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Intro

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1 Preface

This notes is written for basic knowledge on Interior Point Method (IPM). Also see [2], [1]. We begin with Linear Programming (LP), then move onto advanced problem classes.

2 Interior Point Method for Linear Programming

2.1 Elements

We inspect primal dual form of linear programming or linear optimization problems.

$$\begin{aligned} \text{Minimize} &: c^T x \\ \text{s.t. } & Ax = b, x \geq 0 \end{aligned} \tag{1}$$

$$\begin{aligned} \text{Maximize} &: b^T y \\ \text{s.t. } & A^T y + s = c, s \geq 0 \end{aligned} \tag{2}$$

and the following remarks,

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- Feasible region $\mathscr{F}_P, \mathscr{F}_D$ for primal dual respectively. We let $F^\circ_{(\cdot)}$ be the relative interior. Strong duality holds if F°_P, F°_P are nonempty.
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2.1.1 The analytic center

The **analytic center, AC** is defined as the minimizer of **potential function** on a non-empty set Ω.

If Ω is defined (as usual) in form $\Omega = \{x : g(x) \geq 0\}$, assume $\Omega^\circ \neq \emptyset$, the potential function / barrier is,

$$\psi(x) = -\log g^T e \tag{3}$$

$$x \circ v = e, \quad Ax = b \tag{4}$$

Example 1 ψ on half space. For the feasible set $\Omega_y = \{y : c - A^T y \geq 0\}$

$$\begin{aligned} \psi(y) &= -\log s^T e \\ \partial\psi/\partial y = 0 &\Rightarrow -\frac{A}{c - A^T y} = 0 \\ \text{Alternatively} & \\ c - A^T y &= s \\ s \circ u &= e \\ Au &= 0 \end{aligned} \tag{5}$$

For the case with extra equalities (always active) $\Omega_y = \{y : c - A^T y \geq 0\} \bigcap \{y : Ay = b\}$, since the analytic center also lies in the original space, we have y^a, s^a satisfies (5) + $\{y : Ay = b\}$.

2.1.2 The central path

Primal notice the primal potential function and the barrier problem, $\forall \mu \geq 0$

$$\begin{aligned} \text{Minimize} : \psi(x) &= c^T x - \mu \log x^T e \\ \text{s.t. } Ax &= b, x > 0 \end{aligned} \tag{6}$$

The first-order condition,

$$\begin{aligned} x \circ s &= \mu e \\ Ax &= b \\ A^T y + s &= c \end{aligned} \tag{7}$$

depicts the curvature parameterized by μ. The curvature is named as "the central path".

- $\mu \to 0$, $x(\mu)$ solves primal problem that approaches to AC of the optimal face: $\{x : c^T x = z^\star, Ax = b, x \geq 0\}$

- $\mu \to \infty$, $x(\mu)$ approaches to the analytic center (if bounded)

Now it's clear to see the central path is the "path" connecting the AC and optimal solution (if exists).

Dual similarly, we can define the (dual) central path by the log-barrier function,

$$\begin{aligned} \text{Minimize} &: \psi(x) = b^T y + \mu \log s^T e \\ \text{s.t. } & c - A^T y = s, s > 0 \end{aligned} \tag{8}$$

as well as the 1st order condition that coincides with (7).

Primal-Dual A combining view on central path is to see (7) as the parameterization simultaneously on (x, y, s).

- One see duality gap: $c^T x - b^T y = n\mu$
- $x \circ s = \mu e$ is sometimes referred to as perturbed complementary condition (**TCC**).

2.1.3 An overview of IPM

Now it becomes clear to us, the interior point method gradually shrinks $\mu \to 0$ hoping to approach the original problem.

- The outer loop produces the central path $(x(\mu), y(\mu), s(\mu))$ (or a subset of interest)
- The inner subproblem solves the central path system (7)

2.2 Timeline

References

1. Roos, C., Terlaky, T., Vial, J.P.: Interior point methods for linear optimization (2005). Publisher: Springer Science & Business Media
2. Ye, Y.: Interior point algorithms: theory and analysis, vol. 44. John Wiley & Sons (1997)