1 QAP, the problem

QAP, and alternative descriptions, see 1

$$
\begin{aligned}
\min_X f(X) &= \operatorname{tr}(A^\top X B X^\top) \\
&= \operatorname{tr}(X^\top A^\top X B) \qquad\qquad x = \operatorname{vec}(X) \\
&= \langle \operatorname{vec}(X), \operatorname{vec}(A^\top X B)\rangle \\
&= \langle \operatorname{vec}(X), B^\top \otimes A^\top \cdot \operatorname{vec}(X)\rangle \\
&= x^\top (B^\top \otimes A^\top) x \\
\textbf{s.t.} \\
X &\in \Pi_n
\end{aligned}
$$

is the optimization problem on permutation matrices:

$$\Pi_n = \left\{X \in \mathbb{R}^{n\times n} \mid Xe = X^\top e = e, X_{ij} \in \{0,1\}\right\}$$

The convex hull of permutation matrices, the Birkhoff polytope, is defined:

$$D_n = \left\{X \in \mathbb{R}^{n\times n} \mid Xe = X^\top e = e, X \geq 0\right\}$$

for the constraints, also equivalently:

$$\operatorname{tr}(XX^\top) = \langle x, x\rangle_F = n, X \in D_n$$

1.1 Differentiation

$$
\begin{aligned}
&\nabla f = A^\top X B + A X B^\top \\
&\nabla \operatorname{tr}(XX^\top) = 2X
\end{aligned}
$$

2 $\mathcal{L}_p$ regularization

various form of regularized problem:

- $\mathcal{L}_0$: $f(X) + \sigma||X||_0$ is exact to the original problem for efficiently large σ 1, but the problem itself is still NP-hard.
- $\mathcal{L}_p$: also suggested by 1, good in the sense:
  - strongly concave and the global optimizer must be at vertices

- **local optimizer is a permutation matrix** if σ, ϵ satisfies some condition. Also, there is a lower bound for nonzero entries of the KKT points

$$\min_{X \in D_n} F_{\sigma,p,\epsilon}(X) := f(X) + \sigma\|X + \epsilon 1\|_p^p$$

- $\mathcal{L}_2$, and is based on the fact that $\Pi_n = D_n \bigcap \{X : \operatorname{tr}(XX^\top) = n\}$, 2

$$\min_X f(X) + \mu_0 \cdot \operatorname{tr}\left(XX^\top\right)$$

2.1 L_2

2.1.1 naive

$$\begin{aligned} & \operatorname{tr}(A^\top XBX^\top) + \mu_0 \cdot \operatorname{tr}(XX^\top) \\ =& x^\top(B^\top \otimes A^\top + \mu \cdot \mathbf{e}_{n\times n})x \end{aligned}$$

this implies a LD-like method. (but not exactly)

2.1.2 exact penalty

$$\begin{aligned} L_D &= f + \mu_0 \cdot |\operatorname{tr}(XX^T) - n| \\ &= f + \mu_0 \cdot n - \mu_0 \cdot \operatorname{tr}(XX^T) \end{aligned}$$

very likely to become a concave function, cannot be solved by conic solver.

Reference

[1] B. Jiang, Y.-F. Liu, and Z. Wen, "L_p-norm regularization algorithms for optimization over permutation matrices," *SIAM Journal on Optimization*, vol. 26, no. 4, pp. 2284–2313, 2016.

[2] Y. Xia, "An efficient continuation method for quadratic assignment problems," *Computers & Operations Research*, vol. 37, no. 6, pp. 1027–1032, 2010.