Integration of $ln(a+x)ln(c+x)/(f+x)$ with respect to x for complex-valued a, c, and f

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1 Calculation overview

In [1], an extensive series of manipulations is presented that yields an expression for

$$\int_{x_{\ell b}}^{x_{ub}} \frac{\ln(a+x)\ln(c+x)}{f+x} dx \tag{1}$$

involving logarithm, dilogarithm, and trilogarithm functions for the restricted case of pure real a, c, and f. Below, adaptions are made to these manipulations to allow analytical evaluation of (1) for complex-valued a, c, and f.

As a first step, Eq. (1) can be expressed as

$$\int_{y_{\ell b}}^{y_{ub}} dy \left(\ln(a-f) + \ln(1-y) + 2\pi i N_4\right)\left(\ln(c-f) + \ln(1-\xi y) + 2\pi i N_5\right) \Big/ y \tag{2}$$

where

$$y \equiv \frac{f+x}{f-a},$$

$$\xi \equiv \frac{a-f}{c-f}$$

$$N_4 = N_+(a-f, 1-y),$$

$$N_5 = N_+(c-f, 1-\xi y),$$

and

$$N_\pm(\nu, w) \equiv \begin{cases} +1 & s_\pm \leq -\pi \\ -1 & s_\pm > \pi \\ 0 & \text{else} \end{cases} \tag{3}$$

with

$$s_\pm \equiv \mathrm{Arg}(\nu) \pm \mathrm{Arg}(w).$$

The appearance of N_4 and N_5 in Eq. 2 is one consequence of a, c, and f being complex-valued. The calculations required for these new terms are heightened in complexity by the possible variation of N_4 and N_5 with x. In particular, $N_+(\nu, w)$ has a step discontinuity where ν, w, or νw cross the negative real axis. Such discontinuities preserve the equality between the right-hand and left-hand sides of $\ln(\nu w) = \ln(\nu) + \ln(w) + 2\pi i N_+(\nu, w)$ where the logarithm arguments cross the branch cut of $\ln(z)$. For N_4 and N_5, these crossing points are limited as functions of x to single points where, respectively, $\mathrm{Im}(1-y) = 0$ and $\mathrm{Im}(1-\xi y) = 0$. If $\mathrm{Re}(1-y) < 0$ where $1-y$ crosses the real axis, then N_4 varies with x, and similarly for N_5.

All points where N_4 or N_5 are discontinuous are tracked in the code. The terms in Eq. 2 involving N_4 and N_5 are then split into intervals bounded by these points. The values of N_4 and N_5 are determined for a given interval by direct evaluation using Eq. 3 at the interval midpoint.

While N_4 and N_5 account for branch cut crossings in the integrand, such crossings can also produce spurious discontinuities following integration. For the integral of $1/y$, as an example, the $2\pi i$ discontinuity in $\ln(y)$ must be corrected for if the y contour spans the negative real axis, in which case $\int_{y_{\ell b}}^{y_{ub}} y^{-1} dy = \ln(y_{ub}) - \ln(y_{\ell b}) \pm 2\pi i$. If the contour is a straight line, as in the present context, taking $\int_{y_{\ell b}}^{y_{ub}} y^{-1} dy = \ln(y_{ub}/y_{\ell b})$ can resolve this particular complication. However, a similarly simple resolution is not obvious for terms of Eq. (2) involving the dilogarithm.

The dilogarithm $\mathrm{Li}_2(t)$ has a branch cut on $[1, \infty)$, where it is discontinuous by $+2\pi i \ln(t)$ passing from the upper half to the lower half plane [1]. In the definition

$$\mathrm{Li}_2(t) = -\int_0^t \frac{\ln(1-t')}{t'} dt',$$

the t' contour is specified to avoid this branch cut. As a result, if the y contour in Eq. 2 crosses $[1, \infty)$, then $\int_{y_{\ell b}}^{y_{ub}} \ln(1-y) y^{-1} dy = -\mathrm{Li}_2(y_{ub}) + \mathrm{Li}_2(y_{lb}) \pm 2\pi i \ln(y_c)$, where y_c is the crossing point. In the code, the appropriate sign for the correction is determined by evaluating the imaginary part of y at the contour endpoint y_{ub}. The contour is a straight line that crosses $\mathbb{R}$ once, so a positive imaginary part of y_{ub} indicates that the contour passes from the lower to the upper half plane. Functions in the program that apply such corrections, such as `ln_term_corr_A` and `dilog_term_corr_A`, have the descriptor `corr_A` in their names.

Several terms of Eq. 2 can involve both types of branch cut crossing corrections that have been mentioned. For example, if both $1-y$ and $1-\xi y$ cross $\mathbb{R}_-$, then discontinuities in both N_5 and $\mathrm{Li}_2(y)$ impact evaluation of $2\pi i \int_{y_{lb}}^{y_{ub}} N_5 \ln(1-y) y^{-1} dy$. With the locations of both types of discontinuities tracked, it is straightforward to combine the approaches stated above to manage the required corrections. The integration interval $[y_{\ell b}, y_{ub}]$ should be split at the step discontinuity in N_5, and in the sub-interval in which $1-y$ crosses $\mathbb{R}_-$, say at $1-y_c$, a correction $\pm N_5 (2\pi i)^2 \ln(y_c)$ for the $\mathrm{Li}_2(y)$ discontinuity should be added.

In full,

$$\begin{aligned}\int_{x_{\ell b}}^{x_{ub}} \frac{\ln(a+x)\ln(c+x)}{f+x} dx = &\\ &\Bigg((\ln(a-f) + 2\pi i N_4)(\ln(c-f) + 2\pi i N_5)\ln(y) \\ &- (\ln(c-f) + 2\pi i N_5)\mathrm{Li}_2(y) - (\ln(a-f) + 2\pi i N_4)\mathrm{Li}_2(\xi y)\Bigg)_\star \Bigg|_{y_{\ell b}}^{y_{ub}} \\ &+ \int_{y_{\ell b}}^{y_{ub}} \frac{\ln(1-y)\ln(1-\xi y)}{y} dy \end{aligned} \tag{4}$$

where specifically the y contour is a straight line between $y_{\ell b}$ and y_{ub}. The $\star$ here denotes that all the branch cut crossing corrections that have been described above are required for use of this expression. Namely, terms involving correction coefficients N_n should be split up into intervals over which these quantities are constant, and, in all terms, any discontinuity in ln or Li_2 arising from a branch cut crossing should be subtracted out.

2 $\int dy \ln(1-y)\ln(1-\xi y)/y$

The last integral on the right hand side of Eq. 4, $\int_{y_{\ell b}}^{y_{ub}} dy \ln(1-y)\ln(1-\xi y)/y$, requires the trilogarithm. An indirect approach is used in [1] for its evaluation. The first step is as follows:

$$
\begin{aligned}
I &= \int_{y_{\ell b}}^{y_{ub}} dy \log^2\left(\frac{1-\xi y}{1-y}\right) \Big/ y \\
&= \int_{y_{\ell b}}^{y_{ub}} dy \left(\log(1-\xi y) - \log(1-y) + 2\pi i N_1\right)^2 / y \qquad (5)
\end{aligned}
$$

where $N_1 = N_-(1-\xi y, 1-y)$ is a new quantity that is required for complex-valued a, c, or f. Also, $N_-(\nu, w)$ is defined in Eq. 3 and is discontinuous where ν, w, or ν/w lie on the negative real axis. Whereas both N_4 and N_5 are discontinuous as a function of x at maximally a single point, N_1 has up to four such points. Specifically, $1-\xi y$ and $1-y$ each cross the real axis at a single point, $(1-\xi y)/(1-y)$ crosses the real axis at two points, and all of these points are in general distinct.

Terms of Eq. 5 such as $\int_{y_{\ell b}}^{y_{ub}} y^{-1}\log(1-y)N_1 dy$ require special attention if N_1 is discontinuous, specifically if $1-y$ crosses $\mathbb{R}_-$ for this chosen example. For simplicity, say that y_c is the only point in the considered interval at which N_1 is discontinuous. Again specializing to the example term, say specifically that $\mathrm{Im}(y_c) = 0$ and $\mathrm{Re}(1-y_c) < 0$, meaning that the N_1 discontinuity is associated with that in $\log(1-y)$. Lastly, without loss of generality, say $N_1 \neq 0$ for $y \in (y_c, y_{ub}]$ and accordingly $N_1 = 0$ for $y \in [y_{lb}, y_c)$.

In this context, $\int_{y_{\ell b}}^{y_{ub}} y^{-1}\log(1-y)N_1 dy = N_1(\mathrm{Li}_2(y_{ub}) - \mathrm{Li}_2(y_c))$. Here, N_1 is constant on $y \in (y_c, y_{ub}]$ and can be evaluated using Eq. 3 at any point in this interval. However, $\mathrm{Li}_2(y)$ is both discontinuous at y_c and must be evaluated at this point. Namely, $\mathrm{Li}_2(y_c + i\epsilon) \neq \mathrm{Li}_2(y_c - i\epsilon)$, where $i\epsilon$ is an infinitesimal imaginary part. To determine the correct value of $\mathrm{Li}_2(y)$ to use, it is necessary to assess whether the contour segment $y \in (y_c, y_{ub}]$ lies above or below the real line. The choice among $y_c \pm i\epsilon$ for the endpoint should be made such that the contour does not cross the real line. Functions in the program that apply this type of correction: `dilog_term_corr_B` and `ln_terms_corr_B`, for example, feature the descriptor `corr_B` in their names.

Integrals such as

$$
\int_{y_{\ell b}}^{y_{ub}} \frac{\log^2(1-y)}{y} dy
$$

in Eq. 5 also require the introduction of new methods. From [1], for real-valued t,

$$
\begin{aligned}
\mathbb{f}(t) &\equiv \int_0^t \frac{\log^2(1-t')}{t'} dt' \qquad (6) \\
&= \log(t)\log^2(1-t) + 2\log(1-t)\mathrm{Li}_2(1-t) - 2\mathrm{Li}_3(1-t) + 2\mathrm{Li}_3(1), \qquad (7)
\end{aligned}
$$

where

$$\mathrm{Li}_3(t) = \int_0^t \frac{\mathrm{Li}_2(t')}{t'} dt'$$

is the trilogarithm function. To make Eq. 7 valid for complex-valued t, the definition in Eq. 6 can be extended with the specification that the integral contour does not cross any region where Eq. 7 is discontinuous. This extended definition is adopted here. Incidentally, Eq. 7 is continuous across $t \in \mathbb{R}_-$ due to cancellation of discontinuities among its terms, so the integral contour in Eq. 6 can cross $\mathbb{R}_-$.

Equation 7 is discontinuous across $t \in [1, \infty)$. Specifically, the $\pm 2\pi i$ jump in $\log(1-t)$ produces a discontinuity $\Delta\mathbb{f}(t) = \pm 4\pi i \left[\log(t)\log(|1-t|) + \mathrm{Li}_2(1-t)\right]$. As a result, for y integral contours that cross $y \in [1, \infty)$,

$$\int_{y_{\ell b}}^{y_{ub}} \frac{\log^2(1-y)}{y} dy = \mathbb{f}(y_{ub}) - \mathbb{f}(y_{\ell b}) \pm 4\pi i \left[\log(y_c)\log(|1-y_c|) + \mathrm{Li}_2(1-y_c)\right]$$

where y_c is the crossing point, and the sign of the correction depends on whether the contour passes from the lower to the upper half planes or vice versa.

The target expression of this section, $\int_{y_{lb}}^{y_{ub}} dy \ln(1-y)\ln(1-\xi y)/y$, also appears in Eq. 5, although the steps outlined so far only make it as evaluable as I. Lewin [1] makes progress by evaluating I a second time using an expression alternative to Eq. 5. First, a change of variables to $z = (1-\xi y)/(1-y)$ is made, yielding

$$I = \int_{z_{\ell b}}^{z_{ub}} \log^2(z) \left(\frac{1}{\xi - z} - \frac{1}{1-z} \right) dz. \tag{8}$$

For the first term, a second change of variables to $u = 1 - z/\xi = (\xi - 1)/\xi/(1-y)$ is made, leading to

$$T = \int_{z_{\ell b}}^{z_{ub}} \frac{\log^2(z)}{\xi - z} dz = -\int_{u_{\ell b}}^{u_{ub}} \frac{\log^2(\xi(1-u))}{u} du.$$

Proceeding,

$$T = -\int_{u_{\ell b}}^{u_{ub}} \frac{\left(\log(\xi) + \log(1-u) + 2\pi i N_2\right)^2}{u} du, \tag{9}$$

where the term involving $N_2 = N_+(\xi, 1-u)$ is required for complex ξ. As x varies, N_2 can be discontinuous at points where $z = \xi(1-u) = (1-\xi y)/(1-y)$ or $1-u$ cross the real axis. As addressed previously, there are two values of x at which $\mathrm{Im}(\xi(1-u)) = 0$. Also, $\mathrm{Im}(1-u) = 0$ at a single value of x. For real axis crossings specifically on the negative real axis, N_2 is discontinuous.

The term $\int_{u_{\ell b}}^{u_{ub}} u^{-1}\log(1-u) N_2 du$ requires the same treatment as the similar term involving N_1 addressed above. In particular, computation of this integral can involve evaluation of $\mathrm{Li}_2(u)$ on this function's branch cut. Of the two values of $\mathrm{Li}_2(u)$ on the cut, the one that should be used is that which ensures $\mathrm{Li}_2(u)$ is continuous throughout the relevant interval of the integral.

Equating Eq. 5 and Eq. 8 leads to

$$\int_{y_{\ell b}}^{y_{ub}} dy \frac{\log(1-\xi y)\log(1-y)}{y} =$$

$$\frac{1}{2}\Bigg(\mathbb{f}(\xi y) + \mathbb{f}(y) + (2\pi i N_1)^2 \ln(y) - 4\pi i N_1 \mathrm{Li}_2(\xi y) + 4\pi i N_1 \mathrm{Li}_2(y)$$

$$+ (\log^2(\xi) + (2\pi i N_2)^2 + 2\log(\xi) N_2)\log(u) +$$

$$\mathbb{f}(u) - 2(\log(\xi) + N_2)\mathrm{Li}_2(u) - \mathbb{f}(1-z)\Bigg)_{\bigstar}\Bigg|_{y_{\ell b}}^{y_{ub}}. \quad (10)$$

Here, $\star$ has the same significance as for Eq. 4, except discontinuities in $\mathbb{f}$ have to be corrected for in addition to those in ln and Li_2. The following definitions are repeated or made more explicit for convenience,

$$y = \frac{f+k}{f-a}, \quad (11)$$

$$\xi = \frac{a-f}{c-f}, \quad (12)$$

$$z = \left(\frac{f-a}{f-c}\right)\left(\frac{c+k}{a+k}\right) = 1 - \left(\frac{a-c}{f-c}\right)\left(\frac{f+k}{a+k}\right), \quad (13)$$

and

$$u = \frac{a-c}{a+k}. \quad (14)$$

Together, Eq. 4 and Eq. 10 enable computation of $\int_{x_{\ell b}}^{x_{ub}} \ln(a+x)\ln(c+x)/(f+x)dx$.

3 Real axis crossing points

The correction coefficient $N_+(\nu, w)$ defined in Eq. 3 is discontinuous where $\nu,\ w,$ or νw lie on the negative real axis, and similarly for $N_-(\nu, w)$ and $\nu,\ w,$ and ν/w. Listed below are the values of x at which such quantities $g(x) = \nu,\ w,\ \nu w,$ or ν/w relevant to $N_1,\ N_2,\ N_4,$ and N_5 cross the real axis:

$$\mathrm{Im}(y) = 0 \quad \text{at} \quad x_\alpha = \frac{\mathrm{Im}(af^*)}{\mathrm{Im}(f-a)}, \quad (15)$$

$$\mathrm{Im}(\xi y) = 0 \quad \text{at} \quad x_\beta = \frac{\mathrm{Im}(cf^*)}{\mathrm{Im}(f-c)}, \quad (16)$$

$$\mathrm{Im}(u) = 0 \quad \text{at} \quad x_\gamma = \frac{\mathrm{Im}(ac^*)}{\mathrm{Im}(c-a)}, \quad (17)$$

and

$$\mathrm{Im}(z) = 0$$

at points $x_{\delta,\lambda}$ defined by

$$x_{\delta,\lambda}^2 \mathrm{Im}[(c-f)^*(a-f)] + x_{\delta,\lambda}\mathrm{Im}[(c-f)^*(a-f)(a^*+c)] + \mathrm{Im}[(c-f)^*(a-f)ca^*] = 0 \quad (18)$$

Also, Table 1 indicates, for each correction coefficient N_n and for each function $g(x)$ relevant to N_n, the specific points x' from Eq. 15−18 satisfying $\mathrm{Im}(g(x')) = 0$. If $\mathrm{Re}(g(x')) < 0$ at such a point, then N_n is discontinuous at x'. Also, "n.a." indicates that a $g(x)$ is irrelevant for N_n, while "−" indicates that $g(x)$ does not cross the real axis as a function of x. The reason for the inclusion of $x_{\delta,\lambda}$ for N_2 can be appreciated from $\xi(1-u) = z$.

	ν	w	$\mathrm{Im}(\nu) = 0$	$\mathrm{Im}(w) = 0$	$\mathrm{Im}(\nu w) = 0$	$\mathrm{Im}(\nu/w) = 0$
N_1	$1-\xi y$	$1-y$	x_β	x_α	n.a.	x_δ, x_λ
N_2	ξ	$1-u$	−	x_γ	x_δ, x_λ	n.a.
N_4	$a-f$	$1-y$	−	x_α	−	n.a.
N_5	$c-f$	$1-\xi y$	−	x_β	−	n.a.

Table 1: Points x at which a given $N_n = N_\pm(\nu, w)$ may have a step discontinuity. See the text for details.

The branch cut crossing corrections can simplify for pure real a, c, or f. For example, if $\mathrm{Im}(a) = 0$ or $\mathrm{Im}(f) = 0$, then $x_\alpha = -a$ or $x_\alpha = -f$, respectively. In such cases, x_α becomes irrelevant as a branch cut crossing point because, if $x_{\ell b}$ and x_{ub} in Eq. 1 span x_α, then this integral is almost always undefined to begin with. Similar considerations apply for x_β and x_γ. The exceptions alluded to are integrals featuring parameters $a = 1+f$ or $c = 1+f$, which are well-defined if the integral bounds span $-f$. Such integrals are not currently supported by `lnln-pole-int`.

The treatment of $x_{\delta,\lambda}$ for pure real a, c, or f is slightly more complicated but has some similar features as for $x_{\alpha,\beta,\gamma}$. Namely, if $\mathrm{Im}(a) = 0$, then $x_\delta = -a$ and

$$x_\lambda = \frac{-\mathrm{Im}(c(f-a)(f-c)^*)}{\mathrm{Im}((f-a)(f-c)^*)},$$

where the labels δ and λ have been assigned arbitrarily. For the same reason as mentioned in the previous paragraph, the crossing point $x_\delta = -a$ can be ignored in this context. If instead $\mathrm{Im}(f) = 0$, then $x_\delta = -f$ and

$$x_\lambda = \frac{-\mathrm{Im}(a(c-f)(c-a)^*)}{\mathrm{Im}((c-f)(c-a)^*)}.$$

Lastly, if $\mathrm{Im}(c) = 0$, then $x_\delta = -c$ and

$$x_\lambda = \frac{-\mathrm{Im}(a(f-c)(f-a)^*)}{\mathrm{Im}((f-c)(f-a)^*)}.$$

From the expressions for x_λ above, it can be deduced that, if two of a, c, and f are pure real, then $-x_\lambda$ and $-x_\delta$ equal these two parameters. Again, in such contexts, the reasoning outlined above indicates that these crossing points can be ignored.

One more edge case must be considered for computation of $x_{\delta,\lambda}$. Namely, the quadratic coefficient $\mathrm{Im}[(c-f)^*(a-f)]$ in Eq. 18 can be zero. On a side note, when this is true, a, c, and f lie in a straight line in the complex plane. In this case, Eq. 18 only has a single solution x_δ, and $x_\alpha = x_\beta = x_\gamma = x_\delta$.

References

[1] Lewin, Leonard, *Polylogarithms and Associated Functions*. Elsevier, North Holland , New York, NY (1981)