The Chain Rule

This is one of the more difficult topics. We have rules for addition, subtraction, powers, product, quotients, so what else is there?

Answer: Function Composition

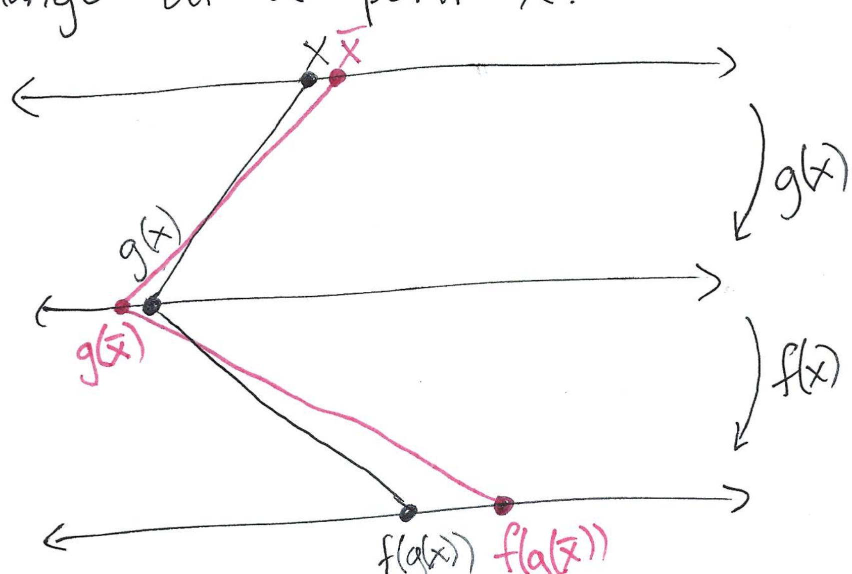
Function composition is essentially "chaining" functions together, hence the name. Recall ~~that~~ the notation:

$$(f \circ g)(x) = f(g(x))$$

How does the derivative act on function composition?

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) \quad \text{[The Chain Rule]}$$

Intuition: Recall the derivative measures the rate of change at a point x.



The changes at each step "compound", the first step differs by $g'(x)$, the second step differs by $f'(g(x))$

$f'(g(x)) \cdot g'(x)$

This rule is the most powerful, but also the hardest to apply. For example, consider $p(x) = (x^2 - x + 1)^{100}$. This is essentially impossible to expand.

However, we can use the chain rule! We $f(x)$ and $g(x)$ such that $p(x) = f(g(x))$.

$$f(x) = x^{100} \qquad f'(x) = 100x^{99}$$
$$g(x) = x^2 - x + 1 \qquad g'(x) = 2x - 1$$

We see that $f(g(x)) = f(x^2 - x + 1) = (x^2 - x + 1)^{100}$

The chain rule says:

$$\begin{aligned} p'(x) = \frac{d}{dx}\left[f(g(x))\right] &= f'(g(x)) \cdot g'(x) \\ &= f'(x^2 - x + 1)(2x - 1) \\ &= \boxed{100(x^2 - x + 1)^{99}(2x - 1)} \end{aligned}$$

It's not as straight-forward as the product/quotient because you need to identify f and g. The general rule is ~~the "inside" function~~ to identify an "inside" expression and a "outside"/"surrounding" expression.

In the previous problem,

$$\underbrace{(x^2 - x + 1)}_{\text{inside expression}}{}^{100} \Big\} \text{ outside expression.}$$

Ex 2) $s(x) = \sqrt{\underbrace{x^2+1}_{\text{inside expression}}} \Big\}$ surrounding expression

$f(x) = \sqrt{x}$

$g(x) = x^2+1$

$f(g(x)) = f(x^2+1) = \sqrt{x^2+1}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$$s'(x) = f'(g(x)) \cdot g'(x)$$
$$= f'(x^2+1) \cdot 2x$$
$$= \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+1}}$$

Ex 3) $f(x) = x \cdot (2x+3)^5$

$$f'(x) = (x)'(2x+3)^5 + x\left[(2x+3)^5\right]'$$
$$= (2x+3)^5 + x\left(10(2x+3)^4\right)$$
$$= (2x+3)^5 + 10x(2x+3)^4$$

$h(x) = x^5$

$g(x) = 2x+3$

$(2x+3)^5 = h(g(x))$

$\left[(2x+3)^5\right]' = h'(g(x)) \cdot g'(x)$

$= 5(2x+3)^4 \cdot 2$

Ex 1 (again))

$$(\underline{x^2-x+1})^{100}$$

$$100(x^2-x+1)^{99}(2x-1)$$

Ex 4) $(x^2-20)\cdot\sqrt{5x+1}$

$\left(\quad\right)^{1/2}$

$(2x)\sqrt{5x+1} + (x^2-20)\left(\sqrt{5x+1}\right)'$

$2x\sqrt{5x+1} + (x^2-20)\,\frac{1}{2}(5x+1)^{-1/2}\cdot(5x+1)'$

$2x\sqrt{5x+1} + \frac{5}{2}(x^2-20)(5x+1)^{-1/2}$

Ex 5) $$\frac{1}{\sqrt{x^2+1}} = \frac{\cancel{\sqrt{x^2+1}\,(1)'} - (1)\left(\sqrt{x^2+1}\right)'}{\left(\sqrt{x^2+1}\right)^2}$$

$$= \frac{-\left(\frac{1}{\cancel{2}}(x^2+1)^{-1/2}\cdot\cancel{2}x\right)}{x^2+1}$$

$$= \frac{-x}{(x^2+1)^{3/2}}$$

Ex 6) $(x^2+9)\cdot\sqrt{4+\frac{1}{x^2}}$

$2x\sqrt{4+\frac{1}{x^2}} + (x^2+9)\left(\sqrt{4+\frac{1}{x^2}}\right)'$

$2x\sqrt{4+\frac{1}{x^2}} + (x^2+9)\,\frac{1}{2}\left(4+\frac{1}{x^2}\right)^{-1/2}\cdot\left(4+\frac{1}{x^2}\right)'$

$$2x\sqrt{4+\frac{1}{x^2}} + \frac{x^2+9}{\cancel{2}\sqrt{4+\frac{1}{x^2}}}\left(-\cancel{2}x^{-3}\right)$$

$$2x\sqrt{4+\frac{1}{x^2}} + \frac{-(x^2+9)}{x^3\sqrt{4+\frac{1}{x^2}}}$$

Ex 7) $\sqrt{\sqrt{x-1}-1}$

$$\frac{1}{2}\left(\sqrt{x-1}-1\right)^{-1/2}\left(\sqrt{x-1}-1\right)'$$

$$\frac{1}{2\sqrt{\sqrt{x-1}-1}}\cdot\left(\frac{1}{2}(x-1)^{-1/2}\cancel{(x-1)'}\right)$$

$$\boxed{\frac{1}{4\sqrt{x-1}\,\sqrt{\sqrt{x-1}-1}}}$$

Ex 8) What is the derivative of $f(\sqrt{x})$?

Chain rule says:

$$\boxed{f'(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}}$$

Ex 9) What is $h'(0)$ if $h(x)=f(g(x))$ and

$f(0)=2$ $\qquad$ $g(0)=3$ $\qquad$?

$f'(3)=-10$ $\qquad$ $g'(0)=3$

$$h'(x)=f'(g(x))\cdot g'(x)$$

$$h'(0)=f'(g(0))g'(0)$$

$$=f'(3)\cdot 3=\boxed{-30}$$

Real-life:

$f(t)$ = wages paid to workers at year t

$g(w)$ = the total cost of workforce if we pay \$$w$ of wages

$g(f(t))$ = total cost of workforce at year t.

Real-life:

$y(d)$ = yield of apples per tree if the trees are planted d feet apart

$p(y)$ = profits per tree given the average yield per tree

$f(d)$ = ~~the numbers we can~~ the number of trees we can plant given that we plant d feet apart

Total profit = # of trees · profit per tree

$$f(d) \cdot p(y(d))$$

Note: Problem (f) requires the chain rule twice.