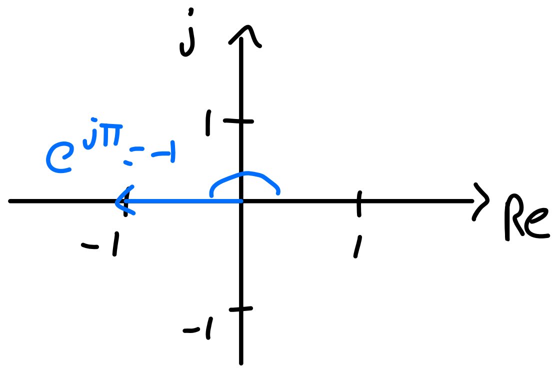
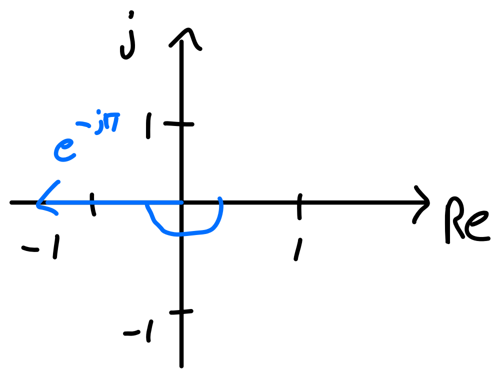
1.1 Express following complex numbers in cartesian form $(x + jy)$:

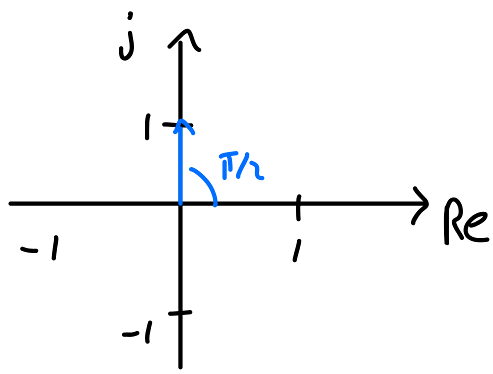
$\frac{1}{2} e^{j\pi} = \frac{1}{2}(-1) = -\frac{1}{2}$



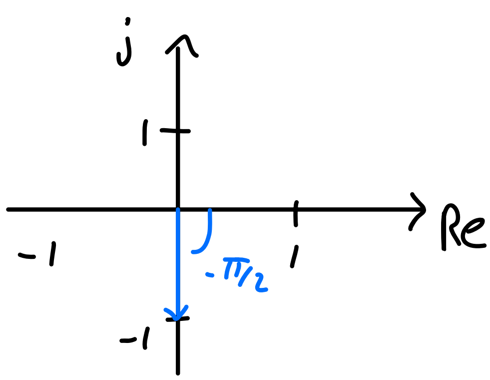
$\frac{1}{2} e^{-j\pi} = \frac{1}{2}(-1) = -\frac{1}{2}$



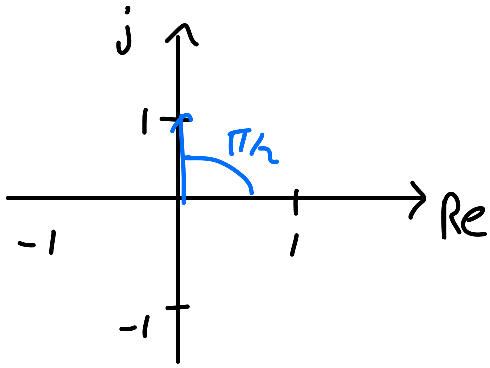
$e^{j\pi/2} = j$



$e^{-j\pi/2} = -j$



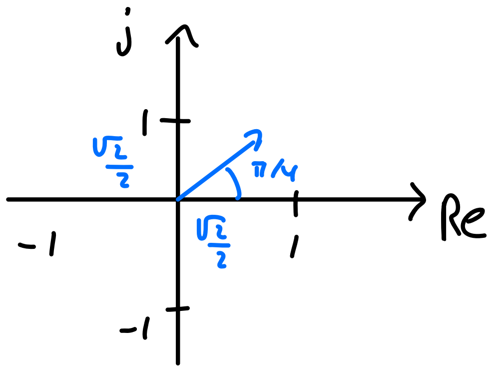
$$e^{j5\pi/2} = j$$



$$\frac{5\pi}{2} - \frac{4\pi}{2} = \frac{\pi}{2}$$

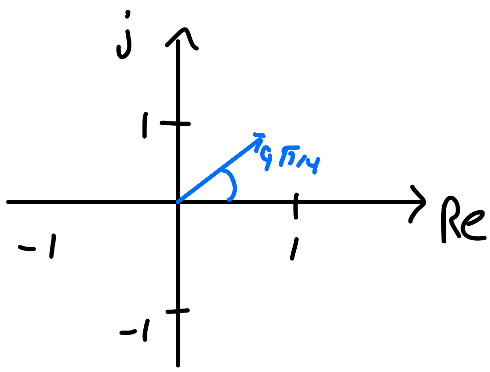
$$\sqrt{2}\, e^{j\pi/4} = \sqrt{2} \cdot \frac{\sqrt{2}}{2}(1+j)$$

$$= \frac{2}{2}(1+j) = 1+j$$



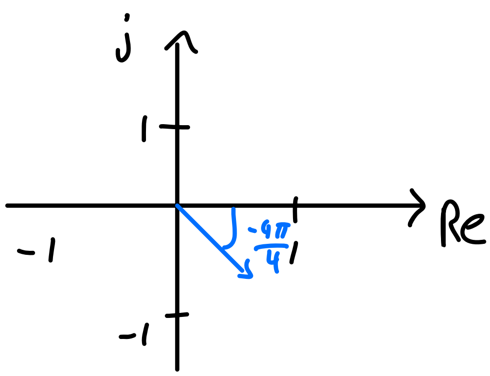
$$\sqrt{2}\, e^{j9\pi/4} = \sqrt{2} \cdot \frac{\sqrt{2}}{2}(1+j)$$

$$= 1+j$$



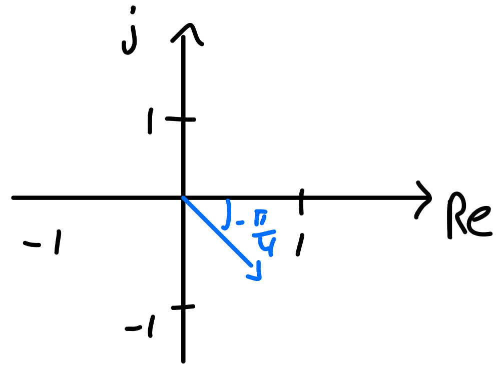
$$\sqrt{2}\, e^{-j9\pi/4} = \sqrt{2} \cdot \frac{\sqrt{2}}{2}(1-j)$$

$$= 1-j$$



$\sqrt{2}\, e^{-j\pi/4} = \sqrt{2} \cdot \frac{\sqrt{2}}{2}(1-j)$

$= 1-j$



$c^2 = 1$

$a^2+b^2 = c^2$

$a^2 = b^2 \to 2a^2 = c^2$

$2a^2 = 1$

$a^2 = \frac{1}{2} \to a = \pm\frac{1}{\sqrt{2}}$

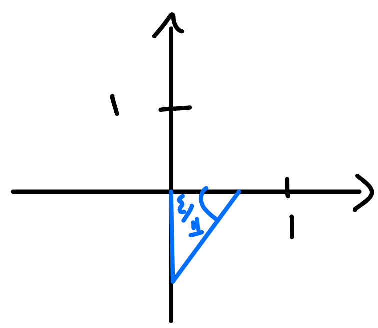
$a = \pm\frac{\sqrt{2}}{2}$

1.2 Express the following in polar form ($re^{j\theta}$, $-\pi < \theta \le \pi$)

$5 \to 5e^0$ $\qquad -3j \to 3e^{-j\pi/2}$

$-2 \to 2e^{j\pi}$

$\frac{1}{2} - j\frac{\sqrt{3}}{2} \to e^{-j\pi/3}$



$1+j \to \sqrt{2}\, e^{j\pi/\cancel{2}4}$ $\qquad (1-j)^2 = 1^2 - 2j - j^2 = 1-2j+1 = 2(1-j)$

$(1-j)^2 \to 2\cancel{\sqrt{2}}\, e^{-j\pi/\cancel{4}2}$

$\frac{1+j}{2} + \frac{j-1}{2} = j$

$j(1-j) = j - j^2 = 1+j \to \sqrt{2}\, e^{j\pi/4}$

$\frac{(1+j)}{(1-j)} = \frac{1}{1-j} + \frac{j}{1-j} = \cancel{\frac{1+j}{1+1}} + \frac{j+j^2}{1+1}$

$= \frac{j-1}{2} = \frac{1}{2}(j-1) \to \frac{\sqrt{2}}{2}\, e^{3\pi/4}$

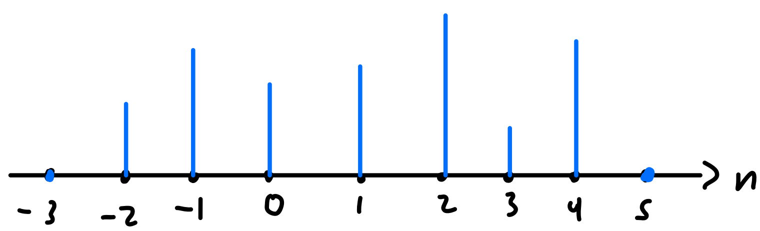
$e^{j\pi/2}$

$$\frac{(\sqrt{2}+j\sqrt{2})}{1+j\sqrt{3}} = \frac{\sqrt{2}}{1+j\sqrt{3}} + \frac{j\sqrt{2}}{1+j\sqrt{3}} \rightarrow \frac{(1-j\sqrt{3})\sqrt{2}}{1-3j^2} + \frac{(1-j\sqrt{3})j\sqrt{2}}{1-3j^2}$$

$$\frac{\sqrt{2} - j\sqrt{6} + j\sqrt{2} - \overset{-1}{\cancel{j^2}}\sqrt{6}}{1+3} = \frac{\sqrt{2}(1+j) - \sqrt{6}(j-1)}{4}$$

$\Rightarrow e^{-j\pi/12}$

1.4 Let $x[n]$ be signal w/ $x[n]=0$ for $n<-2$ and $n>4$. For signals below, determine values of n for which it is guaranteed to be 0



a) $x[n-3]$

$n < -2+3 = 1$

$n > 4+3 = 7$

nonzero in any

$-2 \leq n-3 \leq 4$

$1 \leq n \leq 7$

$\rightarrow$ zero in range

$n < 1$

$n > 7$

b) $x[n+4]$

$n < -2-4 = -6$

$n > 4-4 = 0$

c) $x[-n]$

$n < 2$ $\quad n > 2$

$n > -4$ $\quad n < -4$

d) $x[-n+2]$

non-zero in range

$-2 \le -n+2 \le 4$

$-4 \le -n \le 2$

$4 \ge n \ge 2$

$4 < n \quad \rightarrow \quad n > 4$

$n < 2 \qquad\quad n < 2$

e) $x[-n-2]$

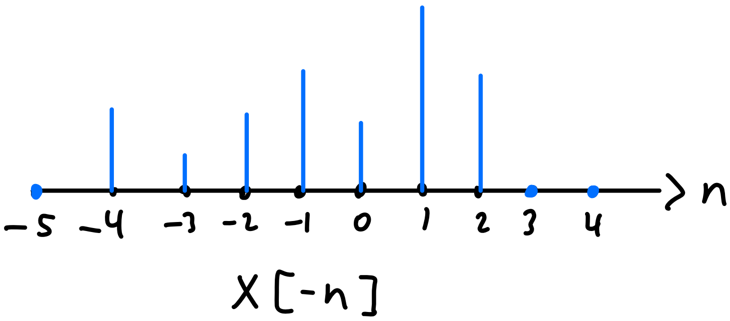
$-2 \le -n-2 \le 4$

$0 \le -n \le 6$

$0 \ge n \ge -6$

$0 < n \quad \rightarrow \quad n > 0$

$n < -6 \qquad\quad n < -6$



1.11 Determine fundamental period of signal $x[n] = 1 + e^{j4\pi n/7} - e^{j2\pi n/5}$

Positive value of T for which $x[t] = x[t+T]$

$\cancel{1 + e^{j4\pi \cdot 35/7} - e^{j2\pi \cdot 35/7} = 1 + e^{j4\pi \cdot 5} - e^{j2\pi \cdot 7}}$

1. Period of $e^{j\frac{4\pi n}{7}}$

Need to satisfy

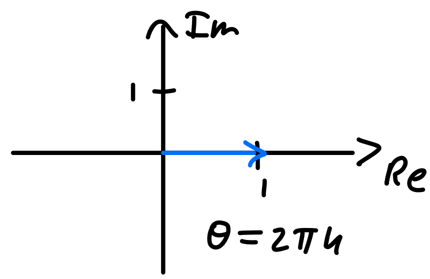
$$e^{j4\pi(n+N_1)/7} = e^{j4\pi n/7}$$

$$e^{j\frac{4\pi n}{7} + j\frac{4\pi N_1}{7}} = e^{j\frac{4\pi n}{7}} \rightarrow e^{j\frac{4\pi n}{7}} e^{j\frac{4\pi N_1}{7}} = e^{j\frac{4\pi n}{7}}$$

$$e^{j\frac{4\pi N_1}{7}} = 1 \rightarrow \frac{4\pi N_1}{7} = 2\pi k$$

$$\frac{2}{7} N_1 = k \rightarrow N_1 = \frac{7}{2} k$$

$e^{j\theta} = 1$



2. Period of $e^{j2\pi n/5}$

$$e^{j\frac{2\pi}{5}(n+N_2)} = e^{j\frac{2\pi}{5}n} \rightarrow e^{j\frac{2\pi}{5}n} \cdot e^{j\frac{2\pi}{5}N_2} = e^{j\frac{2\pi}{5}n}$$

$$e^{j\frac{2\pi}{5}N_2} = 1 \longrightarrow \frac{2\pi}{5}N_2 = 2\pi K \rightarrow \frac{1}{5}N_2 = K \rightarrow N_2 = 5K$$

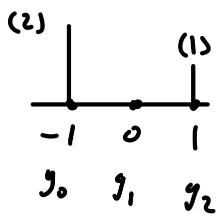
$e^{j\theta} = 1$ when $\theta = 2\pi K,\ K \in \mathbb{Z}$

3. Fundamental Period of $X[n]$ is LCM of individual periods

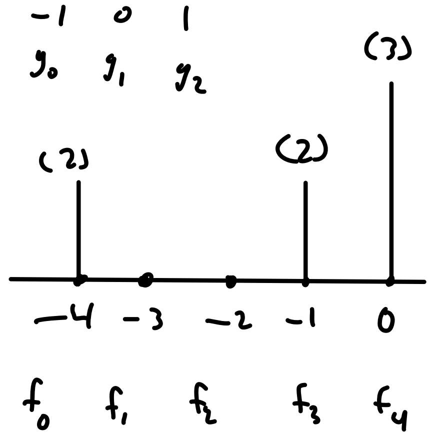
$$LCM(7, 5) = 35$$

Convolution

$$X(n) \longrightarrow \boxed{H} \longrightarrow y(n)$$



$X(n)$



$$f(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4$$
$$g(x) = g_0 + g_1 x + g_2 x^2$$

$$2 + 0 \cdot x + 0 \cdot x^2 + 2x^3 + 3x^4$$
$$2 + 0 \cdot x + x^2$$

$\downarrow$

$$2 + 2x^3 + 3x^4$$
$$\cdot \quad 2 + x^2$$

$$4 + 4x^3 + 6x^4$$
$$2x^2 + 2x^5 + 3x^6$$
$$= 4 + 2x^2 + 4x^3 + 6x^4 + 2x^5 + 3x^6$$

where will the center be? $\longrightarrow$

4	0	2	4	6	2	3
c	x	x^2	x^3	x^4	x^5	x^6

1.6 Determine whether each of the following signals is periodic

b) $x_2[n] = u[n] + u[-n]$

unit step function

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$u[-n] = \begin{cases} 1, & -n \geq 0 \\ 0, & -n < 0 \end{cases} = \begin{cases} 0, & n > 0 \\ 1, & n \leq 0 \end{cases}$$

$$u[n] + u[-n] = \begin{cases} 1, & n \geq 0 \\ 1, & n < 0 \end{cases} = 1 \quad \forall n \in \mathbb{Z}$$

Periodic Signal $\quad x(t) = x(t+T)$

$x_2[n]$ takes on the same value at each time step n. Therefore, it is constant rather than a periodic function

c) $x_3[n] = \sum_{k=-\infty}^{\infty} \{\delta[n-4k] - \delta[n-1-4k]\}$

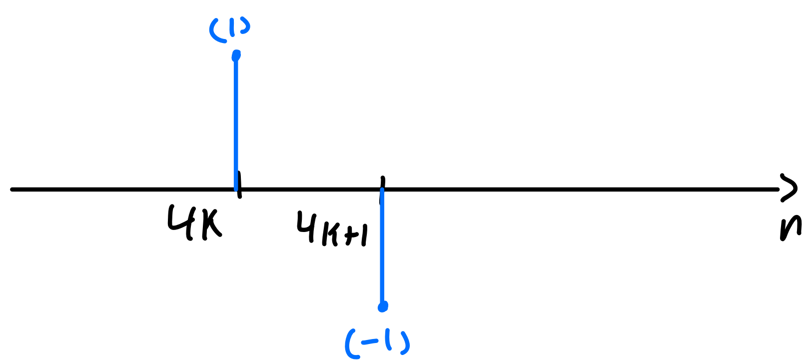
Unit impulse

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$\delta[n-4k] = \begin{cases} 1, & n = 4k \\ 0, & n \neq 4k \end{cases}, \quad \delta[n-1-4k] = \begin{cases} 1, & n = 1+4k \\ 0, & n \neq 1+4k \end{cases}$$

$\delta[n-4k] - \delta[n-1-4k]$

$$x_3[0] = \sum_{k=-\infty}^{\infty} \{\delta[-4k] - \delta[-1-4k]\} = \begin{cases} 1 & k = 0 \\ -1 & k = -1/4 \\ 0 & k \neq 0, k \neq -1/4 \end{cases}$$



Yes, the above is a periodic signal

$$x_3[n] = \sum_{k=-\infty}^{\infty} \{\delta[n-4k] - \delta[n-1-4k])$$

$\delta[n-4k]$: impulse of 1 every $n=4k$, $n = \underbrace{0, 4}_{4}, 8, 12, \dots, 4k$

$\delta[n-1-4k]$: impulse of -1 every $n=4k+1$, $n = \underbrace{1, 5}_{4}, 9, 13, \dots, 4k+1$

Pattern repeats every 4 timesteps in k $\therefore$ $x_3[n]$ is periodic with

Period $T=4$

1.3 Determine values P_x and E_x for each of the following signals

$$E_\infty \triangleq \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \cdot \sum_{n=-\cancel{\infty}\,N}^{\cancel{\infty}\,N} |x[n]|^2$$

d) $x_1[n] = (\frac{1}{2})^n u[n]$

$$E_\infty \triangleq \sum_{n=-\infty}^{\infty} |(\tfrac{1}{2})^n u[n]|^2$$

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \rightarrow \sum_{n=0}^{\infty} |(\tfrac{1}{2})^n|^2 = \sum_{n=0}^{\infty} (\tfrac{1}{2})^{2n} = \sum_{n=0}^{\infty} (\tfrac{1}{4})^n$$

Geometric series w/ common ratio $\frac{1}{4}$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \longrightarrow = \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-\infty}^{\infty} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} (\tfrac{1}{4})^n$$

$$= \frac{1-(\frac{1}{4})^{N+1}}{1-\frac{1}{4}} = \frac{1-(\frac{1}{4})^{N+1}}{\frac{3}{4}} = \frac{4}{3}\left(1-(\tfrac{1}{4})^{N+1}\right) \longrightarrow 0 \text{ as } N\to\infty$$

e) $x_2[n] = e^{j(\frac{\pi}{2}n + \frac{\pi}{8})}$

$$E_\infty = \sum_{n=-\infty}^{\infty} \left| e^{j(\frac{\pi}{2}n + \frac{\pi}{8})} \right|^2$$

$$= \sum_{n=-\infty}^{\infty} 1^2 = \infty$$

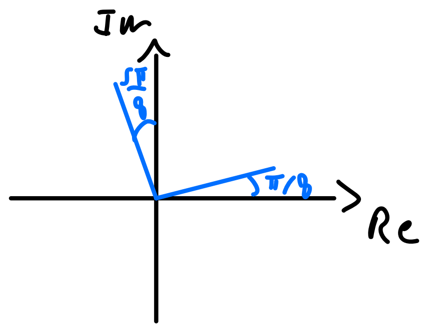
$$E_\infty \triangleq \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_\infty \triangleq \lim_{N \to \infty} \frac{1}{2N+1} \cdot \sum_{n=\cancel{-\infty}\,-N}^{\cancel{\infty}\,N} |x[n]|^2$$

$$\cancel{P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-\infty}^{\infty} \left| e^{j(\frac{\pi}{2}n + \frac{\pi}{8})} \right|^2} = \lim_{N \to \infty} \frac{1}{2N+1} \cdot \infty$$

$\frac{\infty}{\infty}$ indeterminant

$$\cancel{= \frac{\infty}{2\infty + 1} =} \ \frac{1}{2}$$



$$P_\infty = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-\infty}^{\infty} |x[n]|^2 = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| e^{j(\frac{\pi}{2}n + \frac{\pi}{8})} \right|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \underbrace{\cancel{\sum_{n=-N}^{N} 1^2}}_{2N} = \lim_{N \to \infty} \frac{2N}{2N+1} = 1$$

f) $x_3[n] = \cos(\frac{\pi}{4} n)$

$E_\infty \triangleq \sum_{n=-\infty}^{\infty} |x[n]|^2$

$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \cdot \sum_{n=-\infty}^{\infty} |x[n]|$

$E_\infty = \sum_{n=-\infty}^{\infty} |\cos(\frac{\pi}{\cancel{2}\,4} n)|^2 = \sum_{n=-\infty}^{\infty} 1^2 = \infty$

indeterminant use more approach

$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \cdot \sum_{n=-\infty}^{\infty} |\cos(\frac{\pi}{\cancel{2}\,4} n)|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-\infty}^{\infty} 1 = \cancel{\frac{\infty}{2\infty+1} = \frac{1}{2}}$

$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\cos \frac{\pi}{4} n|^2$

$\cos \frac{\pi}{4} n$ oscillates periodically w/ a period of 8. We can find the power by averaging the signal over 1 period

$n=0 \quad \cos(0) = 1$

$n=1 \quad \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$n=2 \quad \cos(\frac{\pi}{2}) = 0$

$n=3 \quad \cos(\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2}$

$n=4 \quad \cos(\pi) = -1$

$n=5 \quad \cos(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$

$n=6 \quad \cos(\frac{3\pi}{2}) = 0$

$n=7 \quad \cos(\frac{7\pi}{4}) = \frac{\sqrt{2}}{2}$

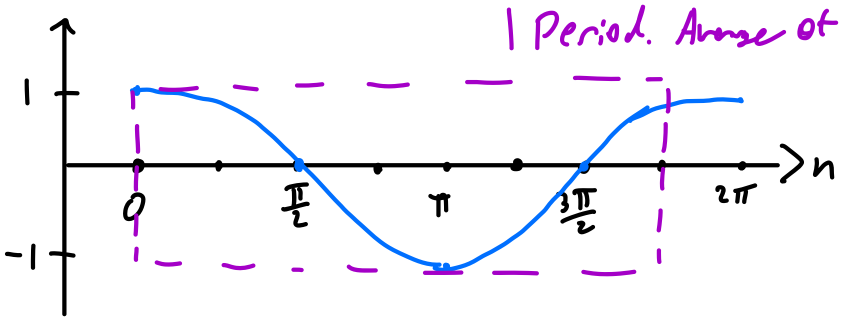
$\rightarrow \frac{1}{8}\left(1^2 + (\frac{\sqrt{2}}{2})^2 + 0^2 + (-\frac{\sqrt{2}}{2})^2 + (-1)^2 + (-\frac{\sqrt{2}}{2})^2 + 0^2 + (\frac{\sqrt{2}}{2})^2\right)$

$= \frac{1}{8}(1 + \frac{2}{4} + \frac{2}{4} + 1 + \frac{2}{4} + \frac{2}{4}) = \frac{1}{8}(2 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}) = \frac{4}{8}$

$= \frac{1}{2}$

$P_\infty = \frac{1}{2}$

1 Period. Average of 1 period will be average of ∞ periods



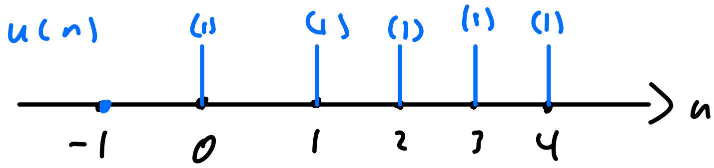
1.7. For each signal given below, determine all the values of the independent variable at which the even part of the signal is guaranteed to be zero.

(a) $x_1[n] = u[n] - u[n-4]$ **(b)** $x_2(t) = \sin(\frac{1}{2}t)$

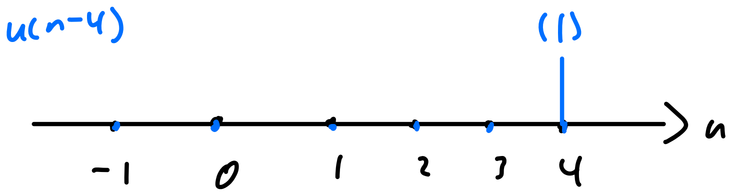
(c) $x_3[n] = (\frac{1}{2})^n u[n-3]$ **(d)** $x_4(t) = e^{-5t}u(t+2)$

a) $X_1(n) = u(n) - u(n-4)$

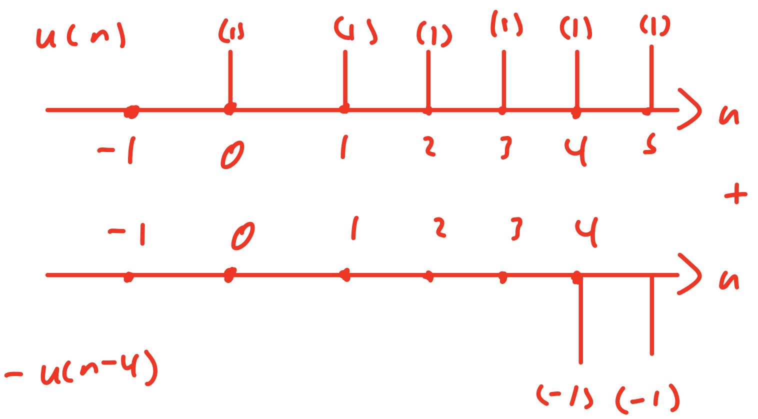
$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



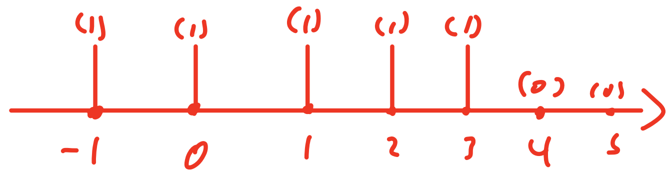
$$u(n-4) = \begin{cases} 1 & n \geq 4 \\ 0 & n < 4 \end{cases}$$



At least part of signal 0 for $n < 4$



$u(n) - u(n-3)$



$$u(n) - u(n-3) = \begin{cases} 1 & n \leq 3 \\ 0 & n > 3 \end{cases}$$

c) $X_3(n) = (\frac{1}{2})^n u(n-3)$

$$u(n-3) = \begin{cases} 1 & n \geq 3 \\ 0 & n < 3 \end{cases}$$

Signal = 0 for $n < 3$

1.9. Determine whether or not each of the following signals is periodic. If a signal is periodic, specify its fundamental period.

(a) $x_1(t) = je^{j10t}$ **(b)** $x_2(t) = e^{(-1+j)t}$ **(c)** $x_3[n] = e^{j7\pi n}$
(d) $x_4[n] = 3e^{j3\pi(n+1/2)/5}$ **(e)** $x_5[n] = 3e^{j3/5(n+1/2)}$

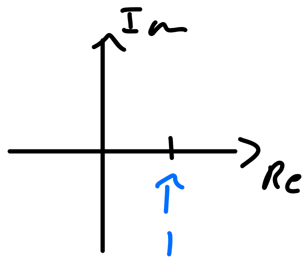
Fundamental frequency of periodic signal,

$x[n] = e^{j\omega_0 n}$ is $\omega_0 = \dfrac{2\pi m}{N}$

c) $x_3(n) = e^{j7\pi n}$ $\rightarrow e^{j7\pi(n+N)} = e^{j7\pi n}$

$e^{j7\pi n} \cdot e^{j7\pi N} = e^{j7\pi n} \rightarrow e^{j7\pi N} = 1$

$2\pi k = 7\pi N \rightarrow N = \frac{2}{7}k$



$e^{j7\pi n} \rightarrow e^{j7\pi(n+N)} = e^{j7\pi n}$

$e^{j7\pi n} e^{j7\pi N} = e^{j7\pi n} \rightarrow e^{j7\pi N} = 1$

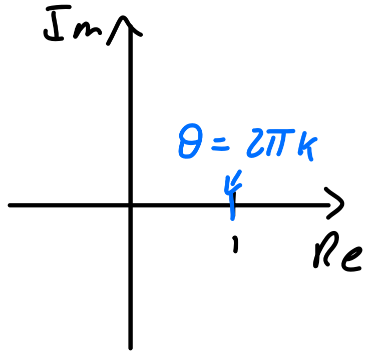
$7\pi N = 2\pi k \rightarrow$ $7\pi = \dfrac{2\pi k}{N}$

$N = 2$ $k = 7 \rightarrow$ fundamental frequency $= 2$

d) $x_4(n) = 3e^{j3\pi(n+\frac{1}{2})/5} \to e^{j3\pi(n+N+\frac{1}{2})/5} = e^{j3\pi(n+\frac{1}{2})/5}$

$e^{j3\pi\frac{n}{5}} e^{j3\pi\frac{N}{5}} e^{j3\pi/10} = e^{j3\pi n} e^{j3\pi/10}$

$e^{j3\pi\frac{N}{5}} = 1 \to 3\pi\frac{N}{5} = 2\pi k \to N = \frac{10}{3}k$



$3e^{j3\pi(n+N+\frac{1}{2})/5} = 3e^{j3\pi(n+1/2)/5} \to e^{j3\pi N/5} = 1$

$\theta = 2\pi k \to 3\frac{\pi}{5}N = 2\pi k$

$3\frac{\pi}{5}N = 2\pi k \to 3\frac{\pi}{5} = \frac{2\pi k}{N} \to N = 10, \; k = 3$

$\frac{2\pi(1)}{10} = \frac{2\pi}{10} = \frac{\pi}{5}$

Fundamental period = 10

e) $X_5(n) = 3e^{j\frac{3}{5}(n+\frac{1}{2})} \to e^{j\frac{3}{5}(n+N+\frac{1}{2})} = e^{j\frac{3}{5}(n+\frac{1}{2})}$

$e^{j\frac{3}{5}N} = 1 \to \frac{3}{5}N = 2k \to N = \frac{10}{3}k$

$3e^{j\frac{3}{5}(n+N+\frac{1}{2})} = 3e^{j\frac{3}{5}(n+1/2)} \to e^{j\frac{3}{5}(N)} = 1$

$\theta = 2\pi k \to \frac{3}{5}N = 2\pi k \to \frac{3}{5} = \frac{2\pi k}{N} \to N = 10\pi, \; k = 3$

No solution since $N \in \mathbb{Z} \Rightarrow$ not periodic

1.12. Consider the discrete-time signal

$$x[n] = 1 - \sum_{k=3}^{\infty} \delta[n-1-k].$$

Determine the values of the integers M and n_0 so that $x[n]$ may be expressed as

$$x[n] = u[Mn - n_0].$$

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases} \qquad u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$X(n) = 1 - \sum_{k=3}^{\infty} \delta(n-1-k)$$

$$X(-1) = 1 - \sum_{k=3}^{\infty} \delta(-1-1-k) \quad 1 \text{ for } k=-2 \text{ but } k \geq 3$$

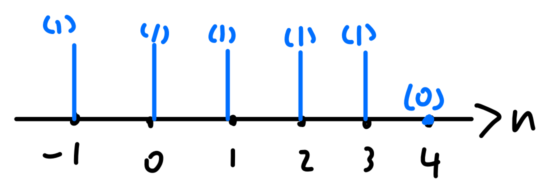
$$= 1 - \sum_{k=3}^{\infty} \delta(-2-k) = 1-0$$

$$X(0) = 1 - \sum_{k=3}^{\infty} \delta(-1-k) = 1-0 \quad 1 \text{ for } k=-1 \text{ but } k\geq 3$$

$$X(1) = 1 - \sum_{k=3}^{\infty} \delta(-k) = 1-0 \quad 1 \text{ for } k=0 \text{ but } k \geq 3$$

$$X(4) = 1 - \sum_{k=3}^{\infty} \delta(4-1-k) = 1-1 = 0 \quad 1 \text{ for } k=3$$

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



$$u(-n) = \begin{cases} 0, & \cancel{n \geq 0} \\ 1, & \cancel{n < 0} \end{cases} = \begin{cases} 0, & n > 0 \\ 1, & n \leq 0 \end{cases}$$

$$u(Mn - n_0) = \begin{cases} 0, & n \geq 4 \\ 1, & n < 4 \end{cases}$$

$$u(-n-n_0) = \begin{cases} 0, & n \geq n_0 \\ 1, & n < n < n_0 \end{cases}$$

$M = -1$

$n_0 = 4$

$$u(-n) = \begin{cases} 0, & n > 0 \\ 1, & n \leq 0 \end{cases}$$

$M = -1$

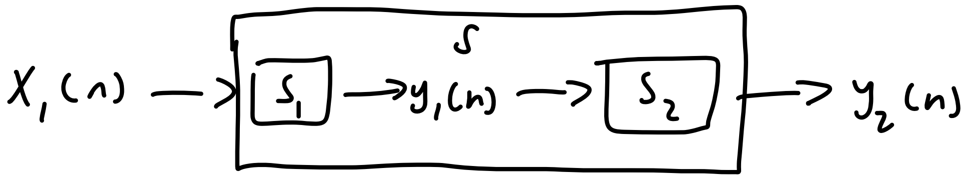
$n_0 = 3$

$$u(-n+3) = \begin{cases} 0, & n > 3 \\ 1, & n \leq 3 \end{cases}$$

1.15: System S w/ input $x(n)$ and output $y(n)$. System Obtained by connecting System S_1 to System S_2.

$$S_1: y_1(n) = 2x_1(n) + 4x_1(n-1)$$

$$S_2: y_2(n) = x_2(n-2) + \frac{1}{2}x_2(n-3)$$

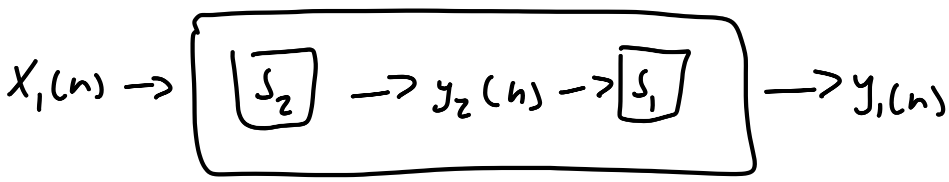


a) Determine input-output relationship for S

$$y_2(n) = y_1(n-2) + \frac{1}{2}y_1(n-3) = 2x_1(n-2) + 4x_1(n-3) + \frac{1}{2}\cdot 2x_1(n-3) + \frac{4}{2}x_1(n-4)$$

$$= 2x_1(n-2) + 4x_1(n-3) + x_1(n-3) + 2x_1(n-4) = 2x_1(n-2) + 5x_1(n-3) + 2x_1(n-4)$$

b) Does input-output relationship of system S change if order in which S_1 and S_2 are connected in series is reversed



$$y_2(n) = x_1(n-2) + \frac{1}{2}x_1(n-3) \rightarrow y_1(n) = 2x_1(n) + 4x_1(n-1)$$

$$y_1(n) = 2y_2(n) + 4y_2(n-1) = 2x_1(n-2) + x_1(n-3) + 4x_1(n-3) + 2x_1(n-4)$$

$$= 2x_1(n-2) + 5x_1(n-3) + 2x_1(n-4)$$

input-output relationship of S does not change depending on order of S_1 and S_2. This satisfies the superposition principle.

1.16 DT-System w/ input $x[n]$ and output $y[n]$.

$$y(n) = x(n)\,x(n-2)$$

a) Is the system memoryless?

b) Determine output of system when input is $A\,\delta(n)$, where A is any real or complex number

c) Is system invertible?

1.18

$$y(n) = \sum_{k=n-n_0}^{n+n_0} x(k) \quad , \quad n_0 \in \mathbb{Z},\ n_0 > 0$$

a) Is the system linear?

- Additivity $\quad x_1(k)+x_2(k) \rightarrow \boxed{S} \rightarrow y_1(k)+y_2(k)$

$$x_1(k)+x_2(k) \rightarrow \boxed{S} \rightarrow \sum_{k=n-n_0}^{n+n_0} (x_1(k)+x_2(k)) = \sum_{k=n-n_0}^{n+n_0} x_1(k) + \sum_{k=n-n_0}^{n+n_0} x_2(k)$$

$$= y_1(n)+y_2(n)$$

- Scaling

$$c\cdot x(k) \rightarrow \boxed{S} \rightarrow \sum_{k=n-n_0}^{n+n_0} c\cdot x(k) = c\cdot \sum_{k=n-n_0}^{n+n_0} x(k) = c\cdot y(n)$$

(b) Is the System TI?

System is TI since it does not change wrt n. The number of summations is always $2n_0$

$$x(n) \to y(n) = \sum_{k=n-n_0}^{n+n_0} x(k)$$

Consider a shifted version $x_1(n) = x(n-n_1)$

$$y_1(n) = \sum_{k=n-n_0}^{n+n_0} x_1(k) = \sum_{k=n-n_0}^{n+n_0} x(k-n_1)$$

$k - n_1 = k_1 \quad k = n-n_1-n_0$ to $n-n_1+n_0$

$$y_1(n) = \sum_{k=n-n_1-n_0}^{n-n_1+n_0} x(k_1)$$

change variable $k_1 \to k$

$$y_1(n) = \sum_{k=n-n_1-n_0}^{n-n_1+n_0} x(k)$$

equivalent

Consider $y_2(n) = y(n-n_1)$

$$y_2(n) = y(n-n_1) = \sum_{k=(n-n_1)-n_0}^{n-n_1+n_0} x(k)$$

$\therefore$ System is time invariant

(c) If $x(n)$ bounded by finite integer B ($|x(n)| < B \; \forall n$), can be shown that $y(n)$ bounded by finite number $C \to$ conclude System is stable.
Express C in terms of B and n_0

$$y(n) = \sum_{k=n-n_0}^{n+n_0} x(k) = 2n_0 \, x(k) < 2n_0 B = C$$

includes start index

$$y(n) = \sum_{k=n-n_0}^{n+n_0} x(k) \to |y(n)| = \left|\sum_{k=n-n_0}^{n+n_0} x(n)\right| = \sum_{k=n-n_0}^{n+n_0} |x(n)| \le \sum_{k=n-n_0}^{n+n_0} B = (2n_0+1)B$$

$\therefore \; y(n) \le (2n_0+1)B$

1.19 Determine whether each system S $x(n) \to \boxed{S} \to y(n)$ is L, TI, or LTI.

b) $y(n) = x^2(n-2)$

Linearity condition

~~Additive~~

function squares input

Relationship between $x(n)$ and $x(n-2)$? Does system cause a shift

$$x \to \boxed{S} \to x^2(n-2)$$

$$x_1 \to \boxed{S} \to x_1^2(n-2) \quad , \quad x_2 \to \boxed{S} \to x_2^2(n-2)$$

$$x_1 + x_2 \to \boxed{S} \to (x_1(n-2) + x_2(n-2))^2 = x_1^2(n-2) + 2x_1(n-2)x_2(n-2) + x_2^2(n-2)$$

$$\neq y_1 + y_2 = x_1^2(n-2) + x_2^2(n-2)$$

~~Scaling~~

$$c \cdot x_1 \to \boxed{S} \to c\, y_1 \to y(c\, x(n)) = (c\, x(n-2))^2 = c^2 x^2(n-2)$$

$$\neq c\, x^2(n-2)$$

Not linear since doesn't satisfy additive property

TI condition

$y_1(n) = x_1^2(n-2) = x^2(n-n_1-2)$ Consider shifted signal $x_1^2(n-2) = x^2(n-n_1-2)$

$y_2(n) = y(n-n_1) = x^2(n-n_1-2)$ Let $y_2(n)$ be the shifted response to signal x

$y_1(n) = x_1^2(n-2)$, $y_1(n-N) = x_1^2(n-N-2)$ Consider shifted version of system

$x_2(n) = x_1(n-N)$ Consider shifted input to system

$y_2(n) = x_2(n) = x_1^2(n-N-2)$ Evaluate system for shifted input

$y_2(n) = y_1(n-N)$ $\therefore$ $y(n)$ is TI

Since the shifted system equals response to shifted input, system is TI

c) $y(n) = x(n+1) - x(n-1)$

Linearity Condition

☑ Additivity $\quad x_1 + x_2 \rightarrow \boxed{S} \rightarrow x_1(n+1) + x_2(n+1) - (x_1(n-1) - x_2(n-1))$

$= x_1(n+1) + x_2(n+1) - x_1(n+1) - x_2(n-1)$

$y_1(n) = x_1(n+1) - x_1(n-1)$, $y_2(n) = x_2(n+1) - x_2(n-1)$

$y_1(n) + y_2(n) = x_1(n+1) - x_1(n-1) + x_2(n+1) - x_2(n-1) = x_1(n+1) + x_2(n+1) - x_1(n-1) - x_2(n-1)$

☑ Scaling $\quad c\,x_1 \rightarrow \boxed{S} \rightarrow c\,y_1$

$c\,x_1 \rightarrow \boxed{S} \rightarrow c\,x_1(n+1) - c\,x_1(n-1)$

$\quad \parallel$

$c\,y_1 = c(x(n+1) - x(n-1)) = c\,x(n+1) - c\,x(n-1)$

? TI Condition

$y_1(n) = x_1(n+1) - x_1(n-1)$ Consider $x_1(n) = x(n-n_1)$

$= x(n-n_1+1) - x(n-n_1-1)$

$y_2(n) = y(n-n_1) = x(n-n_1+1) - x(n-n_1-1)$ Consider $y_2(n) = y(n-n_1)$

The shifted systems match ∴ the system satisfies TI Condition

$\boxed{\text{LTI}}$

Modified Proof

$y_1(n) = x_1(n+1) - x_1(n-1)$, $y_1(n-N) = x_1(n-N+1) - x_1(n-N-1)$ shifted system

$x_2(n) = x_1(n-N)$ shifted input

$y_2(n) = x_2(n+1) - x_2(n-1) = x_1(n-N+1) - x_2(n-N-1)$

$y_2(n) = y_1(n-N)$ Response to shifted input equals shifted system

d) $y(n) = Od\{x(t)\}$ ← what is this function

$y(n) = Od\{x(t)\}$

$y(n) = \frac{1}{2}\{x(n) + x(-n)\}$

Linearity

Additivity

$x_1(n) + x_2(n) \rightarrow \boxed{S} \rightarrow \frac{1}{2}\{x_1(n) + x_2(n) + x_1(-n) + x_2(-n)\}$

$= \frac{1}{2}\{x_1(n) + x_1(-n) + x_2(n) + x_2(-n)\}$

$= \frac{1}{2}\{x_1(n) + x_1(-n)\} + \frac{1}{2}\{x_2(n) + x_2(-n)\} = y_1(n) + y_2(n)$

Homogeneity

$c \cdot x(n) \rightarrow \boxed{S} = \frac{1}{2}\{c \cdot x_1(n) + c \cdot x_2(n)\} = \frac{c}{2}\{x_1(n) + x_2(n)\}$

$= c \cdot y(n)$

System is linear

Time-Invariance

$y_1(n) = \frac{1}{2}\{x_1(n) + x_1(-n)\}$, $y_1(n-N) = \frac{1}{2}\{x_1(n-N) + x_1(-n-N)\}$

$x_2(n) = x_1(n-N)$

$y_2(n) = \frac{1}{2}\{x_2(n) + x_2(-n)\} = \frac{1}{2}\{x_1(n-N) + x_1(-n+N)$

$y_1(n-N) \neq y_2(n)$ ∴ Not TI

Convolution Exercises

Prove Commutative Property:

$x(n) * h(n) = h(n) * x(n)$

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{N=-\infty}^{\infty} x(n-N)\, h(N)$$

Change variable $n-k=N$
$N: -\infty \to N: \infty$, $k = n-N$

$$= \sum_{N=-\infty}^{\infty} h(n)\, x(n-N) = h(n) * x(n)$$

Prove Distributive Property

$x(n) * (h_1(n) + h_2(n)) = x(n) * h_1(n) + x(n) * h_2(n)$

$$x(n) * (h_1(n) + h_2(n)) = \sum_{k=-\infty}^{\infty} x(k)\left(h_1(n-k) \cdot h_2(n-k)\right)$$

$$= \sum_{k=-\infty}^{\infty} x(k) h_1(n-k) + x(k)\, h_2(n-k) = \sum_{k=-\infty}^{\infty} x(k) h_1(n-k) + \sum_{k=-\infty}^{\infty} x(k)\, h_2(n-k)$$

$$= x(n) * h_1(n) + x(n) * h_2(n)$$

Prove Associative Property:

$x(n) * (h_1(n) * h_2(n)) = (x(n) * h_1(n)) * h_2(n)$

$$x(n) * (h_1(n) * h_2(n)) = x(n) * \underbrace{\sum_{k=-\infty}^{\infty} h_1(k)\, h_2(n-k)}_{h(n)}$$

Can I ignore the ℓ shift?

$$= x(n) * h(n) = \sum_{\ell=-\infty}^{\infty} x(\ell)\, h(n-\ell) = \sum_{\ell=-\infty}^{\infty} x(\ell) \sum_{k=-\infty}^{\infty} h_1(k)\, h_2(n-k) \quad ?$$

Example 2.15

$$y(n) - \frac{1}{2}y(n-1) = x(n)$$

$$y(n) = x(n) + \frac{1}{2}y(n-1)$$

$n=0$ $\quad y(0) = x(0) + \frac{1}{2}y(-1) = x(0)$

$n=1$ $\quad y(1) = x(1) + \frac{1}{2}y(0) = x(1) + \frac{1}{2}x(0)$

$n=2$ $\quad y(2) = x(2) + \frac{1}{2}y(1) = x(2) + \frac{1}{2}(x(1) + \frac{1}{2}x(0)) = x(2) + \frac{1}{2}x(1) + \frac{1}{4}x(0)$

$\vdots$

$n=n$ $\quad y(n) = x(n) + \frac{1}{2}y(n-1) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2) + \ldots + \frac{1}{2^n}x(0)$

$$y(n) = \sum_{k=0}^{n} \frac{1}{2^k} x(n-k)$$

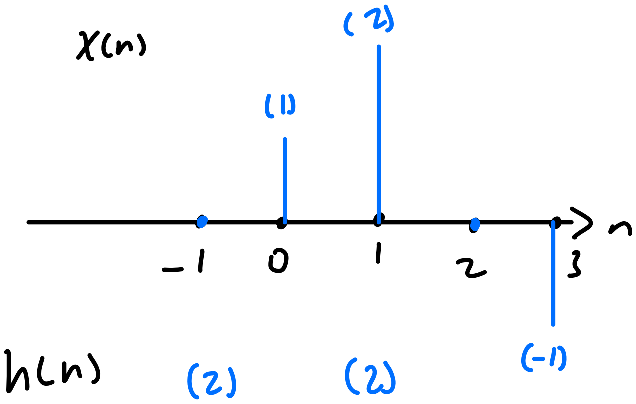
2.1. Let

$$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3] \quad \text{and} \quad h[n] = 2\delta[n+1] + 2\delta[n-1].$$

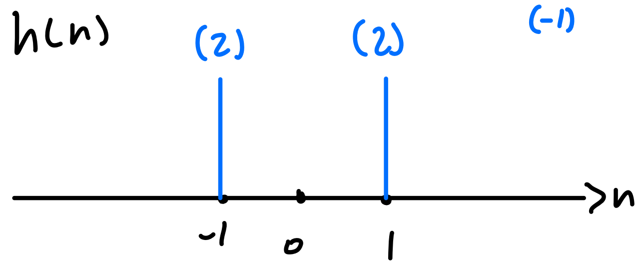
Compute and plot each of the following convolutions:

(a) $y_1[n] = x[n] * h[n]$ **(b)** $y_2[n] = x[n+2] * h[n]$

(c) $y_3[n] = x[n] * h[n+2]$



$$x(n) = \begin{cases} 1, & n=0 \\ 2, & n=1 \\ -1, & n=3 \\ 0, & n \neq 0,1,3 \end{cases}$$

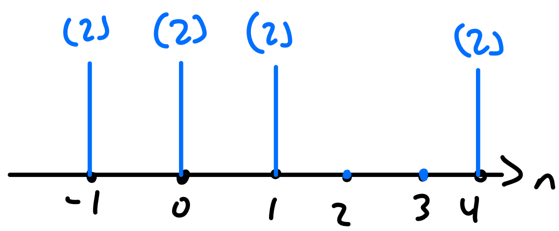


$$h(n) = \begin{cases} 2, & n=-1,1 \\ 0, & n \neq -1,1 \end{cases}$$

a) $y_1(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) = x(0)\, h(n) + x(1)\, h(n-1) + x(3)\, h(n-3)$

$= h(n) + 2h(n-1) - h(n-3) = 2\delta(n+1) + 2\delta(n-1) + 2\delta(n) + \cancel{2\delta(n-2)} - \cancel{2\delta(n-2)} + 2\delta(n-4)$

$= 2\delta(n+1) + 2\delta(n) + 2\delta(n-1) + 2\delta(n-4)$



$$y_1(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) = \sum_{k=-\infty}^{\infty} x(k) \{2\delta(n-k+1) + 2\delta(n-k-1)\}$$

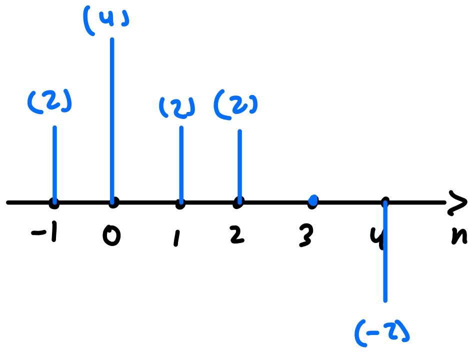
$$= x(0)\{2\delta(n-0+1)+2\delta(n-0-1)\} + x(1)\{2\delta(n-1+1)+2\delta(n-1-1)\} + x(3)\{2\delta(n-3+1)+2\delta(n-3-1)\}$$

$$= 2\delta(n+1) + 2\delta(n-1) + 2(2\delta(n)+2\delta(n-2)) - (2\delta(n-2)+2\delta(n-4))$$

$$= 2\delta(n+1) + 2\delta(n-1) + 4\delta(n) + 4\delta(n-2) - 2\delta(n-2) - 2\delta(n-4)$$

$$= 2\delta(n+1) + 4\delta(n) + 2\delta(n-1) + 4\delta(n-2) - 2\delta(n-2) - 2\delta(n-4)$$

$$= 2\delta(n+1) + 4\delta(n) + 2\delta(n-1) + 2\delta(n-2) - 2\delta(n-4)$$



$$x(n) = \delta(n) + 2\delta(n-1) - \delta(n-3), \quad h(n) = 2\delta(n+1) + 2\delta(n-1)$$

b) $y_2(n) = x(n+2) * h(n)$

$$= \sum_{k=-\infty}^{\infty} x(k+2)\, h(n-k) = \sum_{k=-\infty}^{\infty} x(k+2) \{2\delta(n-k+1) + 2\delta(n-k-1)\}$$

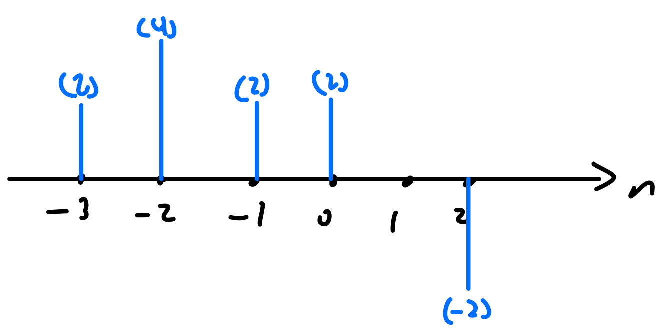
$$x(k+2) = \delta(k+2) + 2\delta(k+1) - \delta(k-1)$$

$$= x(-2+1)\{2\delta(n+2+1)+2\delta(n+2-1)\} + x(-1+1)\{2\delta(n+1+1)+2\delta(n+1-1)\} + x(1+1)\{2\delta(n-1+1)+2\delta(n-1-1)\}$$

$$= 1(2\delta(n+3)+2\delta(n+1)) + 2(2\delta(n+2)+2\delta(n)) - (2\delta(n)+2\delta(n-2))$$

$$= 2\delta(n+3) + 2\delta(n+1) + 4\delta(n+2) + 4\delta(n) - 2\delta(n) - 2\delta(n-2)$$

$$= 2\delta(n+3) + 4\delta(n+2) + 2\delta(n+1) + 2\delta(n) - 2\delta(n-2)$$



$x(n) = \delta(n) + 2\delta(n-1) - \delta(n-3)$, $h(n) = 2\delta(n+1) + 2\delta(n-1)$

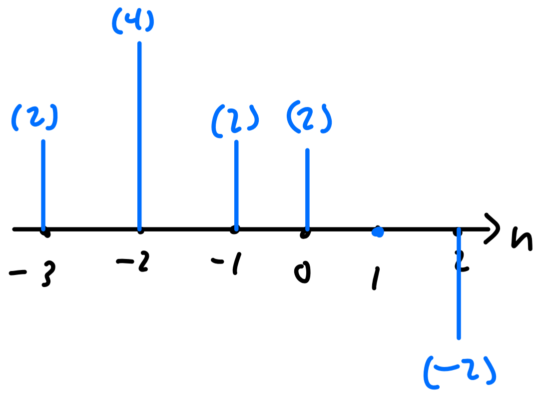
3) $y_1(n) = x(n) * h(n+2)$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k+2) = \sum_{k=-\infty}^{\infty} x(k) \{2\delta(n-k+3) + 2\delta(n-k+1)\}$$

$$= x(0)\{2\delta(n+3) + 2\delta(n+1)\} + x(1)\{2\delta(n+2) + 2\delta(n)\} + x(3)\{2\delta(n) + 2\delta(n-2)\}$$

$$= 2\delta(n+3) + 2\delta(n+1) + 4\delta(n+2) + 4\delta(n) - 2\delta(n) - 2\delta(n-2)$$

$$= 2\delta(n+3) + 4\delta(n+2) + 2\delta(n+1) + 2\delta(n) - 2\delta(n-2)$$



2.2. Consider the signal

$$h[n] = \left(\frac{1}{2}\right)^{n-1} \{u[n+3] - u[n-10]\}.$$

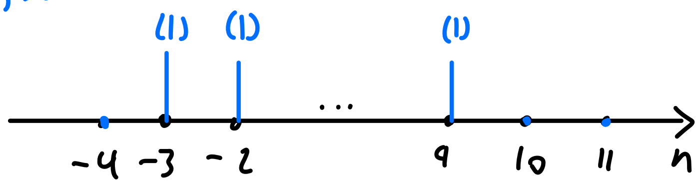
Express A and B in terms of n so that the following equation holds:

$$h[n-k] = \begin{cases} (\frac{1}{2})^{n-k-1}, & A \le k \le B \\ 0, & \text{elsewhere} \end{cases}.$$

$$h(n) = (\tfrac{1}{2})^{n-1} \{u(n+3) - u(n-10)\}$$

$$u(n+3) = \begin{cases} 1, & n \ge -3 \\ 0, & n < -3 \end{cases}, \quad u(n-10) = \begin{cases} 1, & n \ge 10 \\ 0, & n < 10 \end{cases}$$

$u(n+3) - u(n-10)$



	$u(n+3)$		$u(n-10)$
$n=-3$	$u(-3+3) = 1$	$-$	$u(-3-10) = 0 = 0$
	$u(-4+3) = u(-1) = 0$	$-$	$u(-4-10) = 0$
	$u(10+3) = 1$	$-$	$u(10-10) = 1$

$$\cancel{h(n-k) = \begin{cases} (\frac{1}{2})^{n-k-1}, & -3 \le k \le 9 \\ 0, & \text{elsewhere} \end{cases}}$$

$$h(n-k) = \begin{cases} (\frac{1}{2})^{n-k-1}, & -3 \le n-k \le 9 \\ 0, & \text{elsewhere} \end{cases}$$

2.3. Consider an input $x[n]$ and a unit impulse response $h[n]$ given by

$$x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2],$$

$$h[n] = u[n+2].$$

Determine and plot the output $y[n] = x[n] * h[n]$.

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

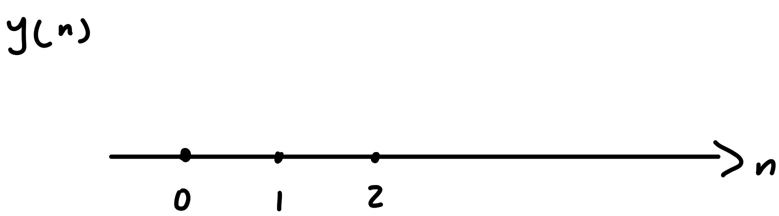
$$= \sum_{k=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^{k-2} u(k-2) \cdot u(n-k+2)$$

$$u(k-2) = \begin{cases} 1, & k-2 \geq 0 \Leftrightarrow k \geq 2 \\ 0, & k-2 < 0 \Leftrightarrow k < 2 \end{cases}$$

$$u(n-k+2) = \begin{cases} 1, & n-k+2 \geq 0 \Leftrightarrow k \leq n+2 \\ 0, & n-k+2 < 0 \Leftrightarrow k > n+2 \end{cases}$$

$$= \sum_{k=2}^{\infty} \left(\tfrac{1}{2}\right)^{k-2} u(n-k+2)$$

$$y(n) = \sum_{k=2}^{n+2} \left(\tfrac{1}{2}\right)^{k-2}$$



$$y(n) = \sum_{k=2}^{n+2} \left(\tfrac{1}{2}\right)^{k-2}$$

Let $m = k-2$

$$= \sum_{m=0}^{n} \left(\tfrac{1}{2}\right)^{m}$$

Geometric series

follows from $\sum_{m=0}^{n} (\cdot)$

$$= \frac{1 - \left(\tfrac{1}{2}\right)^{n+1}}{1 - \tfrac{1}{2}} = 2\left[1 - \left(\tfrac{1}{2}\right)^{n+1}\right], \quad n \geq 0$$

$$= 2\left[1 - \left(\tfrac{1}{2}\right)^{n+1}\right] u(n)$$

why unit step and not unit impulse?

2.4. Compute and plot $y[n] = x[n] * h[n]$, where

$$x[n] = \begin{cases} 1, & 3 \le n \le 8 \\ 0, & \text{otherwise} \end{cases},$$

$$h[n] = \begin{cases} 1, & 4 \le n \le 15 \\ 0, & \text{otherwise} \end{cases}.$$

2.5. Let

$$x[n] = \begin{cases} 1, & 0 \le n \le 9 \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad h[n] = \begin{cases} 1, & 0 \le n \le N \\ 0, & \text{elsewhere} \end{cases},$$

where $N \le 9$ is an integer. Determine the value of N, given that $y[n] = x[n] * h[n]$ and

$$y[4] = 5, \quad y[14] = 0.$$

2.6. Compute and plot the convolution $y[n] = x[n] * h[n]$, where

$$x[n] = \left(\frac{1}{3}\right)^{-n} u[-n-1] \quad \text{and} \quad h[n] = u[n-1].$$

2.7. A linear system S has the relationship

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]g[n-2k]$$

between its input $x[n]$ and its output $y[n]$, where $g[n] = u[n] - u[n-4]$.

(a) Determine $y[n]$ when $x[n] = \delta[n-1]$.
(b) Determine $y[n]$ when $x[n] = \delta[n-2]$.
(c) Is S LTI?
(d) Determine $y[n]$ when $x[n] = u[n]$.

2.13. Consider a discrete-time system S_1 with impulse response

$$h[n] = \left(\frac{1}{5}\right)^n u[n].$$

(a) Find the integer A such that $h[n] - Ah[n-1] = \delta[n]$.

(b) Using the result from part (a), determine the impulse response $g[n]$ of an LTI system S_2 which is the inverse system of S_1.

2.15. Which of the following impulse responses correspond(s) to stable LTI systems?

(a) $h_1[n] = n\cos(\frac{\pi}{4}n)u[n]$ **(b)** $h_2[n] = 3^n u[-n + 10]$

2.18. Consider a causal LTI system whose input $x[n]$ and output $y[n]$ are related by the difference equation

$$y[n] = \frac{1}{4}y[n-1] + x[n].$$

Determine $y[n]$ if $x[n] = \delta[n-1]$.

2.19. Consider the cascade of the following two systems S_1 and S_2, as depicted in Figure P2.19:

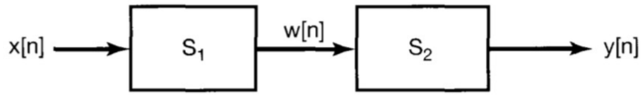


Figure P2.19

S_1 : causal LTI,

$$w[n] = \frac{1}{2}w[n-1] + x[n];$$

S_2 : causal LTI,

$$y[n] = \alpha y[n-1] + \beta w[n].$$

The difference equation relating $x[n]$ and $y[n]$ is:

$$y[n] = -\frac{1}{8}y[n-2] + \frac{3}{4}y[n-1] + x[n].$$

(a) Determine α and β.
(b) Show the impulse response of the cascade connection of S_1 and S_2.