OR-3: Linear Regression with Outlier Selection

A QCP formulation

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0.1 The model

We use the usual notations in a linear regression problem:

0.1.1 Notation

- X - data matrix of n samples, where $X = \begin{bmatrix} x_1^T \\ ... \\ x_n^T \end{bmatrix}, x_i \in \mathbb{R}^m, i = 1, ..., N$
- y - response

0.1.2 Decision

Now we have the regression model:

$$\hat{y} = \beta^T x$$

- β - model coefficients, $\beta \in \mathbb{R}^m$
- q - $0-1$ decision on whether to keep the sample (else classified as one of the **outliers**)
- r - residue of the estimates: $r_i = |y_i - \hat{y}_i|$, or $r \geq ||y - \hat{y}||$, where $||.||$ is the $\mathscr{L}_1$ norm

The model minimizes sum of "absolute selected loss":

$$\begin{aligned}\min_{q,r,\beta} L_{\mathbf{abs}} &= \sum_i r_i \cdot q_i \\ &= \begin{bmatrix} r \\ q \end{bmatrix}^T \begin{bmatrix} 0 & \frac{1}{2}\mathbf{I}_n \\ \frac{1}{2}\mathbf{I}_n & 0 \end{bmatrix} \begin{bmatrix} r \\ q \end{bmatrix}\end{aligned}$$

To make the loss function semi-definite, let:

$$\mathbf{Q} = \lambda \cdot \mathbf{I}_n + \begin{bmatrix} 0 & \frac{1}{2}\mathbf{I}_n \\ \frac{1}{2}\mathbf{I}_n & 0 \end{bmatrix} \in \mathscr{S}_{++}$$

whiling setting scaling parameter λ properly (say 1). We have a binary quadratic programming model

$$\begin{aligned}&\min_{q,r,\beta} L = \begin{bmatrix} r \\ q \end{bmatrix}^T \mathbf{Q} \begin{bmatrix} r \\ q \end{bmatrix} \\ &s.t. \\ &\quad r \geq ||y - X\beta||_1 \\ &\quad \sum_{i=1}^{n} q_i \geq n_0 \qquad q \in \mathscr{B}^n, n_0 \leq n\end{aligned}$$

The second set of constraints is the regularization on number of selections, since it's monotonically better to **unselect** a sample, a lower bound n_0 is needed.

0.2 Quick results

We now choose n_0 experimentally. The computations are done in the following environment.

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- julia 1.4
- gurobi 9.0
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0.2.1 Remark

- Gurobi can do the objective directly as above.
- If you are using Mosek, pls. translate into a conic formulation.

The chart below is the computational results on minimum number of selections vesus R^2