^2 at $t=1$.



the slope of this line is
the ~~derative~~ derivative at
$t=2$. This is larger than
the rate of change at $t=1$.

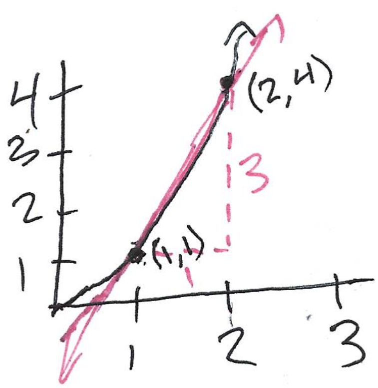
How to compute derivatives?

Using limits!

We'll use better and better approximations of the slope of the tangent line.

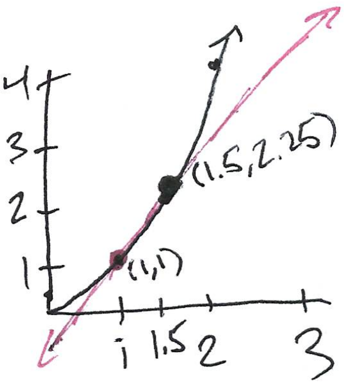
To find an approximation, we compute the slope ~~betwee~~ of the line between $(x, f(x))$ and $(x+h, f(x+h))$ where h is a small positive number.

The derivative of x^2 at $x=1$.



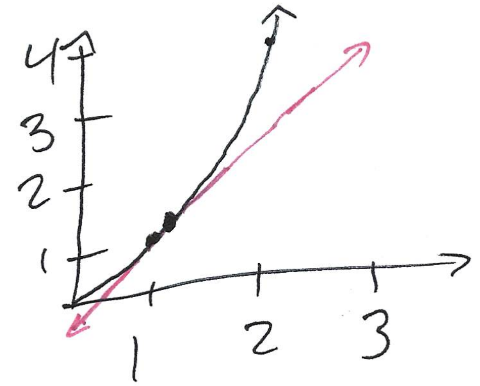
$h=1$

slope $= 3$



$h=.5$

slope $= \dfrac{1.25}{.5} = 2.5$



$h=.001$

slope $\approx$ the slope of the tangent line

Algebraically, we use the slopes of the line between $(x, f(x))$ and $(x+h, f(x+h))$.

$$\frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$$

Finally, we can define the derivative of f at x as $\boxed{\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}}$.

Derivative as a function

We will define a function $f'(x)$ (read as "f prime") which takes a number x and output the derivative of $f(x)$ at x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The domain of $f'(x)$ is all x such that the limit exists.

Two interpretations of the derivative:

1) The rate of change of the function.
2) The slope of the tangent line.

Interpretations in real-life

Physics

$p(t)$ = position at t

$p'(t)$ = change in position
= velocity

$p''(t)$ = change in velocity
= acceleration

Third derivatives are exceedingly rare.

Money

$f(t)$ = money in the bank

$f'(t)$ = change in money
= profit

$f''(t)$ = change in profit
= career growth

Inflation

$f(t)$ = Consumer Price Index

$f'(t)$ = change of average cost
= inflation

$f''(t)$ = change in inflation
(is inflation getting higher?)

Examples

1) $f(x) = x^2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{(\cancel{x^2}+2x\cancel{h}+h^{\cancel{2}}) - \cancel{x^2}}{\cancel{h}}$$

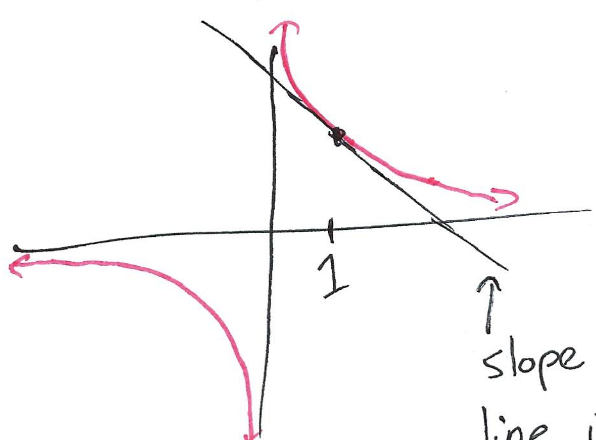
$$= \lim_{h\to 0} 2x + h = \boxed{2x}$$

2) $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)}$$

$$= \lim_{h\to 0} \frac{x - (x+h)}{hx(x+h)} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}x(x+h)}$$

$$= \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$



slope of this tangent line is $f'(1) = -1$.

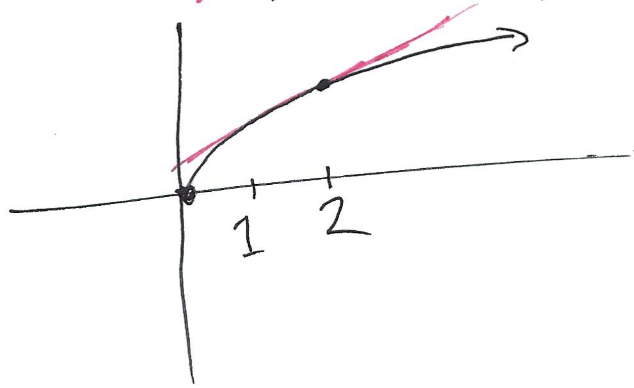
3)

$$f(x) = \sqrt{x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{\overset{F}{(\cancel{x}+h)} + \overset{O}{\cancel{\sqrt{x}\sqrt{x+h}}} - \overset{I}{\cancel{\sqrt{x}\sqrt{x+h}}} \overset{L}{\cancel{-x}}}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$



slope of tangent

= rate of change

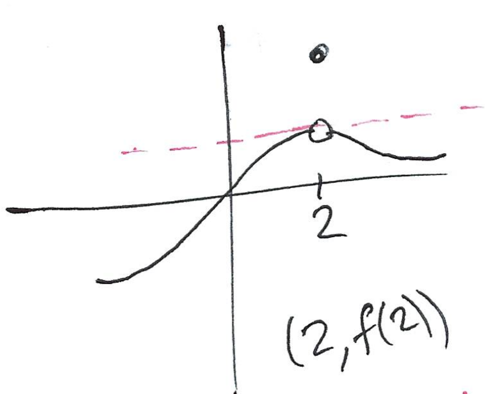
= derivative at 2

$= f'(2) = \frac{1}{2\sqrt{2}} \approx \frac{1}{3}$

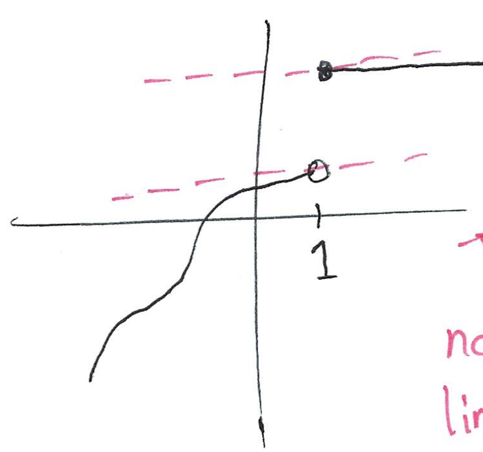
Failure to be differentiable

To be differentiable:

1) The function must be continuous at x



There's no tangent line at $x=2$.

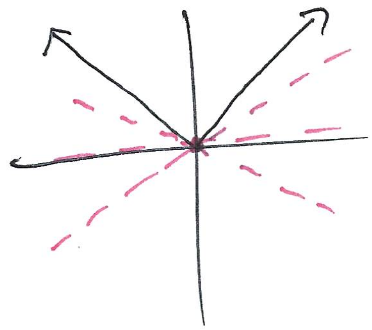


???

There's no tangent line at $x=1$.

2) No sharp ~~one~~ corner at x

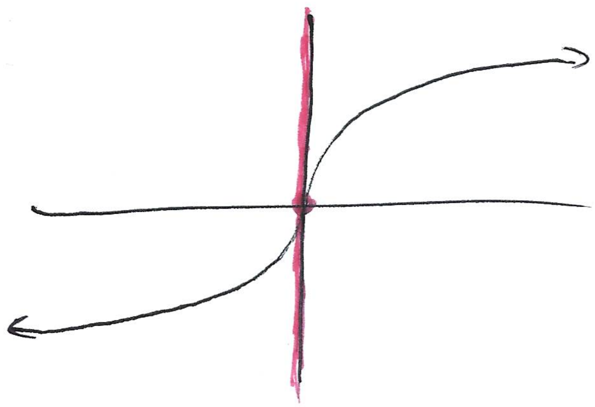
$f(x) = |x|$



There are ~~infinitely~~ too many tangent lines, derivative doesn't work at $x=0$.

3) Infinite slope

$f(x) = \sqrt[3]{x}$



Here $\sqrt[3]{x}$ is not differentiable at $x=0$, because slope of tangent line is undefined.

Notes ~~fo~~ on HW3:

#4) The tangent line goes through the point $(1, f(1)) = (1,6)$.