Course Info | HW3 is available and due tomorrow night
Solutions to worksheet 0 under Resources
Notes under Resources

Limits

<u>Limits</u> are a way of finding a value by taking better and better approximations.

Suppose you're driving a car and your position is modeled by $p(t) = 2t^2$ over the time $[0, 2]$, where is time in seconds and $p(t)$ is in ft/sec. What is our speed at time $t = 1$ sec?

We know how to find average speed $\frac{\text{distance covered}}{\text{time taken}}$
we'll use to approximate our <u>instanteous</u> speed.

We take average speed over smaller and smaller time intervals:

Over $[1,2]$: $\frac{p(2)-p(1)}{2-1} = \frac{2(2)^2 - 2(1)^2}{1} = 6$ ft/sec

Over $[1, 1.5]$: $\frac{p(1.5)-p(1)}{1.5-1} = \frac{2(3/2)^2 - 2(1)^2}{.5}$

$= \left(2\left(\frac{9}{4}\right) - 2\right) \cdot 2 = 4 \cdot \frac{9}{4} - 4 = 5$ ft/sec

Using a computer:

Over $[1, 1.1]$, the average speed is 4.2 ft/sec

$[1, 1.01]$, 4.01 ft/sec

$[1, 1.000001]$ 4.00000201 ft/sec

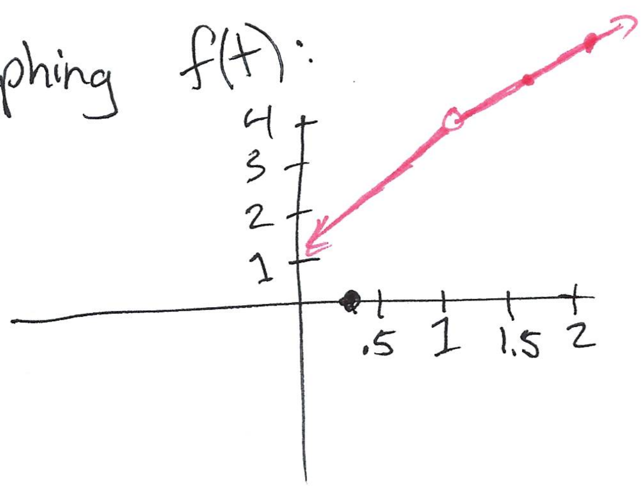
Intuitively, we see that approximations are approaching 4 ft/s.

We can turn this process into a function $f(t)$.

$f(t)$ = finds the average speed over $[1, t]$ $= \dfrac{p(t) - p(1)}{t - 1}$

$= \dfrac{2t^2 - 2}{t - 1}$

Graphing $f(t)$:



$f(1) = \dfrac{2 - 2}{1 - 1} = \cancel{\dfrac{0}{0}}$

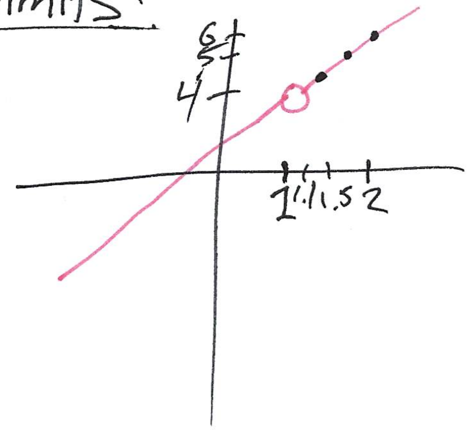
$$\frac{2t^2 - 2}{t - 1} = \frac{2(t^2 - 1)}{t - 1} = \frac{2\cancel{(t-1)}(t+1)}{\cancel{t-1}} = 2t + 2$$

plug-in 1

$2(1) + 2 = 4$ ft/sec

Answer: Our instanteous at $t = 1$ is 4 ft/sec.

How do we actually find limits?



$$\lim_{t\to 1} f(t) = \lim_{t\to 1} \frac{2t^2-2}{t-1}$$

$$= \lim_{t\to 1} \frac{2\cancel{(t-1)}(t+1)}{\cancel{t-1}}$$

$$= \lim_{t\to 1} 2(t+1) = 2(1+1) = 4$$

$$\lim_{x\to 5} \frac{x^2-25}{x-5} = \lim_{x\to 5} \frac{(x+5)\cancel{(x-5)}}{\cancel{x-5}} = \lim_{x\to 5} x+5 = 10$$

$$\lim_{x\to 2} x^2-x-5 = 2^2-2-5 = -3$$

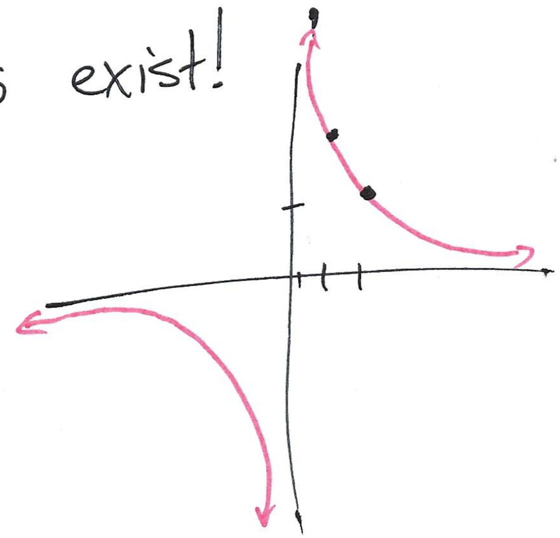
$$\lim_{x\to -2} \frac{x^4-16}{2x^4+x^5} = \lim_{x\to -2} \frac{(x^2-4)(x^2+4)}{x^4(x+2)} = \lim_{x\to -2} \frac{\cancel{(x+2)}(x-2)(x^2+4)}{x^4\cancel{(x+2)}}$$

$$= \lim_{x\to -2} \frac{(x-2)(x^2+4)}{x^4}$$

$$= \frac{-4(4+4)}{(-2)^4}$$

$$= \frac{-32}{16} = \boxed{-2}$$

However, limits do not always exist!



$f(x) = \frac{1}{x}$ $\qquad \lim_{x \to 0} \frac{1}{x}$

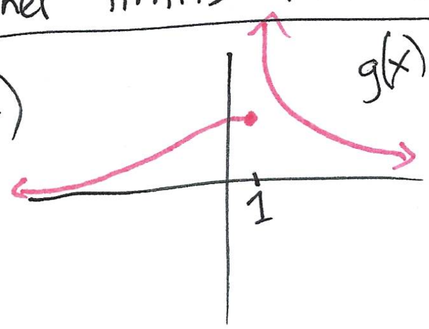
Plug in numbers closer and closer to 0:

x	$\frac{1}{x}$
1	1
1/2	2
1/10	10
1/1000000	1,000,000

$$\lim_{x \to 0} \frac{1}{x} = \text{Does not exist} \quad \underline{\text{DNE}}$$

<u>Other limits that don't exist:</u>

Ex 1)



$$\lim_{x \to 1} g(x) = \text{DNE}$$

Ex 2) $h(x) = \frac{x+12}{x-2}$ $\qquad \lim_{x \to 2} \frac{x+12}{x-2}$

@ $x=3$: $\frac{15}{1}$

@ $x=2.1$: $\frac{14.1}{.1}$

$= 141$

Algebra of limits

$f(x)$ and $g(x)$ are functions

$$\lim_{x\to 1} f(x)+g(x) = \left[\lim_{x\to 1} f(x)\right] + \left[\lim_{x\to 1} g(x)\right]$$

$$\lim_{x\to 1} f(x)\cdot g(x) = \left[\lim_{x\to 1} f(x)\right]\left[\lim_{x\to 1} g(x)\right]$$

$$\lim_{x\to 1} \sqrt{f(x)} = \sqrt{\lim_{x\to 1} f(x)}$$

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x)}$$ as long as $\lim_{x\to 1} g(x) \neq 0$

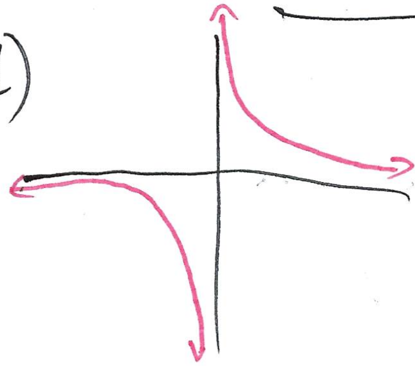
$$\lim_{x\to 1} \frac{f(x)+\sqrt{g(x)}}{g(x)}$$

we know that

$\lim_{x\to 1} f(x) = 1$, $\lim_{x\to 1} g(x) = 4$

$$= \frac{\left[\lim_{x\to 1} f(x)\right] + \sqrt{\lim_{x\to 1} g(x)}}{\lim_{x\to 1} g(x)} = \frac{1+\sqrt{4}}{4} = \frac{3}{4}.$$

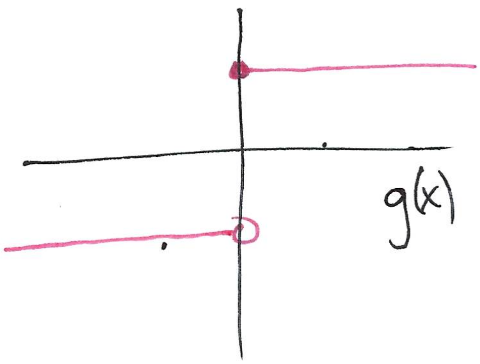
Another type of limit: Limits to infinity

Ex1) 

$f(x) = \frac{1}{x}$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

x	$\frac{1}{x}$
10	$\frac{1}{10}$
10,000	$\frac{1}{10,000}$
$\vdots$	$\vdots$
∞	0

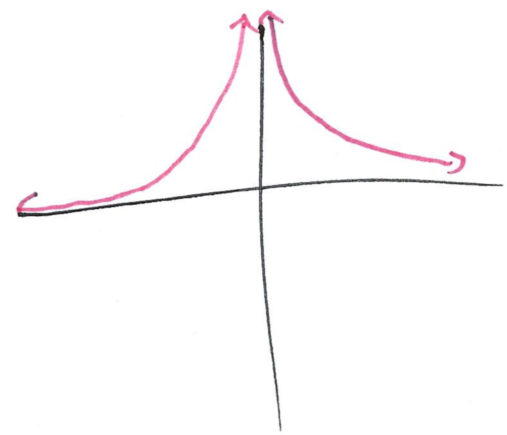
Ex 2) 

$$\lim_{x\to\infty} g(x) = 1$$

$$\lim_{x\to-\infty} g(x) = -1$$

General fact:

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$$

$\frac{1}{x^2}$ 

Ex1) $$\lim_{x\to\infty} \frac{1x^5 + 4x^3 - x + 7}{7x^5 - 8x^2 + 1,000} \frac{(1/x^5)}{(1/x^5)}$$

$$= \lim_{x\to\infty} \frac{1 + \cancel{4/x^2}^{\,0} - \cancel{1/x^4}^{\,0} + \cancel{7/x^5}^{\,0}}{7 - \cancel{8/x^3}_{\,0} + \cancel{1000/x^5}_{\,0}} = \lim_{x\to\infty} \frac{1}{7} = \frac{1}{7}$$

Ex 2)

$$\lim_{x \to \infty} \frac{x^2+3}{x-1} \cdot \frac{(1/x)}{(1/x)} = \lim_{x \to \infty} \frac{x + 3/x}{1 - 1/x}$$

$$= \lim_{x \to \infty} x = \infty$$

Ex 3)

$$\lim_{x \to \infty} \frac{x-1}{x^2+3} \frac{(1/x^2)}{(1/x^2)} = \lim_{x \to \infty} \frac{\cancelto{0}{1/x} - \cancelto{0}{1/x^2}}{1 + \cancelto{0}{3/x^2}}$$

$$= \lim_{x \to \infty} \frac{0}{1} = 0$$

Pattern:

Look at the highest power in the numerator and the denominator

1) If power in numerator is larger than the power in the denominator, then the limit is ∞ (DNE).

2) If the power in the denominator is bigger, then the limit 0.

3) If equal, divide the numbers in front of the highest power.

$$\lim_{x \to \infty} \frac{25x^{100} - x^{50} + 7}{3x^{100} + x^{25} - x^8} = \frac{25}{3}$$

$C(x)$ = cost of making a TV
given that you're making x TV's

$$C(x) = 25 + \frac{x^2 + 1000}{x^3}$$

$$C(1) = 25 + 1{,}001 = \$1{,}026$$

$$C(1{,}000) = 25 + \frac{1{,}000^2 + 1000}{1{,}000^3}$$

$$\approx 25$$

What is the cost of making a TV
as the number of TV's you make go to infinity.

$$P(x) = \text{population of people at time } x = \frac{10{,}000{,}000{,}000\, x^2 + x}{x^2}$$

What is the long-term population?

$$\lim_{x \to \infty} P(x) = 10{,}000{,}000{,}000$$