Class Notes:

- HW and worksheet for today are available
- My plan to complete the HW and worksheets for the week.

Differentiation Rules

We want to do less algebra and fewer limits,

We can write $f'(x)$ as $\frac{d}{dx} f(x)$. $\frac{d}{dx} x^2$

1) If $f(x) = c$ is a constant function, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

So the derivative is zero.

In our new notation:

some number

$$\frac{d}{dx} c = 0.$$

2) "Plays nice" with addition/subtraction

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

$$\frac{d}{dx}(f(x)+g(x)) = \lim_{h\to 0} \frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}$$

$$= \left(\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right) + \left(\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right)$$

$$= \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

3) $\frac{d}{dx}[c \cdot f(x)) = c \cdot \left(\frac{d}{dx} f(x)\right)$

$$\frac{d}{dx} 5x = 5\left(\frac{d}{dx} x\right)$$

4) Finally, there's the power rule

$$\boxed{\frac{d}{dx} x^n = nx^{n-1}}$$

$$\frac{d}{dx} x = 1$$

$$\frac{d}{dx} x^2 = 2x^1$$

Doable $\frac{d}{dx} x^3 = 3x^2$

$$\frac{d}{dx} x^4 = 4x^3$$

$$\vdots$$

$$\frac{d}{dx} x^n = nx^{n-1}$$

Note: This works for any n, not just integers.

$$\frac{d}{dx} \frac{1}{x} = \frac{d}{dx} x^{-1} = (-1)x^{-2}$$

$$= -\frac{1}{x^2}$$

Examples

1) Easily differentiate polynomials.

$$\frac{d}{dx}\left(x^3+2x+5\right) = \frac{d}{dx}x^3 + \frac{d}{dx}2x + \frac{d}{dx}5$$

$$= 3x^2 + 2\frac{d}{dx}x^1 + 0$$

$$= 3x^2 + 2(1)x^0 = 3x^2+2$$

2) $$\frac{d}{dx}\left(x^5+3x^2-2\right) = 5x^4 + 3\cdot 2x^1 + 0$$

$$= 5x^4 + 6x$$

3) $$\frac{d}{dx}\frac{1}{\sqrt{x}} = \frac{d}{dx}x^{-1/2} = -\frac{1}{2}x^{-3/2}$$

$$= \boxed{-\frac{1}{2x^{3/2}}} = -\frac{1}{2(\sqrt[3]{x})}$$

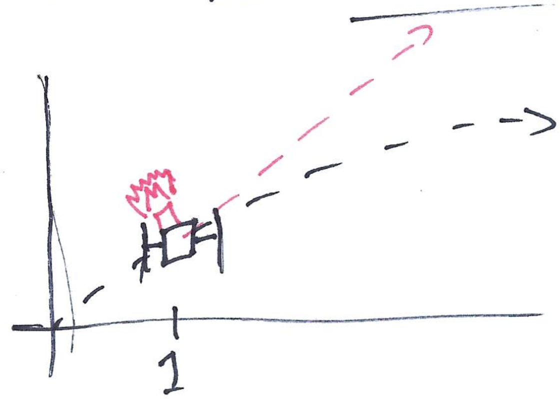
4) $$\frac{d}{dx}\left[x^{-7}+x^{-11}+\sqrt{x}\right]$$

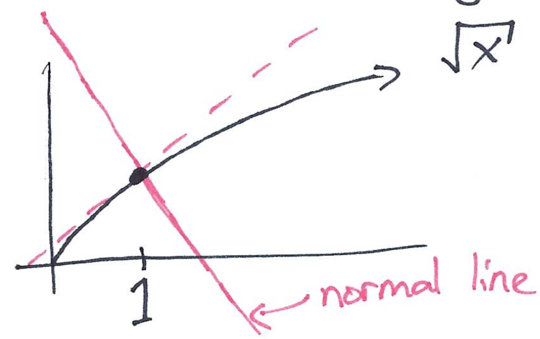
$$= -7x^{-8} + (-11)x^{-12} + \frac{1}{2}x^{-1/2}$$

$$= -\frac{7}{x^8} - \frac{11}{x^{12}} + \frac{1}{2\sqrt{x}}$$

Test

5) Find the line perpendicular to the tangent line of $\sqrt{x}$ at $x=1$, going through the point of tangency. This is the normal line.





Tangent line: $f(x)=\sqrt{x}$ $\quad f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$

$y=\frac{1}{2}x+b$ $\quad f'(1)=\frac{1}{2}$

$1=\frac{1}{2}(1)+b$

$\frac{1}{2}=b$ $\quad y=\frac{1}{2}x+\frac{1}{2}$

Normal line: $y=-2x+b$ $\quad 1=-2(1)+b$

$3=b$

$$\boxed{y=-2x+3}$$

The Product Rule

Recall $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$. However, this is not the case with products.

$$\cancel{\frac{d}{dx}[f(x)g(x)] = \left[\frac{d}{dx}f(x)\right]\left[\frac{d}{dx}g(x)\right]}$$

The true rule is:

$$\frac{d}{dx}[f(x)\,g(x)] = \left[\frac{d}{dx}f(x)\right]\cdot g(x) + f(x)\cdot\left[\frac{d}{dx}g(x)\right]$$

Examples

1) $p(x) = \overbrace{(x^2+x+3)}^{f(x)}\overbrace{(x^7+4)}^{g(x)}$

$$p'(x) = \underbrace{(2x+1)}_{f'(x)}\underbrace{(x^7+4)}_{g(x)} + \underbrace{(x^2+x+3)}_{f(x)}\underbrace{(7x^6+0)}_{g'(x)} \checkmark$$

$$= 2x^8 + 8x + x^7 + 4 + 7x^8 + 7x^7 + 21x^6$$

$$= 9x^8 + 8x^7 + 21x^6 + 8x + 4$$

2) $f(x) = x^3(\sqrt{x} + 8)$

$$\left.\begin{aligned} f'(x) &= 3x^2(\sqrt{x}+8) + x^3\left(\tfrac{1}{2}x^{-1/2} + 0\right) \\ &= 3x^{2.5} + 24x^2 + \tfrac{1}{2}x^{2.5} \\ &= 3.5x^{2.5} + 24x^2 \end{aligned}\right\} \text{Using product rule.}$$

$$\left.\begin{aligned} f(x) &= x^{3.5} + 8x^3 \\ f'(x) &= 3.5x^{2.5} + 24x^2 \end{aligned}\right\} \text{Expanding and then power rule}$$

The Quotient Rule

Similar to the product, it is <u>not</u> the case that

$$\cancel{\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}}.$$

For quotients, we have following rule:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{\overset{\text{Lo}}{g(x)}\overset{\text{dHi}}{f'(x)} - \overset{\text{Hi}}{f(x)}\overset{\text{dLow}}{g'(x)}}{g(x)^2}$$

Square the bottom, and away we go.

Examples

1) $p(x) = \dfrac{x}{2x-4}$

$$p'(x) = \frac{(2x-4)(1) - x(2)}{(2x-4)^2}$$

$$= \frac{2x - 4 - 2x}{(2x-4)^2} = \boxed{\frac{-4}{(2x-4)^2}}$$

2) $q(x) = \dfrac{x^2-1}{x^2+1}$

$$q'(x) = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$$

$$= \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}$$

$$= \frac{4x}{(x^2+1)^2}$$

3) $$\frac{d}{dx}\frac{\sqrt{x}}{x^2+1} = \frac{(x^2+1)(\sqrt{x})' - \sqrt{x}\,(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{(x^2+1)\frac{1}{2\sqrt{x}} - \sqrt{x}\,(2x)}{(x^2+1)^2} \cdot \frac{2\sqrt{x}}{2\sqrt{x}}$$

$$= \frac{(x^2+1) - 2x(2x)}{2\sqrt{x}\,(x^2+1)^2}$$

$$= \boxed{\frac{-3x^2+1}{2\sqrt{x}\,(x^2+1)^2}}$$

Note: For Ex 6, there multiple possible answers.