Centro Federal de Educação Tecnológica de Minas Gerais

Postgraduate Program in Mathematical and Computational Modeling

Complex Systems

Sayama's Textbook Solutions

Breno Martins da Costa Corrêa e Souza (breno.ec@gmail.com)
Stundent

Allbens Atman Picardi Faria (atman@dppg.cefetmg.br)
Teacher

Course
Complex Systems Modeling
Mondays and Wednesdays, 10:40 a.m. to 12:20 p.m.

Av. Amazonas, 7675 - Nova Gameleira, Belo Horizonte - MG, 30510-000
Building 07

Belo Horizonte, 2016

Objective

The goal of this document is to contain solutions to proposed exercises in Sayama's textbook[1] concerning the modeling and analysis of complex systems.

Methods and Materials

Almost all source code is written in Python (`*.py` files), using the environment and methodology presented in the textbook.

Some data is generated by Standart C++ programs (`*.cpp` files). The data is saved in a text file and then analysed with Python source code.

Appendices

All source related to this document can be found at `https://github.com/brenoec/cefetmg.msc.sayama.solutions`.

Solutions

This section will list chapters of the textbook followed by its solutions. Any source code used in a solution will be referenced by its name. Source codes are under `/simulations` folder.

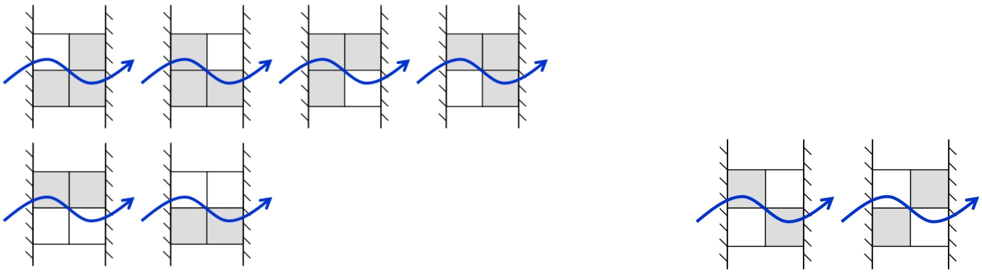
Chapter 12

Celluar Automata II: Analysis

Exercise 12.9

Estimate the critical percolation threshold for the same forest fire model but with von Neumann neighborhoods. Confirm the analytical result by conducting simulations.

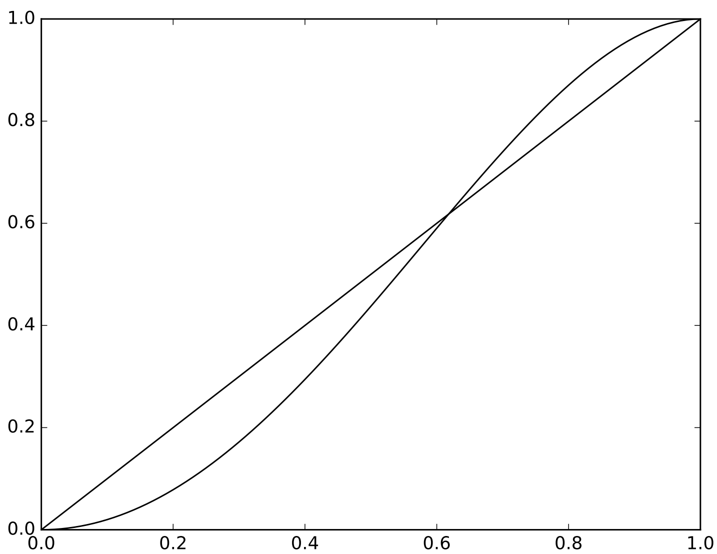
With von Neumann's neighborhood, the system supports percolation in the cases on the left side, and those on the right side are unsupported:



So that the critical percolation threshold is now given by:

$$p_c = {p_c}^4 + 4{p_c}^3(1 - p_c) + 2{p_c}^2(1 - p_c)^2, \tag{1}$$

and the cobweb plot for that equation is:



`12.9-cobweb-plot-for-von-neumann-neighborhood.py`

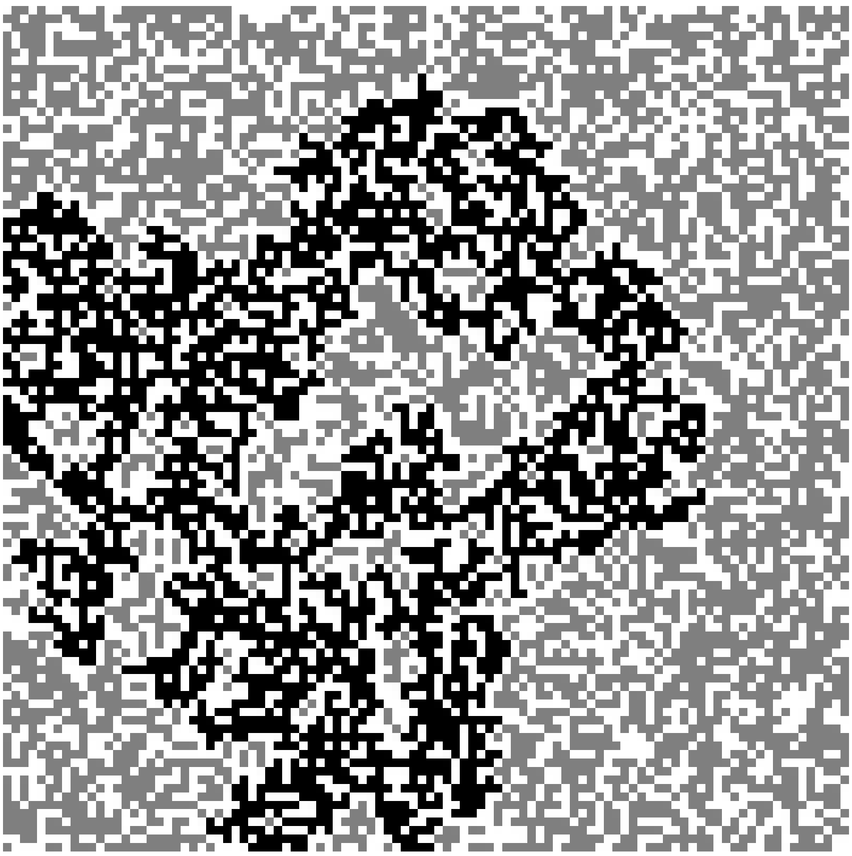
In which is easy to see two asymptotic states possible, $p_\infty = 0$ and $p_\infty = 1$ (the cobweb plot is about relations over scale, not about dynamics over time). There is an unstable equilibrium point around $p = 0.6$. The exact value is given by:

$$p_c = \frac{\sqrt{5}}{2} - \frac{1}{2} \approx 0.6180. \tag{2}$$

To verify this prediction, one could simulate the system a certain number of times for each values of p.

- $p > p_c$: systems should show percolation in most cases;
- $p < p_c$: systems should not show percolation in most cases;
- $p \approx p_c$: systems will show percolation in some cases.

The following figure is a simulation output for $n = 100$ and $p = 0.5875$:



12.9-von-neumann-fire-ca.py

Exercise 12.10

What would happen if the space of the forest fire propagation were 1-D or 3-D? Conduct the renormalization group analysis to see what happens in those cases.

One-diomensional case:

The probability of percolation for one cell is the probability of that cell assume value 1:

$$p_1 = q. \tag{3}$$

Expand our group to two cells, the percolation proccess will occur only if the two cells assume value 1:

$$p_2 = {p_1}^2. \tag{4}$$

In fact, we can generalize the previous equation to:

$$p_{s+1} = {p_s}^2, \tag{5}$$

and conclude that if one cell in our system assume 0, the propagation will be stopped.

Three-diomensional case:

Despite the three dimensions of cells, the probability of percolation for one cell is equal to previous case:

$$p_1 = q. \tag{6}$$

In this case is actually easier to list possibilities that don't allow percolation. The number of possible configurations for the renormalization group analysis made of $2 \times 2 \times 2$ cells is:

$$S = 256, \qquad S = k^{L^D}, \quad k = L = 2, \quad D = 3. \tag{7}$$

Since cells have Moore's neighborhood, any cell is neighbor to the other cells. Imagine the renormalization group as two 2×2 planes, and the percolation proccess from one plane to another. Possibilities that cells are all in one of these two planes prevents the percolation process. To list then:

1 from empty cells;
2 from 4 cells filled;
8 from 3 cells filled;
8 from 1 cell filled;
12 from 2 cells filled;

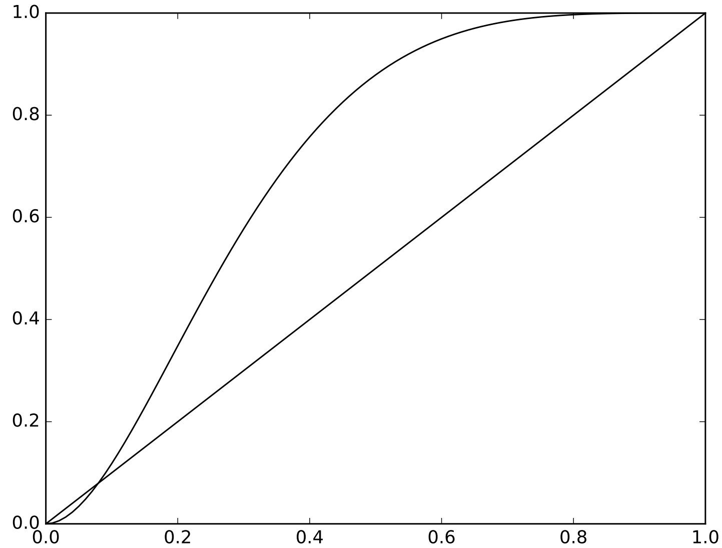
A total of 31 possibilities, that leads to the relation of possibilities in which we have percolation proccess and total number of possibilities:

$$\frac{225}{256} \tag{8}$$

The critical percolation threshold is given by:

$$
\begin{aligned}
p_c = 1 - &\\
& 1 \times (1-p_c)^8 - \\
& 2 \times {p_c}^4 (1-p_c)^4 - \\
& 8 \times {p_c}^3 (1-p_c)^5 - \\
& 8 \times p_c (1-p_c)^7 - \\
& 12 \times {p_c}^2 (1-p_c)^6 .
\end{aligned}
\tag{9}
$$

And the cobweb plot for that equation is:



`12.10-cobweb-plot-for-3-dimensional.py`

In which is easy to see two asymptotic states possible, $p_\infty = 0$ and $p_\infty = 1$ (the cobweb plot is about relations over scale, not about dynamics over time). There is an unstable equilibrium point around $p = 0.1$. The approximate value is:

$$p \approx 0.0794325. \tag{10}$$

Comparnison of cases:

The relation of possibilities in which we have percolation proccess and total number of possibilities for the 1-D, 2-D and 3-D are, respectively:

$$\frac{1}{4} < \frac{9}{16} < \frac{225}{256}. \tag{11}$$

For one-dimensional systems, the critical percolation threshold is 100%, since all cells must assume 1 to percolation to occur. For two-dimensional, the threshold is 38%. Finally, for three-dimensional systems, the threshold is under 8%.

From this we can conclude that as we increase the number of dimensions, we increase the susceptibility to percolation.

Chapter 16

Dynamical Networks I: Modeling

Exercise 16.1

Revise the majority rule dynamical network model developed above so that each node stochastically flips its state with some probability. Then simulate the model and see how its behavior is affected.

References

[1] Hiroki Sayama. *Introduction to the Modeling and Analysis of Complex Systems*. Binghamton University, Suny, 1st edition, 2015.