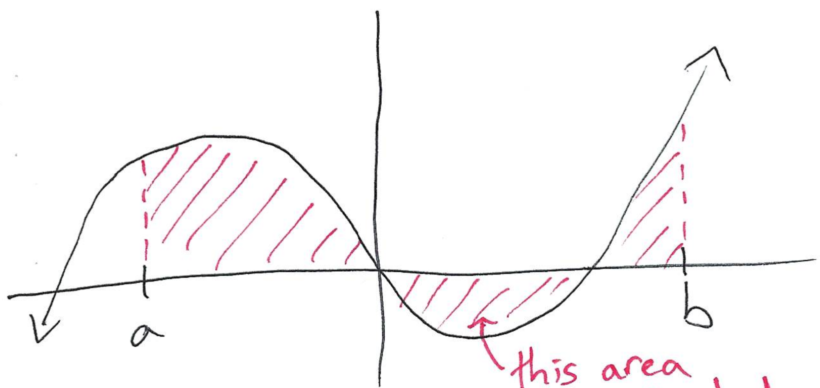
Definite Integral

Given a function $f(x)$, we wish to find the signed area between the graph of f and x-axis. Specifically, given a and b, we want the area of the shaded region:



This is the definite integral, denoted as $\int_a^b f(x)\,dx$.

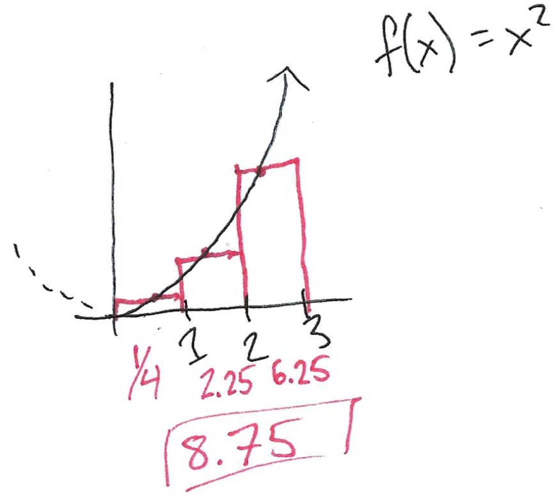
How to calculate?

Method 1: For geometrically simple functions, we can calculate the area using geometric formulas

Triangle: $\frac{b \cdot h}{2}$ Circles: πr^2

Method 2: You approximate the area using rectangles

Ex



$\int_1^3 x^2\,dx$ using rectangles

we can approximate the area under x^2

As a better approximation you could use more rectangles, say with base ½.

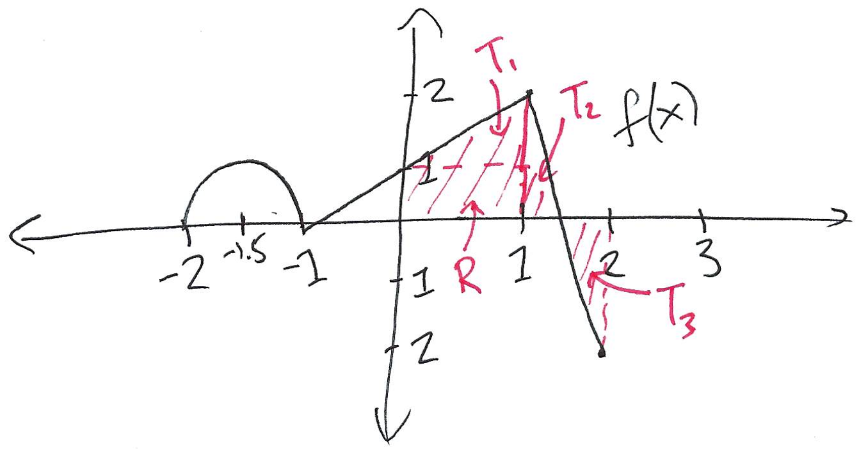
You can always repeat the process with more rectangles to get a better approximation.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\frac{1}{n}\sum_{k=1}^{n-1} f\left(a+\frac{b-a}{n}k\right)}.$$

This is purely formal,
no way to actually calculate
using this limit.

Example

Example using geometry



$$\int_0^2 f(x)\,dx = \underbrace{1\cdot 1}_{R} + \underbrace{\frac{1\cdot 1}{2}}_{T_1} + \underbrace{\frac{2\cdot\frac{1}{2}}{2}}_{T_2} - \underbrace{\frac{2\cdot\frac{1}{2}}{2}}_{T_3}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \boxed{1.5}$$

$$\int_{-2}^{0} f(x)\,dx = \text{Area of semicircle} + \text{Area of small triangle}$$

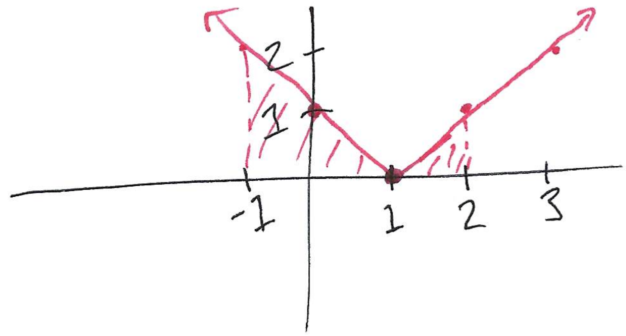
$$= \frac{\pi\left(\frac{1}{2}\right)^2}{2} + \frac{1\cdot 1}{2} = \boxed{\frac{\pi}{8} + \frac{1}{2}}$$

$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx = \frac{\pi}{8} + \frac{1}{2} + \frac{3}{2}$$

$$= \boxed{\frac{\pi}{8} + 2}$$

Ex 2)

$$\int_{-1}^{2} |x-1|\, dx$$



$= \text{Area of bigger Triangle} + \text{Area of smaller triangle}$

$$= \frac{2 \cdot 2}{2} + \frac{1 \cdot 1}{2} = 2 + \frac{1}{2} = \boxed{2.5}$$

Ex 3)

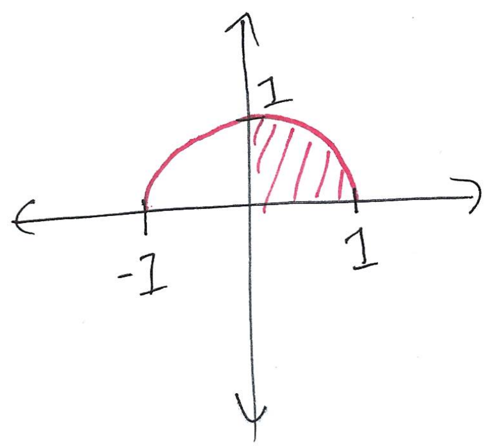
$$\int_{0}^{1} \sqrt{1-x^2}\, dx$$

$y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1$

Formula for a circle



$= \text{Area of quarter-circle}$

$$= \frac{\pi(1)^2}{4} = \boxed{\frac{\pi}{4}}$$

The Fundamental Theorem of Calculus.

Indefinite Integrals = Antiderivatives = $\int f(x)\,dx$

Definite Integrals = Signed area under the graph = $\int_a^b f(x)\,dx$

The Fundamental ~~Calculus~~ Theorem of Calculus (FTC):

$$\int_a^b f(x)\,dx = F(b) - F(a)$$ where $F(x)$ is any antiderivative of $f(x)$.

$$= \boxed{F(x)\Big|_a^b}$$

$F'(x) = f(x)$.

new notation for $F(b) - F(a)$

Ex 1) What is the area under curve of $f(x) = x^2$ over the interval $[0, 1]$?

$$\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1^3}{3} - \frac{0^3}{3} = \boxed{\frac{1}{3}}$$

$\int x^2\,dx = \frac{x^3}{3} + C$, So $F(x) = \frac{x^3}{3}$ is an antiderivative

Ex 2

$$\int_{1}^{3} x\,dx = \frac{x^2}{2}\Big|_{x=1}^{3} = \frac{3^2}{2} - \frac{1^2}{2}$$

↑ ignore these and do indefinite integral

$$= \frac{9}{2} - \frac{1}{2} = \boxed{4}$$

Ex 3

$$\int_1^3 x^2 + e^x\,dx = \frac{x^3}{3} + e^x \Big|_1^3$$

$$= \left(\frac{3^3}{3} + e^3\right) - \left(\frac{1^3}{3} + e^1\right)$$

$$= \boxed{9 + e^3 - \frac{1}{3} - e}$$

Ex 4

$$\int_1^e \frac{x+1}{x}\,dx = \int_1^e \cancel{\frac{x}{x}}^{1} + \frac{1}{x}\,dx$$

$$= x + \ln(x)\Big|_1^e = (e + \ln(e)) - (1 + \ln(1))$$

$$= e + 1 - 1 - 0$$

$$= \boxed{e}$$

Ex 5

$$\int_1^2 \frac{1}{x} - \frac{1}{x^2}\,dx = \ln(x) + \frac{1}{x}\Big|_{x=1}^{2} = \left(\ln(2) + \frac{1}{2}\right) - (\cancel{\ln(1)} + 1)$$

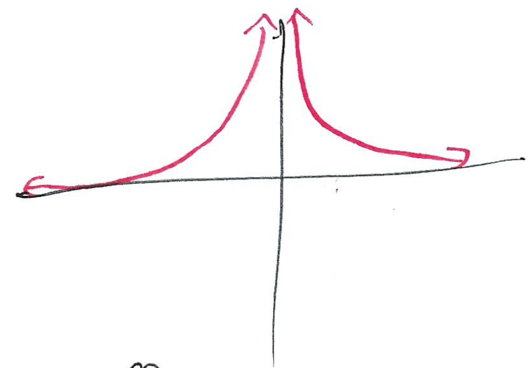
$$= \boxed{\ln(2) - \frac{1}{2}}$$

Non-example

$$\cancel{\int_{-2}^{1} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-2}^{1} = (-1)-\left(-\frac{1}{-2}\right)}$$

$$= -1 - \frac{1}{2} = -1.5$$

$\frac{1}{x^2}$ is not defined at $x=0$ (in fact, it blows up at $x=0$). Based on the graph, the result we got from using FTC makes no sense; this is because $\frac{1}{x^2}$ is not defined at $x=0$.



This is an important caveat to be aware of!

Another use of the FTC:

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$ [Net Change Formula]

$f(t)$ is an antiderivative